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THETA LINE BUNDLES AND THE DETERMINANT OF THE HODGE BUNDLE

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Abstract. We give an expression of the determinant of the push forward of a symmetric line bundle on a complex abelian fibration, in terms of the pull back of the relative dualizing sheaf via the zero section.

0. Introduction

Let $\mathrm{f} : X \longrightarrow B$ be a fibration of abelian varieties with a zero section $s : B \longrightarrow X$. Let $\mathcal{L}$ be a symmetric line bundle on X, trivialized along the zero section, which defines a polarization of type $D = (d_1, \dots, d_g)$ on the fibration. A theorem of Faltings and Chai ([4], Ch. 1, Theorem 5.1) states that $8d^3 \det \mathrm{f}_*\mathcal{L} = -4d^4\, s^*\omega_{X/B}$, where $\omega_{X/B}$ is the relative dualizing sheaf of the fibration and $d := d_1 \cdots d_g$. In this note we show that in the complex analytic category, the above torsion factor can be improved. More specifically, we have

Theorem A. *Let $\mathrm{f} : X \longrightarrow B$ be a fibration of complex abelian varieties of relative dimension g, and let s be the zero section. Let $\mathcal{L}$ be a symmetric line bundle on X, trivialized along the zero section, which defines a polarization of type $D = (d_1, \dots, d_g)$, where $d_1 | \cdots | d_g$ are positive integers. Let $d = d_1 \cdots d_g$. Then $8 \det \mathrm{f}_*\mathcal{L} = -4d\, s^*\omega_{X/B}$, except when $3 | d_g$ and $\gcd(3, d_{g-1}) = 1$, in which case $24 \det \mathrm{f}_*\mathcal{L} = -12d\, s^*\omega_{X/B}$.*

Moreover, when $\mathcal{L}$ is totally symmetric (and therefore d is an even integer), we have

Theorem B. *Keeping the notation of Theorem A, assume in addition that $\mathcal{L}$ is a totally symmetric line bundle on X and that $g \geq 3$. Then $\det \mathrm{f}_*\mathcal{L} = -\frac{d}{2}\, s^*\omega_{X/B}$, except when $3 | d_g$ and $\gcd(3, d_{g-1}) = 1$, in which case $3 \det \mathrm{f}_*\mathcal{L} = -3\frac{d}{2}\, s^*\omega_{X/B}$.*

The theorems are proved by using a refinement of the theta transformation formula, see Propositions 2.1 and 2.2, in order to construct transition functions for $\det \mathrm{f}_*(\mathcal{L})$, see Lemma 3.1.

In the last section, we apply Theorem B to the case of the universal Jacobian variety $\mathrm{f}_{g-1} : \mathcal{J}^{g-1} \longrightarrow \mathcal{M}_g$, where $\mathcal{M}_g$ denotes the moduli space of smooth, irreducible curves of genus $g \geq 3$, without automorphisms. This is an abelian torsor which parametrizes line bundles of degree $g-1$ on the fibers of the universal curve $\psi : \mathcal{C} \longrightarrow \mathcal{M}_g$. On $\mathcal{J}^{g-1}$, there is a canonical theta divisor defined as the push forward of the universal symmetric product of degree $g-1$, via the Abel-Jacobi

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map. We denote by Θ the corresponding line bundle and let $\lambda = \det \psi_* \omega_{\mathcal{C}/\mathcal{M}_g}$ be the determinant of the Hodge bundle. We then have

Theorem C. *In the above notation,* $\det \mathrm{f}_{g-1\,*}(\Theta^{\otimes n}) = \frac{1}{2} n^g (n-1)\lambda$.

We also give an alternative way for proving Theorem C by utilizing special properties of the universal Jacobian varieties.

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1. Abelian varieties and theta functions

We recall in this section some standard theory for complex abelian varieties and theta functions. We follow the book by Lange and Birkenhake [5]. We denote by $X = V/\Lambda$ an abelian variety; V is a $\mathbb{C}$-vector space of dimension g and Λ a $2g$-lattice of maximal rank in V.

Line bundles on abelian varieties. A line bundle on X is determined, up to isomorphism, by an hermitian form H on V such that $\operatorname{Im} H(\Lambda, \Lambda) \subseteq \mathbb{Z}$, and by a semicharacter $\chi : \Lambda \longrightarrow \mathbb{C}_1$ ([5], Ch. 2, §2). We denote by $L(H, \chi)$ a line bundle, up to isomorphism, given by the above data. If $\phi : X' = V'/\Lambda' \longrightarrow X = V/\Lambda$ is a map of abelian varieties, we denote by $\phi_a : V' \longrightarrow V$ and $\phi_r : \Lambda' \longrightarrow \Lambda$ the analytic and the rational representation of ϕ respectively ([5], Ch. 1, §2). Given $L(H, \chi)$ on X, we have that $\phi^* L(H, \chi) = L(\phi_a^* H, \phi_r^* \chi)$.

Let $\mathcal{B}^s$ be the symplectic base of Λ w.r.t. which the alternating form $E := \operatorname{Im} H$ is represented by a matrix $\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$, where D, the polarization type, is an integral diagonal matrix with elements $d_1 | \dots | d_g$. Let Λ_1 (resp. Λ_2) be the lattice spanned by the first (resp. last) g vectors of $\mathcal{B}^s$. Then $\Lambda = \Lambda_1 \oplus \Lambda_2$, and we write $\lambda = \lambda_1 + \lambda_2$ for the corresponding decomposition of $\lambda \in \Lambda$. This induces a decomposition $V = V_1 \oplus V_2$, where $V_i = \Lambda_i \otimes \mathbb{R}$, which is called decomposition of V for H. If $v \in V$, we write $v = v_1 + v_2 \in V_1 \oplus V_2$. We define $\Lambda(H) := \{v \in V : \operatorname{Im} H(v, \Lambda) \subset \mathbb{Z}\}$. Then $\Lambda(H) = \Lambda(H)_1 \oplus \Lambda(H)_2$, where $\Lambda(H)_i := \Lambda(H) \cap V_i$.

We choose a decomposition of V for H. Then we can define a distinguished line bundle $L(H, \chi_0)$, by setting $\chi_0(\lambda) = e(\pi i E(\lambda_1, \lambda_2))$. A characteristic of a line bundle $L(H, \chi)$ is an element $c \in V$, unique up to translations by elements of $\Lambda(H)$, determined by the property $L(H, \chi) = T_c^* L(H, \chi_0)$, where T_c is the translation by c (see [5], Ch. 3, §1).

Period matrices. Let $\mathfrak{h}_g$ denote the Siegel upper half space of dimension g. We fix a polarization type D. A matrix $Z \in \mathfrak{h}_g$ determines a triple $(X_Z, H_Z, \mathcal{B}_Z^s)$, where $X_Z := \mathbb{C}^g/\Lambda_Z$ (with $\Lambda_Z := (Z, D)\mathbb{Z}^{2g}$) is an abelian variety, H_Z is an hermitian form of type D with matrix $(\operatorname{Im} Z)^{-1}$ w.r.t. the standard base of $\mathbb{C}^g$, and $\mathcal{B}^s$ is the symplectic base spanned by the column vectors of the matrix (Z, D); see [5], Ch. 8, §1.

Canonical-classical factor of automorphy. A factor of automorphy is a holomorphic map $f : \Lambda \times V \longrightarrow \mathbb{C}^\times$ satisfying

$$f(\lambda_1 + \lambda_2, v) = f(\lambda_1, \lambda_2 + v) f(\lambda_2, v).$$

Two factors of automorphy f and f' are called equivalent if

$$f'(\lambda, v) = f(\lambda, v) h(v) h(v + \lambda)^{-1}$$

for some holomorphic function $h : V \to \mathbb{C}^\times$. We use the notation $f' = f \star h$ for this situation.

Given an hermitian form H, we denote by B the symmetric form on V associated to H ([5], Ch. 3, Lemma 2.1). Given data (H, χ), we define by $a_{(H,\chi)}(\lambda, v) := \chi(\lambda)\, e(\pi H(v, \lambda) + \frac{\pi}{2} H(\lambda, \lambda))$ the canonical factor of automorphy and by $e_{(H,\chi)}(\lambda, v) := \chi(\lambda)\, e(\pi(H - B)(v, \lambda) + \frac{\pi}{2}(H - B)(\lambda, \lambda))$ the classical factor of automorphy. When the semicharacter is χ_0, we simplify the notation for the above factors of automorphy to a_H and e_H.

Canonical-classical theta functions. A theta function corresponding to a factor of automorphy f is a holomorphic function $\theta : V \longrightarrow \mathbb{C}$ satisfying the functional equation $\theta(\lambda + v) = f(\lambda, v)\theta(v)$. Theta functions corresponding to the canonical (resp. classical) factor of automorphy $a_{(H,\chi)}$ (resp. $e_{(H,\chi)}$) are called canonical (resp. classical) theta functions for $L(H, \chi)$. Let c be a characteristic of $L(H, \chi)$. We define

$$\begin{aligned} \theta^c(v) :=& e(-\pi H(v, c) - \frac{\pi}{2} H(c, c) + \frac{\pi}{2} B(v + c, v + c)) \\ & \cdot \sum_{\lambda_1 \in \Lambda_1} e(\pi(H - B)(v + c, \lambda_1) - \frac{\pi}{2}(H - B)(\lambda_1, \lambda_1)). \end{aligned}$$

We have the following ([5], Ch. 3, §§1 and 2):

i) θ^c is a canonical theta function and $\theta^c(v) = e(-\pi H(v, c) - \frac{\pi}{2} H(c, c))\theta(v + c)$, where $\theta := \theta^0$.

ii) Let $\theta^{c}_{\overline{w}}(v) := a_{(H,\chi)}(w, v)^{-1}\, \theta^c(v + w)$, where $\overline{w} \in K(H) := \Lambda(H)/\Lambda$. The set $\langle \theta^{c}_{\overline{w}} \colon \overline{w} \in K(H)_1 := \Lambda(H)_1/\Lambda_1 \rangle$ forms a base of the canonical theta functions.

Let $Z \in \mathfrak{h}_g$ and let D be a fixed polarization type. Let $X_Z := \mathbb{C}^g/\Lambda_Z$ be the abelian variety corresponding to Z and let $H = H_Z$. Given $v \in \mathbb{C}^g$, we can write uniquely $v = Zv^1 + v^2$, where $v^i \in \mathbb{R}^g$. If $\lambda \in \Lambda_Z$ then it can be written uniquely in the form $\lambda = Z\lambda^1 + \lambda^2$, where $\lambda^1 \in \mathbb{Z}^g$ and $\lambda^2 \in D\mathbb{Z}^g$. Let $L(H, \chi)$ be a line bundle on X_Z of characteristic c w.r.t. the natural decomposition of $\mathbb{C}^g$ determined by Z. We have the following (many of them can be found in [5], Ch. 8, §5; the rest is a straightforward calculation):

1. $H(v, w) = {}^t v\, (\mathrm{Im} Z)^{-1}\, \bar{w}$, $\quad B(v, w) = {}^t v\, (\mathrm{Im} Z)^{-1}\, w$.
   $(H - B)(v, w) = -2i\, {}^t v w^1$, $\quad E(v, w) = {}^t v^1 w^2 - {}^t v^2 w^1$.
2. $e_{(H,\chi)}(\lambda, v) = e(2\pi i\, ({}^t c^1 \lambda^2 - {}^t c^2 \lambda^1) - \pi i\, {}^t \lambda^1 Z \lambda^1 - 2\pi i\, {}^t v \lambda^1)$. Also,

$$a_{(H,\chi)}(\lambda, v) = e(\pi i\, {}^t \lambda^1 \lambda^2 + 2\pi i\, ({}^t c^1 \lambda^2 - {}^t c^2 \lambda^1) + \pi\, {}^t v (\mathrm{Im} Z)^{-1} \bar{\lambda} + \frac{\pi}{2}\, {}^t \lambda (\mathrm{Im} Z)^{-1} \bar{\lambda}).$$

   It is $e_{(H,\chi)} = a_{(H,\chi)} \star h$, where $h(v) = e(\frac{\pi}{2}\, {}^t v (\mathrm{Im} Z)^{-1} v)$.
3. Let $\mathbb{Z}_D$ denote the group $\mathbb{Z}_D := \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_g}$. Then $\Lambda(H)_1 = \{Zv^1 \mid v^1 \in D^{-1}\mathbb{Z}^g\}$, $\quad \Lambda(H)_2 = \{v^2 \mid v^2 \in \mathbb{Z}^g\}$ and $K(H)_1 \cong D^{-1}\mathbb{Z}_D$, $\quad K(H)_2 \cong \mathbb{Z}_D$.

4. Let $c = Zc^1 + c^2$. Then

$$\theta\left[\begin{array}{c} c^1 \\ c^2 \end{array}\right](v,Z) := e(-\frac{\pi}{2}B(v,v) + \pi i\, {}^t c^1 c^2)\, \theta^c(v)$$

is a classical theta function and

$$\theta\left[\begin{array}{c} c^1 \\ c^2 \end{array}\right](v,Z) = \sum_{\lambda^1 \in \mathbb{Z}^g} e\left(\pi i\, {}^t(\lambda^1 + c^1)Z(\lambda^1 + c^1) + 2\pi i\, {}^t(v + c^2)(\lambda^1 + c^1)\right).$$

The set

$$\left\langle \theta\left[\begin{array}{c} c^1 + D^{-1}m \\ c^2 \end{array}\right], m \in \mathbb{Z}_D \right\rangle$$

forms a base of the classical theta functions.

5. Let $c = Zc^1 + c^2 \in \mathbb{C}^g$, $w = Zw^1 + w^2 \in \Lambda(H)$ and $Zs^1 \in \Lambda(H)_1$. Then:
   a) $\theta\left[\begin{array}{c} c^1 + w^1 \\ c^2 \end{array}\right](v,Z) = e(-\frac{\pi}{2}B(v,v) + \pi i\, {}^t c^1 c^2)\, \theta^{c}_{\overline{w}}(v)$.
   b) $\theta\left[\begin{array}{c} c^1 + w^1 \\ c^2 + w^2 \end{array}\right](v,Z) = e(2\pi i\, {}^t(c^1 + w^1)w^2)\, \theta\left[\begin{array}{c} c^1 + w^1 \\ c^2 \end{array}\right](v,Z)$.
   c) $\theta^{c}_{\overline{Zw^1 + w^2}}(v) = \theta^{c}_{\overline{Zw^1}}(v)$.
   d) $\theta^{c+w}(v) = e(-\pi i\, {}^t c^2 w^1 + \pi i\, {}^t(c^1 + w^1)w^2)\, \theta^{c}_{\overline{w}}(v)$.
   e) $\theta^{c+Zs^1}_{\overline{Zw^1}}(v) = e(-\pi i\, {}^t s^1 c^2)\, \theta^{c}_{\overline{Z(w^1+s^1)}}(v)$.

Action of the symplectic group. Let D be a polarization type and let $A_D := \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$ and $I_D := \begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix}$. We set $\Gamma_D := \{R \in M_{2g}(\mathbb{Z}),\ RA_D\, {}^tR = A_D\}$ and $G_D := \{M \in \mathrm{Sp}_{2g}(\mathbb{Q}),\ M = I_D^{-1} R I_D, \text{ for some } R \in \Gamma_D\}$. If $R = \begin{pmatrix} A & B \\ \Gamma & \Delta \end{pmatrix} \in \Gamma_D$ and $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G_D$, then $\alpha = A$, $\beta = BD$, $\gamma = D^{-1}\Gamma$, $\delta = D^{-1}\Delta D$. We have $AD\,{}^t\Delta - BD\,{}^t\Gamma = D$. Also, the matrices $\Gamma D\,{}^t\Delta$, $AD\,{}^tB$, ${}^tAD^{-1}\Gamma$, ${}^t\Delta D^{-1}B$ are symmetric and the matrices $D\,{}^tAD^{-1}$, $D\,{}^tBD^{-1}$, $D\,{}^t\Gamma D^{-1}$, $D\,{}^t\Delta D^{-1}$ are integral.

The group G_D acts on $\mathfrak{h}_g$ by $M(Z) := (\alpha Z + \beta)(\gamma Z + \delta)^{-1}$ ([5], Ch. 8, §1). Two abelian varieties X_Z and $X_{Z'}$ of type D are isomorphic if $Z' = M(Z)$. The isomorphism is given by $\phi(M) : X_Z \longrightarrow X_{M(Z)}$, so that the corresponding map $\phi(M)_r : \Lambda_Z \longrightarrow \Lambda_{M(Z)}$ has matrix $R = {}^tM^{-1}$ w.r.t. the canonical symplectic bases defined by Z and $M(Z)$. Let $j_Z : \mathbb{R}^{2g} \longrightarrow \mathbb{C}^g$ be the isomorphism given by $x \mapsto (Z,1)x$. We have the following diagram ([5], Ch. 8, §8):

(1)
$$\begin{array}{ccccc} \mathbb{R}^{2g} & \xrightarrow{j_Z} & \mathbb{C}^g & \longrightarrow & X_Z \\ \downarrow{\scriptstyle \phi(M)_r} & & \downarrow{\scriptstyle \phi(M)_a} & & \downarrow{\scriptstyle \phi(M)} \\ \mathbb{R}^{2g} & \xrightarrow{j_{M(Z)}} & \mathbb{C}^g & \longrightarrow & X_{M(Z)} \end{array}$$

The map $\phi(M)_a : \mathbb{C}^g \longrightarrow \mathbb{C}^g$ has matrix A^{-1}, where $A = {}^t(\gamma Z + \delta)$, w.r.t. the standard base of $\mathbb{C}^g$. Moreover, $\phi(M)_a^* H_{M(Z)} = H_Z$. We define $M_Z(v) := A^{-1}v$ ($= \phi(M)_a(v)$).

Factors of automorphy and line bundles. Sections and theta functions. A factor of automorphy $f : \Lambda \times V \to \mathbb{C}^\times$ defines an action of Λ on $V \times \mathbb{C}$ given by $\lambda(v, z) := (v + \lambda, f(\lambda, v)z)$. The quotient of $V \times \mathbb{C}$ by this action defines a line bundle L on X, the elements of which we denote by $[v, z]$. If $f' = f \star h$, then for the corresponding line bundles L and L' there exists a canonical isomorphism $\Phi_h : L \longrightarrow L'$ given by $[v, z] \mapsto [v, h(v)^{-1}z]$. Given a map of abelian varieties $\phi : X' \longrightarrow X$, we define $\phi^* f := (\phi_r \times \phi_a)^* f$, which is a factor of automorphy for X'. Then $\phi^* L$ is the line bundle on X' corresponding to $\phi^* f$. If θ is a theta function for f, then $\phi_a^* \theta$ (or $\phi^* \theta$ in a simplified notation) is a theta function for $\phi^* f$.

Sections of the line bundle L correspond to theta functions $\theta : V \to \mathbb{C}$ satisfying the functional equation $\theta(\lambda + v) = f(\lambda, v)\theta(v)$. The relation is the following. Given a section s of L, let $s(x) = [v_s(x), z_s(x)]$. We then define $\theta_s(v) := f(v - v_s(x), v_s(x))\, z_s(x)$. Conversely, given a theta function θ for f, we define $s(x) := [v(x), \theta(v(x))]$, where $v(x)$ is an arbitrary vector which lies over x. If $\phi : X' \longrightarrow X$ is a map as above and $s \in H^0(X, L)$ is a section corresponding to θ, then the section $\phi^* s \in H^0(X', \phi^* L)$ corresponds to $\phi^* \theta$. Suppose $f' = f \star h$. Then given a section $s' \in H^0(X, L')$ corresponding to $\theta_{s'}$, we have that $s := \Phi_h^* s' \in H^0(X, L)$ corresponds to $\theta_s := h(v)\theta_{f'}(v)$.

2. Theta transformation formula

Let $Z \in \mathfrak{h}_g$, and let $L(H_Z, \chi)$ be a line bundle of characteristic c on the abelian variety X_Z. Let $M \in G_D$ and define $Z' := M(Z)$ as in Section 1. Let $\psi = \psi(M) : X_{Z'} \longrightarrow X_Z$ be the inverse of the map $\phi = \phi(M) : X_Z \longrightarrow X_{Z'}$. The line bundle $\psi^* L(H_Z, \chi)$ has type $\psi^* H_Z = H_{Z'}$, semicharacter $\chi' = \psi^* \chi$ and characteristic $M[c]$, with $M[c]^1 = \delta c^1 - \gamma c^2 + \frac{1}{2}(D\gamma\, {}^t\delta)_0$ and $M[c]^2 = -\beta c^1 + \alpha c^2 + \frac{1}{2}(\alpha\, {}^t\beta)_0$ (see [5], Ch. 8, Lemma 4.1, where there is an unfortunate omission of D in the expression of $M[c]^1$). (The $(\ )_0$ stands for the diagonal vector.)

Lemma 2.1. *We have that*

$$\psi^* e_{(H_Z, \chi)} = e_{(H_{Z'}, \chi')} \star h',$$

where $h'(v) = e(\pi i\, {}^t v(\gamma Z + \delta)^{-1}\gamma v)$. *Also,*

$$\phi^* e_{(H_{Z'}, \chi')} = e_{(H_Z, \chi)} \star h,$$

where $h(v) = e(-\pi i\, {}^t v(\gamma Z + \delta)^{-1}\gamma v)$.

Proof. We have that $a_{(H_Z,\chi)} = e_{(H_Z,\chi)} \star h_1$, where $h_1(v) = e(-\frac{\pi}{2}\, {}^t v(\mathrm{Im} Z)^{-1} v)$ (see item 2 in Section 1). Since $\psi^* a_{(H_Z,\chi)} = a_{(H_{Z'},\chi')}$ and $a_{(H_{Z'},\chi')} = e_{(H_{Z'},\chi')} \star h_1'$, where $h_1'(v') = e(-\frac{\pi}{2}\, {}^t v'(\mathrm{Im} Z')^{-1} v')$, by applying ψ^* we get that $h'(v) = \psi^* h_1(v)^{-1}\, h_1'(v')$, i.e., $h'(v) = e(\frac{\pi}{2}\, {}^t v(\mathrm{Im} Z)^{-1} v - \frac{\pi}{2}\, {}^t v'(\mathrm{Im} Z')^{-1} v')$, where $v' := \phi_a(v)$. A straightforward calculation gives the above form for h'. To prove the second formula, we apply ϕ^* to the first one. $\square$

The tuple $B^Z := \langle \theta^{c}_{\overline{ZD^{-1}m}}(v)\,;\, m \in \mathbb{Z}_D \rangle$ forms a base of the canonical theta functions for $L(H_Z, \chi)$ and the tuple $B^{Z'} := \langle \theta^{M[c]}_{\overline{Z'D^{-1}n}}(v')\,;\, n \in \mathbb{Z}_D \rangle$ forms a base of the canonical theta functions for $L(H_{Z'}, \chi')$. On the other hand, the tuple $\psi^* B^Z := \langle \psi_a^* \theta^{c}_{\overline{D^{-1}m}}(v)\,;\, m \in \mathbb{Z}_D \rangle$ also forms a base of the canonical theta functions for $L(H_{Z'}, \chi')$, since $\psi^* L(H_Z, \chi) = L(H_{Z'}, \chi')$ and $\psi^* a_{(H_Z,\chi)} = a_{(H_{Z'},\chi')}$.

Proposition 2.1. *Keeping the above notation, assume that the characteristic* $c \in \frac{1}{2}\Lambda(H_Z)$. *Then the matrix* C, *for which* $\psi^* B^Z = C\, B^{Z'}$, *is of the form* $C = (\det(\gamma Z + \delta))^{-\frac{1}{2}}\ C(M)$, *where* $C(M)$ *depends on* M *and* $\det C(M) = \zeta_8$, *except when* $3|d_g$ *and* $(d_{g-1}, 3) = 1$, *in which case we have* $\det C(M) = \zeta_{24}$ *(by* ζ_m *we denote an* m*-root of unity).*

Proof. Let $G_D^{\rm int} := G_D \cap \mathrm{Sp}_{2g}(\mathbb{Z})$. A matrix M belongs to $G_D^{\rm int}$ if

$$M = \begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix}^{-1} R \begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix},$$

where $R = \begin{pmatrix} A & B \\ \Gamma & \Delta \end{pmatrix} \in \Gamma_D$ and $\Gamma = D\Gamma_1,\ \Gamma_1 \in M_g(\mathbb{Z})$. Therefore, define $\Gamma_D^{\rm int} := \{R = \begin{pmatrix} A & B \\ \Gamma & \Delta \end{pmatrix} \in \Gamma_D, \quad \text{where} \quad \Gamma = D\Gamma_1,\ \Gamma_1 \in M_g(\mathbb{Z})\}$. We have the following lemma:

Lemma 2.2. *The group* Γ_D *is generated by* $\Gamma_D^{\rm int}$ *and the matrix* $J := \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, *and so the group* G_D *is generated by* $G_D^{\rm int}$ *and the matrix* $\begin{pmatrix} 0 & -D \\ D^{-1} & 0 \end{pmatrix}$.

Proof. We use results from [3]. Let $K(D) = D^{-1}\mathbb{Z}_D \oplus \mathbb{Z}_D$. A matrix $R \in \Gamma_D$ acts on $K(D)$ by multiplication by $\begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix} {}^t R \begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix}^{-1}$. By identifying $K(D)$ with $\mathbb{Z}_D \oplus \mathbb{Z}_D$ via the isomorphism given by the matrix $\begin{pmatrix} D & 0 \\ 0 & I \end{pmatrix}$, the action of $R \in \Gamma_D$ on $K(D)$ is given by multiplication by $\bar{D}\, {}^t R \bar{D}^{-1}$, where $\bar{D} := \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}$. One can define on $K(D)$ an alternating form e^D ([5], [3]), and the above action becomes a symplectic action. Let $\mathrm{Sp}(D)$ be the symplectic group of $K(D)$ with respect to e^D. We then have an exact sequence $0 \longrightarrow \Gamma_D(D) \longrightarrow \Gamma_D \xrightarrow{\pi} \mathrm{Sp}(D) \longrightarrow 0$, where $\pi(R) := \bar{D}\, {}^t R \bar{D}^{-1}$ and $\Gamma_D(D) := \{R \in \Gamma_D \mid R = I + \bar{D}A,\ A \in M_{2g}(\mathbb{Z})\}$. Note that $\Gamma_D(D) \subset \Gamma_D^{\rm int}$. It suffices therefore to show that every element of $\mathrm{Sp}(D)$ has a lift to an element of Γ_D which is a product of the matrix J and elements of $\Gamma_D^{\rm int}$.

Following the notation of [3], we have that $A \in L_D$ if $\bar{D}\, {}^t A \bar{D}^{-1} \in \Gamma_D$, where L_D is defined in Section 2 of [3]. In [3], Theorem 2, it is shown that a matrix $A \in \mathrm{Sp}(D)$ has a lift $\tilde{A} \in L_D$ which satisfies a relation

$$\begin{pmatrix} I & 0 \\ c_1 & I \end{pmatrix} \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \begin{pmatrix} a & Dy \\ -D & d \end{pmatrix} \begin{pmatrix} I & b_1 \\ 0 & I \end{pmatrix} \tilde{A} = \begin{pmatrix} I & b_2 \\ 0 & I \end{pmatrix},$$

where y is an integral diagonal matrix and all the matrices belong to L_D. The inverse of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in L_D belongs to L_D and is given by

$$\bar{D} \begin{pmatrix} {}^t d & -{}^t b \\ -{}^t c & {}^t a \end{pmatrix} \bar{D}^{-1}.$$

Therefore A has a lift R in Γ_D which is given by $R = \bar{D}\, {}^t \tilde{A} \bar{D}^{-1}$. But $\begin{pmatrix} I & 0 \\ c & I \end{pmatrix} = J \begin{pmatrix} -I & c \\ 0 & -I \end{pmatrix} J$; hence $R = (JN_1J)N_2N_3N_4(JN_5J)$, where $N_i \in \Gamma_D^{\rm int}$. ∎

As in [7], Ch. II, §5, we can rewrite the formula we want to prove as

$$\begin{aligned}&\langle \theta^{c}_{ZD^{-1}m}(v,Z)\,;\, m\in\mathbb{Z}_D\rangle \sqrt{dv_1\wedge\ldots\wedge dv_g}\\ &\qquad = C(M)\langle \theta^{M[c]}_{Z'D^{-1}n}(v',Z')\,;\, n\in\mathbb{Z}_D\rangle \sqrt{dv'_1\wedge\ldots\wedge dv'_g}\,,\end{aligned}$$

where $v' = {}^t(\gamma Z+\delta)^{-1}v$ and $Z' = (\alpha Z+\beta)(\gamma Z+\delta)^{-1}$. Note that if $c\in\frac{1}{2}\Lambda(H_Z)$, then $M[c]\in\frac{1}{2}\Lambda(H_{Z'})$. We observe that if the formula holds for M_1, $M_2\in G_D$, then it also holds for M_1M_2. It suffices therefore to verify the proposition for the generators. We express the relation $\psi^*B^Z = C\,B^{Z'}$ in terms of classical theta functions, and, by using Lemma 2.1, we get

$$\begin{aligned}&\left\langle \theta\begin{bmatrix} c^1+D^{-1}m\\ c^2\end{bmatrix}(v,Z)\,;\, m\in\mathbb{Z}_D\right\rangle = e(-\pi i\,{}^tM[c]^1M[c]^2+\pi i\,{}^tc^1c^2)\\ &\qquad\cdot e(-\pi i\,{}^tv(\gamma Z+\delta)^{-1}\gamma v)(\det(\gamma Z+\delta))^{-\frac12}\\ &\qquad\cdot C(M)\left\langle\theta\begin{bmatrix} M[c]^1+D^{-1}n\\ M[c]^2\end{bmatrix}(v',Z')\,;\, n\in\mathbb{Z}_D\right\rangle.\end{aligned}\tag{2}$$

Matrices of the form $\begin{pmatrix}0 & -D\\ D^{-1} & 0\end{pmatrix}$. In this case $e(-\pi i\,{}^tv(\gamma Z+\delta)^{-1}\gamma v) = e(-\pi i\,{}^tvZ^{-1}v)$ and $\det(\gamma Z+\delta)=\frac{\det Z}{d}$. We also have $v' = DZ^{-1}v$, $Z'=-DZ^{-1}D$ and $M[c]^1=-D^{-1}c^2$, $M[c]^2 = Dc^1$. Relation (2) in this case becomes

$$\begin{aligned}&\left\langle \theta\begin{bmatrix} c^1+D^{-1}m\\ c^2\end{bmatrix}(v,Z)\,;\, m\in\mathbb{Z}_D\right\rangle = e(-\pi i\,{}^tvZ^{-1}v)e(2\pi i\,{}^tc^1c^2)\\ &\qquad\cdot\left(\frac{\det Z}{d}\right)^{-\frac12}\ C(M)\left\langle\theta\begin{bmatrix} -D^{-1}c^2+D^{-1}n\\ Dc^1\end{bmatrix}(v',Z')\,;\, n\in\mathbb{Z}_D\right\rangle.\end{aligned}\tag{3}$$

As in [7], we apply Fourier transform. Write

$$\theta\begin{bmatrix} c^1+D^{-1}m\\ c^2\end{bmatrix}(v,Z) = \sum_{\lambda\in\mathbb{Z}^g} f(\lambda),$$

where

$$\begin{aligned}f(x) := e(\pi i\,{}^t(x+c^1+D^{-1}m)Z(x+c^1+D^{-1}m)&\\ +2\pi i\,{}^t(v+c^2)(x+c^1+D^{-1}m)).&\end{aligned}$$

Let $\hat f(x) := \int_{\mathbb{R}^g} f(x)\,e(2\pi i\,{}^tx\lambda)\,dx$. We then have

$$\theta\begin{bmatrix} c^1+D^{-1}m\\ c^2\end{bmatrix}(v,Z) = \sum_{\lambda\in\mathbb{Z}^g}\hat f(\lambda).$$

Using [7], Ch. II, Lemma 5.8, by substituting $x' = x+c^1+D^{-1}m$ we get

$$\hat f(\lambda) = e(-2\pi i\,{}^t(c^1+D^{-1}m)\lambda)\ (\det\frac{Z}{i})^{-\frac12}\ e(-\pi i\,{}^t(v+c^2+\lambda)Z^{-1}(v+c^2+\lambda)).$$

Therefore

$$\begin{aligned}&\theta\begin{bmatrix} c^1+D^{-1}m\\ c^2\end{bmatrix}(v,Z) = e(-\pi i\,{}^t(v+c^2)Z^{-1}(v+c^2))\ (\det\frac{Z}{i})^{-\frac12}\\ &\quad\cdot\sum_{\lambda\in\mathbb{Z}^g} e(-2\pi i\,{}^t(c^1+D^{-1}m)\lambda-2\pi i\,{}^t(v+c^2)Z^{-1}\lambda-\pi i\,{}^t\lambda Z^{-1}\lambda).\end{aligned}$$

By substituting $-\lambda = Dk + n$, $n \in \mathbb{Z}_D$, we can rewrite the last sum as

$$\sum_{n\in\mathbb{Z}_D}\sum_{k\in\mathbb{Z}^g} e\left(2\pi i\,{}^t(c^1 + D^{-1}m)(Dk+n) + 2\pi i\,{}^t(v+c^2)Z^{-1}(Dk+n)\right.$$
$$\left. -\pi i\,{}^t(Dk+n)Z^{-1}(Dk+n)\right).$$

A straightforward calculation yields

$$\theta\begin{bmatrix} c^1 + D^{-1}m \\ c^2 \end{bmatrix}(v,Z) = e(-\pi i\,{}^t v Z^{-1} v)\,(\det\frac{Z}{i})^{-\frac{1}{2}}\, e(2\pi i\,{}^t c^1 c^2)$$
$$\cdot \sum_{n\in\mathbb{Z}_D} e(2\pi i\,{}^t m D^{-1} n)\,\theta\begin{bmatrix} -D^{-1}c^2 + D^{-1}n \\ Dc^1 \end{bmatrix}(v',Z'). \tag{4}$$

Comparing relations (3) and (4), we deduce that the matrix $C(M)$ we are asking for has m, n entry equal to $(\frac{d}{i^g})^{-\frac{1}{2}}\, e(2\pi i\,{}^t m D^{-1} n)$. Let $d := \det D$. The matrix $A = (a_{mn})_{m,n\in\mathbb{Z}_D}$, where $a_{mn} := e(2\pi i\,{}^t m D^{-1} n)$, has determinant $\det A = \zeta_4 d^{\frac{d}{2}}$. To see this, we denote by $\mathbb{C}[\mathbb{Z}_{d_i}]$ the $\mathbb{C}$-vector space of dimension d_i "corresponding" to the group $\mathbb{Z}_{d_i}$. Fix the natural base $\langle m,\ m \in \mathbb{Z}_{d_i}\rangle$ and define the map $\phi_i : \mathbb{C}[\mathbb{Z}_{d_i}] \longrightarrow \mathbb{C}[\mathbb{Z}_{d_i}]$ by $\phi_i(m) := \sum_{n\in\mathbb{Z}_{d_i}} e(2\pi i n d_i^{-1} m) n$. Let C_i be the matrix corresponding to ϕ_i. Then $\det C_i = \zeta_4 d_i^{d_i/2}$. Indeed, it is easy to see that $\det(C_i^2) = \pm d_i^{d_i}$. Observe now that A is the matrix corresponding to the tensor product of the maps ϕ_i and so, $\det A = \det C_1^{d/d_1} \cdots \det C_g^{d/d_g} = \zeta_4 d^{d/2}$. To conclude this case, observe that $(\frac{d}{i^g})^{-\frac{d}{2}} = \zeta_8 d^{-\frac{d}{2}}$ and therefore $\det C(M) = \zeta_8$.

Matrices in $G_D^{\mathbf{int}}$. Let $M \in G_D^{\mathrm{int}}$. Then M corresponds to an isomorphism $\psi : X_{Z'} \longrightarrow X_Z$ which is a lift of an isomorphism of principally polarized abelian varieties. In this case, the usual theta transformation formula holds ([5], Ch. 8, §6). Let $a = c + Zw^1$, $c \in \frac{1}{2}\Lambda(H)$, $w^1 = D^{-1}w_1 \in D^{-1}\mathbb{Z}^g$. We denote by $M[\ \ ]_I$ the transformation of the characteristic corresponding to the principal polarization $D = I$. Note that $M[c]_I = M[c] + Z's^1$, where $s^1 := -\frac{D-I}{2}({}^t\gamma\delta)_0$, and so $s^1 = D^{-1}s_1$, with $s_1 \in \mathbb{Z}^g$. We have the following facts ([5], Ch. 8, §§4 and 6).

1. $\psi^*\theta^a(v,Z) = C(Z,M,a)\,\theta^{M[a]_I}(v',Z')$.
2. $C(Z,M,a) = C(Z,M,0)\; e(\pi i E(M[0]_I, A^{-1}a))$, where $A = {}^t(\gamma Z + \delta)$.
3. $C(Z,M,0) = k(M)\; e(\pi i\,{}^t M[0]_I^1 M[0]_I^2)\,\det(\gamma Z+\delta)^{-\frac{1}{2}}$, where $k(M)$ is a ζ_8.

Note that $M[a]_I^1 = M[c]_I^1 + \delta w^1$ and $M[a]_I^2 = M[c]_I^2 - \beta w^1$. The above formulae and the formulae in Section 1 yield that item 1 above becomes

$$e(-\pi i\,{}^t c^2 w^1)\psi^*\theta_{\frac{c}{Zw^1}}(v,Z) = C(Z,M,0)\,e(\pi i E(M[0]_I, A^{-1}a))$$
$$\cdot\, e(-\pi i\,{}^t(\delta w^1)M[c]_I^2 + \pi i\,{}^t M[c]_I^1(-\beta w^1) + \pi i\,{}^t(\delta w^1)(-\beta w^1))\,\theta_{\frac{M[c]_I}{Z'\delta w^1}}(v',Z').$$

Item 5e) of Section 1 gives

$$\theta_{\frac{M[c]_I}{Z'\delta w^1}}(v',Z') = e(-\pi i\,{}^t s^1 M[c]^2)\,\theta_{\frac{M[c]}{Z'(\delta w^1 + s^1)}}(v',Z').$$

Also,

$$M[0]^1_I=\frac{1}{2}(\gamma\,{}^t\delta)_0\in\frac{1}{2}\mathbb{Z}^g,\qquad M[0]^2_I=\frac{1}{2}(\alpha\,{}^t\beta)_0\in\frac{1}{2}\mathbb{Z}^g,$$
$$(A^{-1}a)^1=\delta(c^1+w^1)-\gamma c^2,\qquad (A^{-1}a)^2=-\beta(c^1+w^1)+ac^2,$$
$$M[c]^1_I=\delta c^1-\gamma c^2+\frac{1}{2}(\gamma\,{}^t\delta)_0\in\frac{D^{-1}}{2}\mathbb{Z}^g,$$
$$M[c]^2_I=-\beta c^1+ac^2+\frac{1}{2}(\alpha\,{}^t\beta)_0\in\frac{1}{2}\mathbb{Z}^g.$$

We thus get

(5) $\psi^*\theta^{c}_{\overline{ZD^{-1}w_1}}(v,Z)$
$$=k(M)e(\pi ik)(\pi i\lambda w^1)e(-\pi i\,{}^tw^1\,{}^t\delta\beta w^1)\det(\gamma Z+\delta)^{-\frac{1}{2}}\theta^{M[c]}_{\overline{Z'D^{-1}(\Delta w_1+s_1)}}(v',Z'),$$

where $k={}^tM[0]^1_IM[0]^2_I+{}^tM[0]^1_I(-\beta c^1+ac^2)-{}^tM[0]^2_I(\delta c^1-\gamma c^2)-{}^ts^1M[c]^2$ and $\lambda=-{}^tM[0]^1_I\beta-{}^tM[0]^2_I\delta-{}^tM[c]^2_I\delta-{}^tM[c]^1_I\beta+{}^tc^2$. Observe now that $k\in\frac{1}{4d_g}\mathbb{Z}$ and $\lambda\in\frac{1}{2}\mathbb{Z}$.

Note that when $\gamma\in M_g(\mathbb{Z})$, i.e. $\Gamma=D\Gamma_1$ for some integral matrix Γ_1, the matrix Δ acts as a permutation on $\mathbb{Z}_D$. Indeed, the relation $\Delta D\,{}^tA-\Gamma D\,{}^tB=D$ implies $\Delta(D\,{}^tAD^{-1})=I+\Gamma(D\,{}^tBD^{-1})$ i.e. $\Delta(D\,{}^tAD^{-1})=I+D\Gamma_1(D\,{}^tBD^{-1})$ and so $\Delta A_1=I+DM$ for some integral matrices A_1,M. Hence, Δ induces an epimorphism on $\mathbb{Z}_D$ and so an automorphism.

Relation (5) implies that the matrix $C(M)$ of the proposition has in the $w_1,\Delta w_1+s_1$-entry the value $k(M)e(\pi ik)e(\pi i\lambda D^{-1}w_1)e(-\pi i\,{}^tw_1\,{}^t\Delta D^{-1}Bw_1)$ and any other entries in the w_1 row and Δw_1+s_1 column are zero. To find its determinant, we first note that

(6)
$$\prod_{w_1\in\mathbb{Z}_D}e(\pi i\lambda D^{-1}w_1)=e(\pi i\sum_{i=1}^g\frac{\lambda_i}{d_i}\sum_{w_1\in\mathbb{Z}_D}w_{1,i})=e(\pi i\sum_{i=1}^g\frac{\lambda_i}{d_i}\frac{d}{d_i}\frac{d_i(d_i-1)}{2}).$$

The above sum belongs to $\frac{1}{4}\mathbb{Z}$, and so the product is a ζ_8. Also, the matrix ${}^t\Delta D^{-1}B=D^{-1}(D\,{}^t\Delta D^{-1})B$ is symmetric; let $\alpha_{ij}=\frac{a_{ij}}{d_i},a_{ij}\in\mathbb{Z}$ be its ij-entry. Then

(7) $\prod_{w_1\in\mathbb{Z}_D}e(-\pi i\,{}^tw_1\,{}^t\Delta D^{-1}Bw_1)$
$$=e\left(-\pi i\sum_{i=1}^g\frac{a_{ii}}{d_i}d\frac{(d_i-1)(2d_i-1)}{6}-2\sum_{1\le i<j\le g}\frac{a_{ij}}{d_i}d\frac{(d_i-1)(d_j-1)}{4}\right).$$

The above sums belong to $\frac{1}{2}\mathbb{Z}$, and so the product is a ζ_4, except when $3|d_g$ and $(d_{g-1},3)=1$, in which case it belongs to $\frac{1}{6}\mathbb{Z}$ and the product is a ζ_{12}. To conclude, we have $\det C(M)=\zeta_8$, except when $3|d_g$ and $(d_{g-1},3)=1$, in which case $\det C(M)=\zeta_{24}$. □

Next, for the case of a totally symmetric bundle, note first that such a bundle always has characteristic in $\Lambda(H)$. Moreover, in Lemma 2.1, if $L(H_Z,\chi_0)$ has characteristic in $\Lambda(H)$, then $\psi^*L(H_Z,\chi_0)$ has also characteristic in $\Lambda(H)$. Indeed, in the case of an "even" polarization, we always have that $\chi_0=1$, and so $\psi_r^*\chi_0=1=\chi_0'$.

Proposition 2.2. *Keeping the notation of Proposition 2.1, we assume in addition that $\mathcal{L}$ is totally symmetric and that $g \geq 3$. Then the matrix C, for which $\psi^* B^Z = C\, B^{Z'}$, is of the form $C = (\det(\gamma Z + \delta))^{-\frac{1}{2}}\ C(M)$, where $C(M)$ depends on M and $\det C(M) = 1$, except when $3|d_g$ and $(d_{g-1}, 3) = 1$, in which case we have $\det C(M) = \zeta_3$.*

Proof. The proof is a modification of the proof of Proposition 2.1:

At the end of the subsection "**Matrices of the form** $\begin{pmatrix} 0 & -D \\ D^{-1} & 0 \end{pmatrix}$": For $g \geq 3$ the number $\frac{d}{d_i}$ is a multiple of 4, and so $\det C_i^{\frac{d}{d_i}} = d_i^{\frac{d}{2}}$. Therefore $\det A = d^{\frac{d}{2}}$. Also, for $g \geq 3$ we have $(\frac{d}{i^g})^{-\frac{d}{2}} = d^{-\frac{d}{2}}$ Therefore, $\det C(M) = 1$.

At the end of the subsection "**Matrices in $G_D^{\mathbf{int}}$** ": The sum in relation (6) is an even integer, and so the product is 1. For $g \geq 3$, the right summand in relation (7) is an even integer. The left summand is an even integer, except when $3|d_g$ and $(d_{g-1}, 3) = 1$, in which case it belongs to $\frac{2}{3}\mathbb{Z}$. Therefore the product is 1, except when $3|d_g$ and $(d_{g-1}, 3) = 1$, in which case it is ζ_3. Also $k(M)^d = 1$. Thus, to show that $\det C(M) = 1$ (resp. ζ_3), it suffices to show that the permutation of $\mathbb{Z}_D$ induced by the action of Δ followed by the addition by the vector s_1 has sign 1.

We show first that $\operatorname{sgn}(\Delta) = 1$. Indeed, let $d_i = 2^{k_i} m_i$, with $1 \leq k_1 \leq k_2 \leq \cdots \leq k_g$ and $m_1 | m_2 | \cdots | m_g$ odd integers. Define $\mathbb{Z}_{\mathrm{ev}} := \mathbb{Z}_{2^{k_1}} \oplus \cdots \oplus \mathbb{Z}_{2^{k_g}}$, a group of order n_{ev}, and $\mathbb{Z}_{\mathrm{odd}} := \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_g}$, a group of order n_{odd}. Then $\mathbb{Z}_D = \mathbb{Z}_{\mathrm{ev}} \oplus \mathbb{Z}_{\mathrm{odd}}$. Let $\phi : \mathbb{Z}_D \longrightarrow \mathbb{Z}_D$ be an automorphism. Then $\phi(\mathbb{Z}_{\mathrm{ev}}) = \mathbb{Z}_{\mathrm{ev}}$ and $\phi(\mathbb{Z}_{\mathrm{odd}}) = \mathbb{Z}_{\mathrm{odd}}$. We denote by ϕ_{ev} (resp. ϕ_{odd}) the restriction of ϕ to $\mathbb{Z}_{\mathrm{ev}}$ (resp. to $\mathbb{Z}_{\mathrm{odd}}$). If we interpret ϕ as a linear automorphism of $\mathbb{C}[\mathbb{Z}_D]$, then $\phi = \phi_{\mathrm{ev}} \otimes \phi_{\mathrm{odd}}$, and so $\operatorname{sgn}\phi = \operatorname{sgn}\phi_{\mathrm{ev}}^{n_{\mathrm{odd}}} \operatorname{sgn}\phi_{\mathrm{odd}}^{n_{\mathrm{ev}}}$. But n_{ev} is an even number; hence it suffices to prove the result for $\mathbb{Z}_D = \mathbb{Z}_{\mathrm{ev}}$.

Let E be the matrix which corresponds to the automorphism ϕ_{ev}. We call *elementary transformations of* $\mathbb{Z}_{\mathrm{ev}}$ those which correspond to left or right multiplication by a matrix of one of the following types: 1 in the diagonal and $a_{ij} \in \mathbb{Z}$ in some ij-entry with $j \geq i$; or 1 in the diagonal and $2^{k_i - k_j} a_{ij}$, $a_{ij} \in \mathbb{Z}$, in some ij-entry with $j < i$ (and zero everywhere else). We then claim that by multiplying the matrix $E = (e_{ij})$ with the above type of matrices, we can produce a matrix with all the elements of the last row, except the diagonal one, equal to zero $\bmod 2^{k_g}$ and the i-th element of the last column, with $1 \leq i < g$, zero $\bmod 2^{k_i}$. Indeed, we may first assume that e_{gg} is an odd integer: the determinant of E is an odd number since E defines an automorphism, and so some of the elements of the last row must be odd. If e_{gg} is even, let e_{gj_0}, $j_0 < g$, be the odd element. But then, using an elementary transformation, we can add the j_0-th column to the last one, and so the gg-entry becomes odd. Since $D^{-1}ED$ is an integral matrix, we have that $e_{gj} = 2^{k_g - k_j} m_{gj}$, $m_{gj} \in \mathbb{Z}$. But now the equation $2^{k_g - k_j} e_{gg} x \equiv -2^{k_g - k_j} m_{gj} \bmod 2^{k_g}$ has a solution, and therefore by multipling the matrix E on the right by the elementary matrix which has $2^{k_g - k_j} x$ in the gj-entry, we get that the gj-entry of the product is zero $\bmod 2^{k_g}$. On the other hand, by multiplying on the left by an elementary matrix which has x in the ig-entry, where x is the solution of $c_{gg} x \equiv -c_{ig} \bmod 2^{k_i}$, we get that the ig-entry of the product is zero $\bmod 2^{k_i}$.

A matrix like the one we produced corresponds to an even permutation of $\mathbb{Z}_{\mathrm{ev}}$. Indeed, by writing $\mathbb{Z}_{\mathrm{ev}}$ as a direct sum of two groups, the second of which is the $\mathbb{Z}_{2^{k_g}}$, we get the action is a direct sum of actions. We thus get that the signature of the permutation is one, since both groups are of even order. A similar argument yields that the action is given by the elementary matrices induces an even permutation (here we have to use the hypothesis $g \geq 3$). We therefore get that the permutation given by ϕ_{ev} is an even one.

Finally, the matrix of the permutation of $\mathbb{Z}_D$ induced by addition of the vector s_1 is the tensor product of the matrices corresponding to the permutation of $\mathbb{Z}_{d_i}$ induced by addition of s_i^1. Since the size of all those matrices is an even number, the determinant of the tensor product is 1. This concludes the proof. □

3. Abelian fibrations

Everything we have stated which holds for a fixed abelian variety $X = V/\Lambda$ can be transferred easily over a fibration $X \longrightarrow U$ of abelian varieties of type D, with base U a simply connected Stein manifold (such as $\mathfrak{h}_g$). In this case, the universal covering $\tilde{X}$ of X will take the place of V and the homotopy group $\pi_1(X)$ the place of Λ. When the base is the space $\mathfrak{h}_g$, there exists a universal family $\mathfrak{X} \longrightarrow \mathfrak{h}_g$, with fiber over Z the abelian variety X_Z. It is defined as the quotient of $\mathbb{C}^g \times \mathfrak{h}_g$ by the action of $\Lambda_D = \mathbb{Z}^g \oplus D\mathbb{Z}^g$ given by $l(v, Z) = (v + j_Z(l), Z)$. We have $\pi_1(\mathfrak{X}_D) = \Lambda_D$ and $\tilde{\mathfrak{X}}_D = \mathbb{C}^g \times \mathfrak{h}_g$. Suppose $(c^1, c^2) \in \mathbb{R}^g \oplus \mathbb{R}^g$ and let $c(Z) = Zc^1 + c^2$. For each such $c = c(Z)$, we have on $\mathfrak{X}_D$ a line bundle $\mathcal{L}^c_{\mathfrak{X}}$ corresponding to the classical factor of automorphy $e_c : \Lambda_D \times (\mathbb{C}^g \times \mathfrak{h}_g) \longrightarrow \mathbb{C}^\times$ of characteristic c, given by $e_c(l; v, Z) = e(2\pi i({}^t c^1 \lambda^2 - {}^t c^2 \lambda^1) - \pi\, i\, {}^t\lambda^1 Z \lambda^1 - 2\pi\, i\, {}^t v \lambda^1)$, where $l = (\lambda^1, \lambda^2) \in \Lambda_D = \mathbb{Z}^g \oplus D\mathbb{Z}^g$. Note that e_c, as well as $\mathcal{L}^c_{\mathfrak{X}}$, depends only on the class $(c^1, c^2) \bmod (D^{-1}\mathbb{Z}^g \oplus \mathbb{Z}^g)$.

3.1. **Line bundles on abelian fibrations.** Let $\mathrm{f} : X \longrightarrow B$ be a fibration of abelian varieties and $\mathcal{L}$ a symmetric line bundle on X, trivialized along the zero section, which defines a polarization of type D on each fiber. We denote by $s : B \longrightarrow X$ the zero section and let $S := s(B)$.

We denote by $\tilde{B}$ the universal covering of B. There exist a period map $p : \tilde{B} \longrightarrow \mathfrak{h}_g$ and a representation $\rho : \pi_1(B) \longrightarrow G_D$ of B. The choice of p and ρ is unique, up to the action by a fixed element of G_D. Let $Y := \mathfrak{X}_D \times_{\mathfrak{h}_g} \tilde{B}$, and $\tilde{\mathrm{f}} : Y \longrightarrow \tilde{B}$ the induced map. There is a canonical map $\pi_1 : Y \longrightarrow X$ which makes the following diagram commutative:

(8)
$$\begin{array}{ccccc} X & \xleftarrow{\pi_1} & Y & \xrightarrow{t} & \mathfrak{H}_D \\ \Big\downarrow{\scriptstyle \mathrm{f}} & & \Big\downarrow{\scriptstyle \tilde{\mathrm{f}}} & & \Big\downarrow{\scriptstyle \mathrm{f}_{\mathrm{un}}} \\ B & \xleftarrow{\pi} & \widetilde{B} & \xrightarrow{p} & \mathfrak{h}_g \end{array}$$

For each $\tilde{b} \in \tilde{B}$ and $\sigma \in \pi_1(B)$, the period map p satisfies $p(\sigma\tilde{b}) = \rho(\sigma) \cdot p(\tilde{b})$, where $\cdot$ denotes the action of G_D on $\mathfrak{h}_g$. We use $Z(\tilde{b})$ to denote the matrix $p(\tilde{b})$. If $M = \rho(\sigma)$, then the above relation translates to $Z(\sigma\tilde{b}) = M(Z(\tilde{b}))$, as defined in Section 1.

The group Λ_D acts on $\tilde{B} \times \mathbb{C}^g$ by $l\,(\tilde{b}, v) = (\tilde{b}, v + j_{Z(\tilde{b})}(l))$. The quotient of $\tilde{B} \times \mathbb{C}^g$ by this action, the elements of which we denote by $[\tilde{b}, v]$, is naturally isomorphic to Y, and the canonical map $\tilde{B} \times \mathbb{C}^g/\Lambda_D \longrightarrow \tilde{B}$ is identified with $\tilde{\mathrm{f}}$. The group $\pi_1(B)$ acts on Y by $\sigma\,[\tilde{b}, v] = [\sigma\tilde{b}, M_{Z(\tilde{b})}(v)]$, where $M = \rho(\sigma)$. The action is free and properly discontinuous, the quotient is isomorphic to X, and the canonical map $Y \longrightarrow Y/\pi_1(B)$ is identified with π_1. The above action defines an isomorphism $\phi_\sigma : Y_{\tilde{b}} \longrightarrow Y_{\sigma\tilde{b}}$. When we identify $Y_{\tilde{b}}$ with $X_{Z(\tilde{b})}$ and $Y_{\sigma\tilde{b}}$ with $X_{M(Z(\tilde{b}))}$, the above map becomes the map $\phi(M)$.

Let $\pi_1^*\mathcal{L}$ be the pull back of the line bundle $\mathcal{L}$ to Y. A symmetric line bundle $L(H,\chi)$ always has characteristic $c \in \frac{1}{2}\Lambda(H)$ w.r.t. any decomposition of H, since $\chi(\lambda) = \pm 1$ for all $\lambda \in \Lambda$. Therefore, a characteristic of the restriction of $\pi_1^*\mathcal{L}$ to the fiber over $\tilde{b}$, w.r.t. the decomposition induced by the period map p, is of type $c = Z(\tilde{b})c^1 + c^2$, where $(c^1, c^2) \in \frac{1}{2}(D^{-1}\mathbb{Z}^g \oplus \mathbb{Z}^g)$, and, by continuity, the class $(c^1,c^2) \bmod(D^{-1}\mathbb{Z}^g \oplus \mathbb{Z}^g)$ is independent of the choice of $\tilde{b}$. Note that when $\mathcal{L}_X$ is totally symmetric, then $(c^1, c^2) \in D^{-1}\mathbb{Z}^g \oplus \mathbb{Z}^g$. For each $\sigma \in \pi_1(B)$, the pull back of the isomorphism ϕ_σ defines an isomorphism of the total space of $\pi_1^*\mathcal{L}$. If $M = \rho(\sigma)$, then $\phi_\sigma = \phi(M)$, and so, M "preserves" the characteristic, i.e. $(M[c]^1, M[c]^2) = (c^1, c^2) \bmod(D^{-1}\mathbb{Z}^g \oplus \mathbb{Z}^g)$.

The group Λ_D acts on $\tilde{B} \times \mathbb{C}^g \times \mathbb{C}$ by $l\,(\tilde{b}, v, z) = (\tilde{b}, v + j_{Z(\tilde{b})}(l), e_c(l; v, Z(\tilde{b}))z)$. The quotient $\tilde{\mathcal{L}}^c$ is a line bundle on Y, the elements of which we denote by $[\tilde{b}, v, z]$. By construction, $\tilde{\mathcal{L}}^c = t^*\mathcal{L}^c_{\mathfrak{X}}$. The group $\pi_1(B)$ acts on $\tilde{\mathcal{L}}^c$ by $\sigma\,[\tilde{b}, v, z] = [\sigma\tilde{b}, M_{Z(\tilde{b})}(v), h(v)^{-1}z]$, where $M = \rho(\sigma)$ and h is the function introduced in Lemma 2.1, and its value is taken w.r.t. the element $Z(\tilde{b}) \in \mathfrak{h}_g$ and the matrix M. To see that the action is well defined, one has to use Lemma 2.1, combined with the fact that the action of $\pi_1(B)$ "preserves" the characteristic.

Let $\phi_\sigma : Y_{\tilde{b}} \longrightarrow Y_{\sigma\tilde{b}}$ be the map defined above. We fix the identification $(\pi_1(Y_{\tilde{b}}), \tilde{Y}_{\tilde{b}}) \cong (\Lambda_D, \mathbb{C}^g)$ via the map p. Then, the line bundle $\tilde{\mathcal{L}}^c|_{Y_{\tilde{b}}}$ corresponds to the factor of automorphy e_c and the line bundle $\phi_\sigma^*(\tilde{\mathcal{L}}^c|_{Y_{\sigma\tilde{b}}})$ corresponds to $\phi_\sigma^* e_c$. From Lemma 2.1 we have that $\phi_\sigma^* e_c = e_c \star h$. The action of σ on $\tilde{\mathcal{L}}^c$ then induces an isomorphism $\Phi_\sigma : \tilde{\mathcal{L}}^c|_{Y_{\tilde{b}}} \longrightarrow \phi_\sigma^*(\tilde{\mathcal{L}}^c|_{Y_{\sigma\tilde{b}}})$, which is the canonical isomorphism Φ_h defined in Section 1.

We claim that the quotient of the line bundle $\tilde{\mathcal{L}}^c$ by the above action of $\pi_1(B)$ is a line bundle $\mathcal{L}^c$ on X isomorphic to $\mathcal{L}$. This is a consequence of the see-saw principle. Indeed, the restrictions of $\mathcal{L}$ and $\mathcal{L}^c$ to the fibers of f are isomorphic, since they have the same characteristic. Also, by definition, $\mathcal{L}$ is trivial on the zero section S; the same holds for $\mathcal{L}^c$ since, if $\tilde{S}$ is the lift of S on Y, then the restricted action of $\pi_1(B)$ on $\tilde{S}$ is given by $\sigma[\tilde{b}, 0, z] = [\sigma\tilde{b}, 0, z]$ and therefore the quotient is the trivial bundle. In the following, we identify $\mathcal{L}$ with $\mathcal{L}^c$. Finally, the action of σ defines an isomorphism Ψ_σ of $H^0(Y_{\tilde{b}}, \tilde{\mathcal{L}}^c|_{Y_{\tilde{b}}})$ with $H^0(Y_{\sigma\tilde{b}}, \tilde{\mathcal{L}}^c|_{Y_{\sigma\tilde{b}}})$, which is induced by the map $\Phi_\sigma^*\phi_\sigma^*$. Next we determine the matrix $\tilde{C}^\sigma(\tilde{b})$ of Ψ_σ in terms of given bases.

The functions $\theta\begin{bmatrix} c^1 + D^{-1}m \\ c^2 \end{bmatrix}(v, Z(\tilde{b}))$, $m \in \mathbb{Z}_D$, are theta functions for the classical factor of automorphy $e_c : \Lambda_D \times (\mathbb{C}^g \times p(\tilde{B})) \longrightarrow \mathbb{C}^\times$, and the line bundle $\tilde{\mathcal{L}}^c$ corresponds, by construction, to e_c. Let $\tilde{s}_m$ denote the section of $\tilde{\mathcal{L}}^c$ corresponding to the above theta function. The set $\mathcal{B}^{\tilde{b}} := \langle \tilde{s}_m^{\tilde{b}} := \tilde{s}_m|_{Y_{\tilde{b}}}, m \in \mathbb{Z}_D\rangle$ forms a base of sections of $H^0(Y_{\tilde{b}}, \tilde{\mathcal{L}}^c|_{Y_{\tilde{b}}})$ for every $\tilde{b} \in \tilde{B}$. Let $\mathcal{B}^{\sigma\tilde{b}} := \langle \tilde{s}_n^{\sigma\tilde{b}} := \tilde{s}_n|_{Y_{\sigma\tilde{b}}}, n \in \mathbb{Z}_D\rangle$ be the corresponding base for $H^0(Y_{\sigma\tilde{b}}, \tilde{\mathcal{L}}^c|_{Y_{\sigma\tilde{b}}})$. Let $\mathcal{B}_1^{\tilde{b}} := \langle \Phi_\sigma^*\phi_\sigma^*\tilde{s}_n^{\sigma\tilde{b}}, n \in \mathbb{Z}_D\rangle$; this is also a base for $H^0(Y_{\tilde{b}}, \tilde{\mathcal{L}}^c|_{Y_{\tilde{b}}})$. Then the matrix $\tilde{C}^\sigma(\tilde{b})$ of Ψ_σ in the above bases satisfies the relation $\mathcal{B}_1^{\tilde{b}} = \tilde{C}^\sigma(\tilde{b})\,\mathcal{B}^{\tilde{b}}$.

Let $Z := Z(\tilde{b})$, $Z' := M(Z(\tilde{b}))$ and $v' := M_{Z(\tilde{b})}(v)$. The section $\Phi_\sigma^* \phi_\sigma^* \tilde{s}_m^{\sigma\tilde{b}}$ corresponds to the theta function $h(v)\,\theta\left[\begin{array}{c} c^1 + D^{-1}n \\ c^2 \end{array}\right](v', Z')$. The above relation of bases becomes

$$\begin{aligned}(9)\qquad h(v)\left\langle \theta\left[\begin{array}{c} c^1+D^{-1}n \\ c^2\end{array}\right](v',Z'),\ n\in\mathbb{Z}_D\right\rangle \\ = \tilde{C}^\sigma(\tilde{b})\ \left\langle \theta\left[\begin{array}{c} c^1+D^{-1}m \\ c^2\end{array}\right](v,Z),\ m\in\mathbb{Z}_D\right\rangle.\end{aligned}$$

We write the matrix M in the form $M = \left(\begin{array}{cc}\alpha & \beta\\ \gamma & \delta\end{array}\right)$. Let $c^1 := M[c]^1 + s^1$ and $c^2 := M[c]^2 + s^2$, where $s^1 = D^{-1}s_1$ and $s^2 = s_2$, with $s_1, s_2 \in \mathbb{Z}^g$. Then item 5b) in Section 1 yields

$$\begin{aligned}\theta\left[\begin{array}{c} c^1+D^{-1}n \\ c^2\end{array}\right](v',Z') &= \theta\left[\begin{array}{c} M[c]^1+D^{-1}(n+s_1) \\ M[c]^2+s_2\end{array}\right](v',Z') \\ &= e(2\pi i\,{}^ts^2(M[c]^1+D^{-1}(n+s_1))\,\theta\left[\begin{array}{c} M[c]^1+D^{-1}(n+s_1) \\ M[c]^2\end{array}\right](v',Z').\end{aligned}$$

Using relation (2), we now get

$$\begin{aligned}(10)\quad h(v)\,\theta\left[\begin{array}{c} c^1+D^{-1}n \\ c^2\end{array}\right](v',Z') \\ = e(2\pi i\,{}^ts^2(M[c]^1+s^1)+\pi i\,{}^tM[c]^1M[c]^2-\pi i\,{}^tc^1c^2)\,(\det(\gamma Z+\delta))^{\frac{1}{2}} \\ \cdot\sum_{m\in\mathbb{Z}_D} e(2\pi i\,{}^ts^2D^{-1}n)\,C(M)^{-1}_{s_1+n,m}\,\theta\left[\begin{array}{c} c^1+D^{-1}m \\ c^2\end{array}\right](v,Z).\end{aligned}$$

The number inside the first exponential is of the form $2\pi ik$, where $k \in \frac{1}{4d_g}\mathbb{Z}$. A similar calculation as in relation (6) yields that the matrix A, with $A_{nm} := e(2\pi i\,{}^ts^2D^{-1}n)\,C(M)^{-1}_{s_1+n,m}$, has determinant $\det A = \zeta_2 \det C(M)^{-1}$. Comparing relations (9), (10) and using Proposition 2.1, we conclude that $\det \tilde{C}^\sigma(\tilde{b}) = \zeta_8\,(\det(\gamma Z+\delta))^{\frac{d}{2}}$, except when $3|d_g$ and $gcd(3, d_{g-1}) = 1$, in which case we have $\det \tilde{C}^\sigma(\tilde{b}) = \zeta_{24}\,(\det(\gamma Z+\delta)^{\frac{d}{2}}$.

In the totally symmetric case we have $\prod_{n\in\mathbb{Z}_D^g} e(2\pi i\,{}^ts^2D^{-1}n) = 1$, and the sign of the permutation of $\mathbb{Z}_D$ induced by the action "addition of s_1" is 1. Hence, $\det A = \det C(M)^{-1} = 1$. We therefore get that $\det \tilde{C}^\sigma(\tilde{b}) = (\det(\gamma Z+\delta))^{\frac{d}{2}}$, except when $3|d_g$ and $gcd(3, d_{g-1}) = 1$, in which case we have $\det \tilde{C}^\sigma(\tilde{b}) = \zeta_3\,(\det(\gamma Z+\delta))^{\frac{d}{2}}$.

3.2. **Proof of Theorems A and B.** We cover B by small open analytic sets U^a. We choose W^a to be a lift of U^a on $\tilde{B}$. Let $\pi_a : W^a \longrightarrow U^a$ be the natural isomorphism. For a point $s \in U^a$, we denote by $w^a(s)$ its preimage in W^a. For $s \in U^a$, we define $Z^a(s) := Z(w^a(s))$. Let $\langle U^a, \lambda_1^a(s), \ldots, \lambda_{2g}^a(s)\rangle$ be the choice of a symplectic base on the fibers of $X^a := \mathrm{f}^{-1}(U^a)$, induced by the restriction of the period map p on W^a. For each a, b with $U^{ab} := U^a \cap U^b \neq \emptyset$, there is a matrix $M^{ab} = \left(\begin{array}{cc}\alpha^{ab} & \beta^{ab}\\ \gamma^{ab} & \delta^{ab}\end{array}\right) \in G_D$ relating the two symplectic bases. This matrix has the following interpretation. Given $s \in U^{ab}$, there exists a unique $\sigma_{ab} \in \pi_1(B)$ such that $\sigma_{ab}\,w^a(s) = w^b(s)$ and $M^{ab} = \rho(\sigma_{ab})$. Let $\tilde{C}^{ab}(\tilde{b})$ be the matrix $\tilde{C}^{\sigma_{ab}}(\tilde{b})$ defined in Section 3.1 above. The vector bundle $\mathrm{f}_*\mathcal{L}$ then has transition matrices

$g^{\mathrm{v}}_{ab} : U^{ab} \longrightarrow GL(d)$, where $d = \det D$, defined by $g^{\mathrm{v}}_{ab}(s) := \tilde{C}^{ab}(w^a(s))$. We have thus proven:

Lemma 3.1. *Let* $\mathrm{f} : X \longrightarrow B$ *be a fibration of abelian varieties of relative dimension* g*. Suppose* $\mathcal{L}$ *is the symmetric (resp. totally symmetric and* $g \geq 3$*) line bundle on* X*, and* $\{U^a\}$ *is the trivialization of* B *given above. Then, the transition functions of the line bundle* $\det \mathrm{f}_*\mathcal{L}$ *are given by* $g^{ab}_{\mathcal{L}}(s) = \zeta_8(\det(\gamma^{ab}Z^a(s) + \delta^{ab}))^{\frac{d}{2}}$ *(resp.* $g^{ab}_{\mathcal{L}}(s) = (\det(\gamma^{ab}Z^a(s) + \delta^{ab}))^{\frac{d}{2}}$*), except when* $3|d_g$ *and* $gcd(3, d_{g-1}) = 1$*, in which case we have that* $g^{ab}_{\mathcal{L}}(s) = \zeta_{24}(\det(\gamma^{ab}Z^a(s) + \delta^{ab}))^{\frac{d}{2}}$ *(resp.* $g^{ab}_{\mathcal{L}}(s) = \zeta_3(\det(\gamma^{ab}Z^a(s) + \delta^{ab}))^{\frac{d}{2}}$*).*

The proof of Theorems A and B is now a consequence of the above Lemma 3.1 and the following Lemma 3.2.

Lemma 3.2. *Let* $\mathrm{f} : X \longrightarrow B$ *be a fibration of abelian varieties and* $s : B \longrightarrow X$ *the zero section. Let* $\Omega_{X/B}$ *denote the relative cotangent bundle. Then* $\Omega_{X/B} \cong \mathrm{f}^*E$*, where* $E \cong s^*\Omega_{X/B}$ *is the vector bundle on* B *defined by the transition matrices* $g^{ab}_E := (\gamma^{ab}Z^a(s) + \delta^{ab})^{-1}$*. In particular, for the relative dualizing sheaf of* f *we have that* $\omega_{X/B} \cong \mathrm{f}^*\mu$*, where* $\mu \cong s^*\omega_{X/B}$ *is the line bundle on* B *defined by the transition functions* $g^{ab}_\mu(s) = \det(\gamma^{ab}Z^a(s) + \delta^{ab})^{-1}$*.*

Proof. The period matrix $Z^a(s)$ of X^a_s satisfies

$$\langle \lambda^a_1(s), \dots, \lambda^a_g(s)\rangle = \langle \lambda^a_{g+1}(s), \dots, \lambda^a_{2g}(s)\rangle Z^a(s).$$

Λ_D acts on $\mathbb{C}^g \times U^a$ by $l(v, s) := (v + j_{Z^a(s)}(l), s)$, where $j_{Z^a(s)}(l) := Z^a(s)\lambda^1 + \lambda^2$ and $l = (\lambda^1, \lambda^2)$. There is a canonical isomorphism $\phi_a : X^a \longrightarrow (\mathbb{C}^g \times U^a)/\Lambda_D$ (fibered over U^a) defined on the level of universal coverings by $\tilde{\phi}_a(\lambda^a_{g+i}(s)) = (e_i, s) \in \mathbb{C}^g \times U^a$, $i = 1, \dots, g$, where $\langle e_1, \dots, e_g\rangle$ is the standard base of $\mathbb{C}^g$. Let $\langle z_1, \dots, z_g\rangle$ denote the standard coordinates of $\mathbb{C}^g$. Then dz_i is the dual to e_i. Let $z^a_i := \tilde{\phi}^*_a(z_i \times id)$. Then $\langle dz^a_1, \dots, dz^a_g\rangle$ defines at each point of X^a a base of sections of the fiber of $\Omega_{X/B}|_{X^a}$, and $dz^a_1 \wedge \ldots \wedge dz^a_g$ defines a (nowhere zero) section of $\omega_{X/B}|_{X^a}$. This is because $dz_1 \wedge \ldots \wedge dz_g$ defines a (nowhere zero) section of the relative dualizing sheaf of the fibration $(\mathbb{C}^g \times U^a)/\Lambda_D \longrightarrow U^a$.

We have that $\langle \lambda^b_1, \dots, \lambda^b_{2g}\rangle = \langle \lambda^a_1, \dots, \lambda^a_{2g}\rangle\, {}^tM^{ab}$. Therefore

$$\langle \lambda^b_{g+1}, \dots, \lambda^b_{2g}\rangle = \langle \lambda^a_{g+1}, \dots, \lambda^a_{2g}\rangle\, {}^t(\gamma^{ab}Z^a(s) + \delta^{ab}).$$

By taking dual bases and applying determinants we get that

$$dz^b_1 \wedge \ldots \wedge dz^b_g = \det(\gamma^{ab}Z^a(s) + \delta^{ab})^{-1}\, dz^a_1 \wedge \ldots \wedge dz^a_g.$$

□

4. The Jacobian fibration

We now apply the above considerations to the Jacobian fibration $\mathrm{f} : \mathcal{J} \longrightarrow \mathcal{M}_g$, where $\mathcal{J}$ denotes the universal Jacobian parametrizing line bundles of degree zero on the fibers of the universal curve $\psi : \mathcal{C} \longrightarrow \mathcal{M}_g$. The Picard group of $\mathcal{M}_g$ is freely generated over the integers by the line bundle $\lambda := \det \psi_*\omega_{\mathcal{C}/\mathcal{M}_g}$ [1]. Due to the description of the Jacobian of a curve C as $J^0(C) \cong \frac{H^0(C,\omega_C)^\vee}{H_1(C,\mathbb{Z})}$, one deduces that λ has the same transition functions as $s^*\omega_{\mathcal{J}/\mathcal{M}_g}$; see e.g. [2], Ch. III, Prop. 17.1, where a slightly different notation is used. Hence $\lambda \cong s^*\omega_{\mathcal{J}/\mathcal{M}_g}$.

On the Jacobian fibration $\mathcal{J}$, there is a totally symmetric line bundle $\mathcal{L}$ which restricts to a line bundle of class 2θ on the fibers and is trivial along the zero section. It is defined as the pull back of the Poincaré bundle under the natural map. Theorem B yields that

Corollary 4.1. *With the above notation,*

$$\det \mathrm{f}_*(\mathcal{L}^{\otimes n}) \cong -\frac{(2n)^g}{2} \det \psi_* \omega_{\mathcal{C}/\mathcal{M}_g}.$$

Remark 4.1. One can also prove Corollary 4.1 by using Theorem 5.1 in [4]. The torsion factor in that theorem can be canceled, due to the above mentioned fact about the generator of the Picard group of $\mathcal{M}_g$.

4.1. **Proof of Theorem C.** Let $\mathrm{f}_{g-1} : \mathcal{J}^{g-1} \longrightarrow \mathcal{M}_g$ be the Jacobian fibration of degree $g-1$. We use the following result from [8], [9]. Let $\alpha : \tilde{\mathcal{M}}_g \longrightarrow \mathcal{M}_g$ denote the covering of even theta characteristics in $\mathcal{J}^{g-1}$. It is a covering of degree $2^{g-1}(2^g+1)$. The theta divisor Θ in $\mathcal{J}^{g-1}$ intersects $\tilde{\mathcal{M}}_g$ transversely, and the (set theoretic) intersection projects birationally, via α, to a divisor in $\mathcal{M}_g$ which has class $2^{g-3}(2^g+1)\,\mathrm{c}_1(\lambda)$. On the other hand, the generic point of the intersection corresponds to a line bundle with two sections. By the description of the singularities of the theta divisor, we have that, on such a point, the theta divisor has a singularity of multiplicity 2. Therefore the push-forward, by α, of the (scheme theoretic) intersection of Θ with $\tilde{\mathcal{M}}_g$ is a divisor of class $2^{g-2}(2^g+1)\,\mathrm{c}_1(\lambda)$. We use the following commutative diagram:

(11)
$$\begin{tikzcd}
\mathcal{J} \arrow[d, "\mathrm{f}"'] & \tilde{\mathcal{J}} \arrow[l, "\gamma"'] \arrow[rr, "\phi"] \arrow[dr, "\tilde{\mathrm{f}}"'] & & \tilde{\mathcal{J}}^{g-1} \arrow[r, "\delta"] \arrow[dl, "\tilde{\mathrm{f}}_{g-1}"] & \mathcal{J}^{g-1} \arrow[d, "\mathrm{f}_{g-1}"] \\
\mathcal{M}_g & & \widetilde{\mathcal{M}}_g \arrow[ll, "\alpha"'] \arrow[rr, "\alpha"] & & \mathcal{M}_g
\end{tikzcd}$$

In the diagram we denote by $\tilde{\mathcal{J}}$ and $\tilde{\mathcal{J}}^{g-1}$ the pull back of $\mathcal{J}$ and $\mathcal{J}^{g-1}$ on $\tilde{\mathcal{M}}_g$. By ϕ we denote the etale map of degree 2^{2g} which sends $L \in \tilde{\mathcal{J}}$, sitting over the point $[C,\eta] \in \tilde{\mathcal{M}}_g$, to $L^{\otimes 2} \otimes \eta \in \tilde{\mathcal{J}}^{g-1}$. Let $\tilde{s} : \tilde{\mathcal{M}}_g \longrightarrow \tilde{\mathcal{J}}$ be the zero section and $\sigma : \tilde{\mathcal{M}}_g \longrightarrow \tilde{\mathcal{J}}^{g-1}$ the section which sends $[C,\eta] \mapsto \eta$.

Let $\tilde{\Theta}$ be the line bundle corresponding to the theta divisor on $\tilde{\mathcal{J}}^{g-1}$. Then $\tilde{\Theta} = \delta^*\Theta$, and so $\alpha_*\mathrm{c}_1(\tilde{\mathrm{f}}_{g-1\,*}\tilde{\Theta}^{\otimes n}) = 2^{g-1}(2^g+1)\mathrm{c}_1(\mathrm{f}_{g-1\,*}\Theta^{\otimes n})$. Let $\tilde{\lambda}$ be the determinant of the Hodge bundle of the fibration $\tilde{\mathrm{f}}$. Then $\tilde{\lambda} = \alpha^*\lambda$, and so $\alpha_*\mathrm{c}_1(\tilde{\lambda}) = 2^{g-1}(2^g+1)\mathrm{c}_1(\lambda)$. If $\tilde{\mu} := \sigma^*\tilde{\Theta}$, then $\alpha_*\mathrm{c}_1(\tilde{\mu}) = 2^{g-2}(2^g+1)\,\mathrm{c}_1(\lambda)$. Let $\tilde{\mathcal{L}}$ be the canonical line bundle on $\tilde{\mathcal{J}}$ of Corollary 4.1. Then $\tilde{\mathcal{L}} = \gamma^*\mathcal{L}$. One can see that $\mathrm{c}_1(\tilde{\mathrm{f}}_*\phi^*\tilde{\Theta}^{\otimes n}) = 2^{2g}\mathrm{c}_1(\tilde{\mathrm{f}}_{g-1\,*}\tilde{\Theta}^{\otimes n})$. This is an application of the GRR theorem. One can also see that the restrictions of $\phi^*\tilde{\Theta}$ and $\tilde{\mathcal{L}}^{\otimes 2}$ on the fibers of the map $\tilde{\mathrm{f}}$ are the same. This can be proved by using Proposition 3.5 of Ch. 2 in [5] and Riemann's constant theorem. Therefore, by the see-saw principle, the line bundles $\tilde{\mathcal{L}}^{\otimes 2}$ and $\phi^*\tilde{\Theta}$ are isomorphic up to tensor by the pull back of a line bundle from $\tilde{\mathcal{M}}_g$. Since $\tilde{s}^*\tilde{\mathcal{L}}^{\otimes 2} \cong O$ and $\tilde{s}^*\phi^*\tilde{\Theta} \cong \tilde{\mu}$, we have $\tilde{\mathcal{L}}^{\otimes 2n} \otimes \tilde{\mathrm{f}}^*\tilde{\mu}^{\otimes n} \cong \phi^*\tilde{\Theta}^{\otimes n}$. By applying $\tilde{\mathrm{f}}_*$ and taking c_1, we have $\mathrm{c}_1(\tilde{\mathrm{f}}_*\tilde{\mathcal{L}}^{\otimes 2n}) + (4n)^g n\mathrm{c}_1(\tilde{\mu}) = 2^{2g}\mathrm{c}_1(\tilde{\mathrm{f}}_{g-1\,*}\tilde{\Theta}^{\otimes n})$. Now apply α_* to get

$$\begin{aligned}
&-2^{g-1}(2n)^g\, 2^{g-1}(2^g+1)\mathrm{c}_1(\lambda) + (4n)^g n\, 2^{g-2}(2^g+1)\mathrm{c}_1(\lambda) \\
&\quad = 2^{2g}\, 2^{g-1}(2^g+1)\mathrm{c}_1(\mathrm{f}_{g-1\,*}\Theta^{\otimes n}).
\end{aligned}$$

Therefore $\mathrm{c}_1(\mathrm{f}_{g-1\,*}\Theta^{\otimes n}) = \frac{1}{2}n^g(n-1)\mathrm{c}_1(\lambda)$. Since $\mathrm{Pic}\mathcal{M}_g$ is freely generated by λ [1], this concludes the proof of Theorem C.

4.2. **Alternative proof of Theorem C.** This is an application of the GRR theorem; see also Appendice 2 in [6] for a similar calculation. We keep the notation of section 4.1. In the above diagram (11), let ϕ be the map which sends $L \in \tilde{\mathcal{J}}$, sitting over the point $[C,\eta] \in \tilde{\mathcal{M}}_g$, to $L \otimes \eta \in \tilde{\mathcal{J}}^{g-1}$. By Lemma 3.2 we have $\Omega_{\tilde{\mathcal{J}}/\tilde{\mathcal{M}}_g} \cong \tilde{\mathrm{f}}^* \tilde{s}^* \Omega_{\tilde{\mathcal{J}}/\tilde{\mathcal{M}}_g}$, and since ϕ is an isomorphism, we get

$$\Omega_{\tilde{\mathcal{J}}^{g-1}/\tilde{\mathcal{M}}_g} \cong \tilde{\mathrm{f}}^*_{g-1}\sigma^*\Omega_{\tilde{\mathcal{J}}^{g-1}/\tilde{\mathcal{M}}_g}.$$

We apply GRR to the fibration $\tilde{\mathrm{f}}_{g-1} : \tilde{\mathcal{J}}^{g-1} \longrightarrow \tilde{\mathcal{M}}_g$. It is

$$\mathrm{ch}(\tilde{\mathrm{f}}_{g-1\,!}(\tilde{\Theta}^{\otimes n})) = \tilde{\mathrm{f}}_{g-1\,*}(\mathrm{ch}(\tilde{\Theta}^{\otimes n}) \cdot \mathrm{td}(\Omega^\vee_{\tilde{\mathcal{J}}^{g-1}/\tilde{\mathcal{M}}_g})).$$

We get

$$\mathrm{c}_1(\tilde{\mathrm{f}}_{g-1\,*}\tilde{\Theta}^{\otimes n}) = \frac{n^{g+1}}{(g+1)!}\mathrm{f}_*\mathrm{c}_1^{g+1}(\tilde{\Theta}) - \frac{n^g}{2}\mathrm{c}_1(\tilde{\lambda}).$$

The vanishing of the terms on the right hand side containing the "factor" c_1^k, with $k \le g-1$, in the expansion of $\mathrm{ch}(\tilde{\Theta}^{\otimes n})$, is a consequence of the projection formula and the fact that $\Omega_{\tilde{\mathcal{J}}^{g-1}/\tilde{\mathcal{M}}_g} \cong \tilde{\mathrm{f}}^*_{g-1}E$, where E is a vector bundle; see Lemma 3.2. The form of the term containing the "factor" c_1^{g-1} is due to the Poincaré formula. The appearance of $\tilde{\lambda}$ is a consequence of Corollary 4.1.

Now suppose that, say, $\mathrm{c}_1(\mathrm{f}_{g-1\,*}\Theta^{\otimes n}) = a(n)\mathrm{c}_1(\lambda)$ and $\mathrm{f}_{g-1\,*}\mathrm{c}_1^{g+1}(\Theta) = b\mathrm{c}_1(\lambda)$, where $a(n), b \in \mathbb{Z}$ [1]. Then $\mathrm{c}_1(\tilde{\mathrm{f}}_{g-1\,*}\tilde{\Theta}^{\otimes n}) = a(n)\mathrm{c}_1(\tilde{\lambda})$ and $\tilde{\mathrm{f}}_{g-1\,*}\mathrm{c}_1^{g+1}(\tilde{\Theta}) = b\mathrm{c}_1(\tilde{\lambda})$. We get $a(n) = \frac{n^{g+1}}{(g+1)!}b - \frac{n^g}{2}$. For $n=1$, the above gives that $b = (g+1)!(a(1)+\frac{1}{2})$. But $a(1) = 0$, because the line bundle $\tilde{\mathrm{f}}_{g-1\,*}\tilde{\Theta}$ has by definition a nowhere zero section, and so it is the trivial bundle. Hence $b = \frac{(g+1)!}{2}$, and so $\mathrm{c}_1(\mathrm{f}_{g-1\,*}\Theta^{\otimes n}) = \frac{1}{2}n^g(n-1)\mathrm{c}_1(\lambda)$.

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