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Think co(mpletely)positive ! Matrix properties, examples and a clustered bibliography on copositive optimization — Source link

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Think co(mpletely)positive !
Matrix properties, examples and a clustered bibliography on copositive optimization

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Dedicated to the memory of Reiner Horst

Abstract

Copositive optimization is a quickly expanding scientific research domain with wide-spread applications ranging from global nonconvex problems in engineering to NP-hard combinatorial optimization. It falls into the category of conic programming (optimizing a linear functional over a convex cone subject to linear constraints), namely the cone $\mathcal{C}$ of all completely positive symmetric $n \times n$ matrices (which can be factorized into $FF^\top$, where F is a rectangular matrix with no negative entry), and its dual cone $\mathcal{C}^*$, which coincides with the cone of all copositive matrices (those which generate a quadratic form taking no negative value over the positive orthant). We provide structural algebraic properties of these cones, and numerous (counter-)examples which demonstrate that many relations familiar from semidefinite optimization may fail in the copositive context, illustrating the transition from polynomial-time to NP-hard worst-case behaviour. In course of this development we also present a systematic construction principle for non-attainability phenomena, which apparently has not been noted before in an explicit way. Last but not least, also seemingly for the first time, a somehow systematic clustering of the vast and scattered literature is attempted in this paper.

1 Introduction

Copositive optimization (or copositive programming, coined in [47]) is a special case of conic optimization, which consists of optimizing a linear function over a cone subject to additional linear constraints.

It is well known that the simplest class of hard problems in continuous optimization is that of quadratic optimization problems [214] – to minimize a (possibly indefinite) quadratic form $\mathbf{x}^\top Q\mathbf{x}$ over a polyhedron $\left\{\mathbf{x} \in \mathbb{R}^n_+ : A\mathbf{x} = \mathbf{b}\right\}$. Note that a linear term in the objective function can be removed by an affine transformation of the polyhedron. The number of local, non-global solutions to this problem may be exponential in the number of variables and/or constraints [42].

This class has a close connection to copositive optimization. The so–called *lifting* idea here is to linearize the quadratic form

$$\mathbf{x}^\top Q\mathbf{x} = \text{trace}\,(\mathbf{x}^\top Q\mathbf{x}) = \text{trace}\,(Q\mathbf{x}\mathbf{x}^\top) = Q \bullet \mathbf{x}\mathbf{x}^\top$$

by introducing the new symmetric matrix variable $X = \mathbf{x}\mathbf{x}^\top$ and Frobenius duality $X \bullet Y = \text{trace}\,(XY)$. This technique was mainly applied previously to semidefinite optimization [194]. If $A\mathbf{x} \in \mathbb{R}^m_+$ for all $\mathbf{x} \in \mathbb{R}^n_+$ and $\mathbf{b} \in \mathbb{R}^m_+$, then the linear constraints can be squared, to arrive in a similar way at constraints of the form $A_i \bullet X = b_i^2$.

Now the set of all these X generated by feasible $\mathbf{x}$ is non-convex since rank $(\mathbf{x}\mathbf{x}^\top) = 1$. The convex hull

$$\mathcal{C} = \text{conv}\;\left\{\mathbf{x}\mathbf{x}^\top : \mathbf{x} \in \mathbb{R}^n_+\right\}\;,$$

results in a convex matrix cone called the cone of completely positive matrices since [133]; see [22]. Note that a similar construction dropping nonnegativity constraints leads to

$$\mathcal{P} = \text{conv}\;\left\{\mathbf{x}\mathbf{x}^\top : \mathbf{x} \in \mathbb{R}^n\right\}\;,$$

the cone of positive-semidefinite matrices, the basic set in semidefinite optimization (or semidefinite programming, SDP); see for instance [59].

The first account on copositive optimization goes back to [47], who established a copositive representation of a subclass of particular interest, namely in Standard Quadratic Optimization (StQP). Here the feasible polyhedron is the standard simplex $\Delta = \left\{\mathbf{x} \in \mathbb{R}^n_+ : \sum_i x_i = 1\right\}$: this subclass is also NP-hard from the worst-case complexity but allows for a polynomial-time approximation scheme [46]. There can be exponentially many local non-global solutions (see [61] for the lower bound $30^{\lfloor n/9 \rfloor}$.)

This phenomenon is typical for a problem of Global Optimization. While researchers were aware of this phenomenon since long, the field of Deterministic Global Optimization received a decisive impact by the seminal book [147] by R. Horst and H. Tuy. Soon after, R. Horst co-founded the journal at hand and was its first managing editor, so he can be seen as one of the most influential and driving personalities in Global Optimization.

Now, with J the $n \times n$ all-ones matrix, we have

$$\min\left\{\mathbf{x}^\top Q\mathbf{x} : \mathbf{x} \in \Delta\right\} = \min\left\{Q \bullet X : J \bullet X = 1\,,\ X \in \mathcal{C}\right\}\,. \tag{1}$$

Note that the right-hand problem is convex, so there are no more local, non-global solutions. In addition, the objective function is now linear, and there is just one linear equality constraint. The complexity has been completely pushed into the feasibility condition $X \in \mathcal{C}$, which also shows that there are indeed convex minimization problems which cannot be solved easily.

Duality theory for conic optimization problems requires the dual cone $\mathcal{C}^*$ of $\mathcal{C}$ w.r.t. the Frobenius inner product, which is

$$\mathcal{C}^* = \left\{S \in \mathbf{S}^{n\times n} : S \bullet X \geq 0 \text{ for all } X \in \mathcal{C}\right\}\,,$$

where $\mathbf{S}^{n\times n}$ is the set of symmetric $n \times n$ matrices. Here it can easily be shown that $\mathcal{C}^*$ coincides with the cone of copositive matrices, which justifies terminology:

$$\mathcal{C}^* = \left\{S \in \mathbf{S}^{n\times n} : \mathbf{x}^\top S\mathbf{x} \geq 0 \text{ if } \mathbf{x} \in \mathbb{R}^n_+\right\}\,,$$

i.e., a matrix S is copositive [208] (most probably abbreviating "*co*nditionally *positive*-semidefinite"), if S generates a quadratic form $\mathbf{x}^\top S\mathbf{x}$ taking no negative values over the positive orthant. The dual of the special program (1) over $\mathcal{C}$ above is then

$$\max\left\{y \in \mathbb{R} : S = Q - yJ \in \mathcal{C}^*\right\}\,, \tag{2}$$

a linear objective in just one variable y with the innocent-looking feasibility constraint $S \in \mathcal{C}^*$. This shows that checking membership of $\mathcal{C}^*$ (and, similarly, of $\mathcal{C}$) is already NP-hard, and there are many approaches to algorithmic copositivity detection, we refer to Section 4. More generally, a typical primal-dual pair in copositive optimization (COP) is of the following form:

$$\begin{aligned} &\inf\left\{C \bullet X : A_i \bullet X = b_i\,, i = 1, \dots, m\,,\ X \in \mathcal{C}\right\} \\ \geq\ &\sup\left\{\textstyle\sum_i b_i y_i : \mathbf{y} \in \mathbb{R}^m\,,\ S = C - \sum_i y_i A_i \in \mathcal{C}^*\right\}\,. \end{aligned}$$

The inequality above is just standard weak duality, but observe we have to use inf and sup since – as in general conic optimization – there may be problems with attainability of either or both problems above, and likewise there could be a (finite or infinite) positive duality gap without any further conditions like strict feasibility (Slater's condition). For the above representation of Standard Quadratic Optimization problems, this is not the case:

$$\min\left\{Q \bullet X : J \bullet X = 1\,,\ X \in \mathcal{C}\right\} = \max\left\{y \in \mathbb{R} : S = Q - yJ \in \mathcal{C}^*\right\}\,.$$

But for a similar class arising in many applications, the Multi-Standard Quadratic Optimization problems [57], dual attainability is not guaranteed while the duality gap is zero – an intermediate form between weak and strong duality [234]. We will discuss in detail these phenomena in Section 3. But let us start with collecting a number of elementary properties and counterexamples illustrating the difference between the semidefinite cone $\mathcal{P}$ and the copositive/completely positive cone $\mathcal{C}^*/\mathcal{C}$. This is important for many copositivity detection procedures, and as we saw in (2), the feasibility constraint incorporates most of the hardness in copositive optimization.

Therefore, this paper is organized as follows: Section 2 discusses some algebraic properties of matrices belonging to the copositive or related cones, some of them more or less well known, others apparently never noticed before. Section 3 provides a complete picture of possible attainability/duality gap constellations in primal-dual pairs of copositive programs. Here, apparently for the first time in the literature, we also propose a systematic construction principle for non-attainability phenomena. Finally, also seemingly for the first time, Section 4 strives to provide a rough literature survey by clustering a hopefully large part of copositivity-related publications.

2 Elementary algebraic properties and counterexamples

Here we collect some properties which the cones $\mathcal{C}$ and $\mathcal{C}^*$ share with the more ubiquitous cones $\mathcal{P}$ and $\mathcal{N}$ (of nonnegative matrices), and some other properties which distinguish $\mathcal{C}$ and $\mathcal{C}^*$ from the other cones (and from each other). Some of these properties apparently never have been noticed before in the literature. For the sake of conciseness, in this section (and only here) we will abuse notation by regarding $\mathcal{C}$ as the class of completely positive matrices *of any order*; likewise we use the symbols $\mathcal{C}^*$, $\mathcal{P}$ and $\mathcal{N}$ in this section. So let $\mathcal{N}$ consist of all symmetric matrices with nonnegative entries, let $\mathcal{P} \cap \mathcal{N}$ be the set of *doubly nonnegative* matrices, and finally $\mathcal{P} + \mathcal{N} = \{P + N : P \in \mathcal{P}, N \in \mathcal{N}, P, N \text{ of the same order}\}$. We have the inclusion $\mathcal{C} \subseteq \mathcal{P} \cap \mathcal{N} \subseteq \mathcal{P} + \mathcal{N} \subseteq \mathcal{C}^*$, with equalities $\mathcal{C} = \mathcal{P} \cap \mathcal{N}$ and $\mathcal{P} + \mathcal{N} = \mathcal{C}^*$, if and only if we restrict these classes to matrices of order at most 4, see [22, Thm. 2.4, Rem. 1.10], [96], [133].

One readily observes that taking a principal submatrix of a member of one of these six classes again yields a member of that class (this is called *completeness* in [74]), and also, for A from one of these six classes, every *permutation similar matrix* $P^{-1}AP$ (with P a permutation matrix) and every *positive diagonal congruence* $D^\top AD$ (with D a positive-definite diagonal matrix, see [165]) is again a member of that class.

These properties are all special cases of the following more general one:

Proposition 2.1 (Sandwiching property):

(a) *Let* $\mathcal{K} \in \{\mathcal{P}, \mathcal{N}, \mathcal{C}, \mathcal{P} \cap \mathcal{N}, \mathcal{P} + \mathcal{N}, \mathcal{C}^*\}$.
If $B \in \mathcal{K}$ *and* A *is a rectangular matrix of fitting order with no negative entries, then we have* $A^\top BA \in \mathcal{K}$.

(b) *Let* $\mathcal{K} \in \{\mathcal{P}, \mathcal{N}, \mathcal{C}, \mathcal{P} \cap \mathcal{N}\}$. *Then we have* $\{A, B\} \subset \mathcal{K} \Rightarrow ABA \in \mathcal{K}$.

(c) *The counterpart of (b) is not true for the classes $\mathcal{K} \in \{\mathcal{P}+\mathcal{N}, \mathcal{C}^*\}$.*

Proof. For the case of $\mathcal{K} = \mathcal{C}$, this is stated in [22, Prop. 2.2]. It is as obvious as for the class $\mathcal{K} = \mathcal{P}$, and it is trivial for $\mathcal{K} = \mathcal{N}$. Assertion (b) is a straightforward consequence of $\mathcal{K} \subseteq \mathcal{N}$ and (a), while the claim in (c) is demonstrated by the following example. $\square$

Example 2.1 *With* $A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \in \mathcal{P} \subset \mathcal{C}^*$, $B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \in \mathcal{N} \subset \mathcal{C}^*$, *we have* $ABA^\top = \begin{bmatrix} -3 & 3 \\ 3 & -2 \end{bmatrix} \notin \mathcal{C}^*$. *Note that* $\mathcal{P}+\mathcal{N} = \mathcal{C}^*$ *when restricted to matrices of order* 2.

Similarly, for $\mathcal{K} \in \{\mathcal{P}, \mathcal{N}, \mathcal{C}, \mathcal{P}\cap\mathcal{N}\}$ and $n \in \mathbb{N}$ we have $A \in \mathcal{K} \Rightarrow A^n \in \mathcal{K}$. This follows inductively, for even n directly, for odd n by sandwiching. This is only true for even powers for the classes $\mathcal{P}+\mathcal{N}$ and $\mathcal{C}^*$, as can be shown directly, but not for odd powers, as the following example demonstrates.

Example 2.2 *Let*

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathcal{P}+\mathcal{N}.$$

Then

$$A^3 = \begin{bmatrix} 20 & 21 & -15 \\ 21 & 20 & -15 \\ -15 & -15 & 11 \end{bmatrix} \quad \text{with} \quad [0,2,3]\, A^3\, [0,2,3]^\top = -1\,.$$

Thus $A^3 \notin \mathcal{P}+\mathcal{N}$. *Note that* $\mathcal{P}+\mathcal{N} = \mathcal{C}^*$ *when restricted to matrices of order* 3. *Also the famous* 5×5 *Horn matrix* $H \in \mathcal{C}^* \setminus (\mathcal{P}+\mathcal{N})$ *[133] satisfies* $\mathbf{x}^\top H^3 \mathbf{x} = -3 < 0$ *for* $\mathbf{x} = [2,0,0,2,3]^\top \in \mathbb{R}^5_+$.

The symmetrization $AB+BA$ of the product AB is less well behaved. Of course $A, B \in \mathcal{N} \Rightarrow AB+BA \in \mathcal{N}$, but there is no analogous result for the other matrix classes, as shown below.

Example 2.3 *Let* $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, *which are both in* $\mathcal{C}$, *resulting in* $AB+BA = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$, *with eigenvalues* $1 \pm \sqrt{2}$. *Thus* $AB+BA \notin \mathcal{K}$ *for* $\mathcal{K} \in \{\mathcal{P}, \mathcal{C}, \mathcal{P}\cap\mathcal{N}\}$.

Example 2.4 *For a demonstration of* $A, B \in \mathcal{K}, AB+BA \notin \mathcal{K}$ *for* $\mathcal{K} \in \{\mathcal{P}+\mathcal{N}, \mathcal{C}^*\}$ *let* A *be the matrix from Example 2.2, and take* $B = A^2$.

Table 1: Closure properties of cones for sandwiching, symmetrized products, and posynomials $p(A) = \sum_k c_k A^k$ with $c_k \geq 0$, integer $k \geq 0$.

$\mathcal{K}$	$A, B \in \mathcal{K}$ $\Rightarrow ABA \in \mathcal{K}$	$A, B \in \mathcal{K}$ $\Rightarrow AB + BA \in \mathcal{K}$	$A \in \mathcal{K}$ $\Rightarrow p(A) \in \mathcal{K}$
$\mathcal{P}$	yes: Prop. 2.1	no: Ex. 2.3	yes: follows
$\mathcal{N}$	yes: evident	yes: evident	yes: evident
$\mathcal{P} \cap \mathcal{N}$	yes: Prop. 2.1	no: Ex. 2.3	yes: follows
$\mathcal{P} + \mathcal{N}$	no: Ex. 2.1	no: Ex. 2.4	no[a]: Ex. 2.2
$\mathcal{C}$	yes: Prop. 2.1	no: Ex. 2.3	yes: follows
$\mathcal{C}^*$	no: Ex. 2.1	no: Ex. 2.4	no[a]: Ex. 2.2

[a] but $A^k \in \mathcal{K}$ for $k \in \mathbb{N}$ even (evident)

Example 2.5 *Take* $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. *Then* $A \in \mathcal{C} = \mathcal{P} \cap \mathcal{N}$ *and* $B \in \mathcal{C}^* = \mathcal{P} + \mathcal{N}$, *but* $AB + BA = \begin{bmatrix} -2 & -1 \\ -1 & 4 \end{bmatrix} \notin \mathcal{C}^* = \mathcal{P} + \mathcal{N}$.

Now consider tensor (or Kronecker) products. Recall that for an $m \times m$ matrix A and an $n \times n$ matrix B, this product is defined as the block $mn \times mn$ matrix $A \otimes B = [a_{ij} B]_{i,j=1}^n$.

Proposition 2.2 (tensor product properties):
Let A and B be two symmetric matrices, not necessarily of same size.

(a) *Let $\mathcal{K} \in \{\mathcal{P}, \mathcal{N}, \mathcal{C}, \mathcal{P} \cap \mathcal{N}\}$. Then $\{A, B\} \subset \mathcal{K} \Rightarrow A \otimes B \in \mathcal{K}$.*

(b) *However, this implication is wrong if $\mathcal{K} \in \{\mathcal{P} + \mathcal{N}, \mathcal{C}^*\}$.*

(c) *Let $\mathcal{K} \in \{\mathcal{C}, \mathcal{P} \cap \mathcal{N}\}$. Then $A \in \mathcal{K}$ and $B \in \mathcal{K}^*$ imply $A \otimes B \in \mathcal{K}^*$.*

Proof. (a) is well known; for instance, closure under the tensor product of the completely positive cone has been established already in [22, Prop. 2.3]. We include a simple proof for the readers' convenience here. For $\mathcal{K} = \mathcal{C}$, consider $A = FF^\top$ and $B = GG^\top$, so that $A \otimes B = (FF^\top) \otimes (GG^\top) = (F \otimes G)(F \otimes G)^\top \in \mathcal{C}$, since $F \otimes G$ has no negative entry if neither F nor G have one. Along the same lines one can prove $A, B \in \mathcal{P} \Rightarrow A \otimes B \in \mathcal{P}$, and $A, B \in \mathcal{N} \Rightarrow A \otimes B \in \mathcal{N}$ is evident. Assertion (a) follows then also for $\mathcal{P} \cap \mathcal{N}$. Example 2.6 below illustrates claim (b). To establish (c), observe that (a) implies $A \otimes B \in \mathcal{P} + \mathcal{N}$, if $A \in \mathcal{P} \cap \mathcal{N}$ and $B \in \mathcal{P} + \mathcal{N}$. Finally, we have to show that $A \in \mathcal{C}$ and $B \in \mathcal{C}^*$ implies $A \otimes B \in \mathcal{C}^*$. Indeed, first observe that any $\mathbf{x} \in \mathbb{R}^{mn}_+$ can be written as $\mathbf{x} = \sum_{i=1}^m \mathbf{e}_i \otimes \mathbf{x}_i$ with $\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ the standard basis of $\mathbb{R}^m$ and $\mathbf{x}_i \in \mathbb{R}^n_+$. Let $A = FF^\top \in \mathcal{C}$ with F an $m \times k$

matrix without negative entries, and $B \in \mathcal{C}^*$. Then

$$\begin{array}{rcl} \mathbf{x}^\top(A\otimes B)\mathbf{x} &=& \sum_{i,j}(\mathbf{e}_i\otimes\mathbf{x}_i)^\top(FF^\top\otimes B)(\mathbf{e}_j\otimes\mathbf{x}_j) \\ &=& \sum_{i,j}(\mathbf{e}_i^\top\otimes\mathbf{x}_i^\top)(FF^\top\mathbf{e}_j\otimes B\mathbf{x}_j) \\ &=& \sum_{i,j}(F^\top\mathbf{e}_i)^\top(F^\top\mathbf{e}_j)\otimes\mathbf{x}_i^\top B\mathbf{x}_j\,. \end{array}$$

Now the latter Kronecker factors are scalars, so that the product is the usual scalar one. Hence

$$\begin{array}{rcl} \mathbf{x}^\top(A\otimes B)\mathbf{x} &=& \sum_{i,j}(F^\top\mathbf{e}_i)^\top(F^\top\mathbf{e}_j)(\mathbf{x}_i^\top B\mathbf{x}_j) \\ &=& \sum_{i,j}(F^\top\mathbf{e}_j)^\top(F^\top\mathbf{e}_i)(\mathbf{x}_i^\top B\mathbf{x}_j) \\ &=& \sum_{i,j}\operatorname{trace}\left[(F^\top\mathbf{e}_i)(\mathbf{x}_i^\top B\mathbf{x}_j)(F^\top\mathbf{e}_j)^\top\right] = \operatorname{trace}\left[G^\top BG\right], \end{array}$$

where $G = \sum_j \mathbf{x}_j\mathbf{e}_j^\top F$ is an $n\times k$ matrix without negative entries. Thus

$$\mathbf{x}^\top(A\otimes B)\mathbf{x} = B\bullet GG^\top \ge 0 \quad \text{for all } \mathbf{x}\in\mathbb{R}^{mn}_+\,,$$

where $A\bullet B = \operatorname{trace}(AB)$ denotes the Frobenius inner product of symmetric $n\times n$ matrices A and B. $\square$

Example 2.6 *Let* $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \in \mathcal{P}\subset\mathcal{C}^*$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathcal{N}\subset\mathcal{C}^*$. *Then*

$$A\otimes B = \begin{bmatrix} 0 & 0 & 1 & -1 \\ & & \cdots & \end{bmatrix} \notin \mathcal{C}^*\,.$$

(See also Example 2.7.) It does not help if both factors are the same: Let $C = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in \mathcal{C}^*$, *with* A *and* B *as before. Then* $C\otimes C$ *has* $A\otimes B$ *as one of its principle submatrices, therefore* $C\otimes C\notin\mathcal{C}^*$.

Finally we turn to Hadamard products. For square matrices A, B of the same size, the Hadamard product is defined as $A.B = [a_{ij}b_{ij}]_{i,j}$, so it is bilinear. Also $(A.B)^\top = A^\top.B^\top$ and $A.B = B.A$. We further define the Hadamard power $A^{(n)} = [a_{ij}^n]_{i,j}$, and call *Hadamard posynomial* a function that maps a matrix A to $[p(a_{ij})]_{i,j}$, where p is a polynomial with no negative coefficients.

An important observation is that $A.B$ is a principal submatrix of $A\otimes B$ [145, p.304]. Thus from Proposition 2.2 we conclude

$$A, B\in\mathcal{K} \Rightarrow A.B\in\mathcal{K} \text{ for } \mathcal{K}\in\{\mathcal{P},\mathcal{N},\mathcal{P}\cap\mathcal{N},\mathcal{C}\}\,. \tag{3}$$

In case $\mathcal{K}=\mathcal{P}$, this is known as Schur's theorem, see, e.g., [145, p.309] or [22, Prop.1.7]. The case $\mathcal{K}=\mathcal{C}$ is treated in [22, Cor.2.2]. The following is a counterexample for the case $\mathcal{K}\in\{\mathcal{P}+\mathcal{N},\mathcal{C}^*\}$:

Table 2: Closure properties of cones for Kronecker and Hadamard products, and Hadamard posynomials $p(A) = \sum_k c_k A^{(k)}$ with $c_k \geq 0$.

$\mathcal{K}$	$A, B \in \mathcal{K}$ $\Rightarrow A \otimes B \in \mathcal{K}$	$A, B \in \mathcal{K}$ $\Rightarrow A.B \in \mathcal{K}$	$A \in \mathcal{K}$ $\Rightarrow p(A) \in \mathcal{K}$
$\mathcal{P}$	yes: Prop 2.2	yes: (3)	yes: follows
$\mathcal{N}$	yes: evident	yes: (3)	yes: evident
$\mathcal{P} \cap \mathcal{N}$	yes: Prop 2.2	yes: (3)	yes: follows
$\mathcal{P} + \mathcal{N}$	no[b]: Ex. 2.6	no[c]: Ex. 2.7	yes: Prop. 2.3
$\mathcal{C}$	yes: Prop 2.2	yes: (3)	yes: follows
$\mathcal{C}^*$	no[b]: Ex. 2.6	no[c]: Ex. 2.7	unclear

[b] not even if $A = B$
[c] but $A.A \in \mathcal{K}$ (evident)

Example 2.7 *Both matrices A, B from Example 2.6 belong to $\mathcal{C}^*$, however* $A.B = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \notin \mathcal{C}^*$.

Remark 2.1 *There is no way for concluding $A \otimes B \in \mathcal{C}^*$ from $A.B \in \mathcal{C}^*$, in particular the implication $\{A, B, A.B\} \subset \mathcal{C}^* \Rightarrow A \otimes B \in \mathcal{C}^*$ does not hold. Just take $A = B = C$, with C from Example 2.6.*

Proposition 2.3 (odd Hadamard powers)*:*
If $A \in \mathcal{P} + \mathcal{N}$, and $n = 2k + 1$ with $k \in \mathbb{N}$ is odd, then $A^{(n)} \in \mathcal{P} + \mathcal{N}$.

Proof. Assume $A = P + N$ with $P \in \mathcal{P}$ and $N \in \mathcal{N}$. Then by Schur's theorem, $P^{(n)} \in \mathcal{P}$, and by the monotonicity of odd power functions, $A^{(n)} - P^{(n)} \in \mathcal{N}$. $\square$

We conclude this section by observations on inversion and Schur complements. These two operations leave (the interior of) $\mathcal{P}$ invariant, but it is well known that this is not true for $\mathcal{N}$.

Example 2.8 *The example*

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \in \mathcal{N} \subset \mathcal{C}^* \quad \textit{with} \quad A^{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} \notin \mathcal{C}^*$$

shows that the Schur complement of a positive-definite principal submatrix in a (co)positive matrix need not be (co)positive. Further,

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \in \mathcal{P} \cap \mathcal{N} \quad \textit{but} \quad A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \notin \mathcal{N} \supset \mathcal{C}\,.$$

Note that $\mathcal{P} \cap \mathcal{N} = \mathcal{C}$ when restricted to matrices of order 2.

Table 3: Further closure properties of cones for several products

$\mathcal{K}$	$A \in \mathcal{K}, B \in \mathcal{K}^*$ $\Rightarrow A \otimes B \in \mathcal{K} + \mathcal{K}^*$	$A \in \mathcal{K}, B \in \mathcal{K}^*$ $\Rightarrow A.B \in \mathcal{K} + \mathcal{K}^*$	$A \in \mathcal{K}, B \in \mathcal{K}^*$ $\Rightarrow AB + BA \in \mathcal{K} + \mathcal{K}^*$
$\mathcal{C}$	yes: Prop 2.2(c)	yes: follows	no: Ex. 2.5
$\mathcal{P} \cap \mathcal{N}$	yes: Prop 2.2(c)	yes: follows	no: Ex. 2.5

On the other hand, if the operator norm $\|A\| < 1$ and $A \in \mathcal{C}$, then from Table 1 and closedness of $\mathcal{C}$ we get $(I - A)^{-1} = \sum_{k=0}^{\infty} A^k \in \mathcal{C}$ while $I - A \notin \mathcal{N}$ unless A is diagonal.

3 Duality and attainability in copositive programs

We consider a primal/dual pair of copositive programs, whose primal consists of optimizing a linear function over the intersection of an affine subspace with the completely positive cone: for m symmetric matrices A_i of the order of X, let $\mathbf{A}X = [A_1 \bullet X, \ldots, A_m \bullet X]^\top \in \mathbb{R}^m$, if $X \in \mathcal{C}$, and $\mathbf{b} \in \mathbb{R}^m$. Then

$$p^* = \inf \{ C \bullet X : \mathbf{A}X = \mathbf{b} \,,\; X \in \mathcal{C} \} \tag{4}$$

and with multipliers $\mathbf{y} = [y_1, \ldots y_m]^\top \in \mathbb{R}^m$, the Lagrangian dual function reads

$$\begin{aligned} \Theta_p(\mathbf{y}) &= \inf \{ C \bullet X + \textstyle\sum_{i=1}^m y_i [b_i - A_i \bullet X] : X \in \mathcal{C} \} \\ &= \inf \left\{ (C - \mathbf{A}^\top \mathbf{y}) \bullet X + \mathbf{b}^\top \mathbf{y} : X \in \mathcal{C} \right\} \,, \end{aligned}$$

where $\mathbf{A}^\top \mathbf{y} = \sum_i y_i A_i$ is again a symmetric matrix of the order of X. Obviously, $\Theta_p(\mathbf{y}) > -\infty$ if and only if $C - \mathbf{A}^\top \mathbf{y} \in \mathcal{C}^*$, and then $\Theta_p(\mathbf{y}) = \mathbf{b}^\top \mathbf{y}$. Hence the Lagrangian dual problem reads

$$d^* = \sup \{ \Theta_p(\mathbf{y}) : \mathbf{y} \in \mathbb{R}^m \} = \sup \left\{ \mathbf{b}^\top \mathbf{y} : C - \mathbf{A}^\top \mathbf{y} \in \mathcal{C}^* \,,\; \mathbf{y} \in \mathbb{R}^m \right\} \,, \tag{5}$$

which can be rewritten as a linear optimization problem over the intersection of an affine subspace with the cartesian product cone $\mathbb{R}^m \times \mathcal{C}^*$:

$$d^* = \sup \left\{ \mathbf{b}^\top \mathbf{y} : S + \mathbf{A}^\top \mathbf{y} = C \,,\; (\mathbf{y}, S) \in \mathbb{R}^m \times \mathcal{C}^* \right\} \,. \tag{6}$$

Arranging the multipliers of the constraints in a symmetric matrix U and observing $(\mathbf{A}^\top \mathbf{y}) \bullet U = (\mathbf{A}U)^\top \mathbf{y}$, the Lagrangian dual function of this problem (6) is

$$\begin{aligned} \Theta_d(U) &= \sup \left\{ \mathbf{b}^\top \mathbf{y} + (C - S - \mathbf{A}^\top \mathbf{y}) \bullet U : (\mathbf{y}, S) \in \mathbb{R}^m \times \mathcal{C}^* \right\} \\ &= \sup \left\{ C \bullet U + (\mathbf{b} - \mathbf{A}U)^\top \mathbf{y} - S \bullet U : (\mathbf{y}, S) \in \mathbb{R}^m \times \mathcal{C}^* \right\} \,. \end{aligned}$$

Again, $\Theta_d(U) < +\infty$ if and only if $U \in (\mathcal{C}^*)^* = \mathcal{C}$ and $\mathbf{A}U = \mathbf{b}$, so that the bidual of (4)

$$\inf \left\{ \Theta_d(U) : U = U^\top \right\} = \inf \left\{ C \bullet U : \mathbf{A}U = \mathbf{b}\,,\ U \in \mathcal{C} \right\} = p^*\,,$$

coincides indeed with the primal, as it should be. Of course, weak duality $d^* \leq p*$ always holds for the pair (4) and (5), and Slater's condition applies to guarantee full strong duality (i.e., $d^* = \mathbf{b}^\top \mathbf{y}^*$ is attained for some dually feasible $(\mathbf{y}^*, S^*) \in \mathbb{R}^m \times \mathcal{C}^*$ and coincides with p^*). However, unlike the LP case, primal attainability is not guaranteed for general conic programs. This is well known for semidefinite programs, and there are examples of all sorts of phenomena like positive finite duality gap and/or non-attainability for either the primal or the dual or both. Looking at one such example [139, Ex. 2.2.1], taken from [252], we see that we cannot simply replace the semidefinite cone $\mathcal{P}$ by either $\mathcal{C}$ or $\mathcal{C}^*$, to arrive at suitable examples for copositive programs, which is the main purpose of this section.

First, exclude the standard infeasible/unbounded cases where $p^* = -\infty$ or $d^* = +\infty$.

Next let us examine the case of zero duality gap, i.e., $d^* = p^*$. By above exclusion, we are left with a common finite value. Full strong duality (attainability of both) holds of course under Slater's condition:

$$\{X : \mathbf{A}X = \mathbf{b}\} \cap \operatorname{int} \mathcal{C} \neq \emptyset$$

implies zero duality gap and dual attainability, and

$$\left\{ S : S + \mathbf{A}^\top \mathbf{y} = C \text{ for some } \mathbf{y} \in \mathbb{R}^m \right\} \cap \operatorname{int} \mathcal{C}^* \neq \emptyset$$

implies zero duality gap and primal attainability. But full strong duality also holds for the copositive reformulation of Standard Quadratic Problems (StQPs), as was already observed in [47]. By contrast, failure of dual attainability with $d^* = p^*$ can happen in the general case of reformulation of Multi-StQPs [57]. We complement these observations by two more examples, one where p^* is not attained but d^* is, and a second where both are not attained.

Example 3.1 *This is an adaptation of [139, Ex. 2.2.8] from $\mathcal{P}$ to $\mathcal{C}$ which works: let $n = 2$, $m = 1$, $C \bullet X = x_{11}$, $A_1 \bullet X = x_{12} + x_{21}$ and $b_1 = 2$. Then*

$$d^* = \sup \left\{ 2y_1 : \begin{bmatrix} 1 & -y_1 \\ -y_1 & 0 \end{bmatrix} \in \mathcal{C}^* \right\} = 0$$

is attained for $y_1^* = 0$. Observe that $y_1 = -1$ is also dually feasible, but not optimal. On the other hand, the choice of $x_{11} = \frac{1}{k}$ and $x_{22} = k$ with $x_{12} = 1$ gives a primally feasible X_k with $C \bullet X_k = \frac{1}{k} \searrow 0$ as $k \nearrow \infty$, so that $p^* = d^*$. Obviously, p^* cannot be attained since $x_{11} = 0$ conflicts with $x_{12} = 1$ and $X \in \mathcal{C} \subset \mathcal{P}$.

The next example results from a general principle of constructing non-attainability, starting from a given instance $(\mathbf{A}, \mathbf{b}, C)$ of a copositive program (4) and (5).

Theorem 3.1 (constructing failure in dual attainability):
Let $\mathcal{T}_{\mathrm{d}}(\mathbf{A}, \mathbf{b}, C)$ denote the following new copositive primal problem: augment the $n \times n$ variable matrices X by appending two more rows and columns, to arrive at $(n+2) \times (n+2)$ variable matrices $\bar{X}$; further, define the objective and $m+2$ constraints as follows: $\bar{\mathbf{b}} = [\mathbf{b}^\top, 1, 0]^\top$ and, with $\mathbf{o} \in \mathbb{R}^n$ the zero vector and O the $n \times n$ zero matrix,

$$\bar{C} = \begin{bmatrix} C & \mathbf{o} & \mathbf{o} \\ \mathbf{o}^\top & 0 & -1 \\ \mathbf{o}^\top & -1 & 0 \end{bmatrix} \quad \textit{and} \quad \bar{A}_i = \begin{bmatrix} A_i & \mathbf{o} & \mathbf{o} \\ \mathbf{o}^\top & 0 & 0 \\ \mathbf{o}^\top & 0 & 0 \end{bmatrix}, \ 1 \le i \le m\,,$$

while

$$\bar{A}_{m+1} = \begin{bmatrix} O & \mathbf{o} & \mathbf{o} \\ \mathbf{o}^\top & 1 & 0 \\ \mathbf{o}^\top & 0 & 0 \end{bmatrix} \quad \textit{and} \quad \bar{A}_{m+2} = \begin{bmatrix} O & \mathbf{o} & \mathbf{o} \\ \mathbf{o}^\top & 0 & 0 \\ \mathbf{o}^\top & 0 & 1 \end{bmatrix}.$$

Then $\mathcal{T}_{\mathrm{d}}(\mathbf{A}, \mathbf{b}, C)$ is feasible if and only if (4) is feasible, in which case p^* is attained in (4) for $(\mathbf{A}, \mathbf{b}, C)$ if and only if

$$\bar{p}^* = \inf\left\{\bar{C} \bullet \bar{X} : \bar{A}_i \bullet \bar{X} = \bar{b}_i\,,\ 1 \le i \le m+2\,,\ \bar{X} \in \mathcal{C}\right\} \tag{7}$$

is attained in the primal of $\mathcal{T}_{\mathrm{d}}(\mathbf{A}, \mathbf{b}, C)$. Furthermore, $\mathcal{T}_{\mathrm{d}}(\mathbf{A}, \mathbf{b}, C)$ has the same primal and dual optimal values p^* and d^*, but d^* is not attained: $\bar{p}^* = p^*$ and

$$\bar{d}^* := \sup\left\{\bar{\mathbf{b}}^\top \bar{\mathbf{y}} : \bar{C} - \sum_{i=1}^{m+2} \bar{y}_i \bar{A}_i \in \mathcal{C}^*\,,\ \bar{\mathbf{y}} \in \mathbb{R}^{m+2}\right\} = d^*\,, \tag{8}$$

but $d^* \notin \left\{\bar{\mathbf{b}}^\top \bar{\mathbf{y}} : \bar{C} - \sum_{i=1}^{m+2} \bar{y}_i \bar{A}_i \in \mathcal{C}^*\,,\ \bar{\mathbf{y}} \in \mathbb{R}^{m+2}\right\}$.

Proof. Choose a (4)-feasible sequence $X_k \in \mathcal{C}$ such that $C \bullet X_k \to p^*$ and augment X_k by

$$\bar{X}_k = \begin{bmatrix} X_k & \mathbf{o} & \mathbf{o} \\ \mathbf{o}^\top & 1 & 0 \\ \mathbf{o}^\top & 0 & 0 \end{bmatrix}$$

to get a (7)-feasible sequence with $\bar{C} \bullet \bar{X}_k = C \bullet X_k \to p^*$. Of course, any (7)-feasible $\bar{X}$ contains a leading $n \times n$ principal submatrix X which is (4)-feasible. Further, $\bar{X} \in \mathcal{C}$ with $\bar{X}_{n+2,n+2} = 0$ enforces $\bar{X}_{n+1,n+2} = 0$ so that $\bar{C} \bullet \bar{X} = C \bullet X$. This proves $\bar{p}^* = p^*$, and also the assertions about equivalence of primal feasibility/attainability. Turning towards dual feasibility, and putting $\bar{\mathbf{y}} = [\mathbf{y}^\top, \bar{y}_{m+1}, \bar{y}_{m+2}]^\top$ with $\mathbf{y} \in \mathbb{R}^m$, we have

$$\bar{C} - \sum_{i=1}^{m+2} \bar{y}_i \bar{A}_i = \left[\begin{array}{ccc} C - \sum\limits_{i=1}^{m} y_i A_i & \mathbf{o} & \mathbf{o} \\ \mathbf{o}^\top & -\bar{y}_{m+1} & -1 \\ \mathbf{o}^\top & -1 & -\bar{y}_{m+2} \end{array} \right],$$

so that $\bar{C} - \sum\limits_{i=1}^{m+2} \bar{y}_i \bar{A}_i \in \mathcal{C}^*$ entails $C - \sum\limits_{i=1}^{m} y_i A_i \in \mathcal{C}^*$ and $\bar{y}_{m+1} < 0$, which in turn implies

$$\bar{\mathbf{b}}^\top \bar{\mathbf{y}} = \mathbf{b}^\top \mathbf{y} + \bar{y}_{m+1} < \mathbf{b}^\top \mathbf{y} \le d^*, \tag{9}$$

hence $\bar{d}^* \le d^*$. On the other hand, select a sequence $\mathbf{y}_k \in \mathbb{R}^m$ such that $C - \mathbf{A}^\top \mathbf{y}_k \in \mathcal{C}^*$ and $\mathbf{b}^\top \mathbf{y}_k \to d^*$. Then the sequence $\bar{\mathbf{y}}_k = [\mathbf{y}_k^\top, -\frac{1}{k}, -k]^\top \in \mathbb{R}^{m+2}$ clearly satisfies $\bar{C} - \overline{\mathbf{A}^\top} \bar{\mathbf{y}}_k \in \mathcal{C}^*$ since its lower-right 2×2 block is even positive-semidefinite, and

$$\bar{\mathbf{b}}^\top \bar{\mathbf{y}}_k = \mathbf{b}^\top \mathbf{y}_k - \frac{1}{k} \to d^* \quad \text{as } k \to \infty,$$

which shows $\bar{d}^* \ge d^*$. The strict inequality in (9) implies dual non-attainability.
□

Clearly, $\mathcal{T}_{\mathrm{d}}(\mathbf{A}, \mathbf{b}, C)$ violates Slater's condition even if $(\mathbf{A}, \mathbf{b}, C)$ satisfies it. Any such instance therefore generates, by Theorem 3.1 above, an example of zero duality gap with dual non-attainability. Even more:

Example 3.2 *Choose $(\mathbf{A}, \mathbf{b}, C)$ as in Example 3.1. Then $\mathcal{T}_{\mathrm{d}}(\mathbf{A}, \mathbf{b}, C)$ has zero duality gap but neither of the (finite) optimal values are attained.*

Next we turn to the case to a finite positive duality gap:

$$-\infty < d^* < p^* < +\infty$$

where we need four examples for (non-)attainability.

Example 3.3 *Again, this is an adaptation of an example in [139], now by switching dual with primal. For $n = 3$ and $m = 2$ matrices, let C be such that $C \bullet X = x_{33}$ whereas $A_1 \bullet X = x_{33} + 2x_{12}$ and $A_2 \bullet X = x_{22}$. Further, let $\mathbf{b} = [1, 0]^\top \in \mathbb{R}^2$. Then*

$$p^* = \inf \{x_{33} : x_{33} + 2x_{12} = 1, \, x_{22} = 0, \, X \in \mathcal{C}\} = 1,$$

attained for an $X^* \in \mathcal{C}$ with all $x_{ij}^* = 0$ except $x_{33}^* = 1$. The dual reads

$$d^* = \sup \left\{ y_1 : \left[\begin{array}{ccc} 0 & -y_1 & 0 \\ -y_1 & -y_2 & 0 \\ 0 & 0 & 1 - y_1 \end{array} \right] \in \mathcal{C}^* \right\} = 0,$$

attained for $\mathbf{y}^* = [0, 0]^\top$.

Note that the remedy via eigenspaces for SDPs as suggested in [139] does not seem to apply to copositive programs, the situation appears to be much more difficult.

Anyhow, by Theorem 3.1 we immediately have

Example 3.4 *With $(\mathbf{A}, \mathbf{b}, C)$ as in Example 3.3, $(\tilde{A}, \tilde{\mathbf{b}}, \tilde{C}) = \mathcal{T}_{\rm d}(\mathbf{A}, \mathbf{b}, C)$ has a positive duality gap, and p^* is attained but d^* is not.*

We now provide a direct primal construction as a counterpart to $\mathcal{T}_{\rm d}$, which is slightly simpler:

Theorem 3.2 (constructing failure in primal attainability): *Let $\mathcal{T}_{\rm p}(\mathbf{A}, \mathbf{b}, C)$ denote the following new copositive primal problem: augment the $n \times n$ variable matrices X by appending two more rows and columns, to arrive at $(n+2) \times (n+2)$ variable matrices $\bar{X}$; further, define the objective and $m+1$ constraints as follows: augment A_i by two more zero rows and columns to $\bar{A}_i$ and put $\bar{\mathbf{b}} = [\mathbf{b}^\top, 2]^\top \in \mathbb{R}^{m+1}$ while, with $\mathbf{o} \in \mathbb{R}^n$ the zero vector and O the $n \times n$ zero matrix,*

$$\bar{C} = \begin{bmatrix} C & \mathbf{o} & \mathbf{o} \\ \mathbf{o}^\top & 1 & 0 \\ \mathbf{o}^\top & 0 & 0 \end{bmatrix} \quad \textit{and} \quad \bar{A}_{m+1} = \begin{bmatrix} O & \mathbf{o} & \mathbf{o} \\ \mathbf{o}^\top & 0 & 1 \\ \mathbf{o}^\top & 1 & 0 \end{bmatrix} .$$

Then $\mathcal{T}_{\rm p}(\mathbf{A}, \mathbf{b}, C)$ is dually feasible if and only if (5) is feasible, in which case d^* is attained in (5) if and only if

$$\bar{d}^* := \sup \left\{ \bar{\mathbf{b}}^\top \bar{\mathbf{y}} : \bar{C} - \sum_{i=1}^{m+1} \bar{y}_i \bar{A}_i \in \mathcal{C}^* \, , \; \bar{\mathbf{y}} \in \mathbb{R}^{m+2} \right\} \tag{10}$$

is attained. Furthermore, $\mathcal{T}_{\rm p}(\mathbf{A}, \mathbf{b}, C)$ has the same primal and dual optimal values p^* and d^*, but p^* is not attained: $\bar{d}^* = d^*$ and

$$\bar{p}^* := \inf \left\{ \bar{C} \bullet \bar{X} : \bar{A}_i \bullet \bar{X} = \bar{b}_i \, , \; 1 \le i \le m+1 \, , \, \bar{X} \in \mathcal{C} \right\} = p^* \, , \tag{11}$$

but $p^* \notin \left\{ \bar{C} \bullet \bar{X} : \bar{A}_i \bullet \bar{X} = \bar{b}_i \, , \; 1 \le i \le m+1 \, , \, \bar{X} \in \mathcal{C} \right\}$.

Proof. The constraint $\bar{A}_{m+1} \bullet \bar{X} = 2$ means $x_{n+1,n+2} = 1$ which forces $x_{n+1,n+1} > 0$, so that for any (11)-feasible $\bar{X}$ we have

$$\bar{C} \bullet \bar{X} = C \bullet X + x_{n+1,n+1} > C \bullet X \ge p^* \, , \tag{12}$$

and hence $\bar{p}^* \ge p^*$. If, on the other hand, a sequence $X_k \in \mathcal{C}$ with $\mathbf{A} X_k = \mathbf{b}$ satisfies $C \bullet X_k \to p^*$ and we form $\bar{X}_k$ by

$$\bar{X}_k = \begin{bmatrix} X_k & \mathbf{o} & \mathbf{o} \\ \mathbf{o}^\top & \frac{1}{k} & 1 \\ \mathbf{o}^\top & 1 & k \end{bmatrix} ,$$

then this is a (11)-feasible sequence such that $\lim_{k\to\infty} \bar{C} \bullet \bar{X}_k = p^*$, which shows $\bar{p}^* \le p^*$, and non-attainability of $\bar{p}^*$ follows from the strict inequality in (12). Turning towards dual feasibility, and putting $\bar{\mathbf{y}} = [\mathbf{y}^\top, \bar{y}_{m+1}]^\top$ with $\mathbf{y} \in \mathbb{R}^m$, we have

$$\bar{C} - \sum_{i=1}^{m+1} \bar{y}_i \bar{A}_i = \left[\begin{array}{ccc} C - \sum\limits_{i=1}^{m} y_i A_i & \mathbf{o} & \mathbf{o} \\ \mathbf{o}^\top & 1 & -\bar{y}_{m+1} \\ \mathbf{o}^\top & -\bar{y}_{m+1} & 0 \end{array} \right],$$

so that $\bar{C} - \sum\limits_{i=1}^{m+1} \bar{y}_i \bar{A}_i \in \mathcal{C}^*$ entails $C - \sum\limits_{i=1}^{m} y_i A_i \in \mathcal{C}^*$ and $\bar{y}_{m+1} \le 0$, which in turn implies

$$\bar{\mathbf{b}}^\top \bar{\mathbf{y}} = \mathbf{b}^\top \mathbf{y} + 2\bar{y}_{m+1} \le \mathbf{b}^\top \mathbf{y} \le d^*,$$

and thus $\bar{d}^* \le d^*$. On the other hand, for any $\mathbf{y} \in \mathbb{R}^m$ such that $C - \mathbf{A}^\top \mathbf{y} \in \mathcal{C}^*$ the point $\bar{\mathbf{y}} = [\mathbf{y}^\top, 0]^\top \in \mathbb{R}^{m+1}$ is (10)-feasible with $\bar{\mathbf{b}}^\top \bar{\mathbf{y}} = \mathbf{b}^\top \mathbf{y}$, so that $\bar{d}^* \ge d^*$, and also equivalence of dual feasibility/attainability follows. □

Example 3.5 *Let* $(\mathbf{A}, \mathbf{b}, C)$ *be as in Example 3.3 and consider* $\mathcal{T}_{\mathrm{p}}(\mathbf{A}, \mathbf{b}, C)$. *Then Theorem 3.2 gives an instance with positive finite duality gap where* p^* *is not attained but* d^* *is attained. Applying Theorem 3.1, we finally see that* $\mathcal{T}_{\mathrm{d}}(\mathcal{T}_{\mathrm{p}}(\mathbf{A}, \mathbf{b}, C))$ *has a positive finite duality gap where neither of* p^* *and* d^* *are attained.*

Example 3.6 *The previous effect is also generated by* $\mathcal{T}_{\mathrm{p}}(\tilde{\mathbf{A}}, \tilde{\mathbf{b}}, \tilde{C})$*, where* $(\tilde{\mathbf{A}}, \tilde{\mathbf{b}}, \tilde{C})$ *is the instance from Example 3.4, in other words by considering* $\mathcal{T}_{\mathrm{p}}(\mathcal{T}_{\mathrm{d}}(\mathbf{A}, \mathbf{b}, C))$ *with* $(\mathbf{A}, \mathbf{b}, C)$ *from Example 3.3.*

Now we have to deal with infeasibility of exactly one problem and boundedness of the other one. In this case, the duality gap is infinite, but we still can have attainability or non-attainability in the bounded problem.

Example 3.7 *For* $n = 3$*, consider the constraints* $A_1 \bullet X = 2x_{22} + 2x_{23} = 0 = b_1$ *and* $A_2 \bullet X = 2x_{12} - 2x_{33} = 2 = b_2$*. If* $X \in \mathcal{C}$*, then* $x_{23} \ge 0$ *and* $x_{22} \ge 0$ *imply* $x_{22} = 0$*, hence* $x_{12} = 0$*, hence* $x_{33} = -1 < 0$*, which is absurd. Hence the primal problem is infeasible,* $p^* = +\infty$*. Now choose* $C = O$*. Since*

$$C - \mathbf{A}^\top \mathbf{y} = \left[\begin{array}{ccc} 0 & -y_2 & 0 \\ -y_2 & -2y_1 & -y_1 \\ 0 & -y_1 & 2y_2 \end{array} \right]$$

has a zero in its top-left corner, we infer $y_2 \le 0$ *for any* $\mathbf{y} \in \mathbb{R}^2$ *such that* $C - \mathbf{A}^\top \mathbf{y} \in \mathcal{C}^*$*. Of course,* $\mathbf{y}^* = \mathbf{o}$ *is dually feasible, thus optimal, and* $d^* = 0$ *is attained.*

Table 4: Possible attainability/duality gap constellations in primal-dual pairs of copositive programs. Only the doubly infeasible case $d^* = -\infty, p^* = \infty$ is omitted.

duality gap / attained	zero $d^* = p^* \in \mathbb{R}$	finite positive $-\infty < d^* < p^* < \infty$	infinite $-\infty < d^* < p^* = \infty$	infinite $-\infty = d^* < p^* < \infty$
both attained	StQP [47], strong duality, Slater for both	Ex.3.3	impossible	impossible
p^* attained, d^* not attained	MStQP [57], Slater for dual	Ex.3.4	impossible	Ex.3.9
p^* not attained, d^* attained	Ex.3.1, Slater for primal	Ex.3.5	Ex.3.7	impossible
neither attained	Ex.3.2	Ex.3.6	Ex.3.8	Ex.3.10

Example 3.8 *An application of Theorem 3.1 yields $\mathcal{T}_{\mathrm{d}}(\mathbf{A}, \mathbf{b}, C)$ from the instance $(\mathbf{A}, \mathbf{b}, C)$ in Example 3.7, an instance with $d^* = 0 < p^* = +\infty$ where d^* is not attained.*

To conclude, we need an instance $(\mathbf{A}, \mathbf{b}, C)$ with infeasible dual and bounded primal, where p^* is attained:

Example 3.9 *We keep $\mathbf{A}$ from the instance in Example 3.7 but change $\mathbf{b} = \mathbf{o}$ now. Then any feasible X satisfies $x_{33} = 0$. Now choose C with all zero entries except $c_{33} = -1$. Then $X^* = O \in \mathcal{C}$ is optimal, so $p^* = 0$ is attained. However,*

$$C - \mathbf{A}^\top \mathbf{y} = \begin{bmatrix} 0 & -y_2 & 0 \\ -y_2 & -2y_1 & -y_1 \\ 0 & -y_1 & -1 + 2y_2 \end{bmatrix} \in \mathcal{C}^*$$

is impossible, as $y_2 \leq 0$ must still hold, which implies the absurd $-1 + 2y_2 \leq -1 < 0$ for the lower-right corner entry. Hence $d^* = -\infty$.

Example 3.10 *The last example is generated by $\mathcal{T}_{\mathrm{p}}(\mathbf{A}, \mathbf{b}, C)$ from the instance $(\mathbf{A}, \mathbf{b}, C)$ in Example 3.9. Here $-\infty = d^* < p^* < +\infty$ and p^* is not attained.*

Finally, already in the LP domain we also are familiar with the case $d^* = -\infty$ and $p^* = +\infty$ where both problems are infeasible. Hence, all possible combinations are demonstrated.

4 A clustered bibliography

Of course, any collection of literature references is doomed incomplete (and outdated by the appearance date of printed issues). Nevertheless, we tried to cluster the following lists somehow systematically.

4.1 Surveys, reviews, entries, book chapters

Copositive optimization receives increasing interest in the Operations Research community, and is a rapidly expanding and fertile field of research. While the time may not yet be ripe for writing up the final standard text book in this domain, several authors nonetheless bravely took the challenge of providing an overview, thereby aiming at a rapidly moving target. Recent surveys on copositive optimization are offered by [108] and [43], while [152] and [141] provide reviews on copositivity with less emphasis on optimization. [41] and [71] are entries in the most recent edition of the Encyclopedia of Optimization. Recent book chapters with some character of a survey on copositivity from an optimization viewpoint are [42, Section 1.4] and [68].

4.2 Copositivity checking and properties

To check whether a given matrix is copositive is NP-hard, see [214]; for the completely positive side, see [99], cf. also [21]. There are several algorithmic approaches for this problem, among them recursive methods [2, 7, 25, 69, 82, 83, 88, 105, 172, 188].

On the dual side, an explicit certificate for complete copositivity is given by a non-negative factorization [11, 19, 22, 21, 23, 40, 102, 171, 174, 189, 193, 200, 233, 238, 255, 262, 264].

Other approaches for copositivity checking can be found in [10, 12, 13, 14, 15, 16, 17, 18, 20, 24, 29, 30, 33, 36, 39, 48, 64, 65, 63, 76, 77, 75, 80, 96, 97, 98, 100, 104, 103, 106, 109, 111, 115, 116, 119, 123, 124, 128, 132, 133, 134, 135, 138, 140, 142, 144, 143, 149, 151, 160, 162, 163, 164, 165, 169, 170, 173, 176, 183, 184, 185, 192, 196, 197, 198, 199, 203, 205, 208, 209, 210, 213, 212, 211, 218, 224, 225, 243, 244, 245, 247, 248, 249, 250, 251, 254, 256, 257, 258, 259, 260]. These references also include investigations of (mainly algebraic, but sometimes also geometric) properties of copositive and/or completely positive matrices.

4.3 Role of copositivity in optimization theory

Copositivity occurs in many different optimization contexts. A meanwhile classical role of copositivity is connected to the linear complementarity problem (LCP) [60, 136, 148, 166, 175, 204, 213, 235]. This list also includes references which address feasibility and/or attainability issues.

Other papers dealing with the role of copositivity in theory and application of optimization are [1, 6, 8, 9, 26, 27, 44, 45, 34, 35, 47, 38, 54, 50, 53, 41, 49, 57, 58, 73, 75, 78, 74, 79, 81, 85, 84, 86, 87, 91, 89, 100, 113, 114, 117, 121, 124, 125, 150, 161, 168, 167, 184, 190, 191, 195, 202, 214, 219, 220, 222, 223, 230, 234, 237, 240, 244, 253, 254, 263].

4.4 Copositive programming algorithms

According to Franz Rendl, nobody knows how to solve a copositive program (personal communication). To put it in less categorical terms: there is no state-of-the-art algorithm for copositive optimization. Various attempts have been made to tackle this problem [6, 24, 28, 44, 45, 31, 32, 34, 47, 38, 37, 46, 50, 53, 41, 52, 49, 51, 70, 68, 85, 84, 86, 93, 94, 103, 107, 110, 112, 118, 120, 121, 125, 127, 137, 162, 184, 185, 186, 216, 219, 220, 221, 226, 227, 228, 231, 253, 266]. The apparently most successful procedures up to now employ adaptive simplicial subdivision (see [146] for a good survey on this topic including convergence results): see [65, 66, 261, 265]; related variants for testing copositivity are [48, 243].

4.5 Copositive reformulations and relaxations for hard optimization problems

A considerable part of the success of copositive optimization lies in the versatility of this model, which allows for reformulating many hard problems from several, seemingly unrelated optimization domains. This includes graph theoretical and other discrete optimization models, mixed-integer, (fractional) quadratic problems [1, 28, 55, 34, 35, 47, 38, 37, 46, 50, 53, 41, 49, 67, 68, 93, 91, 94, 89, 107, 111, 112, 126, 129, 130, 131, 149, 185, 186, 202, 216, 217, 219, 220, 221, 226, 227, 228, 230, 231, 232, 239, 240, 241, 242, 244, 253, 261].

4.6 Applications of copositivity

The copositivity concept plays an already classical role in (Linear) Complementarity Problems and connected feasibility questions [106, 113, 114, 132, 213]. An interesting application to Simpson's paradox can be found in [129], while the connection of copositivity with conic geometry and angles is discussed at length in [8, 122, 153, 154, 155, 156, 157, 158, 159, 205, 222, 223, 235, 236].

The central solution concepts in (evolutionary) game theory, evolutionarily and/or neutrally stable strategies, are both refinements of the Nash equilibrium concept. They are closely connected to copositivity, as shown in [56, 28, 38].

Copositive formulations of robust optimization and/or uncertainty modeling are addressed in [125, 216].

Friction and contact problems in rigid body mechanics are treated from a copositive perspective in [3, 4, 5, 168, 167, 182]. Network (stability) problems in queueing, traffic, and reliability are tackled along this approach in [150, 177, 178, 179, 180, 187, 206, 207].

Also in the domain of dynamical systems and optimal control, copositive matrices play an important role, see, e.g. [60, 66, 72, 81, 160, 181, 201, 229, 257], while the articles [62, 191, 215] deal with questions of majorization, under/overestimation and tight bounding.

A few applications of complete positivity can be found in the book [22].

Finally, graph theory application aspects of copositivity (and closely related domains), and those of more general combinatorial optimization character can be found, e.g., in [92, 95, 90, 101, 107, 126, 130, 131, 227, 265], and in the following articles which provide a partial survey in this domain: [186, 246].

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References

[1] P. Amaral, I. M. Bomze, and J. J. Júdice. Copositivity and constrained fractional quadratic problems. Technical Report TR-ISDS 2010-05, University of Vienna, 2010.

[2] L.-E. Andersson, G. Z. Chang, and T. Elfving. Criteria for copositive matrices using simplices and barycentric coordinates. *Linear Algebra Appl.*, 220:9–30, 1995.

[3] M. Anitescu, J. F. Cremer, and F. A. Potra. On the existence of solutions to complementarity formulations of contact problems with friction. In *Complementarity and variational problems (Baltimore, MD, 1995)*, pages 12–21. SIAM, Philadelphia, PA, 1997.

[4] M. Anitescu and F. A. Potra. Formulating dynamic multi-rigid-body contact problems with friction as solvable linear complementarity problems. *Nonlinear Dynam.*, 14(3):231–247, 1997.

[5] M. Anitescu and F. A. Potra. A time-stepping method for stiff multibody dynamics with contact and friction. *Internat. J. Numer. Methods Engrg.*, 55(7):753–784, 2002.

[6] K. M. Anstreicher and S. Burer. D.C. versus copositive bounds for standard QP. *J. Global Optim.*, 33(2):299–312, 2005.

[7] K. M. Anstreicher and S. Burer. Computable representations for convex hulls of low-dimensional quadratic forms. *Math. Program.*, 124(1-2, Ser. B):33–43, 2010.

[8] D. Azé and J.-B. Hiriart-Urruty. Optimal Hoffman-type estimates in eigenvalue and semidefinite inequality constraints. *J. Global Optim.*, 24(2):133–147, 2002.

[9] L. Baratchart, M. Berthod, and L. Pottier. Optimization of positive generalized polynomials under l^p constraints. *J. Convex Anal.*, 5(2):353–379, 1998.

[10] F. Barioli. Decreasing diagonal elements in completely positive matrices. *Rend. Sem. Mat. Univ. Padova*, 100:13–25, 1998.

[11] F. Barioli and A. Berman. The maximal cp-rank of rank k completely positive matrices. *Linear Algebra Appl.*, 363:17–33, 2003.

[12] G. P. Barker. Theory of cones. *Linear Algebra Appl.*, 39:263–291, 1981.

[13] V. J. D. Baston. Extreme copositive quadratic forms. *Acta Arith.*, 15:319–327, 1968/1969.

[14] L. D. Baumert. *Extreme copositive quadratic forms.* PhD thesis, California Institute of Technology, 1965.

[15] L. D. Baumert. Extreme copositive quadratic forms. *Pacific J. Math.*, 19:197–204, 1966.

[16] L. D. Baumert. Extreme copositive quadratic forms. II. *Pacific J. Math.*, 20:1–20, 1967.

[17] A. Berman. *Cones, matrices and mathematical programming.* Springer-Verlag, Berlin, 1973. Lecture Notes in Economics and Mathematical Systems, Vol. 79.

[18] A. Berman. Complete positivity. *Linear Algebra Appl.*, 107:57–63, 1988.

[19] A. Berman and D. Hershkowitz. Combinatorial results on completely positive matrices. *Linear Algebra Appl.*, 95:111–125, 1987.

[20] A. Berman and R. J. Plemmons. *Nonnegative matrices in the mathematical sciences*, volume 9 of *Classics in Applied Mathematics.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1994.

[21] A. Berman and U. G. Rothblum. A note on the computation of the CP-rank. *Linear Algebra Appl.*, 419(1):1–7, 2006.

[22] A. Berman and N. Shaked-Monderer. *Completely positive matrices.* World Scientific Publishing Co. Inc., River Edge, NJ, 2003.

[23] M. W. Berry, M. Browne, A. N. Langville, V. P. Pauca, and R. J. Plemmons. Algorithms and applications for approximate nonnegative matrix factorization. *Comput. Statist. Data Anal.*, 52(1):155–173, 2007.

[24] D. Bertsimas and I. Popescu. Optimal inequalities in probability theory: a convex optimization approach. *SIAM J. Optim.*, 15(3):780–804, 2005.

[25] I. M. Bomze. Remarks on the recursive structure of copositivity. *J. Inform. Optim. Sci.*, 8(3):243–260, 1987.

[26] I. M. Bomze. Copositivity and optimization. In *XII Symposium on Operations Research (Passau, 1987)*, volume 58 of *Methods Oper. Res.*, pages 27–35. Athenäum/Hain/Hanstein, Königstein, 1989.

[27] I. M. Bomze. Copositivity conditions for global optimality in indefinite quadratic programming problems. *Czech. J. Oper. Res.*, 1:7–19, 1992.

[28] I. M. Bomze. Detecting all evolutionarily stable strategies. *J. Optim. Theory Appl.*, 75(2):313–329, 1992.

[29] I. M. Bomze. Checking positive-definiteness by three statements. *Int. J. Math. Educ. Sci. Technol.*, 26(26):289–294, 1995.

[30] I. M. Bomze. Block pivoting and shortcut strategies for detecting copositivity. *Linear Algebra Appl.*, 248:161–184, 1996.

[31] I. M. Bomze. Evolution towards the maximum clique. *J. Global Optim.*, 10(2):143–164, 1997.

[32] I. M. Bomze. Global escape strategies for maximizing quadratic forms over a simplex. *J. Global Optim.*, 11(3):325–338, 1997.

[33] I. M. Bomze. Löwner ordering of partial Schur complements and application to copositivity. Report 98-06, Institut für Statistik, Operations Research und Computerverfahren, Universität Wien, 1998.

[34] I. M. Bomze. On standard quadratic optimization problems. *J. Global Optim.*, 13(4):369–387, 1998.

[35] I. M. Bomze. Copositivity aspects of standard quadratic optimization problems. In *Optimization, dynamics, and economic analysis*, pages 1–11. Physica, Heidelberg, 2000.

[36] I. M. Bomze. Linear-time copositivity detection for tridiagonal matrices and extension to block-tridiagonality. *SIAM J. Matrix Anal. Appl.*, 21(3):840–848, 2000.

[37] I. M. Bomze. Branch-and-bound approaches to standard quadratic optimization problems. *J. Global Optim.*, 22(1-4):17–37, 2002.

[38] I. M. Bomze. Regularity versus degeneracy in dynamics, games, and optimization: a unified approach to different aspects. *SIAM Rev.*, 44(3):394–414, 2002.

[39] I. M. Bomze. Perron-Frobenius property of copositive matrices, and a block copositivity criterion. *Linear Algebra Appl.*, 429(1):68–71, 2008.

[40] I. M. Bomze. Building a completely positive factorization. Technical Report TR-ISDS 2009-06, University of Vienna, 2009.

[41] I. M. Bomze. Copositive optimization. In C. Floudas and P. Pardalos, editors, *Encyclopedia of Optimization*, pages 561–564. Springer, New York, 2009.

[42] I. M. Bomze. Global optimization: a quadratic programming perspective. In *Nonlinear optimization*, volume 1989 of *Lecture Notes in Math.*, pages 1–53. Springer, Berlin, 2010.

[43] I. M. Bomze. Copositive optimization – recent developments and applications. *Europ. J. Oper. Research*, to appear 2011.

[44] I. M. Bomze and G. Danninger. A global optimization algorithm for concave quadratic programming problems. *SIAM J. Optim.*, 3(4):826–842, 1993.

[45] I. M. Bomze and G. Danninger. A finite algorithm for solving general quadratic problems. *J. Global Optim.*, 4(1):1–16, 1994.

[46] I. M. Bomze and E. de Klerk. Solving standard quadratic optimization problems via linear, semidefinite and copositive programming. *J. Global Optim.*, 24(2):163–185, 2002.

[47] I. M. Bomze, M. Dür, E. de Klerk, C. Roos, A. J. Quist, and T. Terlaky. On copositive programming and standard quadratic optimization problems. *J. Global Optim.*, 18(4):301–320, 2000.

[48] I. M. Bomze and G. Eichfelder. Copositivity detection by difference-of-convex decomposition and omega-subdivision. Technical Report TR-ISDS 2010-01, University of Vienna, 2010.

[49] I. M. Bomze, F. Frommlet, and M. Locatelli. Copositivity cuts for improving SDP bounds on the clique number. *Math. Program.*, 124(1-2, Ser. B):13–32, 2010.

[50] I. M. Bomze, F. Frommlet, and M. Rubey. Improved SDP bounds for minimizing quadratic functions over the l^1-ball. *Optim. Lett.*, 1(1):49–59, 2007.

[51] I. M. Bomze and F. Jarre. A note on Burer's copositive representation of mixed-binary QPs. *Optim. Lett.*, 4(3):465–472, 2010.

[52] I. M. Bomze, F. Jarre, and F. Rendl. Quadratic factorization heuristics for copositive programming. *Math. Program. Comput.*, 3(1):37–57, 2011.

[53] I. M. Bomze, M. Locatelli, and F. Tardella. New and old bounds for standard quadratic optimization: dominance, equivalence and incomparability. *Math. Program.*, 115, Ser. A(1):31–64, 2008.

[54] I. M. Bomze and L. Palagi. Quartic formulation of standard quadratic optimization problems. *J. Global Optim.*, 32(2):181–205, 2005.

[55] I. M. Bomze, M. Pelillo, and R. Giacomini. Evolutionary approach to the maximum clique problem: empirical evidence on a larger scale. In *Developments in global optimization (Szeged, 1995)*, volume 18 of *Nonconvex Optim. Appl.*, pages 95–108. Kluwer Acad. Publ., Dordrecht, 1997.

[56] I. M. Bomze and B. M. Pötscher. *Game theoretical foundations of evolutionary stability*, volume 324 of *Lecture Notes in Economics and Mathematical Systems*. Springer-Verlag, Berlin, 1989.

[57] I. M. Bomze and W. Schachinger. Multi-standard quadratic optimization: interior point methods and cone programming reformulation. *Comput. Optim. Appl.*, 45(2):237–256, 2010.

[58] J. M. Borwein. Necessary and sufficient conditions for quadratic minimality. *Numer. Funct. Anal. Optim.*, 5:127–140, 1982.

[59] S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, New York, NY, USA, 2004.

[60] B. Brogliato, A. A. ten Dam, L. Paoli, F. Génot, and M. Abadie. Numerical simulation of finite dimensional multibody nonsmooth mechanical systems. *ASME Applied Mechanics Reviews*, 55(2):107–150, 2002.

[61] M. Broom, C. Cannings, and G. T. Vickers. On the number of local maxima of a constrained quadratic form. *Proc. Roy. Soc. London Ser. A*, 443(1919):573–584, 1993.

[62] M. Buliga. Four applications of majorization to convexity in the calculus of variations. *Linear Algebra Appl.*, 429(7):1528–1545, 2008.

[63] S. Bundfuss. *Copositive Matrices, Copositive Programming, and Applications*. Dissertation, Technische Universität Darmstadt, Darmstadt, Germany, 2009.

[64] S. Bundfuss and M. Dür. Algorithmic copositivity detection by simplicial partition. *Linear Algebra Appl.*, 428(7):1511–1523, 2008.

[65] S. Bundfuss and M. Dür. An adaptive linear approximation algorithm for copositive programs. *SIAM J. Optim.*, 20(1):30–53, 2009.

[66] S. Bundfuss and M. Dür. Copositive Lyapunov functions for switched systems over cones. *Systems Control Lett.*, 58(5):342–345, 2009.

[67] S. Burer. On the copositive representation of binary and continuous nonconvex quadratic programs. *Math. Program.*, 120, Ser. A(2):479–495, 2009.

[68] S. Burer. Copositive programming. In M. F. Anjos and J. B. Lasserre, editors, *Handbook of Semidefinite, Cone and Polynomial Optimization: Theory, Algorithms, Software and Applications*, International Series in Operations Research and Management Science. Springer, New York, (to appear 2011).

[69] S. Burer, K. M. Anstreicher, and M. Dür. The difference between 5×5 doubly nonnegative and completely positive matrices. *Linear Algebra Appl.*, 431(9):1539–1552, 2009.

[70] S. Burer and H. Dong. Separation and relaxation for cones of quadratic forms. Working paper, Dept. of Management Sciences, University of Iowa, Iowa City IA, May 2010.

[71] S. Busygin. Copositive programming. In C. Floudas and P. Pardalos, editors, *Encyclopedia of Optimization*, pages 564–567. Springer, New York, 2009.

[72] M. K. Çamlıbel and J. M. Schumacher. Copositive Lyapunov functions. In V. Blondel and A. Megretski, editors, *Unsolved problems in mathematical systems and control theory*, pages 189–193. Princeton University Press, Princeton, New Jersey, 2004.

[73] L. B. Contesse. Une caractérisation complète des minima locaux. *Numer. Math.*, 34:315–332, 1980.

[74] R. W. Cottle. A field guide to the matrix classes found in the literature of the linear complementarity problem. *Journal of Global Optimization*, 46:571–580, 2010.

[75] R. W. Cottle and J. A. Ferland. Matrix-theoretic criteria for the quasi-convexity and pseudo-convexity of quadratic functions. *Linear Algebra and Appl.*, 5:123–136, 1972.

[76] R. W. Cottle, G. J. Habetler, and C. E. Lemke. On classes of copositive matrices. *Linear Algebra and Appl.*, 3:295–310, 1970.

[77] R. W. Cottle, G. J. Habetler, and C. E. Lemke. Quadratic forms semi-definite over convex cones. In *Proceedings of the Princeton Symposium on Mathematical Programming (Princeton Univ., 1967)*, pages 551–565, Princeton, N.J., 1970. Princeton Univ. Press.

[78] R. W. Cottle, J.-S. Pang, and R. E. Stone. *The linear complementarity problem.* Computer Science and Scientific Computing. Academic Press Inc., Boston, MA, 1992.

[79] J.-P. Crouzeix, A. Hassouni, A. Lahlou, and S. Schaible. Positive subdefinite matrices, generalized monotonicity, and linear complementarity problems. *SIAM J. Matrix Anal. Appl.*, 22(1):66–85, 2000.

[80] J.-P. Crouzeix, J. E. Martínez-Legaz, and A. Seeger. An alternative theorem for quadratic forms and extensions. *Linear Algebra Appl.*, 215:121–134, 1995.

[81] B. Dacorogna. Necessary and sufficient conditions for strong ellipticity of isotropic functions in any dimension. *Discrete Contin. Dyn. Syst. Ser. B*, 1(2):257–263, 2001.

[82] G. Danninger. *Kopositivität symmetrischer Matrizen auf polyhedralen Kegeln.* Dissertation, Universität Wien, Vienna, Austria, 1988.

[83] G. Danninger. A recursive algorithm for determining (strict) copositivity of a symmetric matrix. In *XIV Symposium on Operations Research (Ulm, 1989)*, volume 62 of *Methods Oper. Res.*, pages 45–52. Hain, Frankfurt am Main, 1990.

[84] G. Danninger. Role of copositivity in optimality criteria for nonconvex optimization problems. *J. Optim. Theory Appl.*, 75(3):535–558, 1992.

[85] G. Danninger and I. M. Bomze. Copositivity and nonconvex optimization. In P. Gritzmann, R. Hettich, R. Horst, and E. Sachs, editors, *Operations Research '91*, pages 75–79. Physica-Verlag, 1992.

[86] G. Danninger and I. M. Bomze. Using copositivity for global optimality criteria in concave quadratic programming problems. *Math. Program.*, 62, Ser. A(3):575–580, 1993.

[87] G. Danninger and I. M. Bomze. Generalizing convexity for second order optimality conditions. In S. Komlosi, T. Rapcsak, and S. Schaible, editors, *Proceedings of the Fourth International Workshop on Generalized Convexity,*

Pcs, Hungary, 1992, volume 405 of *Lecture Notes in Economics and Math. Systems*, pages 137–144. Springer, 1994.

[88] D. E. Daykin and R. D. Leitch. Problems and Solutions: Solutions of Advanced Problems: 5867. *Amer. Math. Monthly*, 80(10):1148–1150, 1973.

[89] E. de Klerk. The complexity of optimizing over a simplex, hypercube or sphere: a short survey. *CEJOR Cent. Eur. J. Oper. Res.*, 16(2):111–125, 2008.

[90] E. de Klerk, C. Dobre, and D. V. Pasechnik. Numerical block diagonalization of matrix *-algebras with application to semidefinite programming. *Math. Program.*, to appear 2011.

[91] E. de Klerk, M. Laurent, and P. Parrilo. On the equivalence of algebraic approaches to the minimization of forms on the simplex. In *Positive polynomials in control*, volume 312 of *Lecture Notes in Control and Inform. Sci.*, pages 121–132. Springer, Berlin, 2005.

[92] E. de Klerk, J. Maharry, D. V. Pasechnik, R. B. Richter, and G. Salazar. Improved bounds for the crossing numbers of $K_{m,n}$ and K_n. *SIAM J. Disc. Math.*, 20(1):189–202, 2006.

[93] E. de Klerk and D. V. Pasechnik. Approximation of the stability number of a graph via copositive programming. *SIAM J. Optim.*, 12(4):875–892, 2002.

[94] E. de Klerk and D. V. Pasechnik. A linear programming reformulation of the standard quadratic optimization problem. *J. Global Optim.*, 37(1):75–84, 2007.

[95] E. de Klerk, D. V. Pasechnik, and A. Schrijver. Reduction of symmetric semidefinite programs using the regular *-representation. *Math. Program.*, 109, Ser. B(2-3):613–624, 2007.

[96] P. H. Diananda. On non-negative forms in real variables some or all of which are non-negative. *Proc. Cambridge Philos. Soc.*, 58:17–25, 1962.

[97] P. J. C. Dickinson. An improved characterisation of the interior of the completely positive cone. *Electron. J. Linear Algebra*, 20:723–729, 2010.

[98] P. J. C. Dickinson. Geometry of the copositive and completely positive cones. *J. Math. Anal. Appl*, 380(1):377–395, 2011.

[99] P. J. C. Dickinson and L. Gijben. On the computational complexity of membership problems for the completely positive cone and its dual. Technical Report, Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, The Netherlands, 2011.

[100] B. Ding. *A Parametric Solution for Local and Global Optimization.* PhD thesis, University of Waterloo, Ontario, Canada, 1996.

[101] C. Dobre. *Semidefinite programming approaches for structured combinatorial optimization problems.* PhD thesis, Tilburg University, The Netherlands, 2011.

[102] B. Dong, M. M. Lin, and M. T. Chu. Nonnegative rank factorization via rank reduction, Preprint 2008.

[103] H. Dong. Symmetric tensor approximation hierarchies for the completely positive cone. Working paper, Applied Mathematics and Computational Sciences Program, University of Iowa, Iowa City, IA, 2010.

[104] H. Dong and K. Anstreicher. A note on "5×5 completely positive matrices". *Linear Algebra Appl.*, 433(5):1001–1004, 2010.

[105] H. Dong and K. Anstreicher. Separating doubly nonnegative and completely

positive matrices. Working paper, Dept. of Management Sciences, University of Iowa, Iowa City, IA, 2010.

[106] J. H. Drew, C. R. Johnson, and R. Loewy. Completely positive matrices associated with M-matrices. *Linear and Multilinear Algebra*, 37(4):303–310, 1994.

[107] I. Dukanovic and F. Rendl. Copositive programming motivated bounds on the stability and the chromatic numbers. *Math. Program.*, 121, Ser. A(2):249–268, 2010.

[108] M. Dür. Copositive programming – a survey. In M. Diehl, F. Glineur, E. Jarlebring, and W. Michiels, editors, *Recent Advances in Optimization and its Applications in Engineering*, pages 3–20. Springer Berlin Heidelberg, 2010.

[109] M. Dür and G. Still. Interior points of the completely positive cone. *Electron. J. Linear Algebra*, 17:48–53, 2008.

[110] G. Eichfelder and J. Jahn. Set-semidefinite optimization. *J. Convex Anal.*, 15(4):767–801, 2008.

[111] G. Eichfelder and J. Povh. On reformulations of nonconvex quadratic programs over convex cones by set-semidefinite constraints. Preprint 342, Institut fuer Angewandte Mathematik, Erlangen, 2010.

[112] A. Engau, M. F. Anjos, and I. M. Bomze. Cutting planes and copositive programming for stable set. Technical Report TR-ISOR 2011-04, University of Vienna, 2011.

[113] F. Facchinei and J.-S. Pang. *Finite-dimensional variational inequalities and complementarity problems. Vol. I.* Springer Series in Operations Research. Springer-Verlag, New York, 2003.

[114] F. Facchinei and J.-S. Pang. *Finite-dimensional variational inequalities and complementarity problems. Vol. II.* Springer Series in Operations Research. Springer-Verlag, New York, 2003.

[115] R. W. Farebrother. Necessary and sufficient conditions for a quadratic form to be positive whenever a set of homogeneous linear constraints is satisfied. *Linear Algebra and Appl.*, 16(1):39–42, 1977.

[116] M. Fiedler. Positivity with respect to the round cone. *Mat. Časopis Sloven. Akad. Vied*, 24:155–159, 1974.

[117] A. L. Forsgren, P. E. Gill, and W. Murray. On the identification of local minimizers in inertia-controlling methods for quadratic programming. *SIAM J. Matrix Anal. Appl.*, 12(4):730–746, 1991.

[118] M. Fukuda, M. Yamashita, and M. Kojima. Computational prospects on copositive programming (theory of modeling and optimization). Technical Report 1526, Kyoto University Research Information Repository (Japan), December 2006.

[119] J. W. Gaddum. Linear inequalities and quadratic forms. *Pacific J. Math.*, 8:411–414, 1958.

[120] D. Ge and Y. Ye. On Doubly Positive Semidefinite Programming Relaxations, Working Paper, August 2010.

[121] D. Goeleven and B. Brogliato. Stability and instability matrices for linear evolution variational inequalities. *IEEE Trans. Automat. Control*, 49(4):521–534, 2004.

[122] D. Gourion and A. Seeger. Critical angles in polyhedral convex cones: numer-

ical and statistical considerations. *Math. Program.*, 123, Ser. B(1):173–198, 2010.

[123] M. S. Gowda. Pseudomonotone and copositive star matrices. *Linear Algebra Appl.*, 113:107–118, 1989.

[124] M. S. Gowda. On **Q**-matrices. *Math. Program.*, 49, Ser. A(1):139–141, 1990/91.

[125] E. Guslitser. *Uncertatinty-immunized solutions in linear programming.* Master thesis, Technion, Israeli Institute of Technology, IE&M faculty, 2002.

[126] N. Gvozdenović. *Approximating the Stability Number and the Chromatic Number of a Graph via Semidefinite Programming.* PhD thesis, University of Amsterdam, 2008.

[127] N. Gvozdenović and M. Laurent. The operator Ψ for the chromatic number of a graph. *SIAM J. Optim.*, 19(2):572–591, 2008.

[128] K. P. Hadeler. On copositive matrices. *Linear Algebra Appl.*, 49:79–89, 1983.

[129] P. Hadjicostas. Copositive matrices and Simpson's paradox. *Linear Algebra Appl.*, 264:475–488, 1997.

[130] W. W. Hager and Y. Krylyuk. Graph partitioning and continuous quadratic programming. *SIAM J. Discrete Math.*, 12(4):500–523, 1999.

[131] W. W. Hager and Y. Krylyuk. Multiset graph partitioning. *Math. Methods Oper. Res.*, 55(1):1–10, 2002.

[132] R. H. R. Hahnloser, H. S. Seung, and J.-J. E. Slotine. Permitted and forbidden sets in symmetric threshold-linear networks. *Neural Comput.*, 15:621–638, 2003.

[133] M. Hall, Jr. and M. Newman. Copositive and completely positive quadratic forms. *Proc. Cambridge Philos. Soc.*, 59:329–339, 1963.

[134] S. P. Han and O. L. Mangasarian. A conjugate decomposition of the Euclidean space. *Proc. Nat. Acad. Sci. U.S.A.*, 80(16 i.):5156–5157, 1983.

[135] S. P. Han and O. L. Mangasarian. Conjugate cone characterization of positive definite and semidefinite matrices. *Linear Algebra Appl.*, 56:89–103, 1984.

[136] A. Hassouni, A. Lahlou, and A. Lamghari. Existence theorems for linear complementarity problems on solid closed convex cones. *J. Optim. Theory Appl.*, 126(2):225–246, 2005.

[137] S. Hayashi, T. Yamaguchi, N. Yamashita, and M. Fukushima. A matrix-splitting method for symmetric affine second-order cone complementarity problems. *J. Comput. Appl. Math.*, 175(2):335–353, 2005.

[138] E. Haynsworth and A. J. Hoffman. Two remarks on copositive matrices. *Linear Algebra and Appl.*, 2:387–392, 1969.

[139] C. Helmberg. Semidefinite programming for combinatorial optimization. Report 00–34, ZIB-Report, 2000.

[140] R. Hildebrand. The extremal rays of the 5×5 copositive cone. Working paper, Optimization Online, 2011.

[141] J.-B. Hiriart-Urruty and A. Seeger. A variational approach to copositive matrices. *SIAM Review*, 52(4):593–629, 2010.

[142] A. J. Hoffman and F. Pereira. On copositive matrices with -1,0,1 entries. *J. Combinatorial Theory Ser. A*, 14:302–309, 1973.

[143] L. Hogben. The copositive completion problem: unspecified diagonal entries. *Linear Algebra Appl.*, 420(1):160–162, 2007.

[144] L. Hogben, C. R. Johnson, and R. Reams. The copositive completion problem. *Linear Algebra Appl.*, 408:207–211, 2005.

[145] R. A. Horn and C. R. Johnson. *Topics in Matrix Analysis*. Cambridge University Press, 1991.

[146] R. Horst. On generalized bisection of n-simplices. *Math. Comput.*, 66(218):691–698, 1997.

[147] R. Horst and H. Tuy. *Global Optimization: Deterministic Approaches*. Springer-Verlag, 1990.

[148] J. Hu, J. E. Mitchell, and J.-S. Pang. An LPCC approach to nonconvex quadratic programs. *Math. Program.*, to appear 2011.

[149] Y. Huang and S. Zhang. Complex matrix decomposition and quadratic programming. Technical Report SEEM2005-02, Department of Systems Engineering & Engineering Management, The Chinese University of Hong Kong, 2005.

[150] C. Humes, Jr., J. Ou, and P. R. Kumar. The delay of open Markovian queueing networks: uniform functional bounds, heavy traffic pole multiplicities, and stability. *Math. Oper. Res.*, 22(4):921–954, 1997.

[151] K. D. Ikramov. An algorithm, linear with respect to time, for verifying the copositivity of an acyclic matrix. *Zh. Vychisl. Mat. Mat. Fiz.*, 42(12):1771–1773, 2002. English translation in Comput. Math. Math. Phys. 42 (2002), no. 12, 1701–1703.

[152] K. D. Ikramov and N. V. Savel'eva. Conditionally definite matrices. *J. Math. Sci. (New York)*, 98(1):1–50, 2000.

[153] A. Iusem and A. Seeger. Axiomatization of the index of pointedness for closed convex cones. *Comput. Appl. Math.*, 24(2):245–283, 2005.

[154] A. Iusem and A. Seeger. On pairs of vectors achieving the maximal angle of a convex cone. *Math. Program.*, 104, Ser. B(2-3):501–523, 2005.

[155] A. Iusem and A. Seeger. Measuring the degree of pointedness of a closed convex cone: a metric approach. *Math. Nachr.*, 279(5-6):599–618, 2006.

[156] A. Iusem and A. Seeger. Angular analysis of two classes of non-polyhedral convex cones: the point of view of optimization theory. *Comput. Appl. Math.*, 26(2):191–214, 2007.

[157] A. Iusem and A. Seeger. On convex cones with infinitely many critical angles. *Optimization*, 56(1-2):115–128, 2007.

[158] A. Iusem and A. Seeger. Normality and modulability indices. I. Convex cones in normed spaces. *J. Math. Anal. Appl.*, 338(1):365–391, 2008.

[159] A. Iusem and A. Seeger. Searching for critical angles in a convex cone. *Math. Program.*, 120, Ser. B(1):3–25, 2009.

[160] D. H. Jacobson. *Extensions of linear-quadratic control, optimization and matrix theory*. Academic Press [Harcourt Brace Jovanovich Publishers], London, 1977. Mathematics in Science and Engineering, Vol. 133.

[161] F. Jarre. Burer's key assumption for semidefinite and doubly nonnegative relaxations. *Optim. Lett.*, to appear 2011.

[162] F. Jarre and K. Schmallowsky. On the computation of C^* certificates. *J. Global Optim.*, 45(2):281–296, 2009.

[163] C. R. Johnson and R. Reams. Spectral theory of copositive matrices. *Linear Algebra Appl.*, 395:275–281, 2005.

[164] C. R. Johnson and R. Reams. Constructing copositive matrices from interior matrices. *Electron. J. Linear Algebra*, 17:9–20, 2008.

[165] C. R. Johnson and R. Reams. Scaling of symmetric matrices by positive diagonal congruence. *Linear Multilinear Algebra*, 57(2):123–140, 2009.

[166] P. C. Jones. A note on the Talman, Van der Heyden linear complementarity algorithm. *Math. Program.*, 25(1):122–124, 1983.

[167] J. J. Júdice, M. Raydan, S. S. Rosa, and S. A. Santos. On the solution of the symmetric eigenvalue complementarity problem by the spectral projected gradient algorithm. *Numer. Algorithms*, 47(4):391–407, 2008.

[168] J. J. Júdice, H. D. Sherali, and I. M. Ribeiro. The eigenvalue complementarity problem. *Comput. Optim. Appl.*, 37(2):139–156, 2007.

[169] W. Kaplan. A test for copositive matrices. *Linear Algebra Appl.*, 313(1-3):203–206, 2000.

[170] W. Kaplan. A copositivity probe. *Linear Algebra Appl.*, 337:237–251, 2001.

[171] M. Kaykobad. On nonnegative factorization of matrices. *Linear Algebra Appl.*, 96:27–33, 1987.

[172] G. Kéri. The Sherman-Morrison formula for the determinant and its application for optimizing quadratic functions on condition sets given by extreme generators. In *Optimization theory (Mátraháza, 1999)*, volume 59 of *Appl. Optim.*, pages 119–138. Kluwer Acad. Publ., Dordrecht, 2001.

[173] G. Kimura and A. Kossakowski. A class of linear positive maps in matrix algebras. II. *Open Syst. Inf. Dyn.*, 11(4):343–358, 2004.

[174] N. Kogan and A. Berman. Characterization of completely positive graphs. *Discrete Math.*, 114(1-3):297–304, 1993.

[175] M. Kojima, N. Megiddo, and S. Mizuno. A general framework of continuation methods for complementarity problems. *Math. Oper. Res.*, 18(4):945–963, 1993.

[176] A. Kossakowski. A class of linear positive maps in matrix algebras. *Open Syst. Inf. Dyn.*, 10(3):213–220, 2003.

[177] P. R. Kumar. Scheduling queueing networks: stability, performance analysis and design. In *Stochastic networks*, volume 71 of *IMA Vol. Math. Appl.*, pages 21–70. Springer, New York, 1995.

[178] P. R. Kumar. A tutorial on some new methods for performance evaluation of queueing networks. *IEEE Journal on Selected Areas in Communications*, 13(6):970–980, August 1995.

[179] P. R. Kumar and S. P. Meyn. Stability of queueing networks and scheduling policies. *IEEE Trans. Automat. Control*, 40(2):251–260, 1995.

[180] P. R. Kumar and S. P. Meyn. Duality and linear programs for stability and performance analysis of queuing networks and scheduling policies. *IEEE Trans. Automat. Control*, 41(1):4–17, 1996.

[181] Y. Kwon. On hadamard stability for compressible viscoelastic constitutive equations. *J. Non-Newtonian Fluid Mech.*, 65:151–163, 1996.

[182] C. Lacoursière. Splitting methods for dry frictional contact problems in rigid multibody systems: Preliminary performance results. In *Conference Proceedings from SIGRAD2003*, pages 20–21, Umeå, Sweden, November 2003.

[183] T. J. Laffey and E. Meehan. A refinement of an inequality of Johnson, Loewy and London on nonnegative matrices and some applications. *Electron. J.*

Linear Algebra, 3:119–128, 1998.

[184] J. B. Lasserre. Global optimization with polynomials and the problem of moments. *SIAM J. Optim.*, 11(3):796–817, 2000/01.

[185] J. B. Lasserre. New approximations for the cone of copositive matrices and its dual. *ArXiv e-prints*, (1012.2552), 2010.

[186] M. Laurent and F. Rendl. Semidefinite programming and integer programming. In K. Aardal, G. Nemhauser, and R. Weismantel, editors, *Handbook on Discrete Optimization*, pages 393–514. Elsevier, Amsterdam, 2005.

[187] E. Leonardi, M. Mellia, M. Ajmone Marsan, and F. Neri. On the throughput achievable by isolated and interconnected input-queueing switches under multiclass traffic. In *INFOCOM 2002. Twenty-First Annual Joint Conference of the IEEE Computer and Communications Societies. Proceedings. IEEE*, volume 3, pages 1605 – 1614, 2002.

[188] P. Li and Y. Y. Feng. Criteria for copositive matrices of order four. *Linear Algebra Appl.*, 194:109–124, 1993.

[189] Y. Li, A. Kummert, and A. Frommer. A linear programming based analysis of the CP-rank of completely positive matrices. *Int. J. Appl. Math. Comput. Sci.*, 14(1):25–31, 2004.

[190] J. Q. Liu, T. T. Song, and D. Z. Du. On the necessary and sufficient condition of the local optimal solution of quadratic programming. *Chinese Ann. Math.*, 3:625–630, 1982. (English abstract).

[191] M. Locatelli and F. Schoen. On convex envelopes and underestimators for bivariate functions. Working paper, Optimization Online, 2009.

[192] R. Loewy and H. Schneider. Positive operators on the n-dimensional ice cream cone. *J. Math. Anal. Appl.*, 49:375–392, 1975.

[193] R. Loewy and B.-S. Tam. CP rank of completely positive matrices of order 5. *Linear Algebra Appl.*, 363:161–176, 2003.

[194] L. Lovász and A. Schrijver. Cones of matrices and set-functions and 0-1 optimization. *SIAM Journal on Optimization*, 1:166–190, 1991.

[195] A. Majthay. Optimality conditions for quadratic programming. *Math. Program.*, 1:359–365, 1971.

[196] D. H. Martin. Conditional positivity of quadratic forms in Hilbert space. *SIAM J. Math. Anal.*, 11(6):1047–1057, 1980.

[197] D. H. Martin. Finite criteria for conditional definiteness of quadratic forms. *Linear Algebra Appl.*, 39:9–21, 1981.

[198] D. H. Martin and D. H. Jacobson. Copositive matrices and definiteness of quadratic forms subject to homogeneous linear inequality constraints. *Linear Algebra Appl.*, 35:227–258, 1981.

[199] D. H. Martin, M. J. D. Powell, and D. H. Jacobson. On a decomposition of conditionally positive-semidefinite matrices. *Linear Algebra Appl.*, 39:51–59, 1981.

[200] J. E. Maxfield and H. Minc. On the matrix equation $X'X = A$. *Proc. Edinburgh Math. Soc. (2)*, 13:125–129, 1962/1963.

[201] M. Mesbahi, M. G. Safonov, and G. P. Papavassilopoulos. Bilinearity and complementarity in robust control. In *Advances in linear matrix inequality methods in control*, volume 2 of *Adv. Des. Control*, pages 269–292. SIAM, Philadelphia, PA, 2000.

[202] D. A. Miller and S. W. Zucker. Copositive-plus Lemke algorithm solves polymatrix games. *Oper. Res. Lett.*, 10(5):285–290, 1991.

[203] S. R. Mohan, S. K. Neogy, and A. K. Das. On the classes of fully copositive and fully semimonotone matrices. *Linear Algebra Appl.*, 323(1-3):87–97, 2001.

[204] S. R. Mohan and A. J. J. Talman. Refinement of solutions to the linear complementarity problem. Working paper 1998-78, Tilburg University. Center for Economic Research, 1998.

[205] J.-J. Moreau. Décomposition orthogonale d'un espace hilbertien selon deux cônes mutuellement polaires. *C. R. Acad. Sci. Paris*, 255:238–240, 1962.

[206] J. R. Morrison and P. R. Kumar. New linear program performance bounds for queueing networks. *J. Optim. Theory Appl.*, 100(3):575–597, 1999.

[207] J. R. Morrison and P. R. Kumar. New linear program performance bounds for closed queueing networks. *Discrete Event Dyn. Syst.*, 11(4):291–317, 2001.

[208] T. S. Motzkin. Copositive quadratic forms. Report 1818, National Bureau of Standards, 1952.

[209] T. S. Motzkin. Quadratic forms positive for nonnegative variables not all zero. *Notices Amer. Math. Soc.*, 12:224, 1965.

[210] T. S. Motzkin. Signs of minors. In *Inequalities (Proc. Sympos. Wright-Patterson Air Force Base, Ohio, 1965)*, pages 225–240. Academic Press, New York, 1967.

[211] G. S. R. Murthy and T. Parthasarathy. Fully copositive matrices. *Math. Program.*, 82, Ser. A(3):401–411, 1998.

[212] G. S. R. Murthy, T. Parthasarathy, and G. Ravindran. A copositive Q-matrix which is not $\mathbf{R}_0$. *Math. Program.*, 61, Ser. A(1):131–135, 1993.

[213] K. G. Murty. *Linear complementarity, linear and nonlinear programming*, volume 3 of *Sigma Series in Applied Mathematics*. Heldermann Verlag, Berlin, 1988.

[214] K. G. Murty and S. N. Kabadi. Some NP-complete problems in quadratic and nonlinear programming. *Math. Program.*, 39(2):117–129, 1987.

[215] E. Nadler. Nonnegativity of bivariate quadratic functions on a triangle. *Comput. Aided Geom. Design*, 9(3):195–205, 1992.

[216] K. Natarajan, C. P. Teo, and Z. Zheng. Mixed zero-one linear programs under objective uncertainty: A completely positive representation. *Operations Research*, to appear 2011.

[217] J.-S. Pang. Computing generalized nash equilibria. Working paper, Department of Mathematical Sciences, The John Hopkins University, 2002.

[218] P. A. Parrilo. Semidefinite programming based tests for matrix copositivity. In *Proceedings of the 39th IEEE Conference on Decision and Control*, volume 96, pages 4624–4629, Sidney, December 2000.

[219] P. A. Parrilo. *Structured Semidefinite Programs and Semi-algebraic Geometry Methods in Robustness and Optimization.* PhD thesis, California Institute of Technology, Pasadena, CA, May 2000.

[220] P. A. Parrilo. Semidefinite programming relaxations for semialgebraic problems. *Math. Program.*, 96, Ser. B(2):293–320, 2003.

[221] J. Peña, J. Vera, and L. F. Zuluaga. Computing the stability number of a

graph via linear and semidefinite programming. *SIAM J. Optim.*, 18(1):87–105, 2007.

[222] A. Pinto da Costa and A. Seeger. Numerical resolution of cone-constrained eigenvalue problems. *Comput. Appl. Math.*, 28(1):37–61, 2009.

[223] A. Pinto da Costa and A. Seeger. Cone-constrained eigenvalue problems: theory and algorithms. *Comput. Optim. Appl.*, 45(1):25–57, 2010.

[224] I. Pólik and T. Terlaky. A survey of the S-lemma. *SIAM Rev.*, 49(3):371–418, 2007.

[225] G. Pólya. Über positive Darstellung von Polynomen. *Vierteljahresschrift der Naturforschenden Gesellschaft in Zürich*, 73:141–145, 1928.

[226] J. Povh and F. Rendl. A copositive programming approach to graph partitioning. *SIAM J. Optim.*, 18(1):223–241, 2007.

[227] J. Povh and F. Rendl. Copositive and semidefinite relaxations of the quadratic assignment problem. *Discrete Optim.*, 6(3):231–241, 2009.

[228] J. C. Preisig. Copositivity and the minimization of quadratic functions with nonnegativity and quadratic equality constraints. *SIAM J. Control Optim.*, 34(4):1135–1150, 1996.

[229] R. X. Qian and C. L. DeMarco. An approach to robust stability of matrix polytopes through copositive homogeneous polynomials. *IEEE Trans. Automat. Control*, 37(6):848–852, 1992.

[230] M. Queiroz, J. J. Júdice, and C. Humes, Jr. The symmetric eigenvalue complementarity problem. *Math. Comp.*, 73(248):1849–1863, 2004.

[231] A. J. Quist, E. de Klerk, C. Roos, and T. Terlaky. Copositive relaxation for general quadratic programming. *Optim. Methods Softw.*, 9(1-3):185–208, 1998.

[232] T. E. S. Raghavan and Z. Syed. Computing stationary Nash equilibria of undiscounted single-controller stochastic games. *Math. Oper. Res.*, 27(2):384–400, 2002.

[233] L. Salce and P. Zanardo. Completely positive matrices and positivity of least squares solutions. *Linear Algebra Appl.*, 178:201–216, 1993.

[234] W. Schachinger and I. Bomze. A conic duality Frank-Wolfe-type theorem via exact penalization in quadratic optimization. *Math. Oper. Res.*, 34(1):83–91, 2009.

[235] A. Seeger. Eigenvalue analysis of equilibrium processes defined by linear complementarity conditions. *Linear Algebra Appl.*, 292(1-3):1–14, 1999.

[236] A. Seeger and M. Torki. On eigenvalues induced by a cone constraint. *Linear Algebra Appl.*, 372:181–206, 2003.

[237] A. Seeger and M. Torki. Local minima of quadratic forms on convex cones. *J. Global Optim.*, 44(1):1–28, 2009.

[238] N. Shaked-Monderer. Minimal CP rank. *Electron. J. Linear Algebra*, 8:140–157, 2001.

[239] M. Shibata, N. Yamashita, and M. Fukushima. The extended semidefinite linear complementarity problem: a reformulation approach. In *Nonlinear analysis and convex analysis (Niigata, 1998)*, pages 326–332. World Sci. Publ., River Edge, NJ, 1999.

[240] H. C. Simpson and S. J. Spector. On copositive matrices and strong ellipticity for isotropic elastic materials. *Arch. Rational Mech. Anal.*, 84(1):55–68, 1983.

[241] M. V. Solodov. Stationary points of bound constrained minimization reformulations of complementarity problems. *J. Optim. Theory Appl.*, 94(2):449–467, 1997.

[242] M. V. Solodov. Some optimization reformulations of the extended linear complementarity problem. *Comput. Optim. Appl.*, 13(1-3):187–200, 1999.

[243] J. Sponsel, S. Bundfuss, and M. Dür. Testing copositivity using semidefinitness, Working paper, 2009.

[244] J. F. Sturm and S. Zhang. On cones of nonnegative quadratic functions. *Math. Oper. Res.*, 28(2):246–267, 2003.

[245] O. Taussky. Positive definite matrices. In O. Shisha, editor, *Inequalities*, pages 309–319. Academic Press, New York, 1967.

[246] M. Tawarmalani and N. V. Sahinidis. Global optimization of mixed-integer nonlinear programs: a theoretical and computational study. *Math. Program.*, 99, Ser. A(3):563–591, 2004.

[247] H. Väliaho. Criteria for copositive matrices. *Linear Algebra Appl.*, 81:19–34, 1986.

[248] H. Väliaho. Testing the definiteness of matrices on polyhedral cones. *Linear Algebra Appl.*, 101:135–165, 1988.

[249] H. Väliaho. Almost copositive matrices. *Linear Algebra Appl.*, 116:121–134, 1989.

[250] H. Väliaho. Quadratic-programming criteria for copositive matrices. *Linear Algebra Appl.*, 119:163–182, 1989.

[251] H. Väliaho. A new proof for the criss-cross method for quadratic programming. *Optimization*, 25(4):391–400, 1992.

[252] L. Vandenberghe and S. Boyd. Semidefinite programming. *SIAM Review*, 38:49–95, 1996.

[253] C. Weninger. *Kopositivität und quadratische Optimierung.* Dissertation, Universität Wien, Vienna, Austria, 1993.

[254] A. C. Williams. Boundedness relations for linear constraint sets. *Linear Algebra and Appl.*, 3:129–141, 1970.

[255] S. Xiang and S. Xiang. Notes on completely positive matrices. *Linear Algebra Appl.*, 271:273–282, 1998.

[256] J. Xu and Y. Yao. A complete algorithm for determining copositive matrices. *ArXiv e-prints*, (1011.2039), 2010.

[257] V. A. Yakubovich. S-procedure in nonlinear control theory. *Vestnik Leningradskogo Universiteta, Ser. Matematika, Series 1*, 13:62–77, 1971. English translation in Vestnik Leningrad Univ., 1977, vol. 4, pp. 73–93.

[258] S. Yang and X. Li. Algorithms for determining the copositivity of a given symmetric matrix. *Linear Algebra Appl.*, 430(2-3):609–618, 2009.

[259] S. Yang, C. Xu, and X. Li. A note on algorithms for determining the copositivity of a given symmetric matrix. *J. Inequal. Appl.*, pages Art. ID 498631, 10, 2010.

[260] B. Ycart. Extrémales du cône des matrices de type non négatif, à coefficients positifs ou nuls. *Linear Algebra Appl.*, 48:317–330, 1982.

[261] E. A. Yıldırım. On the accuracy of uniform polyhedral approximations of the copositive cone. *Optim. Methods Softw.*, to appear 2011.

[262] D. A. Yopp and R. D. Hill. On completely copositive and decomposable

linear transformations. *Linear Algebra Appl.*, 312(1-3):1–12, 2000.

[263] Y. Yuan. On a subproblem of trust region algorithms for constrained optimization. *Math. Program.*, 47, Ser. A(1):53–63, 1990.

[264] X. D. Zhang and J. S. Li. Factorization index for completely positive graphs. *Acta Math. Sin. (Engl. Ser.)*, 18(4):823–832, 2002.

[265] J. Žilinskas and M. Dür. Depth-first Simplicial Partition for Copositivity Detection, with an Application to MaxClique. *Optim. Methods Softw.*, to appear 2011.

[266] L. F. Zuluaga, J. Vera, and J. Peña. LMI approximations for cones of positive semidefinite forms. *SIAM J. Optim.*, 16(4):1076–1091, 2006.