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# THIN ELASTIC FILMS: THE IMPACT OF HIGHER ORDER PERTURBATIONS

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**Abstract.** The asymptotic behavior of an elastic thin film penalized by a van der Wals type interfacial energy is investigated when both its thickness and the magnitude of the additional energy vanish in the limit. Keeping track of both mid-plane and out-of-plane deformations (through the introduction of the Cosserat vector), the resulting behavior strongly depends upon the ratio between thickness and interfacial energy.

**1. Introduction.** Thin films and coatings, which are increasingly used for their outstanding mechanical properties, are also the topic of an increasing literature, especially since it was recognized in [16] that elastic membranes could be derived as variational limits of 3d elastic energies for domains with a vanishing thickness. That paper paved the way for many studies that adopt the viewpoint of Γ-convergence in dealing with dimensional reduction.

It is thought that very thin films, in part because of their polycrystalline nature, are quite sensitive to possible interfacial effects, and this has motivated several studies that investigate the impact of a van der Waals type interfacial energy on their behavior. The first such study in the variational framework was conducted in [4]. There, the authors

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add a fixed interfacial energy of the form $\kappa \int_{\omega\times(-\frac{\varepsilon}{2},\frac{\varepsilon}{2})} |D_{ij}u|^2\,dx$ to the potential energy $\int_{\omega\times(-\frac{\varepsilon}{2},\frac{\varepsilon}{2})} W(Du)\,dx$ of the thin elastic domain; they obtain in the limit a 2d energy density which depends on the deformation $u$ (or rather $D_i u,\ i=1,2$) of the mid-surface, together with the Cosserat vector $b$ which describes both transverse shear and normal compression in the thickness (think of the limit transformation as being of the form $u+\varepsilon\int_0^{x_3} b(x_1,x_2,s)\,ds$). Because their emphasis is on martensitic materials, they obtain that way a thin film which admits exact energy minimizing interfaces between austenitic and martensitic phases, whereas the corresponding bulk material must generically finely twin the phases.

Their analysis does not allow a correlation between the strength of the interfacial energy and the thickness of the sample. That issue was subsequently addressed in [18]. In that paper the interfacial energy is allowed to tend to 0, and a micro-structural parameter is also added. The author then investigates the different regimes that correspond to the relative strengths of the three vanishing parameters: the thickness, the strength of the interfacial energy, and the size of the heterogeneities. The analysis is however restricted to the mid-surface of the film, or, in other words, the Cosserat vector is a priori minimized out of the computed energy.

In a different direction, the paper [5] investigates both mid-surface and cross-sectional behavior in the absence of interfacial energy, but this analysis only accounts for the bending moment – the average of the Cosserat vector through the cross section – and it is conjectured that the behavior becomes nonlocal if the actual Cosserat vector is kept in the formulation in lieu of its average.

In this work, we propose to introduce interfacial energy as in [4], [18], while tracking down the cross-sectional behavior as in [5], but without averaging through the cross section. We then exhibit a membrane whose constitutive behavior critically depends upon the strength of the vanishing interfacial energy.

Specifically, the domain is of the form $\omega\times(-\frac{\varepsilon}{2},\frac{\varepsilon}{2})$, $\omega$ being its mid-section (a 2d set) and $\varepsilon$ its thickness. We add to the elastic energy

$$\int_{\omega\times(-\frac{\varepsilon}{2},\frac{\varepsilon}{2})} W(Du)\,dx$$

an interfacial energy of the form

$$\varepsilon^\gamma \int_{\omega\times(-\frac{\varepsilon}{2},\frac{\varepsilon}{2})} |D^2u|^2\,dx,$$

where $D^2u$ is the Hessian matrix associated to the transformation vector $u : \omega\times(-\frac{\varepsilon}{2},\frac{\varepsilon}{2}) \to \mathbb{R}^3$ and $\gamma>0$. It proves to be more convenient to re-scale the resulting energy minimization problem onto a fixed domain of thickness 2 through a $\frac{1}{\varepsilon}$-dilation of the transverse variable. The problem, cast in the variational framework, consists in studying the $\Gamma$-limit, in an appropriate topology, of

$$\int_{\omega\times(-\frac{1}{2},\frac{1}{2})} \left[W\left(D_p u|\frac{1}{\varepsilon}D_3u\right) + \varepsilon^\gamma\left(|D_p^2u|^2+\frac{1}{\varepsilon^2}|D_{p3}u|^2+\frac{1}{\varepsilon^4}|D_{33}u|^2\right)\right] dx, \tag{1.1}$$

where $D_p$ stands for the gradient in the plane of the film; i.e., the differential operators in (1.1) are defined as

$$D_p := (D_1, D_2), \quad D_p^2 := (D_{11}, D_{22}, D_{12}), \quad D_{p3} := (D_{13}, D_{23}).$$

If $u_\varepsilon$ is an approximate minimizer of that energy (for adequate boundary conditions and external forces), then an appropriate topology will keep track of the limits of both $\{u_\varepsilon\}$ and $\left\{\frac{1}{\varepsilon} D_3 u_\varepsilon\right\}$.

We now give a brief and nontechnical description of our main results. In all cases, provided that the elastic density has polynomial growth – which of course excludes the setting of hyperelasticity for which $W$ blows up as $\det Du$ vanishes – the limit kinematics is identical as far as the limit of $u_\varepsilon$ is concerned: the limit field $u$ is independent of the transverse variable $x_3$; it represents the transformation of the mid-surface of the membrane. In the case $\gamma < 2$, the interfacial energy is strong enough to rigidify the cross section (the Cosserat vector is $b$ independent of $x_3$) and to decouple the mid-plane deformation from the bending moment (the average of $b$ over the thickness). Thus, the resulting energy is merely the lower semi-continuous envelope of $\int_{\omega\times\left(-\frac{1}{2},\frac{1}{2}\right)} W(D_p u|b)\, dx$.

If $\gamma = 2$, then the behavior drastically depends on the form of the Cosserat vector $b$. If it is independent of $x_3$, then the resulting energy once again treats $D_p u$ and $b$ as independent fields and the result is that obtained for $\gamma < 2$. If $b$ does depend on $x_3$, then the resulting energy is more involved and we strongly suspect that it is nonlocal.

When $\gamma > 2$, the interfacial energy is weak, yet the result is still that obtained for $\gamma < 2$, with an $x_3$-dependent Cosserat vector; this is a bit surprising, and we cannot explain why it should be so!

The plan of the paper is as follows. Section 2 is devoted to general properties of the limit energy. In Section 3 we study the $\Gamma$-convergence of the family of functionals (1.1) for all $\gamma > 0$ and characterize the limit energy for $\gamma \neq 2$. Section 4 addresses the critical case $\gamma = 2$.

If the Cosserat vector is $x_3$-independent, then we obtain an explicit local integral representation for the $\Gamma$-limit. Otherwise, that is, when the Cosserat vector is $x_3$-dependent, the characterization of the $\Gamma$-limit leads us to the auxiliary functional

$$\begin{aligned} \inf\Bigg\{ &\liminf_{n\to\infty} \int_{A\times I} \left( W\left(D_p u_n \,|b_n\right) + |D_3 b_n|^2 \right) dx : \\ &\{u_n\} \subset W^{1,q}(\Omega;\mathbb{R}^3), \{b_n\} \subset L^q\left(\Omega;\mathbb{R}^3\right) \text{ with } D_3 b_n \in L^2\left(\Omega;\mathbb{R}^3\right), \\ &u_n \rightharpoonup u \text{ in } W^{1,q}(\Omega;\mathbb{R}^3),\ b_n \rightharpoonup b \text{ in } L^q(\Omega;\mathbb{R}^3) \Bigg\}. \end{aligned}$$

We have been unable to obtain a local integral representation for this functional.

Note that, stricto sensu, our results can only be compared to those of [18] in that, upon minimizing the resulting energy over all admissible Cosserat vectors, we should recover the results in [18], provided we drop the dependence of the energy considered in that paper upon the micro-structural parameter. This is the object of Remark 4.6.

**2. Preliminaries.** We start with some notation. In the remainder of the paper, $\omega$ is an open, bounded, connected subset of $\mathbb{R}^2$ with Lipschitz boundary, so that, in particular, all Sobolev extension theorems apply. We define $\mathcal{A}(\omega)$ as the class of all open subsets of $\omega$. Points in $\omega$ will be designated by $x_\alpha$ or $x_\beta$ unless there is some ambiguity. Also, $Q'$ will denote the unit square $(-\frac{1}{2},\frac{1}{2})^2$, while $Q := Q' \times (-\frac{1}{2},\frac{1}{2})$; $C$ will denote a generic positive constant, so that e.g. $C = 2C$. For any $x_\alpha^0 \in \mathbb{R}^2$, $\delta > 0$, we define $Q'(x_\alpha^0,\delta)$ to be the square $x_\alpha^0 + (-\frac{\delta}{2},\frac{\delta}{2})^2$. We set $I := \left(-\frac{1}{2},\frac{1}{2}\right)$.

Finally, $\to$ will denote strong convergence, while $\rightharpoonup$ and $\overset{*}{\rightharpoonup}$ will stand for weak and weak* convergence, respectively.

For $\varepsilon$, $\gamma > 0$ and $1 < q < \infty$ consider the functional

$$E_\varepsilon^\gamma : W^{1,q}\left(\Omega;\mathbb{R}^3\right) \times \mathcal{A}(\omega) \to [0,\infty]$$

defined by

$$E_\varepsilon^\gamma(u;A) := \int_{A\times I} W\left(D_p u | \frac{1}{\varepsilon} D_3 u\right) dx + \varepsilon^\gamma \int_{A\times I}\left(|D_p^2 u|^2 + \frac{1}{\varepsilon^2}|D_{p3}u|^2 + \frac{1}{\varepsilon^4}|D_{33}u|^2\right) dx$$

if $u \in W^{2,2}(\Omega;\mathbb{R}^3)$, and $E_\varepsilon^\gamma(u;A) := \infty$ otherwise. Here

$$\Omega := \omega \times I,$$

and the elastic energy density $W : \mathbb{R}^{3\times 3} \to [0,\infty)$ satisfies the following hypothesis:

$(H_1)$ $W$ is continuous and there exists $C > 0$ such that

$$W(F) \geq \frac{1}{C}|F|^q - C$$

for all $F \in \mathbb{R}^{3\times 3}$.

THEOREM 2.1 (Compactness). Assume that $W$ satisfies condition $(H_1)$. Let $\varepsilon_n \to 0^+$ and let $\{u_n\} \subset W^{2,2}(\Omega;\mathbb{R}^3)$ be such that

$$\sup_n E_{\varepsilon_n}^\gamma(u_n;\Omega) < \infty.$$

Then there exist a subsequence $\{u_{n_k}\}$, $u \in W^{1,q}\left(\Omega;\mathbb{R}^3\right)$, with $D_3 u = 0$ $\mathcal{L}^3$ a.e. in $\Omega$, and $b \in L^q\left(\Omega;\mathbb{R}^3\right)$ such that

$$u_{n_k} \rightharpoonup u \text{ in } W^{1,q}(\Omega;\mathbb{R}^3), \quad \frac{1}{\varepsilon_{n_k}} D_3 u_{n_k} \rightharpoonup b \text{ in } L^q(\Omega;\mathbb{R}^3). \tag{2.1}$$

Moreover, if $\gamma < 2$, then $D_3 b = 0$ $\mathcal{L}^3$ a.e. in $\Omega$, while if $\gamma = 2$, then $D_3 b \in L^2\left(\Omega;\mathbb{R}^3\right)$.

As a consequence of the previous theorem, for every $\gamma > 0$ the natural ambient space for the limit energy is given by

$$\mathcal{V}^\gamma := \big\{(u,b) \in W^{1,q}\left(\Omega;\mathbb{R}^3\right) \times L^q\left(\Omega;\mathbb{R}^3\right) : D_3 u = 0 \ \mathcal{L}^3 \text{ a.e. in } \Omega, \\ D_3 b = 0 \ \mathcal{L}^3 \text{ a.e. in } \Omega \text{ if } \gamma < 2,\ D_3 b \in L^2\left(\Omega;\mathbb{R}^3\right) \text{ if } \gamma = 2\big\}. \tag{2.2}$$

In what follows we identify functions $u \in W^{1,q}\left(\Omega;\mathbb{R}^3\right)$ such that $D_3u = 0$ $\mathcal{L}^3$ a.e. in $\Omega$ with functions in $W^{1,q}(\omega;\mathbb{R}^3)$, and, similarly, functions $b \in L^q\left(\Omega;\mathbb{R}^3\right)$ such that $D_3b = 0$ $\mathcal{L}^3$ a.e. in $\Omega$ with functions in $L^q\left(\omega;\mathbb{R}^3\right)$.

In view of (2.1) the appropriate notion of $\Gamma$-convergence in our setting is the following: Let $\varepsilon_n \to 0^+$. We say that a functional

$$E_-^\gamma : \mathcal{V}^\gamma \times \mathcal{A}(\omega) \to [0,\infty]$$

is the $\Gamma$-$\liminf$ of the sequence of functionals $\{E_{\varepsilon_n}^\gamma\}$ with respect to the weak convergence in $W^{1,q}\left(\Omega;\mathbb{R}^3\right) \times L^q\left(\Omega;\mathbb{R}^3\right)$ if for every $(u,b) \in \mathcal{V}^\gamma$ and $A \in \mathcal{A}(\omega)$,

$$E_-^\gamma(u,b;A) = \inf\left\{\liminf_{n\to\infty} E_{\varepsilon_n}^\gamma(u_n;A) :\ u_n \in W^{1,q}\left(\Omega;\mathbb{R}^3\right),\ u_n \rightharpoonup u \text{ in } W^{1,q}\left(\Omega;\mathbb{R}^3\right),\right.$$
$$\left.\frac{1}{\varepsilon_n} D_3 u_n \rightharpoonup b \text{ in } L^q\left(\Omega;\mathbb{R}^3\right)\right\},$$

and we write

$$E_-^\gamma = \Gamma\text{-}\liminf_{n\to\infty} E_{\varepsilon_n}^\gamma.$$

Since $E_{\varepsilon_n}^\gamma(u_n;A) = \infty$ if $u_n \notin W^{2,2}\left(\Omega;\mathbb{R}^3\right)$, it is clear that we may write

$$E_-^\gamma(u,b;A) = \inf\left\{\liminf_{n\to\infty} E_{\varepsilon_n}^\gamma(u_n;A) :\ u_n \in W^{2,2}\left(\Omega;\mathbb{R}^3\right),\ u_n \rightharpoonup u \text{ in } W^{1,q}\left(\Omega;\mathbb{R}^3\right),\right.$$
$$\left.\frac{1}{\varepsilon_n} D_3 u_n \rightharpoonup b \text{ in } L^q\left(\Omega;\mathbb{R}^3\right)\right\}.$$

A standard diagonalization argument, together with the coercivity hypothesis $(H_1)$, yields the following result, whose proof we omit. We remark that this argument holds only for $q > 1$.

Proposition 2.2. Assume that $W$ satisfies condition $(H_1)$. Then for any open subset $A \subset \omega$, the functional $E_-^\gamma(\cdot,\cdot;A)$ is sequentially lower semi-continuous with respect to the weak convergence in $W^{1,q}(\Omega;\mathbb{R}^3) \times L^q(\Omega;\mathbb{R}^3)$.

Similarly, we say that a functional

$$E_+^\gamma : \mathcal{V}^\gamma \times \mathcal{A}(\omega) \to [0,\infty]$$

is the $\Gamma$-$\limsup$ of the sequence of functionals $\{E_{\varepsilon_n}^\gamma\}$ with respect to the weak-weak convergence in $W^{1,q}\left(\Omega;\mathbb{R}^3\right) \times L^q\left(\Omega;\mathbb{R}^3\right)$ if for every $(u,b) \in \mathcal{V}^\gamma$ and $A \in \mathcal{A}(\omega)$,

$$E_+^\gamma(u,b;A) = \inf\left\{\limsup_{n\to\infty} E_{\varepsilon_n}^\gamma(u_n;A) :\ u_n \in W^{2,2}\left(\Omega;\mathbb{R}^3\right),\ u_n \rightharpoonup u \text{ in } W^{1,q}\left(\Omega;\mathbb{R}^3\right),\right. \tag{2.3}$$
$$\left.\frac{1}{\varepsilon_n} D_3 u_n \rightharpoonup b \text{ in } L^q\left(\Omega;\mathbb{R}^3\right)\right\},$$

and we write

$$E_+^\gamma = \Gamma\text{-}\limsup_{n\to\infty} E_{\varepsilon_n}^\gamma.$$

We say that the sequence $\{E^\gamma_{\varepsilon_n}\}$ $\Gamma$-*converges* to a functional $E^\gamma$ if the $\Gamma$-$\liminf$ and $\Gamma$-$\limsup$ coincide, and we write

$$E^\gamma = \Gamma\text{-}\lim_{n\to\infty} E^\gamma_{\varepsilon_n}.$$

DEFINITION 2.3. The functional $E^\gamma$ is said to be the $\Gamma$-limit of the *family* of functionals $\{E^\gamma_\varepsilon\}_{\varepsilon>0}$ with respect to weak convergence in $W^{1,q}\left(\Omega;\mathbb{R}^3\right)\times L^q\left(\Omega;\mathbb{R}^3\right)$ if for *every* sequence $\varepsilon_n \to 0^+$ we have that

$$E^\gamma = \Gamma\text{-}\lim_{n\to\infty} E^\gamma_{\varepsilon_n},$$

and we write

$$E^\gamma = \Gamma\text{-}\lim_{\varepsilon\to 0} E^\gamma_\varepsilon.$$

We conclude this section with two results which will be useful in the sequel. For a proof we refer to [14], [15].

LEMMA 2.4 (Decomposition). Let $E\subset\mathbb{R}^N$ be a bounded Lebesgue measurable set and let $\{u_n\}$ be a sequence of functions uniformly bounded in $L^q\left(E;\mathbb{R}^d\right)$, $1\le q<\infty$. For $r>0$ consider the truncation $\tau_r:\mathbb{R}^d\to\mathbb{R}^d$ defined by

$$\tau_r(z) := \begin{cases} z & \text{if } \ |z|\le r, \\ \dfrac{z}{|z|}r & \text{if } \ |z|>r. \end{cases}$$

Then there exists a subsequence of $\{u_n\}$ (not relabeled) and an increasing sequence of numbers $r_n\to\infty$ such that the truncated sequence $\{\tau_{r_n}\circ u_n\}$ is $q$-equi-integrable, and

$$\left|\{x\in E:\ u_n(x)\neq (\tau_{r_n}\circ u_n)(x)\}\right|\to 0.$$

THEOREM 2.5. Let $\Omega\subset\mathbb{R}^N$ be an open bounded set, let $1<q<\infty$, and let $\{u_n\}\subset W^{1,q}(\Omega;\mathbb{R}^d)$ be a sequence of functions converging weakly in $W^{1,q}(\Omega;\mathbb{R}^d)$ to some function $u_0\in W^{1,q}(\Omega;\mathbb{R}^d)$. Then there exist a subsequence $\{u_{n_k}\}$ and a sequence $\{v_k\}\subset W^{1,q}(\mathbb{R}^N;\mathbb{R}^d)$ such that $\{v_k\}$ converges to $u_0$ weakly in $W^{1,q}(\Omega;\mathbb{R}^d)$, $v_k=u_0$ in a neighborhood of $\partial\Omega$,

$$\left|\{x\in\Omega:\ v_k(x)\neq u_{n_k}(x)\}\right|\to 0 \quad \text{as } k\to\infty \tag{2.4}$$

and $\{|\nabla v_k|^q\}$ is equi-integrable.

**3. $\Gamma$-convergence.** In this section, under standard $q$-growth and coercivity conditions on $W$, we prove that the family of functionals $\{E^\gamma_\varepsilon\}_{\varepsilon>0}$ $\Gamma$-converges to the functional $H^\gamma$ defined as follows: for $(u,b)\in\mathcal{V}^\gamma$ and $A\in\mathcal{A}(\omega)$,

$$H^\gamma(u,b,A) := \inf\left\{\liminf_{n\to\infty}\int_{A\times I} W\left(D_p u_n\,|b_n\right)dx:\ \{u_n\}\subset W^{1,q}(\Omega;\mathbb{R}^3),\right.$$
$$\left.\{b_n\}\subset L^q\left(\Omega;\mathbb{R}^3\right),\ u_n\rightharpoonup u \text{ in } W^{1,q}(\Omega;\mathbb{R}^3),\ b_n\rightharpoonup b \text{ in } L^q(\Omega;\mathbb{R}^3)\right\}$$

if $\gamma \neq 2$, and

$$H^2(u,b,A) := \inf\Bigg\{ \liminf_{n\to\infty} \int_{A\times I} \left[ W\left(D_p u_n \,|b_n\right) + |D_3 b_n|^2 \right] dx : \{u_n\} \subset W^{1,q}(\Omega;\mathbb{R}^3), \\ \{b_n\} \subset L^q\left(\Omega;\mathbb{R}^3\right) \text{ with } D_3 b_n \in L^2\left(\Omega;\mathbb{R}^3\right), \\ u_n \rightharpoonup u \text{ in } W^{1,q}(\Omega;\mathbb{R}^3),\ b_n \rightharpoonup b \text{ in } L^q(\Omega;\mathbb{R}^3) \Bigg\}$$

if $\gamma = 2$.

In the remainder of the paper we assume that condition $(H_1)$ is strengthened as follows:
$(H_1)'$ $W : \mathbb{R}^{3\times 3} \to [0,\infty)$ is continuous and there exists $C > 0$ such that

$$\frac{1}{C}|F|^q - C \leq W(F) \leq C\left(1 + |F|^q\right)$$

for all $F \in \mathbb{R}^{3\times 3}$.

THEOREM 3.1 ($\Gamma$-convergence). Assume that condition $(H_1)'$ is satisfied. Then the family of functionals $\{E_\varepsilon^\gamma\}_{\varepsilon>0}$ $\Gamma$-converges to the functional $H^\gamma$.

*Proof.* Fix a sequence $\varepsilon_n \to 0^+$ and let

$$E_-^\gamma := \Gamma\text{-}\liminf_{n\to\infty} E_{\varepsilon_n}^\gamma.$$

Lower bound. We claim that

$$E_-^\gamma(u,b;A) \geq H^\gamma(u,b,A) \tag{3.1}$$

for all $(u,b) \in \mathcal{V}^\gamma$ and $A \in \mathcal{A}(\omega)$.

Fix $(u,b) \in \mathcal{V}^\gamma$ and $A \in \mathcal{A}(\omega)$, and consider any sequence $\{u_n\} \subset W^{2,2}\left(\Omega;\mathbb{R}^3\right)$ such that $u_n \rightharpoonup u$ in $W^{1,q}(\Omega;\mathbb{R}^3)$, $\frac{1}{\varepsilon_n} D_3 u_n \rightharpoonup b$ in $L^q(\Omega;\mathbb{R}^3)$. If $\gamma \neq 2$, then

$$\liminf_{n\to\infty} E_{\varepsilon_n}^\gamma(u_n;A) \geq \liminf_{n\to\infty} \int_{A\times I} W\left(D_p u_n \left| \frac{1}{\varepsilon_n} D_3 u_n\right.\right) dx \\ \geq H^\gamma(u,b,A),$$

while if $\gamma = 2$,

$$\liminf_{n\to\infty} E_{\varepsilon_n}^2(u_n;A) \geq \liminf_{n\to\infty} \int_{A\times I} \left[ W\left(D_p u_n \left| \frac{1}{\varepsilon_n} D_3 u_n\right.\right) + \left|\frac{1}{\varepsilon_n} D_{33} u_n\right|^2 \right] dx \\ \geq H^2(u,b,A).$$

By taking the infimum over all such sequences $\{u_n\}$ in both cases we obtain (3.1).

Upper bound. We claim that for every $(u,b) \in \mathcal{V}^\gamma$ and for all $A \in \mathcal{A}(\omega)$,

$$E_-^\gamma(u,b;A) \leq \int_{A\times I} W\left(D_p u \,|b\right) dx \tag{3.2}$$

if $\gamma \neq 2$, while

$$E_-^2(u,b;A) \leq \int_{A\times I} \left( W\left(D_p u \,|b\right) + |D_3 b|^2 \right) dx. \tag{3.3}$$

Extend $u$ as a $W^{1,q}\left(\mathbb{R}^3;\mathbb{R}^3\right)$ function with

$$\|u\|_{W^{1,q}(\mathbb{R}^3;\mathbb{R}^3)} \leq C\|u\|_{W^{1,q}(\Omega;\mathbb{R}^3)}$$

and $D_3 u = 0$ $\mathcal{L}^3$ a.e. in $\mathbb{R}^3$. Extend also $b$ as an $L^q\left(\mathbb{R}^3;\mathbb{R}^3\right)$ function with

$$\|b\|_{L^q(\mathbb{R}^3;\mathbb{R}^3)} \leq C\|b\|_{L^q(\Omega;\mathbb{R}^3)},$$

and $D_3 b = 0$ $\mathcal{L}^3$ a.e. in $\mathbb{R}^3$ if $\gamma < 2$, and $D_3 b \in L^2\left(\mathbb{R}^3;\mathbb{R}^3\right)$ with

$$\|D_3 b\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \leq C\|D_3 b\|_{L^2(\Omega;\mathbb{R}^3)}$$

if $\gamma = 2$.

Consider a mollifier of the type

$$\varphi_j\left(x\right) := \frac{1}{\delta_j^3}\varphi\left(\frac{x}{\delta_j}\right),$$

where $\delta_j > 0$ and $\varphi \in C_c^\infty\left(\mathbb{R}^3\right)$ with $\int_{\mathbb{R}^3}\varphi\left(x\right)\,dx = 1$. Set

$$u_{nj}\left(x\right) := \left(u * \varphi_j\right)\left(x\right) + \varepsilon_n \int_0^{x_3} \left(b * \varphi_j\right)\left(x_\alpha, s\right)\,ds \quad \text{for } x \in \mathbb{R}^3.$$

Since $D_3\left(u * \varphi_j\right) = D_3 u * \varphi_j = 0$ (recall that $D_3 u = 0$ $\mathcal{L}^3$ a.e. in $\mathbb{R}^3$) it follows that $D_3 u_{n,j} = \varepsilon_n b * \varphi_j$, and so for every fixed $j$,

$$\frac{1}{\varepsilon_n} D_3 u_{n,j} = b * \varphi_j.$$

Moreover, for $i = 1, 2$,

$$D_i u_{n,j} = D_i\left(u * \varphi_j\right) + \varepsilon_n \int_0^{x_3} \left(b * D_i\varphi_j\right)\left(x_\alpha, s\right)\,ds,$$

and so

$$\left|D_i u_{n,j} - D_i\left(u * \varphi_j\right)\right| \leq \frac{C\varepsilon_n}{\delta_j}\|b\|_{L^q(\mathbb{R}^3;\mathbb{R}^3)}, \tag{3.4}$$

where we have used the fact that, by the Hölder inequality, for all $x \in \Omega$ we have

$$\begin{aligned} \left|\left(b * D_i\varphi_j\right)\left(x\right)\right| &= \left|\int_{\mathbb{R}^3} b\left(y\right) D_i\varphi_j\left(x - y\right)\,dy\right| \\ &\leq \|b\|_{L^q(\mathbb{R}^3;\mathbb{R}^3)}\|D_i\varphi_j\|_{L^{q'}(\mathbb{R}^3;\mathbb{R}^2)} \leq \frac{C}{\delta_j}\|b\|_{L^q(\Omega;\mathbb{R}^3)}. \end{aligned} \tag{3.5}$$

Hence for every fixed $j$ the sequence $\{u_{n,j}\}$ is admissible for $E_{-}^{\gamma}(u * \varphi_j, b * \varphi_j; A)$, and we have

$$\begin{aligned} E_{-}^{\gamma}(u*\varphi_j, b*\varphi_j; A) &\le \liminf_{n\to\infty} E_{\varepsilon_n}^{\gamma}(u_{nj}; A) \\ &\le \lim_{n\to\infty} \int_{A\times I} W\left(D_p u * \varphi_j + \frac{C\varepsilon_n}{\delta_j} O(1) \,|b*\varphi_j\right) dx \\ &\quad + \liminf_{n\to\infty} \int_{A\times I} \varepsilon_n^{\gamma}\left(\left|D_p^2 u_{n,j}\right|^2 + \frac{1}{\varepsilon_n^2}\left|D_{p3}u_{n,j}\right|^2 + \frac{1}{\varepsilon_n^4}\left|D_{33}u_{n,j}\right|^2\right) dx \\ &= \int_{A\times I} W\left(D_p u * \varphi_j \,|b*\varphi_j\right) dx \qquad (3.6) \\ &\quad + \liminf_{n\to\infty} \int_{A\times I} \varepsilon_n^{\gamma}\left(\left|D_p^2 u_{n,j}\right|^2 + \frac{1}{\varepsilon_n^2}\left|D_{p3}u_{n,j}\right|^2 + \frac{1}{\varepsilon_n^4}\left|D_{33}u_{n,j}\right|^2\right) dx, \end{aligned}$$

where in the last equality we have used the Lebesgue Dominated Convergence Theorem together with the continuity of $W$. We claim that

$$\lim_{n\to\infty} \int_{A\times I} \varepsilon_n^{\gamma}\left(\left|D_p^2 u_{n,j}\right|^2 + \frac{1}{\varepsilon_n^2}\left|D_{p3}u_{n,j}\right|^2 + \frac{1}{\varepsilon_n^4}\left|D_{33}u_{n,j}\right|^2\right) dx = 0 \qquad (3.7)$$

if $\gamma \neq 2$, while

$$\begin{aligned} &\lim_{n\to\infty} \int_{A\times I} \varepsilon_n^{2}\left(\left|D_p^2 u_{n,j}\right|^2 + \frac{1}{\varepsilon_n^2}\left|D_{p3}u_{n,j}\right|^2 + \frac{1}{\varepsilon_n^4}\left|D_{33}u_{n,j}\right|^2\right) dx \qquad (3.8) \\ &= \int_{A\times I} \left|D_3 b * \varphi_j\right|^2 dx \end{aligned}$$

if $\gamma = 2$.

For $i,\ k = 1, 2$,

$$\begin{aligned} D_{ik}u_{n,j}(x) &= (D_i u * D_k\varphi_j)(x) + \varepsilon_n \int_0^{x_3} (b * D_{ik}\varphi_j)(x_\alpha, s)\, ds, \\ D_{k3}u_{n,j}(x) &= \varepsilon_n (b * D_k\varphi_j)(x), \end{aligned}$$

and so, reasoning as in (3.5),

$$\begin{aligned} \left\|D_p^2 u_{n,j}\right\|_\infty &\le \frac{C}{\delta_j}\|D_p u\|_{L^q(\Omega;\mathbb{R}^{3\times 2})} + \frac{C\varepsilon_n}{\delta_j^2}\|b\|_{L^q(\Omega;\mathbb{R}^3)}, \\ \left\|D_{p3}u_{n,j}\right\|_\infty &\le \frac{C\varepsilon_n}{\delta_j}\|b\|_{L^q(\Omega;\mathbb{R}^3)}, \end{aligned}$$

which clearly imply that

$$\lim_{n\to\infty} \int_{A\times I} \varepsilon_n^{\gamma}\left(\left|D_p^2 u_{n,j}\right|^2 + \frac{1}{\varepsilon_n^2}\left|D_{p3}u_{n,j}\right|^2\right) dx = 0.$$

To estimate the last term in (3.7) note that if $\gamma < 2$, then

$$D_{33}u_{n,j} = \varepsilon_n D_3 b * \varphi_j \equiv 0$$

since $D_3 b = 0$ $\mathcal{L}^3$ a.e. in $\mathbb{R}^3$, while if $\gamma > 2$, then

$$D_{33}u_{n,j}(x) = \varepsilon_n (b * D_3\varphi_j)(x),$$

and so

$$\|D_{33}u_{n,j}\|_\infty \le \frac{C\varepsilon_n}{\delta_j}\|b\|_{L^q(\Omega;\mathbb{R}^3)}.$$

In turn

$$\varepsilon_n^{\gamma-4}\|D_{33}u_{n,j}\|_\infty^2 \le \frac{C\varepsilon_n^{\gamma-2}}{\delta_j^2}\|b\|_{L^q(\Omega;\mathbb{R}^3)}^2,$$

which yields

$$\lim_{n\to\infty}\int_{A\times I}\varepsilon_n^{\gamma-4}\left|D_{33}u_{n,j}\right|^2\,dx=0.$$

Finally, if $\gamma=2$, then

$$D_{33}u_{n,j}=\varepsilon_n D_3 b*\varphi_j$$

and thus

$$\lim_{n\to\infty}\int_{A\times I}\frac{1}{\varepsilon_n^2}\left|D_{33}u_{n,j}\right|^2\,dx=\int_{A\times I}\left|D_3 b*\varphi_j\right|^2\,dx.$$

Hence (3.7) and (3.8) hold, and by (3.6) we deduce that

$$\begin{aligned}&E_-^\gamma(u*\varphi_j,b*\varphi_j;A)\\&\le\begin{cases}\int_{A\times I}W\left(D_p u*\varphi_j\,|b*\varphi_j\right)\,dx & \text{if } \gamma\neq 2,\\ \int_{A\times I}\left[W\left(D_p u*\varphi_j\,|b*\varphi_j\right)+\left|D_3 b*\varphi_j\right|^2\right]dx & \text{if } \gamma=2.\end{cases}\end{aligned}$$

Since $u*\varphi_j\to u$ in $W^{1,q}\left(\Omega;\mathbb{R}^3\right)$, $b*\varphi_j\to b$ in $L^q\left(\Omega;\mathbb{R}^3\right)$, and when $\gamma=2$ also $D_3 b*\varphi_j\to D_3 b$ in $L^q\left(\Omega;\mathbb{R}^3\right)$, letting $j\to\infty$ in the previous inequality and using Proposition 2.2 and $(H_1)'$ we obtain (3.2) and (3.3).

Fix $(u,b)\in\mathcal{V}^\gamma$ and $A\in\mathcal{A}(\omega)$ and let $\{u_j\}\subset W^{1,q}(\Omega;\mathbb{R}^3)$ converge weakly to $u$ in $W^{1,q}(\Omega;\mathbb{R}^3)$, and $\{b_j\}\subset L^q(\Omega;\mathbb{R}^3)$ converge weakly to $b$ in $L^q(\Omega;\mathbb{R}^3)$. Using Proposition 2.2, (3.2) and (3.3) we have that

$$E_-^\gamma(u,b;A)\le\liminf_{j\to\infty}E_-^\gamma(u_j,b_j;A)\le\liminf_{j\to\infty}\int_{A\times I}W\left(D_p u_j\,|b_j\right)\,dx$$

if $\gamma\neq 2$, and

$$\begin{aligned}E_-^\gamma(u,b;A)&\le\liminf_{j\to\infty}E_-^\gamma(u_j,b_j;A)\\&\le\liminf_{j\to\infty}\int_{A\times I}\left[W\left(D_p u_j\,|b_j\right)+\left|D_3 b_j\right|^2\right]\,dx.\end{aligned}$$

Taking the infimum over all such sequences $\{u_j\}$ and $\{b_j\}$ yields

$$E_-^\gamma\,(u,b;A)\le H^\gamma\,(u,b,A)$$

for all $(u,b)\in\mathcal{V}^\gamma$ and $A\in\mathcal{A}(\omega)$. Together with (3.1) this yields

$$\Gamma\text{-}\lim_{n\to\infty}E_{\varepsilon_n}^\gamma=H^\gamma.$$

In turn, given the arbitrariness of the sequence $\varepsilon_n\to 0^+$, by Definition 2.3 we obtain that

$$\Gamma\text{-}\lim_{\varepsilon\to 0^+}E_\varepsilon^\gamma=H^\gamma,$$

and this completes the proof. □

Thus it remains to identify the functional $H^\gamma$. We consider separately the two ranges $\gamma \neq 2$ and $\gamma = 2$.

THEOREM 3.2 ($\gamma \neq 2$). Assume that $\gamma \neq 2$ and that condition $(H_1)'$ is satisfied. Then for all $(u, b) \in \mathcal{V}^\gamma$ and $A \in \mathcal{A}(\omega)$,

$$H^\gamma(u, b, A) = \int_{A \times I} (Q_3 \times C_3)[W](D_p u(x_\alpha) | b(x))\, dx,$$

where $(Q_3 \times C_3)[W]$ is the cross quasiconvexification-convexification of $W$,

$$(Q_3 \times C_3)[W]\left(\overline{F} | z\right) := \inf\left\{ \int_Q W\left(\overline{F} + D_p\varphi(x) | z + \phi(x)\right) dx\,, \right.$$
$$\left. \varphi \in W_0^{1,q}\left(Q; \mathbb{R}^3\right), \phi \in L^q\left(Q; \mathbb{R}^3\right), \int_Q \phi(x)\, dx = 0 \right\}$$

for $\overline{F} \in \mathbb{R}^{2\times 3}$, $z \in \mathbb{R}^3$.

*Proof.* The proof of this result is standard, and so we omit it (see e.g. [13]). □

REMARK 3.3. Note that if $\gamma \neq 2$, then

$$H^\gamma(u, b, A) = \int_{A \times I} (Q_2 \times C_2)[W](D_p u(x_\alpha) | b(x))\, dx, \tag{3.9}$$

where $(Q_2 \times C_2)[W]$ is defined as

$$(Q_2 \times C_2)[W]\left(\overline{F} | z\right) := \inf\left\{ \int_{Q'} W\left(\overline{F} + D_p\varphi(x_\alpha) | z + \phi(x_\alpha)\right) dx_\alpha\,, \right.$$
$$\left. \varphi \in W_0^{1,q}\left(Q'; \mathbb{R}^3\right), \phi \in L^q\left(Q'; \mathbb{R}^3\right), \int_{Q'} \phi(x_\alpha)\, dx_\alpha = 0 \right\}$$

for $\overline{F} \in \mathbb{R}^{2\times 3}$, $z \in \mathbb{R}^3$.

To see this we begin by observing that in the definition of $(Q_3 \times C_3)[W]$ (resp. of $(Q_2 \times C_2)[W]$) $Q$-periodic functions in $W^{1,q}\left(Q; \mathbb{R}^3\right)$ (resp. $Q'$-periodic functions in $W^{1,q}(Q'; \mathbb{R}^3)$) can also be used in lieu of $W_0^{1,q}(Q; \mathbb{R}^3)$-functions (resp. $W_0^{1,q}(Q'; \mathbb{R}^3)$-functions) (see [3], Conjecture 3.7 and Theorem 3.1). Hence

$$(Q_3 \times C_3)[W](\overline{F}|z) \leq (Q_2 \times C_2)[W](\overline{F}|z).$$

Conversely, for $\varepsilon > 0$ find $\varphi$, $\phi$ admissible in the definition of $(Q_3 \times C_3)[W]\left(\overline{F}, z\right)$ and such that

$$(Q_3 \times C_3)[W]\left(\overline{F}, z\right) \geq \int_Q W\left(\overline{F} + D_p\varphi(x) | z + \phi(x)\right) dx - \varepsilon.$$

Then

$$
\begin{aligned}
&(Q_3 \times C_3)\,[W]\left(\overline{F}, z\right) \\
&\geq \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{Q'} W\left(\overline{F} + D_p\varphi(x)|\overline{z}(x_3) + \phi(x) - \int_{Q'} \phi(y_\alpha, x_3)\,dy_\alpha\right)\,dx_\alpha\,dx_3 - \varepsilon \\
&\geq \int_{-\frac{1}{2}}^{\frac{1}{2}} (Q_2 \times C_2)\,[W](\overline{F}|\overline{z}(x_3))\,dx_3 - \varepsilon \\
&\geq (Q_2 \times C_2)\,[W]\left(\overline{F}|\int_{-\frac{1}{2}}^{\frac{1}{2}} \overline{z}(x_3)\,dx_3\right) - \varepsilon \\
&= (Q_2 \times C_2)\,[W]\left(\overline{F}|z\right) - \varepsilon,
\end{aligned}
$$

where

$$\overline{z}(x_3) := z + \int_{Q'} \phi(y_\alpha, x_3)\,dy_\alpha,$$

and where the last inequality holds by Jensen's inequality, because $(Q_2 \times C_2)[W](F|\cdot)$ is convex and $\phi$ has zero average over $Q$. Given the arbitrariness of $\varepsilon$ we conclude that

$$(Q_3 \times C_3)\,[W](\overline{F}|z) = (Q_2 \times C_2)\,[W](\overline{F}|z), \tag{3.10}$$

which, together with the previous theorem, yields (3.9). In particular, if $W$ is independent of $z$, then we recover the well-known identity

$$Q_3 W(\overline{F}) = Q_2 W(\overline{F}). \tag{3.11}$$

**4. The critical case $\gamma = 2$.** In this section we study the critical case $\gamma = 2$. We recall that

$$
\begin{aligned}
H^2(u, b, A) := \inf\Bigg\{ &\liminf_{n\to\infty} \int_{A\times I} \left(W\left(D_p u_n\,|b_n\right) + |D_3 b_n|^2\right)\,dx : \\
&\{u_n\} \subset W^{1,q}(\Omega; \mathbb{R}^3), \{b_n\} \subset L^q\left(\Omega; \mathbb{R}^3\right) \text{ with } D_3 b_n \in L^2\left(\Omega; \mathbb{R}^3\right), \\
&u_n \rightharpoonup u \text{ in } W^{1,q}(\Omega; \mathbb{R}^3),\ b_n \rightharpoonup b \text{ in } L^q(\Omega; \mathbb{R}^3)\Bigg\}
\end{aligned}
$$

for all $(u, b) \in \mathcal{V}^2$ and $A \in \mathcal{A}(\omega)$, where

$$
\begin{aligned}
\mathcal{V}^2 := \big\{(u, b) \in W^{1,q}\left(\Omega; \mathbb{R}^3\right) \times L^q\left(\Omega; \mathbb{R}^3\right) :\ &D_3 u = 0\ \mathcal{L}^3 \text{ a.e. in } \Omega, \\
&D_3 b \in L^2\left(\Omega; \mathbb{R}^3\right)\big\}.
\end{aligned}
$$

We begin by characterizing $H^2(u, b, A)$ for functions $b$ which do not depend on $x_3$. Note that in this case $(u, b) \in \mathcal{V}^2$ may be identified with a function in $W^{1,q}(\omega; \mathbb{R}^3) \times L^q\left(\omega; \mathbb{R}^3\right)$.

Theorem 4.1. Assume that condition $(H_1)'$ is satisfied. Then for all $(u, b) \in \mathcal{V}^2$ with $D_3 b = 0\ \mathcal{L}^3$ a.e. in $\Omega$, and for all $A \in \mathcal{A}(\omega)$,

$$H^2(u, b, A) = \int_A (Q_2 \times C_2)\,[W]\left(D_p u\left(x_\alpha\right)|b\left(x_\alpha\right)\right)\,dx_\alpha.$$

*Proof.* It is clear that

$$H^2(u,b,A) \le \inf\Big\{ \liminf_{n\to\infty} \int_A W\left(D_p u_n(x_\alpha)\,|b_n(x_\alpha)\right) dx_\alpha : \{u_n\} \subset W^{1,q}(\omega;\mathbb{R}^3),$$
$$\{b_n\} \subset L^q\left(\omega;\mathbb{R}^3\right),\ u_n \rightharpoonup u \text{ in } W^{1,q}(\omega;\mathbb{R}^3),\ b_n \rightharpoonup b \text{ in } L^q(\omega;\mathbb{R}^3)\Big\}$$

since the infimum on the right-hand side is taken over a smaller class of sequences (i.e., those independent of $x_3$). By standard results (see e.g [13]) the right-hand side coincides with

$$\int_A (Q_2 \times C_2)[W]\left(D_p u(x_\alpha)\,|b(x_\alpha)\right) dx_\alpha,$$

and so we have

$$H^2(u,b,A) \le \int_A (Q_2 \times C_2)[W]\left(D_p u(x_\alpha)\,|b(x_\alpha)\right) dx_\alpha.$$

On the other hand, since $W \ge (Q_3 \times C_3)[W]$, by classical lower semi-continuity results we deduce that

$$H^2(u,b;A) \ge \int_A (Q_3 \times C_3)[W]\left(D_p u(x_\alpha)\,|b(x_\alpha)\right) dx_\alpha,$$

and now it suffices to recall (3.10). □

When the function $b$ depends on the $x_3$ variable the situation is significantly more involved, and the representation obtained in Theorem 4.4 below is considerably less explicit.

For $u \in W^{1,q}(\Omega;\mathbb{R}^3)$, $b \in L^q\left(\Omega;\mathbb{R}^3\right)$ with $D_3 b \in L^2\left(\Omega;\mathbb{R}^3\right)$ and $A \in \mathcal{A}(\omega)$ define

$$F(u,b;A) := \int_{A\times I} \left[W\left(D_p u\,|b\right) + |D_3 b|^2\right] dx.$$

Similar arguments to those used in the proof of Theorem 4.2 below may be found in [10], Th. 1, page 24 (see also Chapter 11 in [6] for an alternative proof based on the De Giorgi Slicing Lemma).

THEOREM 4.2. Assume that condition $(H_1)'$ is satisfied. Then for every $(u,b) \in \mathcal{V}^2$ the set function $H^2(u,b;\cdot)$ is the trace of a Radon measure absolutely continuous with respect to $\mathcal{L}^2\lfloor\omega$.

*Proof.* **Step 1:** Fix $(u,b) \in \mathcal{V}^2$. We claim that

$$H^2(u,b;A_1) \le H^2(u,b;A_2) + H^2(u,b;A_1 \setminus \overline{A_3}) \tag{4.1}$$

for all $A_1$, $A_2$, $A_3 \in \mathcal{A}(\omega)$, with $A_3 \subset\subset A_2 \subset A_1$.

Without loss of generality we may assume that the right-hand side of the previous inequality is finite.

Fix $\eta > 0$ and find $\{u_n\}$, $\{v_n\} \subset W^{1,q}(\Omega;\mathbb{R}^3)$ converging weakly to $u$ in $W^{1,q}(\Omega;\mathbb{R}^3)$ and $\{b_n\}$, $\{z_n\} \subset L^q\left(\Omega;\mathbb{R}^3\right)$ converging weakly to $b$ in $L^q(\Omega;\mathbb{R}^3)$ such that $D_3 b_n$, $D_3 z_n \in$

$L^2\left(\Omega;\mathbb{R}^3\right)$ for all $n\in\mathbb{N}$, and

$$\lim_{n\to\infty} F\left(u_n,b_n;\left(A_1\setminus\overline{A_3}\right)\right)\le H^2(u,b;A_1\setminus\overline{A_3})+\eta, \tag{4.2}$$

$$\lim_{n\to\infty} F\left(v_n,z_n;A_2\right)\le H^2(u,b;A_2)+\eta, \tag{4.3}$$

$$\sup_n\left(F\left(u_n,b_n;\left(A_1\setminus\overline{A_3}\right)\right)+F\left(v_n,z_n;A_2\right)\right)<\infty. \tag{4.4}$$

For every $v\in W^{1,q}(\Omega;\mathbb{R}^3)$ and $z\in L^q\left(\Omega;\mathbb{R}^3\right)$ with $Dz\in L^2\left(\Omega;\mathbb{R}^3\right)$ and for every Borel set $E\subset\omega$ define

$$\mathcal{G}(v,z;E):=\int_{E\times I}\left(1+|D_\alpha v|^q+|z|^q+|D_3 z|^2\right)dx.$$

Due to the coercivity hypothesis $(H_1)'$ and (4.4) we may extract a bounded subsequence from the sequence of measures $\nu_j:=\mathcal{G}(u_{n_j},b_{n_j};\cdot)+\mathcal{G}(v_{n_j},b_{n_j};\cdot)$ restricted to $A_2\setminus\overline{A_3}$ converging $\star$-weakly to some Radon measure $\nu$ defined on $A_2\setminus\overline{A_3}$.

Find $t>0$ so small such that the set

$$S_t:=\{x_\alpha\in A_2:\operatorname{dist}(x_\alpha,\overline{A_3})=t\}$$

is nonempty and $\nu\left(S_t\right)=0$. For $\delta>0$ define

$$L_\delta:=\{x_\alpha\in A_2:\ \operatorname{dist}(x_\alpha,S_t)<\delta\}.$$

Choose $\delta$ so small that $L_\delta\subset A_2\setminus\overline{A_3}$. Consider a smooth cut-off function $\varphi_\delta\in C_0^\infty(A_2;[0,1])$ such that $\varphi_\delta=1$ in

$$\{x_\alpha\in A_2:\ \operatorname{dist}(x_\alpha,\partial A_3)<t-\delta\}$$

and $\varphi_\delta=0$ in

$$\{x_\alpha\in A_2:\ \operatorname{dist}(x_\alpha,\partial A_3)>t+\delta\},$$

with

$$\|D_p\varphi_\delta\|_{L^\infty(\omega)}\le C/\delta.$$

Define

$$\tilde{u}_j(x):=(1-\varphi_\delta(x_\alpha))u_{n_j}(x)+\varphi_\delta(x_\alpha)v_{n_j}(x),$$
$$\tilde{b}_j(x):=(1-\varphi_\delta(x_\alpha))b_{n_j}(x)+\varphi_\delta(x_\alpha)z_{n_j}(x).$$

Clearly $\tilde{u}_j\rightharpoonup u$ in $W^{1,q}(\Omega;\mathbb{R}^3)$, $\tilde{b}_j\rightharpoonup b$ in $L^q(\Omega;\mathbb{R}^3)$ as $j\to\infty$. By the growth condition $(H_1)'$, we have the estimate

$$F\left(\tilde{u}_j,\tilde{b}_j;A_1\right)\le F(u_{n_j},b_{n_j};A_1\setminus\overline{A_3})+F(v_{n_j},z_{n_j};A_2)$$
$$+C\left(\mathcal{G}(u_{n_j},b_{n_j};L_\delta)+\mathcal{G}(v_{n_j},z_{n_j};L_\delta)+\frac{1}{\delta^q}\int_{L_\delta\times I}|u_{n_j}-v_{n_j}|^q\,dx\right).$$

Passing to the limit as $j\to\infty$ in the previous inequality and using (4.2) and (4.3), we have

$$H^2(u,b;A_1)\le H^2(u,b;A_1\setminus\overline{A_3})+H^2(u,b;A_2)+2\eta+C\nu(\overline{L_\delta}),$$

and letting $\delta$ go to zero we obtain

$$\begin{aligned} H^2(u,b;A_1) &\le H^2(u,b;A_2) + H^2(u,b;A_1\setminus\overline{A_3}) + 2\eta + C\nu(S_t) \\ &= H^2(u,b;A_2) + H^2(u,b;A_1\setminus\overline{A_3}) + 2\eta. \end{aligned}$$

It suffices to let $\eta \to 0^+$.

**Step 2:** In view of (4.1) and $(H_1)'$ it follows from standard arguments that the set function $H^2(u,b;\cdot)$ is the trace of a Borel measure (see [2] and Theorem 2.6 in [7]). Moreover we have

$$H^2(u,b;A) \le C\int_{A\times I}\left[1+|D_pu|^q+|b|^q+|D_3b|^2\right]\,dx,$$

and thus $H^2(u,b;\cdot)$ is absolutely continuous with respect to $\mathcal{L}^2$. ☐

As an immediate consequence of the previous theorem we have

$$H^2(u,b;A) = \int_A \frac{dH^2(u,b;\cdot)}{d\mathcal{L}^2}(x_\alpha)\,dx_\alpha,$$

where $\frac{dH^2(u,b;\cdot)}{d\mathcal{L}^2}$ is the Radon-Nikodym derivative of $H^2(u,b;\cdot)$ with respect to the Lebesgue measure on $\mathbb{R}^2$. In order to identify $\frac{dH^2(u,b;\cdot)}{d\mathcal{L}^2}$ we introduce the functional

$$\overline{\mathcal{W}}:\mathbb{R}^{2\times 3}\times W^{1,2}(I;\mathbb{R}^3)\to[0,\infty)$$

defined for $\overline{F}\in\mathbb{R}^{2\times 3},\ b\in W^{1,2}(I;\mathbb{R}^3)$, as follows:

$$\begin{aligned} &\overline{\mathcal{W}}(\overline{F}|b) := \int_{-\frac12}^{\frac12}|D_3b(x_3)|^2\,dx_3 \\ &+\inf_{\varphi,\,\psi}\left\{\int_Q\left(W\left(\overline{F}+D_p\varphi(x)|b(x_3)+\psi\left(x\right)\right)+|D_3\psi(x)|^2\right)\,dx:\right. \\ &\varphi\in W^{1,q}(Q;\mathbb{R}^3),\ \varphi(\cdot,x_3)\ Q'\text{-periodic for }\mathcal{L}^1\text{ a.e. }x_3, \\ &\left.\psi\in L^q(Q;\mathbb{R}^3),\ D_3\psi\in L^2(Q;\mathbb{R}^3),\ \int_{Q'}\psi(x_\alpha,x_3)\,dx_\alpha=0\text{ for }\mathcal{L}^1\text{ a.e. }x_3\right\}. \end{aligned} \tag{4.5}$$

Note that

$$\begin{aligned} &\overline{\mathcal{W}}(\overline{F}|b) := \inf_{\varphi,\,\psi}\left\{\int_Q\left(W(\overline{F}+D_\alpha\varphi(x)|b(x_3)+\psi(x))+|D_3b(x_3)+D_3\psi(x)|^2\right)\,dx:\right. \\ &\varphi\in W^{1,q}(Q;\mathbb{R}^3),\ \varphi(\cdot,x_3)\ Q'\text{-periodic for }\mathcal{L}^1\text{ a.e. }x_3, \\ &\left.\psi\in L^q(Q;\mathbb{R}^3),\ D_3\psi\in L^2(Q;\mathbb{R}^3),\ \int_{Q'}\psi(x_\alpha,x_3)\,dx_\alpha=0\text{ for }\mathcal{L}^1\text{ a.e. }x_3\right\}. \end{aligned}$$

Indeed the equality holds because the admissible test functions $\psi$ satisfy

$$\int_{Q'}D_3\psi(x_\alpha,x_3)\,dx_\alpha=0\text{ for }\mathcal{L}^1\text{ a.e. }x_3.$$

REMARK 4.3. $\overline{\mathcal{W}}(\cdot,\cdot)$ is upper semi-continuous on $\mathbb{R}^{2\times 3} \times W^{1,2}(I;\mathbb{R}^3)$ equipped with its strong topology. Indeed, assume that $\overline{F}_j \to \overline{F}$ and $b_j \to b$ in $W^{1,2}(I;\mathbb{R}^3)$ and consider, for a fixed $\eta > 0$, $\varphi$, $\psi$ such that

$$\overline{\mathcal{W}}(\overline{F}, b) + \eta \geq \int_Q \left( W(\overline{F} + D_p\varphi | b + \psi) + |D_3 b + D_3\psi|^2 \right) dx.$$

Then $\varphi$, $\psi$ are admissible test functions in the definition of $\overline{\mathcal{W}}(F_j|b_j)$, so that, in view of the continuous character of $W$,

$$\begin{aligned}\limsup_{j\to\infty} \overline{\mathcal{W}}(F_j|b_j) &\leq \limsup_{j\to\infty} \int_Q \left( W(\overline{F}_j + D_p\varphi|b_j + \psi) + |D_3 b_j + D_3\psi|^2 \right)\, dx \\ &= \int_Q \left( W(\overline{F} + D_p\varphi|b + \psi) + |D_3 b + D_3\psi|^2 \right)\, dx \\ &\leq \overline{\mathcal{W}}(\overline{F}, b) + \eta.\end{aligned}$$

The result is obtained by letting $\eta$ tend to 0.

THEOREM 4.4. Assume that condition $(H_1)'$ is satisfied. Then for all $(u, b) \in \mathcal{V}^2$ and $A \in \mathcal{A}(\omega)$,

$$H^2(u, b; A) = \int_A \overline{\mathcal{W}}(D_p u(x_\alpha)|b(x_\alpha, \cdot))\, dx_\alpha, \tag{4.6}$$

where $\overline{\mathcal{W}}$ is defined in (4.5).

*Proof of the lower bound.* **Step 1:** Let $\{u_n\} \subset W^{1,q}(\Omega;\mathbb{R}^3)$ and $\{b_n\} \subset L^q(\Omega;\mathbb{R}^3)$ with $D_3 b_n \in L^2(\Omega;\mathbb{R}^3)$ be such that $u_n \rightharpoonup u$ in $W^{1,q}(\Omega;\mathbb{R}^3)$, $b_n \rightharpoonup b$ in $L^q(\Omega;\mathbb{R}^3)$, and

$$\lim_{n\to\infty} \int_{A\times I} (W(D_p u_n|b_n) + |D_3 b_n|^2)\, dx < \infty. \tag{4.7}$$

Apply Lemma 2.4 and Theorem 2.5 to obtain subsequences $\{u_{n_k}\}$, $\{b_{n_k}\}$, a sequence $\{v_k\} \subset W^{1,q}(\mathbb{R}^3;\mathbb{R}^3)$ and a sequence $m_k \nearrow \infty$ such that $\{v_k\}$ converges to $u$ weakly in $W^{1,q}(\Omega;\mathbb{R}^3)$,

$$|\{x \in \Omega : v_k(x) \neq u_{n_k}(x) \text{ or } \tau_{m_k}(b_{n_k})(x) \neq b_{n_k}(x)\}| \to 0 \quad \text{as } k \to \infty, \tag{4.8}$$

and $\{|Dv_k|^q\}$ and $\{|\tau_{m_k}(b_{n_k})|^q\}$ are equi-integrable. Here

$$\tau_{m_k}(z) := \begin{cases} z & \text{if } |z| \leq m_k, \\ \frac{z}{|z|} m_k & \text{if } |z| > m_k. \end{cases}$$

Define $z_k := \tau_{m_k}(b_{n_k})$ and note that

$$\int_{A\times I} |D_3 z_k|^2\, dx \leq \int_{A\times I} |D_3 b_{n_k}|^2\, dx. \tag{4.9}$$

Indeed, for $\mathcal{L}^3$ a.e. $x \in A \times I$ such that $|b_{n_k}(x)| > m_k$ we have

$$\begin{aligned}D_3 z_k(x) &= \frac{m_k}{|b_{n_k}(x)|} \left( \mathbb{I} - \frac{b_{n_k}(x)}{|b_{n_k}(x)|} \otimes \frac{b_{n_k}(x)}{|b_{n_k}(x)|} \right) D_3 b_{n_k}(x) \\ &= \frac{m_k}{|b_{n_k}(x)|} \left( D_3 b_{n_k}(x) - \left( D_3 b_{n_k}(x) \cdot \frac{b_{n_k}(x)}{|b_{n_k}(x)|} \right) \frac{b_{n_k}(x)}{|b_{n_k}(x)|} \right)\end{aligned}$$

and so

$$|D_3 z_k(x)|^2 \leq |D_3 b_{n_k}(x)|^2 - \left( D_3 b_{n_k}(x) \cdot \frac{b_{n_k}(x)}{|b_{n_k}(x)|} \right)^2 .$$

Moreover $z_k \rightharpoonup b$ in $L^q(\Omega;\mathbb{R}^3)$. To see this let $\phi \in L^\infty(\Omega;\mathbb{R})$. Then

$$\begin{aligned}
\left| \int_\Omega z_k \phi \, dx - \int_\Omega b_{n_k} \phi \, dx \right| &= \left| \int_{\{z_k \neq b_{n_k}\}} (z_k - b_{n_k}) \phi \, dx \right| \\
&\leq \|\phi\|_\infty \int_{\{z_k \neq b_{n_k}\}} (|z_k| + |b_{n_k}|) \, dx \\
&\leq 2 \|\phi\|_\infty \int_{\{|b_{n_k}| > m_k\}} |b_{n_k}| \, dx \\
&\leq 2 \|\phi\|_\infty \mathcal{L}^3 (\{|b_{n_k}| > m_k\})^{\frac{1}{q'}} \|b_{n_k}\|_{L^q} \\
&\leq 2 \|\phi\|_\infty \left( \frac{1}{m_k} \right)^{\frac{q}{q'}} \|b_{n_k}\|_{L^q}^q \to 0
\end{aligned}$$

as $k \to \infty$ because $m_k \to \infty$ and $\{b_n\}$ is bounded in $L^q(\Omega;\mathbb{R}^3)$ with $q > 1$.

From (4.7) and (4.9),

$$\begin{aligned}
\infty &> \lim_{n\to\infty} \int_{A\times I} (W(D_p u_n | b_n) + |D_3 b_n|^2) \, dx \\
&= \lim_{k\to\infty} \int_{A\times I} (W(D_p u_{n_k} | b_{n_k}) + |D_3 b_{n_k}|^2) \, dx \\
&\geq \limsup_{k\to\infty} \left( \int_{\{v_k = u_{n_k},\, z_k = b_{n_k}\}} W(D_p v_k | z_k) \, dx + \int_{A\times I} |D_3 z_k|^2 \, dx \right) \qquad (4.10) \\
&= \limsup_{k\to\infty} \int_{A\times I} \left( W(D_p v_k | z_k) + |D_3 z_k|^2 \right) \, dy,
\end{aligned}$$

where in the last equality we have used (4.8), the growth condition $(H_1)'$ and the equi-integrability of $\{|\nabla v_k|^q\}$ and $\{|z_k|^q\}$.

We now invoke the de la Vallé-Poussin criterion to find a (nonnegative) function $\Phi$ such that

$$\lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty \qquad (4.11)$$

and

$$\{\Phi(|Dv_k|^q + |z_k|^q)\} \text{ is bounded in } L^1.$$

By (4.7), (4.9), and extracting a subsequence, if necessary, we may assume that

$$D_3 z_k \rightharpoonup D_3 b \text{ in } L^2(\Omega;\mathbb{R}^3) \qquad (4.12)$$

and that, since

$$\begin{aligned}
\mu_k &:= (W(Dv_k | z_k) + |D_3 z_k|^2) \mathcal{L}^3 \lfloor (A \times I) \, , \\
\lambda_k &:= \Phi(|Dv_k|^q + |z_k|^q) \mathcal{L}^3 \lfloor (A \times I)
\end{aligned}$$

are bounded sequences of nonnegative finite Radon measures, there exist nonnegative finite Radon measures $\mu$, $\lambda$ on $A \times I$ such that the subsequences of $\{\mu_k\}$ and $\{\lambda_k\}$ – still indexed by $k$ with no loss of generality – satisfy

$$\mu_k \overset{*}{\rightharpoonup} \mu, \quad \lambda_k \overset{*}{\rightharpoonup} \lambda \text{ in } \mathcal{M}(A \times I).$$

Denote by $\hat{\mu}$ and $\hat{\lambda}$ the finite Radon measures on $A$ defined as

$$\hat{\mu}(B) := \mu(B \times I), \quad \hat{\lambda}(B) := \mu(B \times I)$$

for all Borel sets $B \subset A$. We will show below that the Radon-Nikodym derivative of $\hat{\mu}$ with respect to the Lebesgue measure on $\mathbb{R}^2$ satisfies

$$\frac{d\hat{\mu}}{d\mathcal{L}^2}(x_\alpha) \geq \overline{\mathcal{W}}(D_p u(x_\alpha)|b(x_\alpha, \cdot)) \tag{4.13}$$

for $\mathcal{L}^2$ a.e. every point $x_\alpha \in A$.

Note that if (4.13) holds, then from (4.10),

$$\begin{aligned}
&\lim_{n\to\infty} \int_{A\times I} (W(D_p u_n|b_n) + |D_3 b_n|^2)\, dx \\
&\quad \geq \limsup_{k\to\infty} \int_{A\times I} \left(W(D_p v_k|z_k) + |D_3 z_k|^2\right)\, dy \\
&\quad = \limsup_{k\to\infty} \mu_k(A \times I) \\
&\quad \geq \hat{\mu}(A) \geq \int_A \frac{d\hat{\mu}}{d\mathcal{L}^2}(x_\alpha)\, dx_\alpha \geq \int_A \overline{\mathcal{W}}(D_p u(x_\alpha)|b(x_\alpha, \cdot))\, dx_\alpha.
\end{aligned}$$

Taking the infimum over all admissible sequences $\{u_n\}$ and $\{b_n\}$ we obtain

$$H^2(u, b; A) \geq \int_A \overline{\mathcal{W}}(D_p u(x_\alpha)|b(x_\alpha, \cdot))\, dx_\alpha,$$

which proves the lower bound in (4.6).

**Step 2:** It can be shown that, up to the extraction of a subsequence,

$$z_k(\cdot, x_3) \rightharpoonup b(\cdot, x_3) \text{ in } L^q(A; \mathbb{R}^3) \text{ for all } x_3 \in I$$

and for any Borel subset $B \subset A$ and for all $x_3 \in I$,

$$\sup_k \left| \int_B z_k(x_\alpha, x_3)\, dx_\alpha \right| < \infty.$$

The proof is standard and for the convenience of the reader the argument is provided in Lemma 5.1 in the Appendix.

We now address the proof of (4.13).

Since $D_3 u = 0$ $\mathcal{L}^3$ a.e. in $\Omega$, identifying $u$ with a function in $W^{1,q}(\omega; \mathbb{R}^3)$ for $\mathcal{L}^2$ a.e. $x_\alpha^0 \in A$ we have

$$\lim_{\delta\to 0^+} \frac{1}{\delta^{2+q}} \int_{Q'(x_\alpha,\delta)} |u(x_\alpha) - u(x_\alpha^0) - D_p u(x_\alpha^0)(x_\alpha - x_\alpha^0)|^q\, dx_\alpha = 0. \tag{4.14}$$

Moreover, viewing $b$ as a Bochner integrable function, that is, an element of

$$L^q\left(A; L^q\left(I; \mathbb{R}^3\right)\right)$$

(see [17]), for $\mathcal{L}^2$ a.e. $x_\alpha^0 \in A$ we have

$$\lim_{\delta\to 0^+}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\frac{1}{\delta^2}\int_{Q'(x_\alpha^0,\delta)} b(x_\alpha,x_3)\,dx_\alpha - b(x_\alpha^0,x_3)\right|^q dx_3 = 0. \tag{4.15}$$

Fix a point $x_\alpha^0 \in A$ which satisfies (4.14), (4.15), and such that

$$b(x_\alpha^0,\cdot)\in W^{1,2}\left(I;\mathbb{R}^3\right) \tag{4.16}$$

and

$$\frac{d\hat{\mu}}{d\mathcal{L}^2}(x_\alpha^0) \text{ and } \frac{d\hat{\lambda}}{d\mathcal{L}^2}(x_\alpha^0) \text{ exist and are finite.}$$

We claim that (4.13) holds at $x_\alpha^0$.

Consider a sequence $\{\delta_j\}$, with $\delta_j \to 0^+$ such that

$$\mu\left(\partial\left(Q'(x_\alpha^0,\delta_j)\times I\right)\right) = \lambda\left(\partial\left(Q'\left(x_\alpha^0,\delta_j\right)\times I\right)\right) = 0.$$

From the definition of $\hat{\mu}$ and $\hat{\lambda}$ together with that of the Radon-Nikodym derivative (see e.g. [11], Section 1.6), we obtain

$$\begin{aligned}\frac{d\hat{\mu}}{d\mathcal{L}^2}(x_\alpha^0) &= \lim_{j\to\infty}\frac{\hat{\mu}(Q'(x_\alpha^0,\delta_j))}{(\delta_j)^2} = \lim_{j\to\infty}\frac{\mu\left(Q'(x_\alpha^0,\delta_j)\times I\right)}{(\delta_j)^2}\\ &= \lim_{j\to\infty}\lim_{k\to\infty}\frac{1}{(\delta_j)^2}\int_{Q'(x_\alpha^0,\delta)\times I}(W(D_p v_k(x)\,|z_k(x)) + |D_3 z_k(x)|^2)\,dx\\ &= \lim_{j\to\infty}\lim_{k\to\infty}\int_Q (W(D_p v_{k,j}(y)\,|z_{k,j}(y)) + |D_3 z_{k,j}(y)|^2)\,dy,\end{aligned}$$

where for $y\in Q$,

$$v_{k,j}(y) := \frac{v_k(x_\alpha^0+\delta_j y_\alpha,y_3)-u(x_\alpha^0)}{\delta_j},\quad z_{k,j}(y) := z_k(x_\alpha^0+\delta_j y_\alpha,y_3),$$

and also, for later use,

$$u_0(y) := D_p u(x_\alpha^0)\cdot y_\alpha,\quad b_0(y_3) := b(x_\alpha^0,y_3).$$

Similarly,

$$\begin{aligned}\frac{d\hat{\lambda}}{d\mathcal{L}^2}(x_\alpha^0) &= \lim_{j\to\infty}\frac{\hat{\lambda}(Q'(x_\alpha^0,\delta_j))}{(\delta_j)^2} = \lim_{j\to\infty}\frac{\lambda\left(Q'(x_\alpha^0,\delta_j)\times\left(-\frac{1}{2},\frac{1}{2}\right)\right)}{(\delta_j)^2}\\ &= \lim_{j\to\infty}\lim_{k\to\infty}\frac{1}{(\delta_j)^2}\int_{Q'(x_\alpha^0,\delta)\times I}\Phi(|Dv_k(x)|^q + |z_k(x)|^q)\,dx\\ &= \lim_{j\to\infty}\lim_{k\to\infty}\int_Q \Phi(|Dv_{k,j}|^q + |z_{k,j}|^q)\,dy.\end{aligned}$$

Note that, since $v_k \to u$ in $L^q(\Omega;\mathbb{R}^3)$ and by (4.14), we have

$$\begin{aligned}
&\lim_{j\to\infty}\lim_{k\to\infty}\int_Q |v_{k,j}(y)-u_0\,(y)|^q\;dy\\
&=\lim_{j\to\infty}\lim_{k\to\infty}\frac{1}{(\delta_j)^{2+q}}\int_{Q'(x_\alpha^0,\delta_j)\times I}|v_k(x)-u(x_\alpha^0)-D_pu(x_\alpha^0)\cdot(x_\alpha-x_\alpha^0)|^q\,dx\\
&=\lim_{j\to\infty}\frac{1}{(\delta_j)^{2+q}}\int_{Q'(x_\alpha^0,\delta_j)}|u(x_\alpha)-u(x_\alpha^0)-D_pu(x_\alpha^0)\cdot(x_\alpha-x_\alpha^0)|^q\,dx_\alpha=0.
\end{aligned}$$

On the other hand, in view of (5.9), for all $y_3\in I$,

$$\begin{aligned}
\lim_{k\to\infty}\int_{Q'} z_{k,j}(y_\alpha,y_3)\,dy_\alpha &= \lim_{k\to\infty}\frac{1}{(\delta_j)^2}\int_{Q'(x_\alpha^0,\delta_j)} z_k(x_\alpha,y_3)\,dx_\alpha\\
&=\frac{1}{(\delta_j)^2}\int_{Q'(x_\alpha^0,\delta_j)} b(x_\alpha,y_3)\,dx_\alpha,
\end{aligned}$$

and so by (5.7) it follows from Lebesgue's Dominated Convergence Theorem that

$$\begin{aligned}
&\lim_{k\to\infty}\int_{-\frac12}^{\frac12}\left|\int_{Q'} z_{k,j}(y_\alpha,y_3)\,dy_\alpha-b_0(y_3)\right|^q dy_3\\
&=\int_{-\frac12}^{\frac12}\left|\frac{1}{(\delta_j)^2}\int_{Q'(x_\alpha^0,\delta_j)} b(x_\alpha,x_3)\,dx_\alpha-b(x_\alpha^0,x_3)\right|^q dx_3.
\end{aligned}$$

By (4.15) we have

$$\lim_{j\to\infty}\lim_{k\to\infty}\int_{-\frac12}^{\frac12}\left|\int_{Q'} z_{k,j}(y_\alpha,y_3)\,dy_\alpha-b_0(y_3)\right|^q dy_3=0.$$

By a standard diagonalization argument, we may extract subsequences $v_j := v_{k_j,j}$ and $z_j := z_{k_j,j}$ such that

$$\infty>\frac{d\hat\mu}{d\mathcal{L}^2}(x_\alpha^0)=\lim_{j\to\infty}\int_Q (W(D_pv_j|z_j)+|D_3z_j|^2)\,dy, \tag{4.17}$$

$$\lim_{j\to\infty}\int_Q |v_j-u_0|^q\;dy=0, \tag{4.18}$$

$$\lim_{j\to\infty}\int_{-\frac12}^{\frac12}\left|\int_{Q'} z_j(y_\alpha,y_3)\,dy_\alpha-b_0(y_3)\right|^q dy_3=0, \tag{4.19}$$

and

$$\infty>\sup_j\int_Q \Phi(|Dv_j|^q+|z_j|^q)\,dy. \tag{4.20}$$

Note that $\{|Dv_j|^q\}$ and $\{|z_j|^q\}$ are still equi-integrable in view of (4.11) and (4.20). Moreover, by applying Theorem 2.5 and reasoning as in (4.10) once more, we can assume, without loss of generality, that $v_j = u_0$ in a neighborhood of $\partial Q$.

For $y\in Q$,

$$\begin{aligned}
z_j\,(y)&=b_0(y_3)+\left(z_j\,(y)-\int_{Q'} z_j(w_\alpha,y_3)\,dw_\alpha\right)+\left(\int_{Q'} z_j(w_\alpha,y_3)\,dw_\alpha-b_0(y_3)\right)\\
&=: b_0(y_3)+\psi_j\,(y)+\bar z_j\,(y_3)\,,
\end{aligned}$$

and note that

$$\int_{Q'} \psi_j(y_\alpha, y_3)\, dy_\alpha = 0 \text{ for all } y_3 \in I. \tag{4.21}$$

It follows that

$$\int_{Q'} D_3\psi_j(y_\alpha, y_3)\, dy_\alpha = D_3 \left( \int_{Q'} \psi_j(y_\alpha, y_3)\, dy_\alpha \right) = 0 \text{ for all } y_3 \in I.$$

In turn

$$\begin{aligned} \int_Q |D_3 z_j|^2\, dy &\geq \int_Q |D_3\left(b_0 + \psi_j\right)|^2\, dy + 2\int_Q D_3\left(b_0+\psi_j\right) \cdot D_3\bar{z}_j\left(y_3\right)\, dy \\ &= \int_Q |D_3\left(b_0 + \psi_j\right)|^2\, dy + 2\int_Q D_3 b_0 \cdot D_3\bar{z}_j\left(y_3\right)\, dy. \end{aligned} \tag{4.22}$$

We claim that

$$\bar{z}_j \rightharpoonup 0 \text{ in } W^{1,2}\left(I; \mathbb{R}^3\right). \tag{4.23}$$

If the claim holds, then letting $j \to \infty$ in the previous inequality yields

$$\limsup_{j\to\infty} \int_Q |D_3 z_j|^2\, dy \geq \limsup_{j\to\infty} \int_Q |D_3\left(b_0+\psi_j\right)|^2\, dy. \tag{4.24}$$

To prove (4.23) note that, up to a subsequence, from (4.17) and (4.19) we may assume that $\bar{z}_j\left(y_3\right) \to 0$ for $\mathcal{L}^1$ a.e. $y_3 \in I$ and that

$$\sup_j \int_Q |D_3 z_j|^2\, dy < \infty.$$

Hence by the Hölder Inequality,

$$\begin{aligned} \int_{-\frac{1}{2}}^{\frac{1}{2}} |D_3\bar{z}_j\left(y_3\right)|^2\, dy_3 &\leq 2\int_{-\frac{1}{2}}^{\frac{1}{2}} \left[ \left| D_3 \int_{Q'} z_j(w_\alpha, y_3)\, dw_\alpha \right|^2 + |D_3 b_0(y_3)|^2 \right] dy_3 \\ &= 2\int_{-\frac{1}{2}}^{\frac{1}{2}} \left[ \left| \int_{Q'} D_3 z_j(w_\alpha, y_3)\, dw_\alpha \right|^2 + |D_3 b_0(y_3)|^2 \right] dy_3 \\ &\leq 2\int_{-\frac{1}{2}}^{\frac{1}{2}} \left[ \int_{Q'} |D_3 z_j(w_\alpha, y_3)|^2\, dw_\alpha + |D_3 b_0(y_3)|^2 \right] dy_3 \end{aligned}$$

and so also by (4.16),

$$\sup_j \int_{-\frac{1}{2}}^{\frac{1}{2}} |D_3\bar{z}_j\left(y_3\right)|^2\, dy_3 < \infty.$$

By extracting a further subsequnce, if necessary, we have shown (4.23).

Fix $\varepsilon > 0$. Since $\{|Dv_j|^q\}$ and $\{|\psi_j|^q\}$ are equi-integrable, there exists $L > 1$ such that

$$\sup_j \int_{\{|Dv_j|+|b_0+\psi_j|>L\}} W(D_p v_j | b_0 + \psi_j)\, dy \leq \varepsilon, \tag{4.25}$$

where we have used $(H_1)'$. In view of the uniform continuity of $W$ on $\overline{B_{3\times 3}\left(0, L+1\right)}$ there exists $\delta = \delta\left(\varepsilon\right) \in (0,1)$ such that

$$|W\left(F\right) - W\left(G\right)| \leq \varepsilon \tag{4.26}$$

for all $F$, $G \in \mathbb{R}^{3\times 3}$ with $|F - G| \leq \delta$ and $|F|$, $|G| \leq L+1$.

By (4.23) and the Ascoli-Arzelá Theorem, $\bar{z}_j \to 0$ uniformly. Hence for all $j$ sufficiently large, $\|\bar{z}_j\|_{L^\infty} \leq \delta$, and so by (4.26) we have

$$\begin{aligned}\int_Q W(D_p v_j|z_j)\,dy &= \int_Q W(D_p v_j|b_0+\psi_j+\bar{z}_j)\,dy \\ &\geq \int_{Q\cap\{|Dv_j|+|b_0+\psi_j|\leq L\}} W(D_p v_j|b_0+\psi_j+\bar{z}_j)\,dy \\ &\geq \int_{Q\cap\{|Dv_j|+|b_0+\psi_j|\leq L\}} W(D_p v_j|b_0+\psi_j)\,dy-\varepsilon \\ &\geq \int_Q W(D_p v_j|b_0+\psi_j)\,dy-2\varepsilon,\end{aligned}$$

where in the last inequality we have used (4.25).

In turn, using also (4.17), (4.24) we have that

$$\frac{d\hat{\mu}}{d\mathcal{L}^2}(x_\alpha^0) \geq \limsup_{j\to\infty}\int_Q W(D_p v_j|b_0+\psi_j)\,dy+\int_Q |D_3\left(b_0+\psi_j\right)|^2\,dy-2\varepsilon.$$

Since by construction $\varphi_j := v_j - u_0$ and $\psi_j$ are admissible functions in the definition of $\overline{\mathcal{W}}(D_p u(x_\alpha^0)|b(x_\alpha^0,\cdot))$ (see (4.21)), it follows that

$$\frac{d\hat{\mu}}{d\mathcal{L}^2}(x_\alpha^0) \geq \overline{\mathcal{W}}(D_p u(x_\alpha^0)|b(x_\alpha^0,\cdot))-2\varepsilon,$$

and letting $\varepsilon \to 0^+$ the proof of (4.13) is complete. □

We now prove the upper bound.

*Proof of the upper bound.* Fix $(u,b)\in\mathcal{V}^2$. As usual, we identify $u$ with a function in $W^{1,q}\left(\omega;\mathbb{R}^3\right)$.

**Step 1:** We first prove the upper bound

$$H^2(u,b;A) \leq \int_A \overline{\mathcal{W}}(D_p u(x_\alpha)|b(x_\alpha,\cdot))\,dx_\alpha$$

when $u = \overline{F}x_\alpha + c$ for some $\overline{F}\in\mathbb{R}^{3\times 2}$, $c\in\mathbb{R}^3$, and $b\in W^{1,2}\left(I;\mathbb{R}^3\right)$. For $\eta>0$ fixed, choose $\varphi\in W^{1,q}(Q;\mathbb{R}^3)$, $\psi\in L^q(Q;\mathbb{R}^3)$, $D_3\psi\in L^2(Q;\mathbb{R}^3)$, with $\varphi(\cdot,x_3)$ $Q'$-periodic and $\int_{Q'}\psi(x_\alpha,x_3)\,dx_\alpha = 0$ for all $x_3$, such that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}|D_3 b(x_3)|^2\,dx_3+\int_Q\left(W(\overline{F}+D_p\varphi|b+\psi)+|D_3\psi|^2\right)\,dx \leq \overline{\mathcal{W}}(\overline{F}|b)+\eta. \qquad (4.27)$$

Extend $\varphi(\cdot,x_3)$ and $\psi(\cdot,x_3)$ periodically with period $Q'$, and for $x\in\Omega$ define

$$u_n(x_\alpha,x_3) := \overline{F}x_\alpha + c + \frac{1}{n}\varphi(nx_\alpha,x_3), \quad b_n(x_\alpha,x_3) := b(x_3)+\psi(nx_\alpha,x_3).$$

Then, by Fubini's Theorem, $u_n \rightharpoonup u$ in $W^{1,q}(\Omega;\mathbb{R}^3)$, $b_n \rightharpoonup b$ in $L^q(\Omega;\mathbb{R}^3)$. Recalling the definition of $H^2(u,b;A)$, we have

$$H^2(u,b;A) \leq \liminf_{n\to\infty}\int_{A\times I}(W(D_p u_n|b_n)+|D_3 b_n|^2)\,dx. \qquad (4.28)$$

We now estimate the right-hand side of (4.28). Since, for $\mathcal{L}^1$ a.e. $x_3 \in I$ the function $\left(W(D_p u_n|b_n) + |D_3 b_n|^2\right)(\cdot, x_3)$ is $Q'$-periodic, then it converges weakly in $L^1(A)$ to its mean, that is, to

$$\int_{Q'} \left(W(\overline{F} + D_p\varphi(x_\alpha, x_3)|b(x_3) + \psi(x_\alpha, x_3)) + |D_3 b(x_3) + D_3\psi(x_\alpha, x_3)|^2\right) dx_\alpha.$$

Lebesgue's Dominated Convergence Theorem implies that

$$\begin{aligned}&\lim_{n\to\infty}\int_{A\times I}(W(D_p u_n|b_n) + |D_3 b_n|^2)\,dx\\ &= \mathcal{L}^2(A)\int_Q \left(W(\overline{F} + D_p\varphi|b+\psi) + |D_3 b + D_3\psi|^2\right) dx,\end{aligned}$$

which, in view of (4.27), (4.28), finally yields

$$H^2(u, b; A) \leq \mathcal{L}^2(A)[\overline{\mathcal{W}}(\overline{F}|b) + \eta].$$

Letting $\eta$ tend to 0, we conclude

$$H^2(u, b; A) \leq \mathcal{L}^2(A)\overline{\mathcal{W}}(\overline{F}|b). \tag{4.29}$$

**Step 2:** Assume now that there exists a partition $A_1, ..., A_N$ of $A$ such that

$$u(x_\alpha) = \sum_{i=1}^N \left(\overline{F}_i x_\alpha + c_i\right)\chi_{A_i}(x_\alpha), \quad b(x) = \sum_{i=1}^N b_i(x_3)\chi_{A_i}(x_\alpha), \tag{4.30}$$

for some $N \in \mathbb{N}$, $\overline{F}_i \in \mathbb{R}^{3\times 2}$, $c_i \in \mathbb{R}^3$ and $b_i \in W^{1,2}\left(I; \mathbb{R}^3\right)$, $i = 1, ..., N$. By (4.29), for all $i = 1, ..., N$,

$$H^2(\overline{F}_i x_\alpha + c_i, b_i; A_i) \leq \mathcal{L}^2(A_i)\overline{\mathcal{W}}(\overline{F}_i|b_i).$$

In view of Theorem 4.2, $H^2(u, b; \cdot)$ is a measure; thus

$$H^2(u, b; A) = \sum_{i=1}^N H^2(\overline{F}_i x_\alpha + c_i, b_i; A_i) \leq \sum_{i=1}^N \mathcal{L}^2(A_i)\overline{\mathcal{W}}(\overline{F}_i|b_i) = \int_A \overline{\mathcal{W}}(D_\alpha u|b)\,dx_\alpha.$$

**Step 3:** Finally, if $(u, b) \in \mathcal{V}^2$ is of general form, then we observe that there exists a sequence $\{b_j\}$ as in (4.30) such that $b_j \to b$ in $L^q(\Omega; \mathbb{R}^3)$, $D_3 b_j \to D_3 b$ in $L^2(\Omega; \mathbb{R}^3)$ (for the convenience of the reader a detailed proof may be found in Lemma 5.2 in the Appendix).

By further refining, if necessary, the partition of $\omega$, it is possible to approximate $u$ strongly in $W^{1,q}(\omega; \mathbb{R}^3)$ by piecewise affine functions $u_j$ so that $\{(u_j, b_j)\}$ satisfies (4.30).

But $H^2(\cdot, \cdot; A)$ is lower semi-continuous, so by the previous inequality,

$$H^2(u, b; A) \leq \liminf_{j\to\infty} H^2(u_j, b_j; A) \leq \liminf_{j\to\infty}\int_A \overline{\mathcal{W}}(D_\alpha u_j|b_j)\,dx_\alpha.$$

Since $\overline{\mathcal{W}}$ is upper semi-continuous (see Remark 4.3), by Fatou's lemma and $(H_1)'$ we obtain

$$H^2(u, b; A) \leq \int_A \overline{\mathcal{W}}(D_\alpha u|b)\,dx_\alpha.$$

□

REMARK 4.5. (i) By the very definition of $(Q_3 \times C_3)[W]$ for any $\overline{F} \in \mathbb{R}^{2\times 3}$ and $b \in W^{1,2}(I;\mathbb{R}^3)$,

$$\overline{\mathcal{W}}(\overline{F}|b) \geq \int_{-\frac{1}{2}}^{\frac{1}{2}} (Q_3 \times C_3)[W](\overline{F}|b(x_3))\, dx_3 + \int_{-\frac{1}{2}}^{\frac{1}{2}} |D_3 b(x_3)|^2\, dx_3.$$

On the other hand, taking $\varphi = \psi \equiv 0$ in the definition of $\overline{\mathcal{W}}(\overline{F}|b)$, we get

$$\overline{\mathcal{W}}(\overline{F}|b) \leq \int_{-\frac{1}{2}}^{\frac{1}{2}} W(\overline{F}|b(x_3))\, dx_3 + \int_{-\frac{1}{2}}^{\frac{1}{2}} |D_3 b(x_3)|^2\, dx_3.$$

It follows that if $W$ is cross quasiconvex-convex, i.e. if $W = (Q_3 \times C_3)[W]$, then

$$\overline{\mathcal{W}}(\overline{F}|b) = \int_{-\frac{1}{2}}^{\frac{1}{2}} W(\overline{F}|b(x_3))\, dx_3 + \int_{-\frac{1}{2}}^{\frac{1}{2}} |D_3 b(x_3)|^2\, dx_3.$$

(ii) In view of (i) we conclude that for all $(u,b) \in \mathcal{V}^2$ and $A \in \mathcal{A}(\omega)$,

$$H^2(u,b;A) \geq \int_{A\times I} \left[(Q_3 \times C_3)[W](D_\alpha u|b) + |D_3 b|^2\right] dx,$$

with equality if $W$ is cross quasiconvex-convex.

(iii) Note that if we define for $\overline{F} \in \mathbb{R}^{2\times 3}$, $b \in W^{1,2}(I;\mathbb{R}^3)$,

$$\widetilde{\mathcal{W}}(\overline{F}|b) := \inf_{\varphi,\,\psi} \Big\{ \int_Q \left( W\left(\overline{F} + D_\alpha \varphi(x)|b(x_3) + \psi(x)\right) + |D_3 \psi(x)|^2 \right) dx :$$
$$\varphi \in W^{1,q}(Q;\mathbb{R}^3),\ \varphi(\cdot, x_3)\ Q'\text{-periodic for } \mathcal{L}^1 \text{ a.e. } x_3,$$
$$\psi \in L^q(Q;\mathbb{R}^3),\ D_3\psi \in L^2(Q;\mathbb{R}^3),\ \int_{Q'} \psi(x_\alpha, x_3)\, dx_\alpha = 0 \text{ for } \mathcal{L}^1 \text{ a.e. } x_3 \Big\},$$

then from the previous theorem we have

$$H^2(u,b;A) = \int_A \widetilde{\mathcal{W}}(D_\alpha u(x_\alpha)|b(x_\alpha,\cdot))\, dx_\alpha + \int_{A\times I} |D_3 b|^2\, dx,$$

for all $(u,b) \in \mathcal{V}^2$ and $A \in \mathcal{A}(\omega)$.

As mentioned in the Introduction, we compare our results with those in [18].

REMARK 4.6. Shu's results extend beyond the identification of the $\Gamma$-limit with respect to the weak convergence in $W^{1,q}(\Omega;\mathbb{R}^3)$ of energies of the type

$$I_\varepsilon(u) := \int_{\omega\times I} \left[ W\left(D_p u|\frac{1}{\varepsilon} D_3 u\right) + k_\varepsilon \left( |D_p^2 u|^2 + \frac{1}{\varepsilon^2}|D_{p3} u|^2 + \frac{1}{\varepsilon^4}|D_{33} u|^2 \right) \right] dx,$$

for different regimes of $k_\varepsilon$ and considering even $x$-dependent bulk energies $W$ in the context of homogenization. In particular, he showed that

$$\inf \Big\{ \liminf_{n\to\infty} I_{\varepsilon_n}(u_n) : \{u_n\} \subset W^{1,q}(\Omega;\mathbb{R}^3),\ \varepsilon_n \to 0^+,$$
$$u_n \rightharpoonup u \text{ in } W^{1,q}(\Omega;\mathbb{R}^3) \Big\} = \int_\omega Q_2 \overline{W}(D_p u(x_\alpha))\, dx_\alpha,$$

where for $\overline{F} \in \mathbb{R}^{2\times 3}$,

$$\overline{W}(\overline{F}) := \inf_{z\in\mathbb{R}^3} W(\overline{F}|z),$$

i.e., $\overline{W}$ is the membrane energy density obtained in [16].

In the present work, $k_\varepsilon := \varepsilon^\gamma$ and admissible sequences are additionally constrained, in that for a fixed Cosserat vector $b$ we impose that

$$\frac{1}{\varepsilon_n} D_3 u_n \rightharpoonup b \text{ in } L^q(\Omega;\mathbb{R}^3).$$

To reconcile Shu's results with Theorem 3.1, we must prove that

$$\inf_b H^\gamma(u,b;\omega) = \int_\omega Q_2\overline{W}\left(D_p u\left(x_\alpha\right)\right)\, dx_\alpha.$$

This is confirmed in the proposition below.

Define

$$\mathcal{B}^\gamma := \left\{ b \in L^q\left(\Omega;\mathbb{R}^3\right) :\, D_3 b = 0 \ \mathcal{L}^3 \text{ a.e. in } \Omega \text{ if } \gamma < 2, \quad D_3 b \in L^2\left(\Omega;\mathbb{R}^3\right) \text{ if } \gamma = 2 \right\}.$$

PROPOSITION 4.7. Assume that condition $(H_1)'$ is satisfied. Then for every $u \in W^{1,q}\left(\omega;\mathbb{R}^3\right)$ and $\gamma > 0$,

$$\inf_{b\in\mathcal{B}^\gamma} H^\gamma(u,b;\omega) = \int_\omega Q_2\overline{W}\left(D_p u\left(x_\alpha\right)\right)\, dx_\alpha.$$

*Proof.* By the definition of $H^\gamma$ and standard lower semi-continuity results,

$$\begin{aligned} &H^\gamma\left(u,b,\omega\right) \\ &\quad\geq \inf\left\{ \liminf_{n\to\infty} \int_\Omega Q_3\overline{W}\left(D_p u_n\right)\, dx :\, \{u_n\} \subset W^{1,q}(\Omega;\mathbb{R}^3), u_n \rightharpoonup u \text{ in } W^{1,q}(\Omega;\mathbb{R}^3) \right\} \\ &\quad\geq \int_\Omega Q_3\overline{W}\left(D_p u\left(x_\alpha\right)\right)\, dx = \int_\omega Q_2\overline{W}\left(D_p u\left(x_\alpha\right)\right)\, dx_\alpha, \end{aligned}$$

where in the last equality we used formula (3.11) in Remark 3.3.

To prove the converse inequality, and in view of Theorems 3.2 and 4.1, we observe that

$$\begin{aligned} \inf_{b\in\mathcal{B}^\gamma} H^\gamma(u,b;\omega) &\leq \inf\left\{ H^\gamma(u,b;\omega) :\, b \in\mathcal{B}^\gamma,\, D_3 b = 0 \right\} \\ &= \inf_{b\in L^q(\omega;\mathbb{R}^3)} \int_\omega \left(Q_2 \times C_2\right)[W]\left(D_p u\left(x_\alpha\right)|b\left(x_\alpha\right)\right)\, dx_\alpha, \end{aligned}$$

and thus it suffices to show that for every $\varepsilon > 0$,

$$\int_\omega Q_2\overline{W}\left(D_p u\left(x_\alpha\right)\right)\, dx_\alpha \geq \int_\omega \left(Q_2 \times C_2\right)[W]\left(D_p u\left(x_\alpha\right)|b\left(x_\alpha\right)\right)\, dx_\alpha - \varepsilon \tag{4.31}$$

for some $b \in L^q\left(\omega;\mathbb{R}^3\right)$.

**Step 1:** Assume first that $u$ is affine with $D_p u = \overline{F} \in \mathbb{R}^{2\times 3}$, and let $\varphi \in C_c^\infty\left(\omega; \mathbb{R}^3\right)$ be such that

$$Q_2\overline{W}\left(\overline{F}\right) \geq \frac{1}{\mathcal{L}^2(\omega)} \int_\omega \overline{W}\left(\overline{F} + D_p\varphi(x_\alpha)\right)\, dx_\alpha - \frac{\varepsilon}{2\mathcal{L}^2(\omega)}$$
$$\geq \frac{1}{\mathcal{L}^2(\omega)} \int_\omega W\left(\overline{F} + D_p\varphi(x_\alpha)\,|b(x_\alpha)\right)\, dx_\alpha - \frac{\varepsilon}{\mathcal{L}^2(\omega)}$$

for some $b \in L^q\left(\omega; \mathbb{R}^3\right)$, where in the last inequality we used the definition of $\overline{W}$, Aumann's Measurable Selection Theorem, and the coercivity condition in $(H_1)'$.

Writing

$$b = \frac{1}{\mathcal{L}^2(\omega)} \int_\omega b(y_\alpha)\, dy_\alpha + \left(b - \frac{1}{\mathcal{L}^2(\omega)} \int_\omega b(y_\alpha)\, dy_\alpha\right),$$

it now follows that

$$Q_2\overline{W}\left(\overline{F}\right) \geq (Q_2 \times C_2)[W]\left(\overline{F}\left|\frac{1}{\mathcal{L}^2(\omega)} \int_\omega b(y_\alpha)\, dy_\alpha\right.\right) - \frac{\varepsilon}{\mathcal{L}^2(\omega)},$$

where we invoke the invariance of domain property for the definition of $(Q_2 \times C_2)[W]$, and we prove (4.31) for affine functions $u$.

**Step 2:** Suppose now that $u$ is piecewise affine with

$$D_p u = \sum_{i=1}^k \overline{F}_i \chi_{\omega_i} \quad \mathcal{L}^2 \text{ a.e. in } \omega,$$

for some $k \in \mathbb{N}$, $\overline{F}_i \in \mathbb{R}^{2\times 3}$, and some open, mutually disjoint Lipschitz sets $\omega_i$, with $i = 1, \ldots, k$.

By Step 1 find constant vectors $b_i \in \mathbb{R}^3$ such that

$$Q_2\overline{W}\left(\overline{F}_i\right) \geq (Q_2 \times C_2)[W]\left(\overline{F}_i\,|b_i\right) - \frac{\varepsilon}{\mathcal{L}^2(\omega_i)}$$

for all $i = 1, \ldots, k$. Setting

$$b = \sum_{i=1}^k b_i \chi_{\omega_i} \in L^\infty\left(\omega; \mathbb{R}^3\right),$$

we deduce (4.31).

**Step 3:** For a general $u \in W^{1,q}\left(\omega; \mathbb{R}^3\right)$ we consider a sequence $\{u_n\}$ of piecewise affine functions as in Step 2 such that $u_n \to u$ in $W^{1,q}\left(\omega; \mathbb{R}^3\right)$. For every $n$ let $\{b_n\} \subset L^\infty\left(\omega; \mathbb{R}^3\right)$ satisfy

$$\int_\omega Q_2\overline{W}\left(D_p u_n(x_\alpha)\right) dx_\alpha \geq \int_\omega (Q_2 \times C_2)[W]\left(D_p u_n(x_\alpha)\,|b_n(x_\alpha)\right) dx_\alpha - \varepsilon. \tag{4.32}$$

Using $(H_1)'$ it is easy to prove that

$$\frac{1}{C}\left|\overline{F}\right|^q - C \leq Q_2\overline{W}\left(\overline{F}\right) \leq C\left(1 + \left|\overline{F}\right|^q\right)$$

for all $\overline{F} \in \mathbb{R}^{2\times 3}$ and

$$\frac{1}{C}\left(\left|\overline{F}\right|^q + |z|^q\right) - C \leq (Q_2 \times C_2)[W]\left(\overline{F}\,|z\right) \leq C\left(1 + \left|\overline{F}\right|^q + |z|^q\right)$$

for all $\overline{F} \in \mathbb{R}^{2\times 3}$ and $z \in \mathbb{R}^3$. Hence by (4.32) the sequence $\{b_n\}$ is bounded in $L^q\left(\omega;\mathbb{R}^3\right)$ and so, up to the extraction of a subsequence, not relabelled, $\{b_n\}$ converges weakly in $L^q\left(\omega;\mathbb{R}^3\right)$ to some function $b$. Standard lower semi-continuity results, together with the continuity of $Q_2\overline{W}\left(\overline{F}\right)$, yield (4.31). □

## 5. Appendix.

LEMMA 5.1. Let $\{z_k\} \subset L^q\left(\Omega;\mathbb{R}^3\right)$ and $\{D_3 z_k\} \subset L^2\left(\Omega;\mathbb{R}^3\right)$ be such that

$$z_k \rightharpoonup b \text{ in } L^q(\Omega;\mathbb{R}^3), \tag{5.1}$$

$$D_3 z_k \rightharpoonup D_3 b \text{ in } L^2(\Omega;\mathbb{R}^3). \tag{5.2}$$

Then, up to the possible extraction of a subsequence,

$$z_k(\cdot,x_3) \rightharpoonup b(\cdot,x_3) \text{ in } L^q(\omega;\mathbb{R}^3) \text{ for all } x_3 \in I,$$

and for any Borel subset $B \subset \omega$ and for all $x_3 \in I$,

$$\sup_k \left| \int_B z_k\left(x_\alpha,x_3\right)\, dx_\alpha \right| < \infty.$$

*Proof.* Since $z_k \rightharpoonup b$ in $L^q(\Omega;\mathbb{R}^3)$, by Fubini's Theorem and Fatou's Lemma,

$$\infty > \liminf_{k\to\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_A \left|z_k\left(x_\alpha,x_3\right)\right|^q dx_\alpha dx_3 \geq \int_{-\frac{1}{2}}^{\frac{1}{2}} \liminf_{k\to\infty} \int_A \left|z_k\left(x_\alpha,x_3\right)\right|^q dx_\alpha dx_3$$

and so

$$\liminf_{k\to\infty} \int_A \left|z_k\left(x_\alpha,x_3\right)\right|^q dx_\alpha < \infty$$

for $\mathcal{L}^1$ a.e. $x_3 \in I$. Therefore we may find $\overline{x}_3$ and a subsequence (not relabelled) such that

$$\sup_k \int_A \left|z_k\left(x_\alpha,\overline{x}_3\right)\right|^q dx_\alpha < \infty, \quad \int_A \left|b\left(x_\alpha,\overline{x}_3\right)\right|^q dx_\alpha < \infty,$$

where in the latter inequality we used again Fubini's Theorem, and

$$z_k(\cdot,\overline{x}_3) \rightharpoonup \overline{b}(\cdot) \text{ in } L^q(A;\mathbb{R}^3). \tag{5.3}$$

Standard slicing arguments, together with the Sobolev Embedding Theorem, yield

$$z_k\left(x_\alpha,\cdot\right),\, b\left(x_\alpha,\cdot\right) \in W^{1,2}\left(I;\mathbb{R}^3\right) \tag{5.4}$$

for all $k \in \mathbb{N}$ and for $\mathcal{L}^2$ a.e. $x_\alpha \in A$; therefore for all $k \in \mathbb{N}$, for $\mathcal{L}^2$ a.e. $x_\alpha \in A$, and for *all* $x_3 \in I$, it follows that

$$z_k\left(x_\alpha,x_3\right) = z_k\left(x_\alpha,\overline{x}_3\right) + \int_{\overline{x}_3}^{x_3} D_3 z_k\left(x_\alpha,s\right)\, ds, \tag{5.5}$$

$$b\left(x_\alpha,x_3\right) = b\left(x_\alpha,\overline{x}_3\right) + \int_{\overline{x}_3}^{x_3} D_3 b\left(x_\alpha,s\right)\, ds. \tag{5.6}$$

By (5.5) and the choice of $\overline{x}_3$, for any Borel subset $B \subset A$ and for all $x_3 \in I$ we have

$$\left|\int_B z_k(x_\alpha, x_3)\, dx_\alpha\right| = \left|\int_B z_k(x_\alpha, \overline{x}_3)\, dx_\alpha + \int_{\overline{x}_3}^{x_3}\int_B D_3 z_k(x_\alpha, s)\, dx_\alpha ds\right| \tag{5.7}$$
$$\leq \sup_i \left(\int_A |z_i(x_\alpha, \overline{x}_3)|\, dx_\alpha + \int_{A\times I} |D_3 z_i(x)|\, dx\right) < \infty.$$

Next we claim that $\overline{b}(\cdot) = b(\cdot, \overline{x}_3)$. To see this, we first observe that by (5.5), (5.3), and (5.2), for every $\phi \in L^\infty(A;\mathbb{R})$ and for all $x_3 \in I$,

$$\lim_{k\to\infty}\int_A z_k(x_\alpha, x_3)\,\phi(x_\alpha)\, dx_\alpha \tag{5.8}$$
$$= \lim_{k\to\infty}\left[\int_A z_k(x_\alpha, \overline{x}_3)\,\phi(x_\alpha)\, dx_\alpha + \int_{\overline{x}_3}^{x_3}\int_A D_3 z_k(x_\alpha, s)\,\phi(x_\alpha)\, dx_\alpha ds\right]$$
$$= \int_A \overline{b}(x_\alpha)\,\phi(x_\alpha)\, dx_\alpha + \int_{\overline{x}_3}^{x_3}\int_A D_3 b(x_\alpha, s)\,\phi(x_\alpha)\, dx_\alpha ds.$$

In turn, for $\phi \in L^\infty(A;\mathbb{R})$ and $\varphi \in L^\infty(I;\mathbb{R})$, (5.8) implies

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_A b(x_\alpha, x_3)\,\phi(x_\alpha)\,\varphi(x_3)\, dx_\alpha dx_3$$
$$= \lim_{k\to\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_A z_k(x_\alpha, x_3)\,\phi(x_\alpha)\,\varphi(x_3)\, dx_\alpha dx_3$$
$$= \lim_{k\to\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}}\varphi(x_3)\left[\int_A z_k(x_\alpha, \overline{x}_3)\,\phi(x_\alpha)\, dx_\alpha + \int_{\overline{x}_3}^{x_3}\int_A D_3 z_k(x_\alpha, s)\,\phi(x_\alpha)\, dx_\alpha ds\right] dx_3$$
$$= \int_{-\frac{1}{2}}^{\frac{1}{2}}\varphi(x_3)\left[\int_A \overline{b}(x_\alpha)\,\phi(x_\alpha)\, dx_\alpha + \int_{\overline{x}_3}^{x_3}\int_A D_3 b(x_\alpha, s)\,\phi(x_\alpha)\, dx_\alpha ds\right] dx_3,$$

where we have used the Lebesgue Dominated Convergence Theorem, which can be applied since

$$\left|\varphi(x_3)\left[\int_A z_k(x_\alpha, \overline{x}_3)\,\phi(x_\alpha)\, dx_\alpha + \int_{\overline{x}_3}^{x_3}\int_A D_3 z_k(x_\alpha, s)\,\phi(x_\alpha)\, dx_\alpha ds\right]\right|$$
$$\leq C\,\|\varphi\|_\infty\,\|\phi\|_\infty \sup_i\left(\int_A |z_i(x_\alpha, \overline{x}_3)|\, dx_\alpha + \int_{A\times I}|D_3 z_i(x)|\, dx\right) < \infty$$

for $\mathcal{L}^1$ a.e. $x_3 \in I$. By the arbitrariness of $\phi \in L^\infty(A;\mathbb{R})$ we conclude that for $\mathcal{L}^2$ a.e. $x_\alpha \in A$,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} b(x_\alpha, x_3)\,\varphi(x_3)\, dx_3 = \int_{-\frac{1}{2}}^{\frac{1}{2}}\varphi(x_3)\left[\overline{b}(x_\alpha) + \int_{\overline{x}_3}^{x_3} D_3 b(x_\alpha, s)\, ds\right] dx_3,$$

which, using now the arbitrariness of $\varphi \in L^\infty(I;\mathbb{R})$ and (5.4), yields

$$b(x_\alpha, x_3) = \overline{b}(x_\alpha) + \int_{\overline{x}_3}^{x_3} D_3 b(x_\alpha, s)\, ds$$

for all $x_3 \in I$. It now follows from (5.6) that $\overline{b}(x_\alpha) = b(x_\alpha, \overline{x}_3)$ for $\mathcal{L}^2$ a.e. $x_\alpha \in A$.

Hence by (5.6) and (5.8),

$$z_k(\cdot, x_3) \rightharpoonup b(\cdot, x_3) \text{ in } L^q(A;\mathbb{R}^3) \text{ for all } x_3 \in I. \tag{5.9}$$

□

LEMMA 5.2. If $b \in L^q\left(\Omega;\mathbb{R}^3\right)$ and $D_3 b \in L^2\left(\Omega;\mathbb{R}^3\right)$, then there exists a sequence $\{b_j\}$ of the form

$$b_j(x) = \sum_{i=1}^{N_j} b_i^{(j)}(x_3)\chi_{A_i^{(j)}}(x_\alpha), \tag{5.10}$$

where $N_j \in \mathbb{N}$ and $b_i^{(j)} \in W^{1,2}\left(I;\mathbb{R}^3\right)$, $i = 1, ..., N_j$, such that $b_j \to b$ in $L^q\left(\Omega;\mathbb{R}^3\right)$ and $D_3 b_j \to D_3 b$ in $L^2\left(\Omega;\mathbb{R}^3\right)$.

*Proof.* Extend $b$ to $\mathbb{R}^3$ as follows:

$$\overline{b}(x) := \begin{cases} b(x) & \text{if } x \in \Omega, \\ b\left(x_\alpha, \frac{1}{2}\right) & \text{if } x_\alpha \in \omega \text{ and } x_3 \geq \frac{1}{2}, \\ b\left(x_\alpha, -\frac{1}{2}\right) & \text{if } x_\alpha \in \omega \text{ and } x_3 \leq -\frac{1}{2}, \\ 0 & \text{otherwise,} \end{cases}$$

and let $\rho_\varepsilon$ be a family of standard mollifiers in $\mathbb{R}^3$. Clearly $\rho_\varepsilon * \overline{b} \to b$ in $L^q\left(\Omega;\mathbb{R}^3\right)$ and $D_3\left(\rho_\varepsilon * \overline{b}\right) \to D_3 b$ in $L^2\left(\Omega;\mathbb{R}^3\right)$, and so it suffices to obtain the desired approximation result in the case where, in addition, the target $b$ belongs to $C^\infty\left(\mathbb{R}^3\right)$.

For $j \in \mathbb{N}$ consider a partition of $\mathbb{R}^2$ into squares $\left\{Q'_{j,n}\right\}_{n\in\mathbb{N}}$ of area $\frac{1}{j^2}$ and define

$$b_j(x) := j^2 \int_{Q'_{j,n}} b(y_\alpha, x_3)\, dy_\alpha \text{ for } x \in Q'_{j,n} \times I.$$

Obviously $b_j$ is of the form (5.10), and using the uniform continuity of $b$ and of $D_3 b$ on compact sets of $\mathbb{R}^3$ it is easy to see that $b_j \to b$ in $L^q\left(\Omega;\mathbb{R}^3\right)$ and $D_3 b_j \to D_3 b$ in $L^2\left(\Omega;\mathbb{R}^3\right)$. □

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