

# Theta Constants, Riemann Surfaces and the Modular Group

An Introduction with Applications to  
Uniformization Theorems, Partition Identities  
and Combinatorial Number Theory

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