Thin-lens approximation for radial gradient-index lenses

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Abstract. Expressions for the Seidel aberrations of thin radial gradient-index (GRIN) lenses having a similar structure with the corresponding expressions for thin homogeneous lenses are obtained. By enabling a direct analysis of the relationship between aberrations and lens data, the thin-lens approximation provides new insight into the possibilities and limitations of aberration correction for simple optical systems using radial gradients. It is found for instance that, as in the homogeneous case, within the frame of this approximation the astigmatism of single radial GRIN lenses cannot be corrected if spherical aberration and coma are corrected or if the aperture stop is located at the lens.

Subject terms: gradient-index optics; aberrations; optical design.

Paper RMA-10 received Aug. 8, 1995; revised manuscript received Oct. 28, 1995; accepted for publication Nov. 17, 1995.

1 Introduction

A necessary condition for obtaining a successful optical design is the control of the primary aberrations of the system. Only optical systems where both Seidel and chromatic aberrations can be reduced to acceptable values are capable of producing an image of good quality. For homogeneous lenses, it is known, however, that the control of these aberrations is subject to certain basic limitations. For simple optical systems, a useful tool for obtaining insight into the possibilities and limitations of aberration correction is the thin-lens approximation.

Since for homogeneous optical systems the lens thicknesses are generally a less effective degree of freedom for controlling aberrations than the surface curvatures and the air spaces between the lenses, the aberration coefficients of a homogeneous lens having a finite but not too large thickness are of the same order of size and show approximately the same variation with the other lens parameters, as in the case of the corresponding lens considered to have zero thickness.[1] On the other hand, setting the lens thickness equal to zero in the exact paraxial and aberration formulas enables a considerable simplification of the corresponding expressions for the entire lens. Even if the results of the thin-lens approximation differ to some extent in absolute magnitude from the exact results, the thin-lens approximation yields a useful qualitative insight into the general properties of the lens aberrations. Thus, in early design stages, the thin-lens theory enables the designer to investigate whether a proposed system layout is capable of yielding the required degree of aberration correction.

The use of gradients in optical systems leads to new possibilities for aberration correction. If the required values of the refractive index parameters are in the producible range, a considerable reduction of the number of elements, as compared to homogeneous designs of same performance, becomes possible.[2,3]

For a better insight into the additional possibilities of aberration correction provided by the gradients, a formalism having the same simplicity as the homogeneous thin-lens theory is desirable. While some thin-lens expressions have been derived for various special cases,[4] no general thin-lens theory for gradient-index lenses, similar to that for homogeneous lenses, is currently known in the literature. For radial gradients, thin-lens expressions are frequently used only for lens power, Petzval curvature, and axial color. Thus, it has been shown that it is in principle possible to control simultaneously Petzval curvature and axial color of a single radial gradient-index (RGRIN) lens.[2,3] The purpose of this paper is the generalization of the thin-lens approximation for all Seidel aberration coefficients of RGRIN lenses.

Since for RGRIN lenses the gradient medium can have a major contribution to power and primary aberrations, the lens thickness must be regarded as a design parameter as significant as other parameters. Therefore, setting the lens thickness equal to zero is not adequate, because the contributions stemming from transfer through the gradient medium would be lost.

In this paper, it is shown that the thin-lens approximation can be extended for RGRIN lenses to include the gradient medium contributions. The starting point is the expressions of the Seidel aberrations of RGRIN lenses having finite thickness derived in an earlier paper[5] and reviewed in Sec. 2. After a brief discussion of the paraxial approximation in Sec. 3, expressions for the Seidel aberrations for the entire thin lens that have the same structure as the homogeneous thin-lens formulas are obtained in Sec. 4. The approximate expressions, which are considerably simpler than the exact Seidel formulas, are obtained as the sum of the lowest order nonvanishing terms in a power series expansion with respect to the lens thickness in the transfer contributions with the corresponding contributions of the two end faces.

 0091-3286/96/$6.00

The thin-lens formulas indicate which parameters of the RGRIN lens are effective for controlling a given primary aberration. For instance, it is found that within the framework of the thin-lens approximation (i.e., unless the lens becomes very thick or the gradient very strong), an aplanatic single RGRIN lens or system of such lenses in contact cannot also be corrected for astigmatism. Thus, for RGRIN lenses we have the same basic limitation as in the homogeneous case.

As will be discussed in detail elsewhere, comparison with the results obtained with exact Seidel expressions shows that the approximate formulas derived in this paper provide a very good description of the effects of the refractive index parameters on the Seidel aberrations of the lens. These formulas are useful, e.g., for determining the shape of curves giving the change of various aberrations if power is transferred between surfaces and medium or for understanding specific features of simple systems optimized by ray tracing.

2 Seidel Aberrations of RGRIN Lenses of Finite Thickness

In this section, the formulas giving the Seidel aberrations of an RGRIN lens of finite thickness and the paraxial relations needed for their computation are summarized.

Consider an RGRIN lens having the refractive index distribution

$$n^2(r^2)=n_0^2(1-kr^2+N_4k^2r^4+\cdots), \tag{1}$$

where n_0 is the refractive index on the optical axis. Only terms of order $\leqslant 4$ determining the Seidel aberrations are written in Eq. (1). The special case where the quadratic refractive index coefficient k vanishes (shallow gradients) is discussed separately in Sec. 4.

As in the homogeneous case, the Seidel aberrations can be calculated from the paraxial marginal and chief ray data at the lens surfaces. Using the same notation as in Ref. 5, the paraxial marginal and chief ray heights are denoted h and m, the corresponding marginal and chief ray slopes are denoted u and w, and the incidence angles for the marginal and chief rays are denoted i and j. The sign convention adopted for u and w is that their signs are the opposite of those of the corresponding direction cosines.

The Seidel aberrations of the RGRIN lens are sums of three types of contributions: (1) ordinary and (2) inhomogeneous contributions at the two end faces and (3) transfer contributions from the RGRIN medium.[6]

First, for a surface of curvature ρ, the ordinary contributions to spherical aberration S_1, coma S_2, astigmatism S_3, Petzval curvature P_s, and distortion S_4 are given by

$$\begin{aligned}
S_1&=(n_0i)^2h\Delta(u/n_0),\\
S_2&=n_0in_0jh\Delta(u/n_0),\\
S_3&=(n_0j)^2h\Delta(u/n_0),\\
P_s&=-\rho H^2\Delta(1/n_0),\\
S_4&=(n_0j)^2m\Delta(u/n_0)+n_0jH\Delta(w/n_0),
\end{aligned} \tag{2}$$

where n_0i and n_0j are the paraxial refraction invariants

$$n_0i=n_0h\rho-n_0u, \quad n_0j=n_0m\rho-n_0w, \tag{3}$$

and H is the paraxial system invariant

$$H=mn_0u-hn_0w. \tag{4}$$

By Δ (#) we denote the difference between the values after and prior to refraction or the transfer of the quantity in the brackets. Note that Eqs. (2) can be obtained from the corresponding relations in the homogeneous case[7] simply by replacing n by n_0.

Second, the inhomogeneous surface contributions, denoted by an asterisk, are

$$\begin{aligned}
S_1^*&=-2h^4\rho\Delta(n_0k),\\
S_2^*&=-2h^3m\rho\Delta(n_0k),\\
S_3^*&=-2h^2m^2\rho\Delta(n_0k),\\
S_4^*&=-2hm^3\rho\Delta(n_0k).
\end{aligned} \tag{5}$$

The inhomogeneous surface contribution to Petzval curvature vanishes.

Third, the transfer contributions of an RGRIN medium of thickness d to the Seidel coefficients read[5]

$$\begin{aligned}
T_1&=n_0de_1^2(1-3N_4/2)+n_0(1+N_4)\Delta(hu^3)\\
&\quad-5n_0N_4e_1\Delta(hu)/2,\\
T_2&=n_0de_1e_2(1-3N_4/2)+n_0(1+N_4)\Delta(hu^2w)\\
&\quad-5n_0N_4e_2\Delta(hu)/2-N_4H\Delta(u^2),\\
T_3&=n_0de_2^2(1-3N_4/2)+n_0(1+N_4)\Delta(huw^2)\\
&\quad-5n_0N_4e_3\Delta(hu)/2-2N_4H\Delta(uw)-N_4P_T/2,\\
T_4&=n_0de_2e_3(1-3N_4/2)+n_0(1+N_4)\Delta(hw^3)\\
&\quad-5n_0N_4e_3\Delta(mu)/2-N_4H\Delta(w^2)/2,\\
P_T&=kdH^2/n_0,
\end{aligned} \tag{6}$$

where T_i, $i=1,2,3,4$, denote spherical aberration, coma, astigmatism, and distortion, and P_T denotes Petzval curvature. At transfer through the RGRIN medium, in addition to the system invariant of Eq. (4), the quantities

$$\begin{aligned}
e_1&=kh^2+u^2,\\
e_2&=khm+uw,\\
e_3&=km^2+w^2,
\end{aligned} \tag{7}$$

also remain unchanged.

Expressions for paraxial ray tracing through RGRIN media can be found from the solutions for the refractive index distribution [Eq. (1)] of the differential equations describing the ray propagation in inhomogeneous media.[8] We adopt the following notation: quantities after transfer are denoted by a prime whereas quantities before transfer are left unprimed.

For $k>0$ (positive gradients) by writing $g=k^{1/2}$ the marginal ray parameters are given by

$$u'=u\cos gd+hg\sin gd,$$
$$h'=-\frac{u}{g}\sin gd+h\cos gd, \tag{8}$$

and those of the chief ray by

$$w'=w\cos gd+mg\sin gd,$$
$$m'=-\frac{w}{g}\sin gd+m\cos gd. \tag{9}$$

Similarly, for $k<0$ (negative gradients) we have

$$u'=u\cosh gd-h\hat{g}\sinh gd,$$
$$h'=-\frac{u}{\hat{g}}\sinh\hat{g}d+h\cosh\hat{g}d,$$
$$w'=w\cosh\hat{g}d-m\hat{g}\sinh\hat{g}d,$$
$$m'=-\frac{w}{\hat{g}}\sinh\hat{g}d+m\cosh\hat{g}d, \tag{10}$$

where $\hat{g}=(-k)^{1/2}$. The difference between the transfer formulas in the two cases $k>0$ and $k<0$ disappears if the trigonometric and hyperbolic functions in Eqs. (8) to (10) are expanded into power series. Setting

$$E_1(kd^2)=\cos gd,\quad E_2(kd^2)=\frac{\sin gd}{gd}, \tag{11}$$

we have, e.g., for the marginal ray

$$u'=uE_1(kd^2)+hkdE_2(kd^2),$$
$$h'=-udE_2(kd^2)+hE_1(kd^2). \tag{12}$$

From the series expansions of Eqs. (11) it can be observed that E_1 and E_2 depend indeed only on $g^2=k$:

$$E_1(kd^2)=1-\frac{1}{2!}\,kd^2+\frac{1}{4!}\,(kd^2)^2-\cdots\,,$$
$$E_2(kd^2)=1-\frac{1}{3!}\,kd^2+\frac{1}{5!}\,(kd^2)^2-\cdots\,. \tag{13}$$

Thus, for $k=0$ (shallow gradients) we have $E_1=E_2=1$ and Eqs. (8) and (9) become the well-known transfer equations for a homogeneous medium. For instance, for the marginal ray we obtain

$$u'=u,\quad h'=h-ud. \tag{14}$$

3 Paraxial Approximation for Thin RGRIN Lenses

To derive the thin-lens expressions for the Seidel aberrations, a systematic procedure for generalizing the thin-lens approximation for radial gradients is developed in this section. This procedure is first applied for regaining well-known paraxial results. In the next section, the same procedure leads to the simplified expressions of the Seidel coefficients of a thin RGRIN lens.

For a homogeneous lens having curvatures ρ_1 and ρ_2, the thin-lens approximation is valid if the lens thickness d is small in comparison with the radii of the two end surfaces

$$|\rho_1|d\ll 1,\quad |\rho_2|d\ll 1. \tag{15}$$

In this case, the change inside the lens of the ray height given by the second of Eqs. (14) is also small and can be neglected. Thus, at transfer we have

$$\Delta h=0. \tag{16}$$

Consider a thin RGRIN lens situated in air and having curvatures ρ_1 and ρ_2. To include the gradient medium contributions into the thin-lens theory note that the thin RGRIN lens can be regarded as being composed from three thin lenses in contact: two homogeneous thin lenses of refractive index n_0 having each a plane face and the other end face of curvatures ρ_1 and ρ_2, respectively, and a thin Wood lens. (A Wood lens is an RGRIN lens with plane end faces.) If the lens thickness d is much smaller than the focal length of each of the three ‘‘components,’’ then the power of the thin RGRIN lens is the sum

$$\varphi=\varphi_h+\varphi_g \tag{17}$$

of the powers of its ‘‘homogeneous’’ and ‘‘gradient’’ parts. Equation (17) can be easily generalized for an arbitrary system of thin lenses in contact, situated in air

$$\varphi=\sum\ \varphi_i\,, \tag{18}$$

where the index i denotes the various homogeneous and gradient contributions of the components to the power of the lens system. In Eq. (17), the power of the ‘‘homogeneous’’ part is given by

$$\varphi_h=\varphi_1+\varphi_2\,, \tag{19}$$

where we have

$$\varphi_1=(n_0-1)\rho_1\,,\quad \varphi_2=-(n_0-1)\rho_2\,. \tag{20}$$

To determine the expression of φ_g consider a thin Wood lens having a thickness much smaller than its focal length

$$d\ll|f_W|. \tag{21}$$

As shown later, the preceding approximation is equivalent to the condition

$$|k|d^2 \ll 1. \tag{22}$$

With the condition of Eq. (22), Eqs. (13) become $E_1 = E_2 = 1$ and the transfer Eqs. (12) for the marginal ray read

$$u' = u + kdh, \tag{23}$$

and

$$h' = h - ud. \tag{24}$$

Consider an axial object point situated at infinity. Let u_1 and u_1' be the marginal ray slopes at the first surface before and after refraction and let u_2 and u_2' be the same quantities at the second surface. From the second of Eqs. (14) we obtain

$$h - u_2' f_w = 0. \tag{25}$$

Since the end surfaces are plane, we have $u_1' = u_1 = 0$, and consequently

$$\frac{h}{f_W} = u_2' = n_0 u_2 = n_0 kdh. \tag{26}$$

The second equality in Eq. (26) comes from refraction at the second end face and the third one follows from Eq. (23). Thus, we arrive at the well-known relation for the focal length of the Wood lens[8,9]

$$f_W = (n_0 kd)^{-1}. \tag{27}$$

The power $\varphi_g = f_W^{-1}$ is then given by

$$\varphi_g = n_0 kd. \tag{28}$$

The relationships between positions and sizes of object and image of the thin RGRIN lens or system of thin lenses in contact are the same as in the homogeneous case. For a given value of the transverse magnification β, the positions s and s' of object and image are given by

$$-s = \frac{1}{\varphi}(1 - \beta^{-1}), \quad s' = \frac{1}{\varphi}(1 - \beta), \tag{29}$$

where s and s' are positive if the object (image) is situated to the right of the lens and negative otherwise. Eliminating β in Eqs. (29) yields the well-known relationship between position of object and image of a thin lens

$$\frac{1}{s'} = \varphi + \frac{1}{s}. \tag{30}$$

The condition of Eq. (22), which follows immediately from the condition of Eq. (21) and Eq. (27), is the additional condition that, together with the conditions of Eqs. (15), defines the thin-lens approximation for radial gradients. In this approximation, the gradient contribution to power [Eq. (28)] has been obtained by keeping in the paraxial transfer equations only the lowest order nonvanishing terms in the lens thickness, which are the linear terms.

The same procedure will be applied for deriving the thin-lens expressions for the Seidel aberrations, where only lowest order terms will be kept in both the surface and the transfer contributions. Thus, in the transfer contributions of Eqs. (6) the linear terms in d will be kept. On the other hand, the surface contributions of Eqs. (2) and (5) can be handled as in the homogeneous case by assuming that the paraxial ray heights within the lens do not change [Eq. (16)].

4 Seidel Aberrations of Thin RGRIN Lenses

Equations (2), (5), and (6), giving the contributions of refraction and transfer to the Seidel aberrations, can provide numerical values for the various aberration coefficients, but offer only a limited insight into the correction possibilities for the entire lens or for the entire optical system. To obtain additional insight, in this section simplified expressions for the Seidel aberrations of the entire RGRIN lens are derived based on the developed thin-lens approximation. To sum up the surface and transfer contributions, appropriate thin-lens variables are introduced. With the resulting formulas, the correction possibilities of the Seidel aberrations as well as their limitations are briefly discussed.

Consider first the transfer contributions of positive or negative radial gradients (i.e., nonzero k) to the Seidel aberrations. A closer inspection of Eqs. (6) to (9) shows that the transfer contributions vanish for $d = 0$. By substituting the invariants [Eqs. (7)] and the paraxial transfer Eqs. (23) and (24) into Eqs. (6), we obtain for the transfer contributions polynomials of fourth order in the lens thickness d. As in the paraxial case, we keep only the lowest order terms in d, which are the linear terms. These calculations were performed by means of computer algebra. Using Eq. (28), the results are

$$\begin{aligned}
T_1 &= \varphi_g[h^4 k(1 - 4N_4) + 5h^2 u^2], \\
T_2 &= \varphi_g\left[h^3 mk(1 - 4N_4) + 5h^2 uw + 2hu\,\frac{H}{n_0}\right], \\
T_3 &= \varphi_g\left[h^2 m^2 k(1 - 4N_4) + 5h^2 w^2 + 4hw\,\frac{H}{n_0}\right], \\
T_4 &= \varphi_g\left[hm^3 k(1 - 4N_4) + 5hmw^2 + mw\,\frac{H}{n_0}\right].
\end{aligned} \tag{31}$$

The Petzval curvature for transfer can be immediately rewritten as

$$P_T = \varphi_g\,\frac{H^2}{n_0^2}. \tag{32}$$

Note that all expressions in Eqs. (31) and (32) are proportional to the gradient power φ_g, i.e., to the product kd.

To derive the expressions for the aberrations of the entire thin lens, it is convenient to consider first the case where the aperture stop is situated at the lens and then to allow for arbitrary stop positions by means of stop-shift formulas. Expressions of stop-shift formulas, which are valid for arbitrary rotationally symmetric optical systems, can be found, e.g., in Ref. 7.

If the stop is at the lens, we have in all formulas $m=0$. The invariants of Eqs. (3) and (4) become

$$n_0 j = n_0 m\rho - n_0 w = -n_0 w,$$
$$H = m n_0 u - h n_0 w = -h n_0 w. \tag{33}$$

Thus, the angles j and w can be written as

$$n_0 j = \frac{H}{h} = -n_0 w, \tag{34}$$

and all quantities related to the chief ray can be removed from the aberration expressions.

The surface contributions [Eqs. (2) and (5)] then become

$$S_1 = (n_0 i)^2 h \Delta(u/n_0),$$
$$S_2 = n_0 i H \Delta(u/n_0),$$
$$S_3 = \frac{H^2}{h}\,\Delta(u/n_0), \tag{35}$$
$$P_S = -\rho H^2 \Delta(1/n_0),$$
$$S_4 = n_0 j H n_0 w \Delta(1/n_0^2) = -\frac{H^3}{h^2}\,\Delta(1/n_0^2),$$

and

$$S_1^* = -2h^4 \rho \Delta(n_0 k), \quad S_2^* = S_3^* = S_4^* = 0. \tag{36}$$

The transfer contributions [Eqs. (31)] read

$$T_1 = \varphi_g[h^4 k(1-4N_4) + 5h^2 u^2],$$
$$T_2 = \varphi_g\left[5h^2 uw + 2hu\,\frac{H}{n_0}\right] = -3\varphi_g hu\,\frac{H}{n_0},$$
$$T_3 = \varphi_g\left[5h^2 w^2 + 4hw\,\frac{H}{n_0}\right] = \varphi_g\left(\frac{H}{n_0}\right)^2, \tag{37}$$
$$T_4 = 0.$$

As in the homogeneous thin-lens theory,[1] in the summation of the various contributions to the total aberration coefficients it is assumed that the height h of the marginal ray does not change within the lens. Before performing the summation, however, two new thin lens variables are introduced: a variable conveniently expressing the marginal ray inclination u and a bending variable giving the two surface curvatures ρ_1 and ρ_2 for a given value of the homogeneous power φ_h. The new variables are defined such that in the limiting case of a homogeneous medium, the present thin lens formulas reduce to the thin lens formulas first derived by Argentieri (see Refs. 10 and 11). For homogeneous media, other definitions of the thin lens variables are also possible, e.g., the Coddington variables.[1,7] For radial gradients, however, the Coddington variables are less appropriate, because one of them becomes indefinite in the special case where both end faces are plane (Wood lens). The Argentieri variables are

$$\psi = \frac{1+\beta}{1-\beta}\,\varphi_h, \quad \varpi = \rho_1 + \rho_2 - \psi. \tag{38}$$

In the subsequent calculations, the indices 1 and 2 denote quantities related to the first and second end surfaces.

First, the marginal ray inclination is expressed in all aberration formulas through the variable ψ. Observing that

$$s u_1 = h = s' u_2', \tag{39}$$

it follows from Eq. (30) that the total change of u produced by the thin lens is given by

$$u_2' - u_1 = h\varphi = h(\varphi_h + \varphi_g). \tag{40}$$

Since we have

$$u_1 = \beta u_2', \tag{41}$$

we can also write

$$\psi = \frac{1+\beta}{1-\beta}\,\varphi_h = \frac{u_2'+u_1}{u_2'-u_1}\,\varphi_h = \frac{u_2'+u_1}{h(\varphi_g+\varphi_h)}\,\varphi_h, \tag{42}$$

obtaining an alternative expression for the first Argentieri variable [Eq. (38)]:

$$\psi = \left(\frac{1}{s'} + \frac{1}{s}\right)\frac{\varphi_h}{\varphi_g + \varphi_h}. \tag{43}$$

From Eq. (42), we obtain

$$u_2' + u_1 = h\psi\left(1 + \frac{\varphi_g}{\varphi_h}\right). \tag{44}$$

Equations (40) and (44) form a linear system of equations with unknowns u_1 and u_2'. The solution reads

$$u_2' = \frac{1}{2}\,h\left[\psi\left(1 + \frac{\varphi_g}{\varphi_h}\right) + \varphi_h + \varphi_g\right],$$
$$u_1 = \frac{1}{2}\,h\left[\psi\left(1 + \frac{\varphi_g}{\varphi_h}\right) - \varphi_h - \varphi_g\right]. \tag{45}$$

Equations (45) give the marginal ray inclinations at the two end surfaces, outside the thin lens. The values at the end surfaces, inside the lens (i.e., after refraction at the first surface and prior to refraction at the second surface) are then

$$u_1' = \frac{1}{n_0}\,[u_1 + (n_0-1)h\rho_1], \quad u_2 = \frac{1}{n_0}\,[u_2' + (n_0-1)h\rho_2]. \tag{46}$$

We must now decide what value should be assigned to the quantity u in the first two of Eqs. (37). In the derivation of these equations, u was assumed to be the marginal ray inclination in the gradient medium prior to transfer $u = u_1'$. We do not have, however, any reason to prefer the value prior to the transfer to the value after transfer u_2. Thus, in Eqs. (37), we set

$$u = \frac{u_1' + u_2}{2}. \tag{47}$$

As can be seen from Eq. (23), u_2 differs from u_1' by a linear term in d. Thus, the replacement [Eq. (47)] is consistent with the approximation used for deriving Eqs. (37)—that only lowest order nonvanishing terms are kept—because these equations are already linear in d.

Second, the surface curvatures are expressed through the Argentieri variables. From Eqs. (19), (20), and (38), we obtain a linear system of equations for ρ_1 and ρ_2

$$\rho_1 - \rho_2 = \frac{\varphi_h}{n_0 - 1}, \quad \rho_1 + \rho_2 = \varpi + \psi, \tag{48}$$

having the solutions

$$\rho_1 = \frac{1}{2}\left(\varpi + \psi + \frac{\varphi_h}{n_0 - 1}\right), \qquad \rho_2 = \frac{1}{2}\left(\varpi + \psi - \frac{\varphi_h}{n_0 - 1}\right). \tag{49}$$

After substituting Eqs. (45), (46), and (49), Eq. (47) and the quantities $n_0 i$ and $\Delta(u/n_0)$ appearing in the surface contributions [Eqs. (35)] at the two end surfaces,

$$(n_0 i)_1 = h\rho_1 - u_1, \quad (n_0 i)_2 = h\rho_2 - u_2', \tag{50}$$

$$\Delta(u/n_0)_1 = u_1'/n_0 - u_1, \quad \Delta(u/n_0)_2 = u_2' - u_2/n_0, \tag{51}$$

come to be expressed through the Argentieri variables [Eqs. (38)].

Finally, we sum up for each aberration coefficient the surface and transfer contributions, obtaining for the entire RGRIN lens the expressions for spherical aberration Γ_1, coma Γ_2, astigmatism Γ_3, Petzval curvature P, and distortion Γ_4. Note first that the quantities

$$\Delta(n_0 k), \Delta(1/n_0), \Delta(1/n_0^2)$$

have equal magnitudes but opposite signs at the two end surfaces. Thus, for distortion, the sum vanishes because the two surface contributions cancel each other out:

$$\Gamma_4 = 0. \tag{52}$$

The well-known expression of Petzval curvature[2] follows immediately as

$$P = H^2\left(\frac{\varphi_h}{n_0} + \frac{\varphi_g}{n_0^2}\right). \tag{53}$$

It can be observed that the Petzval curvature of the RGRIN lens can be corrected by choosing

$$\varphi_g = -n_0 \varphi_h. \tag{54}$$

In the case of astigmatism, $\Gamma_3 = S_{3,1} + S_{3,2} + T_3$, it follows from Eqs. (35), (51), (40), (23), and (28) that

$$S_{3,1} + S_{3,2} = \frac{H^2}{h}\left[u_2' - u_1 - \frac{1}{n_0}(u_2 - u_1')\right] = H^2\left(\varphi_h + \varphi_g - \frac{\varphi_g}{n_0^2}\right),$$

and from Eqs. (37) that

$$\Gamma_3 = H^2(\varphi_h + \varphi_g) = H^2 \varphi. \tag{55}$$

Thus, we arrive at the same remarkable result as in the homogeneous case: If the stop is at the lens, the astigmatism of a thin RGRIN lens of nonzero power has a fixed value and cannot be corrected by any of the lens parameters.

For spherical aberration $\Gamma_1 = S_{1,1} + S_{1,2} + S_{1,1}^* + S_{1,2}^* + T_1$ and coma $\Gamma_2 = S_{2,1} + S_{2,2} + T_2$, the summation of the surface and transfer contributions expressed through the Argentieri variables is straightforward, but lengthy, and has therefore been performed by means of computer algebra. In the resulting sums, only the linear terms in φ_g (i.e., the linear terms in d) are kept. For the total coma

$$\Gamma_2 = B = B_h + B_g, \tag{56}$$

in addition to the ordinary contribution given by the expression of Argentieri,[10,11]

$$B_h = \frac{\varphi_h H h^2}{2}\left(\frac{n_0 + 1}{n_0}\varpi - \psi\right), \tag{57}$$

we have an additional gradient contribution

$$B_g = \frac{\varphi_g H h^2}{2}\left[\left(\frac{n_0 - 1}{n_0}\right)^2 \varpi - 3\frac{n_0 + 1}{n_0}\psi\right]. \tag{58}$$

Thus, for a given value of the total power, the coma of the thin RGRIN lens can be corrected by transferring power either between surfaces (ordinary bending) or between a surface and the RGRIN medium (gradient bending).

The total spherical aberration,

$$\Gamma_1 = A = A_h + A_g + A_0^* + A_1^*, \tag{59}$$

can be written as a sum of four terms: the ordinary Argentieri term

$$A_h = \frac{\varphi_h h^4}{4}\left[\frac{n_0 + 2}{n_0}\varpi^2 - 2\varpi\psi + \left(\frac{n_0}{n_0 - 1}\right)^2 \varphi_h^2\right] \tag{60}$$

and three terms due to the gradient. The first term

$$A_g=\varphi_g h^4\left[\frac{\varpi^2(n_0-1)(3n_0-1)}{2n_0^2}+\frac{\varpi\psi(n_0-3)}{n_0}+\frac{7\psi^2}{4}+\frac{(3n_0+1)\varphi_h^2}{4(n_0-1)}\right] \quad (61)$$

has the same nature as the gradient contribution to coma [Eq. (58)]. The second term

$$A_0^*=-2\varphi_h h^4\,\frac{n_0 k}{n_0-1} \quad (62)$$

does not depend on the lens thickness and is due to the asphericlike effect of the RGRIN medium on the surface contributions to spherical aberration. The last of these terms

$$A_1^*=\varphi_g h^4 k(1-4N_4) \quad (63)$$

is the only term in the aberration expressions [Eqs. (52) to (63)] that depends on the fourth-order refractive-index coefficient N_4. Thus, by changing N_4 it is possible to correct spherical aberration without affecting the power or the correction state of other Seidel or chromatic paraxial aberrations.

For small values of the quadratic coefficient k, the refractive index distribution [Eq. (1)] must be replaced with

$$n^2(r^2)=n_0^2(1-kr^2+\epsilon r^4+\cdots), \quad (64)$$

where the fourth-order coefficient is now denoted by

$$\epsilon=N_4k^2. \quad (65)$$

Thus, in the special case of the shallow gradients ($k=0$), Eq. (63) becomes

$$A_1^*=-4dn_0\epsilon h^4, \quad (66)$$

while all other gradient contributions to spherical aberration, coma, and Petzval curvature vanish:

$$A_g=A_0^*=B_g=0. \quad (67)$$

Thus, because of Eq. (66), the Seidel aberrations of a thin lens with a shallow radial gradient are equivalent to those of a homogeneous aspheric lens.

As can be seen from the preceding equations, if the refractive index parameters k and N_4 are regarded as variables and the required values are in the producible range, the use of radial gradients creates design possibilities that are not available for homogeneous lenses. Because of the gradient contributions to the Seidel aberrations, which depend [excepting Eq. (62)] on the gradient power and therefore on d, the thickness is also more effective for controlling aberrations than in the homogeneous case.

Finally, when the stop is situated at a certain distance from the lens, the expressions for the Seidel aberrations can be immediately derived from the above expressions by means of stop-shift formulas:[7]

$$\Gamma_1=A,\quad \Gamma_2=B+\vartheta A,\quad \Gamma_3=H^2\varphi+2\vartheta B+\vartheta^2A,$$

$$P=H^2\left(\frac{\varphi_h}{n_0}+\frac{\varphi_g}{n_0^2}\right), \quad (68)$$

$$\Gamma_4=3\vartheta H^2\varphi+\vartheta P+3\vartheta^2B+\vartheta^3A,$$

where the stop-shift parameter ϑ is given by the ratio of the chief and marginal ray heights at the lens:

$$\vartheta=\frac{m}{h}. \quad (69)$$

For systems of thin lenses separated by air spaces, the Seidel aberrations can then be determined following the same procedures as in the homogeneous case.

Equations (68) show that, within the domain of validity of the thin lens approximation [conditions of Eqs. (15) and (22)], for a given value of the total power, the five Seidel coefficients are expressed only through four independent quantities: A, B, P, and ϑ. Therefore, it is not possible to correct simultaneously all five Seidel aberrations. For instance, for a thin RGRIN lens with nonzero power, if spherical aberration and coma are corrected ($A=B=0$) or if the stop is at the lens ($\vartheta=0$), then astigmatism is proportional to the lens power and is therefore uncorrected. The same limitation also occurs for a group of thin lenses in contact. This limitation, which is well known from the homogeneous case, cannot be removed by the use of gradients.

Numerical comparisons of the Seidel aberrations computed with exact and thin-lens formulas have shown that the qualitative conclusions drawn by thin-lens analysis apply fairly well for RGRIN lenses having typical values of the lens parameters, as long as the thickness and quadratic coefficient k are not too large. Results quite different from those predicted by thin-lens analysis have been obtained only for lenses, where the conditions of Eqs. (15) and (22) are severely violated.

A first example of a single radial GRIN lens where third-order spherical aberration, coma, and astigmatism are simultaneously corrected has been given by Moore.[12] With the present notation, the corresponding lens parameters are: $R_1=0.333$, $R_2=-0.190$, $d=0.175$, $n_0=1.522$, $k=-12.920$, $N_4=2.788$, $\beta=0$, and $f=1$. Astigmatism could be corrected because the absolute value of the radius R_2 of the second surface is nearly as small as d and because the absolute value of k is very large ($kd^2=-0.395$ is therefore not negligibly small).

Simultaneous correction of spherical aberration, coma, and astigmatism is known also for patented objectives consisting of a single RGRIN lens. The parameters of such an objective, which has been optimized for a numerical aperture of 0.5, are given in Table 52 of Ref. 13: $R_1=1.697$, $R_2=-1.356$, $d=1.560$, $n_0=1.50$, $k=0.340$, $N_4=0.208$, $\beta=0$, and $f=1$. Correction of astigmatism is possible because of the large value of the lens thickness. Note that d is larger than the effective focal length f and that $kd^2=0.827$ has the order of magnitude of unity.

5 Conclusions

For thin RGRIN lenses [i.e., for lenses satisfying the conditions Eqs. (15) and (22)] expressions of the Seidel aberrations of the entire lens were obtained [Eqs. (52), (53), (55) to (63), (68), and (69)] in which the lens parameters appear either directly or through simple intermediate variables [Eqs. (28), (38), and (43)]. With these expressions, a reasonably accurate qualitative description of the behavior of Seidel aberrations was obtained also for RGRIN lenses of finite thickness, provided that the thickness and the quadratic refractive index coefficient are not too large. The analysis of the possibilities and limitations of aberration correction with RGRIN lenses is thus considerably simplified as compared with exact Seidel formalism or direct optimization with ray tracing.

Acknowledgments

The author wishes to thank Professor J. Kross from the Institute of Optics, Technical University of Berlin, Germany, for his suggestion to pursue this research and for fruitful discussions on this topic.

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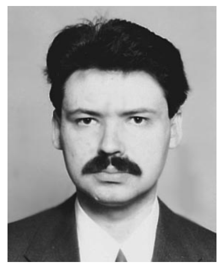

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