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Thickness optimisation of a bi-stratum structure

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Abstract

This paper deals with the thickness optimisation of layers of a bi-stratum composite structure having infinite lateral dimensions. The composite is considered as a one-dimensional structure. Using a direct integration of the behaviour equation an analytical solution can be obtained without expensive calculation time. This approach is based on the techniques which are applied to acoustic analysis of multi-layer structures and can be extended to calculate a general composite laminate structure. The advantage of such a method is a short calculation time.

The objective of this paper is to minimise the Von Mises stress in average with respect to the position of the interface between two layers. The analytical solution is obtained in a similar way as for analysis problem. This analytical solution is compared with the numerical results obtained by using the finite element system ANSYS. A very good correlation between analytical and numerical solutions is observed.

1 Introduction

This paper deals with the thickness optimisation of layers of a bi-stratum composite structure having infinite lateral dimensions. The composite is considered as a one-dimensional structure. Using a direct integration of the behaviour equation an analytical solution can be obtained without expensive calculation time. This approach is based on the techniques which are applied to acoustic analysis of multi-layer structures and can be extended to calculate a

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2 Notations

- z direction orthogonal to the plane of layers of laminated composite and e_z is a unit vector in this direction; x and y are in-plane orthogonal axes (see fig.II.1).
- The origin of coordinate system is placed on the inferior layer. H is the total thickness of composite, e_1 et e_2 are the thickness of each layer (see figure 1).
- u is a displacement vector, σ represents a stress tensor and ε denotes a strain tensor :

$$\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{1}$$

- The materials properties used are :
  ρ - mass density, g - gravity force and λ, μ - Lame coefficients

- The symbol div denotes divergence operator :

$$(div\sigma)_i = \frac{\partial \sigma_{ix}}{\partial x} + \frac{\partial \sigma_{iy}}{\partial y} + \frac{\partial \sigma_{iz}}{\partial z} \quad (i = x, y \ ou \ z) \tag{2}$$

3 Hypotheses

H1) The bi-layer composite is considered to have infinite dimensions in x and y directions.
H2) The layers are homogeneous, elastic and isotropic.
H3) The hypothesis of small perturbations is assumed and static case is considered.
H4) The contact between layers is perfect (there is neither sliding nor unsticking).
H5) The lower surface of the inferior layer is fixed, and a uniform pressure in the direction of gravity is applied on the upper surface of the composite (see fig. 1).
H6) The gravity forces are taken into account in order to have a quadratic objective function.

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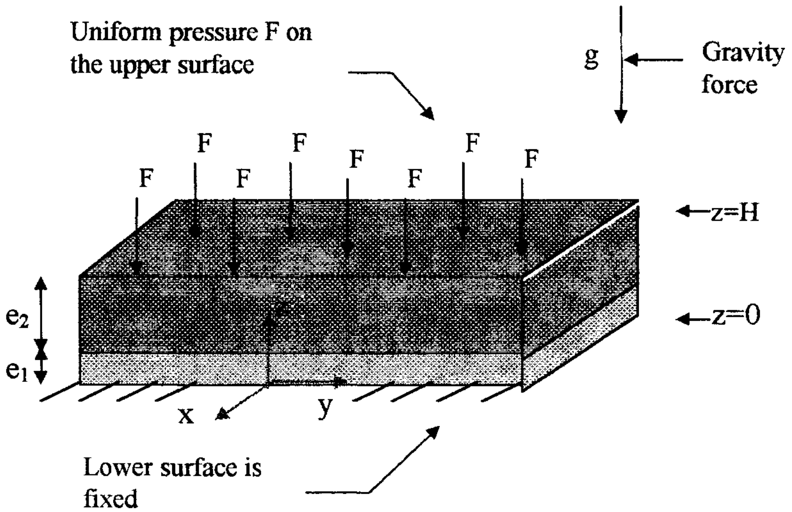

Figure 1

4 Displacement and stress analysis

The hypothesis H1 concerning the geometry of the composite implies that the displacement and the strains should satisfy the following conditions :

$$u_x = u_y = 0 \; ; \; u_z(z) \neq 0 \tag{3}$$

$$\varepsilon_{xx}(u) = \varepsilon_{yy}(u) = \varepsilon_{xy}(u) = \varepsilon_{xz}(u) = \varepsilon_{yz}(u) = 0 \; ; \; \varepsilon_{zz}(u) = \frac{\partial u_z}{\partial z} \tag{4}$$

The stresses can be obtained by using the hypothesis H2 (elastic and isotropic materials) :

$$\sigma_{xx} = \sigma_{yy} = \lambda \frac{\partial u_z}{\partial z} \; ; \; \sigma_{zz} = (\lambda + 2\mu) \frac{\partial u_z}{\partial z} \; ; \; \sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0 \tag{5}$$

After the substitution of these stresses into the equilibrium equation :

$$div\, \sigma - \rho g\, e_z = 0 \tag{6}$$

The differential equation governing the displacements in each layer is determined :

$$\frac{d^2 u_z}{dz^2}(z) = \frac{\rho g}{\lambda + 2\mu} \tag{7}$$

Where λ et μ are independent of z because the layers are homogeneous (hypothesis H2).
The solution of (7) is very easy to obtain :

$$u_z(z) = \frac{1}{2} \frac{\rho g}{\lambda + 2\mu} z^2 + Az + B \tag{8}$$

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where A and B are integration constants and will be calculated using the exterior and interface boundary conditions.
Using (5) the stresses are calculated in the following way :

$$\sigma_{xx}=\sigma_{yy}=\lambda\left\{\frac{\rho g}{\lambda+2\mu}z+A\right\}\ ;\ \sigma_{zz}=\rho g z+(\lambda+2\mu)A\ ;\ \sigma_{xy}=\sigma_{xz}=\sigma_{yz}=0 \quad (9)$$

and the von Mises stress is given by :

$$\sigma_{vm}=\left|\sigma_{xx}-\sigma_{zz}\right|=2\mu\left|\frac{\rho g}{\lambda+2\mu}z+A\right| \quad (10)$$

Having calculated A, one can prove that

$$\frac{\rho g}{\lambda+2\mu}z+A\leq 0$$

and then the von Mises stress is expressed as :

$$\sigma_{vm}=-2\mu\left(\frac{\rho g}{\lambda+2\mu}z+A\right) \quad (11)$$

The displacement u_z (8) is a quadratic function of z and the stresses (7) are linear because the gravity forces have been taken into account.
In order to determine the solution for the displacement and stresses the integration constants should be calculated. There are in total four constants ; it means two constants for each of two layers. They are calculated using the boundary conditions (H5) and the interface conditions (H4). This involve :

$$u_z(0)=0\ ;\ \sigma_{zz}(H)=-F\ ;\ u_z^1(e_1)=u_z^2(e_1)\ ;\ \sigma_{zz}^1(e_1)=\sigma_{zz}^2(e_1) \quad (12)$$

where the signs 1 and 2 denote the number of the layer.
The four conditions (12) lead to a system of four linear equations which can be easily solved to obtain four constants.

In the case of a laminar composite of n layers, the code CASTAN has been developed. This code adopts the same computing method presented above for a two-layer composite.

5 Comparison with the results obtained by using ANSYS

In order to validate our approach the computations with ANSYS code have been realised. Let us consider a two-layer composite. This structure is composed of two plates; one is of steel and another is of aluminium. The thickness of each plate is 2cm and in-plane dimensions are 200cm. In this way the hypothesis H1 is satisfied.
The mesh of this structure is done using solid twenty-node brick elements (solid 95 in ANSYS). There are eight divisions in the thickness and 10x10 divisions in the plane of the structure. The bottom surface is fixed and an uniform pressure of 100Pa is applied on the upper surface. The low magnitude of this pressure is

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chosen in order to reduce the influence of the linear component in the expression of displacement (8). If the linear component is dominant then the objective function will be quasi-linear and it will not be interesting to test the robustness of the optimization algorithms. The low magnitude of pressure implies the small values of displacements and stresses (tables 1 and 2).

The comparison of the analytical results with the numerical computations by ANSYS is established for nine nodes (in the thickness) in the center of the composite. In such a way the influence of the boundary conditions is reduced and the hypothesis H1 becomes more true. The figures 2 to 5 show the distribution of displacement u_z, σ_{xx} and σ_{zz} stresses, and von Mises stress in z-direction in the center of the composite. The displacement u_z seems to be quadratic in each layer and the stresses are linear.
One can say that the displacement u_z and the stress σ_{zz} are continuous at the interface between two layers . This fact is in conformity with the hypothesis H4. On the contrary, σ_{xx} stress and von Mises stress are discontinuous. This can be explained by the fact that there is any continuity condition on σ_{xx}, and the von Mises stress is a combination of σ_{xx} (discontinuous) and σ_{zz} (continuous).

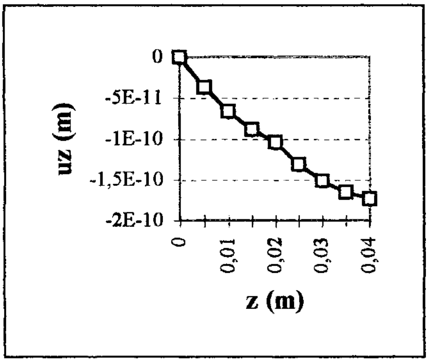
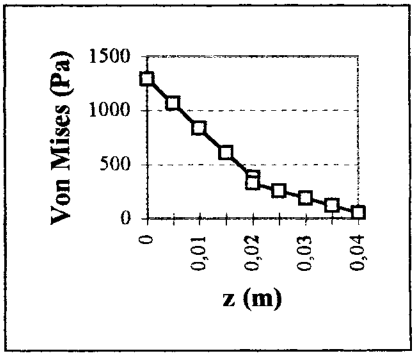


Figure 2



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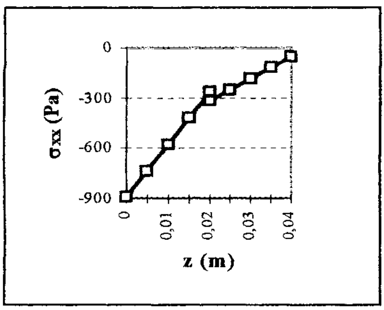

Figure 4

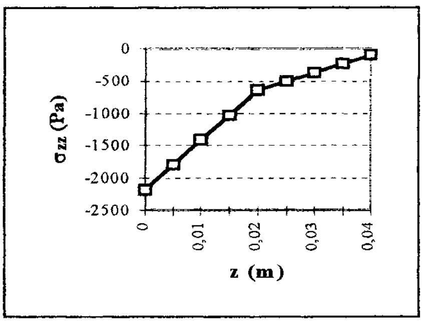

Figure 5

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In the table 1, the analytical (CASTAN) and numerical (ANSYS) results are given. The table 2 gives Von Mises. For the discontinuous von Mises stress two values are given on the interface (z=2cm). In general, the differences between analytical and numerical results are very small (of order 10^{-3} %). A good correlation of these results proves that the established hypotheses are valid.

z (m)	CASTAN (Pa)	ANSYS (Pa)	Error (%)
0	1291,73	1291,76	-2,32E-03
5,0E-03	1063,96	1063,95	9,40E-04
1,0E-02	836,19	836,19	0
1,5E-02	608,42	608,43	-1,64E-03
2,0E-02	380,65	380,66	-2,63E-03
2,0E-02	326,54	326,55	-3,06E-03
2,5E-02	257,59	257,6	-3,88E-03
3,0E-02	188,64	188,65	-5,30E-03
3,5E-02	119,69	119,7	-8,35E-03
4,0E-02	50,75	50,75	0

table 1 : Von Mises stress

6 Optimization

Now the objective is to minimize an average value of the von Mises stress. The objective function is defined in the following manner :

$$\overline{\sigma_{vm}} = \frac{1}{H} \int_0^H \sigma_{vm}(z) dz \tag{13}$$

The total thickness H of the composite is constant and we have :

$$e_1 + e_2 = H \tag{14}$$

where e_1 and e_2 are the thickness of two layers.

The optimization problem is stated in the following way

$$\begin{cases} \underset{e_1, e_2}{Min}\left[\overline{\sigma_{vm}}(e_1, e_2)\right] \\ Objective\ function : \overline{\sigma_{vm}} = \frac{1}{H} \int_0^H \sigma_{vm}(z) dz \\ design\ variables : e_1 \geq 0, e_2 \geq 0 \\ constraints\ : e_1 + e_2 = H \end{cases} \tag{15}$$

The variable e_2 can be eliminated using constraint equation and after introducing a new normalized variable

$$X = \frac{e_1}{H} \tag{16}$$

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The objective function can be rewritten as

$$\underset{X \in [0,1]}{Min} \left[\overline{\sigma_{vm}} (X, 1-X) \right] \tag{17}$$

This means that our optimization problem consists in finding the best position of the interface. Finally, it is possible to obtain an analytical expression of the objective function. The integral (13) can be easily calculated since $\sigma_{vm}(z)$ is a linear function (see (11)). After analytical integration we obtain the average von Mises stress which is a quadratic function of X :

$$\overline{\sigma_{vm}}(X) = a\,X^2 + bX + c \tag{18}$$

where the coefficients a, b and c are given by the following formulas :

$$\begin{cases} a = gH\left[\rho_2 \dfrac{\mu_2}{\lambda_2+2\mu_2} - (2\rho_2 - \rho_1)\dfrac{\mu_1}{\lambda_1+2\mu_1}\right] \\ b = 2\,(F + \rho_2 gH)\left[\dfrac{\mu_1}{\lambda_1+2\mu_1} - \dfrac{\mu_2}{\lambda_2+2\mu_2}\right] ; \; c = (2F + \rho_2 gH)\dfrac{\mu_2}{\lambda_2+2\mu_2} \end{cases} \tag{19}$$

In these formulas, we have only a quotient $\mu/(\lambda+2\mu)$ which is dependent only on the Poisson's coefficient. This means that the Young's moduli have any influence on the solution of our optimization problem.

If the gravity force is neglected (g=0) the coefficient a is equal to zero and the objective function becomes linear. We haven taken into account the gravity force (g =0) in order to have a quadratic objective function and test the optimization procedure of ANSYS for this function.

Since the objective function is quadratic there are five cases of the position of minimum :

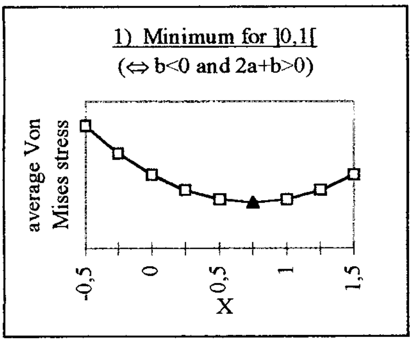

Figure 6

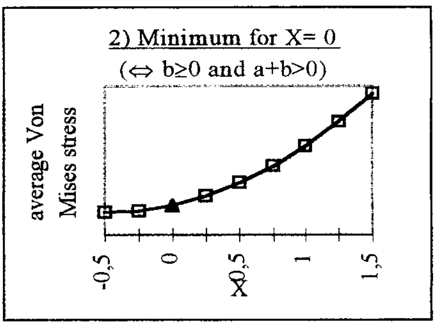

Figure 7

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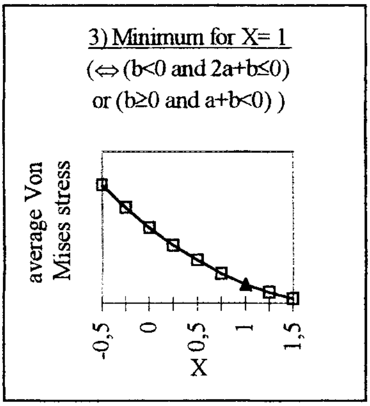

Figure 8

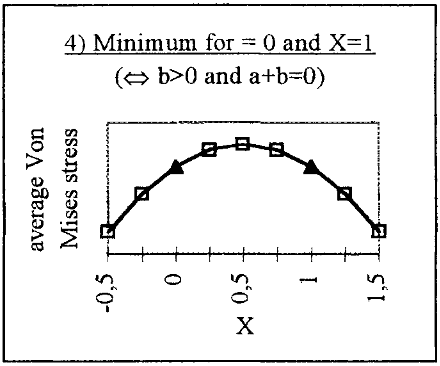

Figure 9

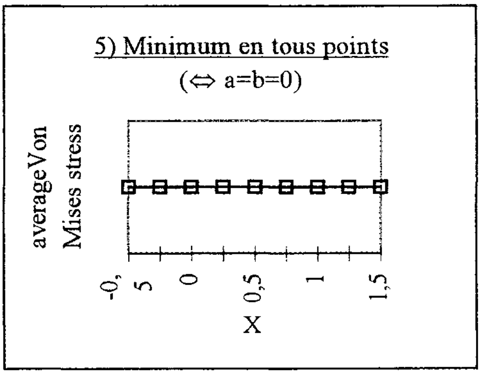

Figure 10

In the first case, the minimum corresponds to X=-b/(2a) where the derivative is null. In the next three cases, the minimum is reached for X=0 or X=1, it means on the boundary limits. The last case is not interesting since it corresponds to a composite with two layers of the same material. In this case, the average von Mises stress does not depend on the position of the interface.

The first four cases have been tested using optimization possibilities of ANSYS code. In each layer, there were ten by ten solid brick elements "solid 95". So in total the mesh compounded 200 elements. It was sufficient to use only one row of elements in each layer because the element "solid 95" involves shape functions of second order, and even with few elements in the thickness it is possible to approximate correctly the displacements u_z.
In the first case, one layer was of steel material, and the second layer was of a fictitious material for which the Poisson's coefficient was 0.25 and the mass varied from 3500kg/m^3 to 7700kg/m^3. The objective of this test was to compare the analytical and numerical (ANSYS) results concerning the position of the minimum. The results are presented in the table 2.

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Mass (kg/m^3)	Position of min. (ANSYS)	Position of min. (analytical)	Error (%)
3500	0,09953	0,09945	-0,08389
5000	0,19157	0,19148	-0,04437
6000	0,30415	0,30409	-0,01973
6500	0,39425	0,39430	0,01268
6900	0,49738	0,49751	0,02714
7200	0,60493	0,60516	0,03885
7400	0,69908	0,69947	0,05650
7580	0,80670	0,80719	0,06074

table 2 : minimum reached between 0 and 1

The comparative error is smaller than 0.1%. The correlation between analytical approach and numerical computations by finite elements is very good. This is certainly due to the parabolic shape of the objective function. The optimization procedure used in ANSYS [1] consists in solving an approximated problem (sub-problem) in which the objective function starting from randomly generated points (radom design). In our problem, the objective function is quadratic. So its approximation is exactly the same. This fact explains a very good correlation between analytical and numerical results. The more complex cases (involving e.g. geometrical or material non-linearities) should be studied in order to verify the robustness of optimization capabilities of ANSYS.
In the cases where the minimum is reached on the boundary of admissible domain (table 2) the error of the numerical solution relating to the analytical solution is about 1.2%. The thickness of one of two layers tend towards zero and the solid elements in this layer become degenerated. This fact does not enable a good and rapid convergence to the optimal solution.

For information, the case where the minimum is reached for X=0 corresponds to a composite with one layer of steel and the other of aluminium. The minimum located in 1 is obtained for a composite with one layer of steel and another layer of the material for which the Poisson's coefficient is equal to 0.25 and the mass is equal to 7850kg/m^3. Finally, when the minimum is in 0 and in 1 the associated composite has one layer of steel and the other of the material corresponding to the Poisson's coefficient is equal to 0.33 and the mass is equal to 9235.2kg/m^3.
In the case where the objective function has two simultaneous analytical minima in 0 and in 1 the iterative procedure in ANSYS can converge towards one or the other. It depends on the set of points generated randomly by the procedure "random design". For example, starting with a random set of ten points the procedure converges towards the minimum in 0. If the point neighbouring zero is eliminated and the procedure "sub-problem" is restarted the convergence processus will give the minimum in 1.

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7 Conclusions and future developments

In this paper we have presented a method for analysis and optimization of two-layered elastic composite. The simplified hypotheses (infinite in-plane dimensions orthogonal to the thickness) have been assumed in order to obtain the analytical solutions and compare them with the numerical finite element computations using ANSYS code. The objective of the paper was to test and validate the optimization methods for the simple cases before passing to more complex structures (like strongly heterogeneous composite structures in the domain of non-linear behaviour).
It will be interesting to optimize an heterogeneous composite structure using the method developed in this paper and compare the results with another approach using the homogeneization techniques ([2], [3], [5] , [7]). This comparison could give some information about the precision of the homogeneization methods coupled with the optimization algorithms.
It will be also interesting to develop an analytical approach for the multilayered composite structures in the range of the large deformations. In this cases the objective function would be no longer quadratic but strongly non-linear.
These two developments concerning the homogeneization techniques and the non-linear behaviour for optimization of multilayered composite structures (with the same hypotheses assumed in this paper, i.e. one dimensional problem) should be realized before passing to the 3D problem.

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