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Third cumulant Stein approximation for Poisson stochastic integrals

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Abstract

We derive Edgeworth-type expansions for Poisson stochastic integrals, based on cumulant operators defined by the Malliavin calculus. As a consequence we obtain Stein approximation bounds for stochastic integrals, which are based on third cumulants instead of the L^3 norm term found in the literature. The use of the third cumulant results into a convergence rate faster than the classical Berry-Esseen rate on certain examples.

Key words: Stein approximation; Malliavin calculus; Poisson stochastic integral; cumulants; Edgeworth expansions.
Mathematics Subject Classification: 60H07, 62E17, 60H05.

1 Introduction

Edgeworth expansions have been derived on the Wiener space in [9], [2], [4], using a construction of cumulant operators based on the inverse L^{-1} of the Ornstein-Uhlenbeck operator [11]. This approach extends the results of [10], [13] on Stein approximation, Berry-Esseen bounds and the fourth moment theorem. Related Edgeworth type expansions have also been derived for the Itô-Skorohod integral $\delta(u)$ of a process u on the Wiener space in [18].

In this paper we derive Edgeworth type expansions of the form

$$E\big[\delta(u)g(\delta(u))\big] = E\big[\|u\|^2_{L^2(\mathbb{R}_+)}g'(\delta(u))\big] + \sum_{k=2}^{n} E\big[g^{(k)}(\delta(u))\Gamma^u_{k+1}\mathbf{1}\big] + E\big[g^{(n+1)}(\delta(u))R^u_n\big] \tag{1.1}$$

for the compensated Poisson stochastic integral $\delta(u) = \int_0^\infty u_t d(N_t - t)$ of an adapted process $(u_t)_{t\in\mathbb{R}_+}$ with respect to a standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$, where Γ^u_k and R^u_n are respectively a cumulant type operator and a remainder term defined using the derivation operators of the Malliavin calculus on the Poisson space, see Proposition 3.1.

From (1.1), in Proposition 4.1 and Corollary 4.2 we deduce Stein approximation bounds of the form

$$d(\delta(u),\mathcal{N}) \le E\big[\big|1 - \|u\|^2_{L^2(\mathbb{R}_+)}\big|\big] + E\left[\left|\int_0^\infty u_s^3 ds + \left\langle u, D\int_0^\infty u_t^2 dt\right\rangle\right|\right] + 2E\big[|R^u_1|\big],$$

and

$$\begin{aligned} d(\delta(u),\mathcal{N}) \quad &\le \quad |1 - \mathrm{Var}[\delta(u)]| + \sqrt{\mathrm{Var}\big[\|u\|^2_H\big]} + E\left[\left|\int_0^\infty u_s^3 ds + \left\langle u, D\int_0^\infty u_t^2 dt\right\rangle\right|\right] \\ &\quad + 2E\big[|R^u_1|\big], \end{aligned} \tag{1.2}$$

when the process u is adapted with respect to the Poisson filtration, where D is a gradient operator acting on Poisson functionals, $\mathcal{N} \simeq \mathcal{N}(0,1)$ is a standard Gaussian random variable and

$$d(F,G) := \sup_{h\in\mathcal{L}} |\mathrm{E}[h(F)] - \mathrm{E}[h(G)]|$$

is the Wasserstein distance between the laws of two random variables F and G, where $\mathcal{L}$ denotes the class of 1-Lipschitz functions on $\mathbb{R}$.

In Section 5 we present examples of adapted processes $(u_t)_{t\in\mathbb{R}_+}$ for which (1.2) holds, see Proposition 5.1 and Corollary 5.2, in relation with classical examples such as the normalized sequence $((T_k - k)/\sqrt{k})_{k\geq 1}$, where $(T_k)_{k\geq 1}$ is the sequence of jump times of the standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$.

In particular, when f is a differentiable deterministic function we obtain bounds of the form

$$d\left(\int_0^\infty f(t)d(N_t - t), \mathcal{N}\right) \leq \left|1 - \|f\|^2_{L^2(\mathbb{R}_+)}\right| + \left|\int_0^\infty f^3(t)dt\right| + 2\|f\|_{L^2(\mathbb{R}_+)} \int_0^\infty t|f'(t)|^2 dt, \tag{1.3}$$

depending on the regularity of the function f, see Corollary 5.3. This alternative approach, which is based on derivation operators, replaces the $L^3(\mathbb{R}_+)$ norm of f in the classical Stein bound

$$d\left(\int_0^\infty f(t)d(N_t - t), \mathcal{N}\right) \leq \left|1 - \|f\|^2_{L^2(\mathbb{R}_+)}\right| + \int_0^\infty |f^3(t)|dt, \tag{1.4}$$

see Corollary 3.4 of [14], with the third cumulant $\kappa_3^f = \int_0^\infty f^3(t)dt$, by removing the inner absolute value in the integral.

The main reason for the appearance of an L^3 norm in (1.4) instead of the third cumulant $\kappa_3^f = \int_0^\infty f^3(t)dt$ lies with the use of finite difference operators and the replacement of the chain rule of derivation with a Taylor expansion bound, see Theorem 3.1 of [14] and § 4.2 of the recent survey [3]. In the present paper, the use of derivation operators allows us instead to use the third cumulant $\kappa_3^f = \int_0^\infty f^3(t)dt$ in (1.3).

Taking f_k of the form

$$f_k(t) := \frac{1}{\sqrt{k}} g(t/k), \qquad k \geq 1,$$

where $g \in \mathcal{C}^1(\mathbb{R})$ is such that $\|g\|_{L^2(\mathbb{R}_+)} = 1$ and $\int_0^\infty g^3(t)dt = 0$, (1.3) shows that

$$d\left(\int_0^\infty f_k(t)d(N_t - t), \mathcal{N}\right) \leq \frac{2}{k}\int_0^\infty t|g'(t)|^2 dt, \qquad k \geq 1,$$

see (5.4) below, while Corollary 3.4 of [14] only yields the standard Berry-Esseen rate, see (5.6). This similarly improves on the bound

$$d\left(\int_0^\infty f(t)d(N_t - t), \mathcal{N}\right) \leq |1 - \|f\|^2_{L^2(\mathbb{R}_+)}| + \sqrt{\int_0^\infty |f'(t)|^2 \left(\int_0^t f(s)ds\right)^2 dt} \tag{1.5}$$

obtained using derivation operators on the Poisson space in Corollary 4.4 of [20], which also yields the standard Berry-Esseen rate in this case, see (5.5) below.

In Section 2 we recall some background material on the Malliavin calculus and cumulant operators for stochastic integrals on the Poisson space. In Section 3 we derive Edgeworth type expansions, based on a family of cumulant operators that are associated to the process u and specially defined for the Skorohod integral operator δ. In Section 4 we derive Stein type approximation bounds for stochastic integrals, and in Section 5 we consider adapted and deterministic examples.

2 Malliavin operators

Let $(T_k)_{k\geq 1}$ denote the sequence of jump times of a standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ with unit intensity and generating the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ on a probability space $(\Omega, \mathcal{F}_\infty, P)$, with $T_0 := 0$. The gradient operator D defined on random functionals

$$F \in \mathcal{S} := \big\{ F = f(T_1, \ldots, T_n) \ : \ f \in \mathcal{C}_b^1(\mathbb{R}^n) \big\},$$

as

$$D_t F = - \sum_{k=1}^n \mathbf{1}_{[0,T_k]}(t) \frac{\partial f}{\partial x_k}(T_1, \ldots, T_n),$$

has the derivation property, cf. [7], [5], [15]. The operator D defines the Sobolev spaces $\mathbb{D}_{p,1}$ with the Sobolev norms

$$\|F\|_{\mathbb{D}_{p,1}} := \|F\|_{L^p(\Omega)} + \|DF\|_{L^p(\Omega,H)}, \qquad F \in \mathcal{S}.$$

$p > 1$, where $H := L^2(\mathbb{R}_+)$.

Covariant derivative

In addition to the operator D, we will also need the following notion of covariant derivative, see [19] and references therein. In the sequel we let $W_{2,1}(\mathbb{R}_+)$ denote the Sobolev space of weakly differentiable functions on $\mathbb{R}_+$ such that

$$\|f\|_{W_{2,1}}^2 := \int_0^\infty |f(t)|^2 dt + \int_0^\infty t |f'(t)|^2 dt < \infty.$$

Definition 2.1 *Let the operator* $\widetilde{\nabla}$ *be defined on*

$$u \in \mathcal{U} := \left\{ \sum_{i=1}^n h_i F_i \ : \ F_i \in \mathcal{S}, \ h_i \in W_{2,1}(\mathbb{R}_+), \ i = 1, \ldots, n, \ n \geq 1 \right\},$$

as

$$\widetilde{\nabla}_s u_t := D_s u_t - \dot{u}_t \mathbf{1}_{[0,t]}(s), \qquad s, t \in \mathbb{R}_+,$$

where $\dot{u}_t$ *denotes the time derivative of* $t \longmapsto u_t$ *with respect to* t*.*

By closability, the operator $\widetilde{\nabla}$ extends to the Sobolev spaces $\widetilde{\mathbb{D}}_{p,1}(H)$ of processes, $p \geq 1$, by the Sobolev norms

$$\|u\|_{\widetilde{\mathbb{D}}_{p,1}(H)} := \|u\|_{L^p(\Omega, W_{2,1}(\mathbb{R}_+))} + \|Du\|_{L^p(\Omega, H \otimes H)}, \qquad u \in \mathcal{U}.$$

We note that when a process $u \in \widetilde{\mathbb{D}}_{p,1}(H)$ is $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$-adapted we have

$$\widetilde{\nabla}_s u_t = D_s u_t = 0, \qquad s > t. \tag{2.1}$$

Definition 2.2 *Given* $k \geq 1$ *and* $u \in \widetilde{\mathbb{D}}_{2,1}(H)$ *we define the operator power* $(\widetilde{\nabla} u)^k$ *in the sense of matrix powers with continuous indices, as*

$$(\widetilde{\nabla} u)^k h_s = \int_0^\infty \cdots \int_0^\infty (\widetilde{\nabla}_{t_k} u_s \widetilde{\nabla}_{t_{k-1}} u_{t_k} \cdots \widetilde{\nabla}_{t_1} u_{t_2}) h_{t_1} dt_1 \cdots dt_k, \quad s \in \mathbb{R}_+, \quad h \in H.$$

In particular, for $h \in W_{2,1}(\mathbb{R}_+)$ and $v \in \mathcal{U}$ we have

$$(\widetilde{\nabla} h) v_s = \int_0^\infty v_t \widetilde{\nabla}_t h(s) dt = -\dot{h}(s) \int_0^\infty v_t \mathbf{1}_{[0,s]}(t) dt = -\dot{h}(s) \int_0^s v_t dt, \quad s \in \mathbb{R}_+. \tag{2.2}$$

We note that when $h \in W_{2,1}(\mathbb{R}_+)$ is a deterministic function, (2.2) can be iterated to show that

$$\begin{aligned}
(\widetilde{\nabla} h)^k f_s &= \int_0^\infty \cdots \int_0^\infty \widetilde{\nabla}_{t_k} h(s) \widetilde{\nabla}_{t_{k-1}} h(t_k) \cdots \widetilde{\nabla}_{t_1} h(t_2) f(t_1) dt_1 \cdots dt_k \\
&= (-1)^k \dot{h}(s) \int_0^\infty \cdots \int_0^\infty \mathbf{1}_{[0,s]}(t_k) \mathbf{1}_{[0,t_k]}(t_{k-1}) \dot{h}(t_k) \cdots \mathbf{1}_{[0,t_2]}(t_1) \dot{h}(t_2) f(t_1) dt_1 \cdots dt_k \\
&= (-1)^k \dot{h}(s) \int_0^s \dot{h}(t_k) \int_0^{t_k} \cdots \dot{h}(t_2) \int_0^{t_2} f(t_1) dt_1 \cdots dt_k, \qquad s \in \mathbb{R}_+, \quad k \geq 1,
\end{aligned}$$

with the bound

$$\|(\widetilde{\nabla} h)^k f\|_{L^2(\mathbb{R}_+)} \leq \|\widetilde{\nabla} h\|^k_{L^2(\mathbb{R}_+^2)} \|f\|_{L^2(\mathbb{R}_+)} = \left(\int_0^\infty t |h'(t)|^2 dt \right)^{k/2} \|f\|_{L^2(\mathbb{R}_+)}.$$

This also yields

$$\begin{aligned}\langle(\widetilde{\nabla}h)^k f,h\rangle &= (-1)^k\int_0^\infty h(s)\dot{h}(s)\int_0^s\int_0^{t_k}\cdots\int_0^{t_2}\dot{h}(t_k)\cdots\dot{h}(t_2)f(t_1)dt_1\cdots dt_k\\ &= \frac{1}{2}(-1)^{k-1}\int_0^\infty h^2(t_k)\dot{h}(t_k)\int_0^{t_k}\cdots\int_0^{t_2}\dot{h}(t_{k-1})\cdots\dot{h}(t_2)f(t_1)dt_1\cdots dt_k\\ &= \frac{1}{(k+1)!}\int_0^\infty h^{k+1}(t_1)f(t_1)dt_1, \qquad f\in L^2(\mathbb{R}_+), \quad k\geq 1,\end{aligned}$$

and in particular, for $h\in W_{2,1}(\mathbb{R}_+)$ we have $h\in\bigcap_{p=2}^\infty L^p(\mathbb{R}_+)$, with

$$\int_0^\infty |h^k(t)|dt \leq (k-1)!\langle|(\widetilde{\nabla}|h|)^{k-2}h|,|h|\rangle \leq (k-1)!\|h\|_{W_{2,1}}^k, \qquad k\geq 2.$$

Note that due to (2.1), when the process $u\in\widetilde{\mathbb{D}}_{p,1}(H)$ is $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-adapted, we have

$$(\widetilde{\nabla}u)^k v_s = \int_0^s\int_0^{t_k}\cdots\int_0^{t_2}(\widetilde{\nabla}_{t_k}u_s\widetilde{\nabla}_{t_{k-1}}u_{t_k}\cdots\widetilde{\nabla}_{t_1}u_{t_2})v_{t_1}dt_1\cdots dt_k, \quad s\in\mathbb{R}_+,$$

and as a consequence, the process $((\widetilde{\nabla}u)^k v_s)_{s\in\mathbb{R}_+}$ is also $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-adapted. In the sequel we simply denote $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_H = \langle\cdot,\cdot\rangle_{L^2(\mathbb{R}_+)}$.

Lemma 2.3 *Letting $n\geq 1$ and $u\in\widetilde{\mathbb{D}}_{2,1}(H)$, we have*

$$\langle(\widetilde{\nabla}u)^n u,u\rangle = \frac{1}{(n+1)!}\int_0^\infty u_t^{n+2}dt + \sum_{k=2}^{n+1}\frac{1}{k!}\left\langle(\widetilde{\nabla}u)^{n+1-k}u, D\int_0^\infty u_t^k dt\right\rangle. \qquad (2.3)$$

Proof. Using the adjoint $\widetilde{\nabla}^*u$ of $\widetilde{\nabla}u$ on H given by

$$(\widetilde{\nabla}^*u)v_s := \int_0^\infty(\widetilde{\nabla}_s u_t)v_t dt, \qquad s\in\mathbb{R}_+, \quad v\in H,$$

with the duality relation

$$\langle v,(\widetilde{\nabla}^*u)h\rangle = \langle(\widetilde{\nabla}u)v,h\rangle, \qquad h,v\in H,$$

we will show by induction on $1\leq k\leq n+1$ that

$$\begin{aligned}(\widetilde{\nabla}^*u)^n u_{t_0} &= \int_0^\infty\cdots\int_0^\infty u_{t_n}\widetilde{\nabla}_{t_0}u_{t_1}\widetilde{\nabla}_{t_1}u_{t_2}\cdots\widetilde{\nabla}_{t_{n-1}}u_{t_n}dt_1\cdots dt_n\\ &= \sum_{i=2}^k\frac{1}{i!}\int_0^\infty\cdots\int_0^\infty\widetilde{\nabla}_{t_0}u_{t_1}\cdots\widetilde{\nabla}_{t_{n-i}}u_{t_{n+1-i}}D_{t_{n+1-i}}u_{t_{n+2-i}}^i dt_1\cdots dt_{n-i}\end{aligned}$$

$$+\frac{1}{k!}\int_0^\infty\cdots\int_0^\infty u^k_{t_{n+1-k}}\widetilde{\nabla}_{t_0}u_{t_1}\cdots\widetilde{\nabla}_{t_{n-k}}u_{t_{n+1-k}}dt_1\cdots dt_{n+1-k}. \quad (2.4)$$

This relation holds for $k=1$. Next, assuming that the identity (2.4) holds for some $k\in\{1,\ldots,n\}$, and using the relation

$$\widetilde{\nabla}_{t_{n-k}}u_{t_{n+1-k}} = D_{t_{n-k}}u_{t_{n+1-k}} - \mathbf{1}_{[0,t_{n+1-k}]}(t_{n-k})\dot{u}_{t_{n+1-k}}, \quad t_{n-k},t_{n+1-k}\in\mathbb{R}_+,$$

we have

$$\begin{aligned}
(\widetilde{\nabla}^*u)^n u_{t_0} &= \sum_{i=2}^k\frac{1}{i!}\int_0^\infty\cdots\int_0^\infty\widetilde{\nabla}_{t_0}u_{t_1}\cdots\widetilde{\nabla}_{t_{n-i}}u_{t_{n+1-i}}D_{t_{n+1-i}}u^i_{t_{n+2-i}}dt_1\cdots dt_{n+2-i}\\
&\quad+\frac{1}{k!}\int_0^\infty\cdots\int_0^\infty u^k_{t_{n+1-k}}\widetilde{\nabla}_{t_0}u_{t_1}\cdots\widetilde{\nabla}_{t_{n-k}}u_{t_{n+1-k}}dt_1\cdots dt_{n+1-k}\\
&= \sum_{i=2}^k\frac{1}{i!}\int_0^\infty\cdots\int_0^\infty\widetilde{\nabla}_{t_0}u_{t_1}\cdots\widetilde{\nabla}_{t_{n-i}}u_{t_{n+1-i}}D_{t_{n+1-i}}u^i_{t_{n+2-i}}dt_1\cdots dt_{n+2-i}\\
&\quad+\frac{1}{k!}\int_0^\infty\cdots\int_0^\infty u^k_{t_{n+1-k}}\widetilde{\nabla}_{t_0}u_{t_1}\cdots\widetilde{\nabla}_{t_{n-k-1}}u_{t_{n-k}}D_{t_{n-k}}u_{t_{n+1-k}}dt_1\cdots dt_{n+1-k}\\
&\quad-\frac{1}{k!}\int_0^\infty\cdots\int_0^\infty\int_{t_{n-k}}^\infty \dot{u}_{t_{n+1-k}}u^k_{t_{n+1-k}}\widetilde{\nabla}_{t_0}u_{t_1}\cdots\widetilde{\nabla}_{t_{n-1-k}}u_{t_{n-k}}dt_1\cdots dt_{n+1-k}\\
&= \sum_{i=2}^k\frac{1}{i!}\int_0^\infty\cdots\int_0^\infty\widetilde{\nabla}_{t_0}u_{t_1}\cdots\widetilde{\nabla}_{t_{n-i}}u_{t_{n+1-i}}D_{t_{n+1-i}}u^i_{t_{n+2-i}}dt_1\cdots dt_{n+2-i}\\
&\quad+\frac{1}{(k+1)!}\int_0^\infty\cdots\int_0^\infty\widetilde{\nabla}_{t_0}u_{t_1}\cdots\widetilde{\nabla}_{t_{n-k-1}}u_{t_{n-k}}D_{t_{n-k}}u^{k+1}_{t_{n+1-k}}dt_1\cdots dt_{n+1-k}\\
&\quad-\frac{1}{(k+1)!}\int_0^\infty\cdots\int_0^\infty\widetilde{\nabla}_{t_0}u_{t_1}\cdots\widetilde{\nabla}_{t_{n-k-1}}u_{t_{n-k}}\int_{t_{n-k}}^\infty(u^{k+1}_t)'dtdt_1\cdots dt_{n-k}\\
&= \sum_{i=2}^{k+1}\frac{1}{i!}\int_0^\infty\cdots\int_0^\infty\widetilde{\nabla}_{t_0}u_{t_1}\cdots\widetilde{\nabla}_{t_{n-i}}u_{t_{n+1-i}}D_{t_{n+1-i}}u^i_{t_{n+2-i}}dt_1\cdots dt_{n+2-i}\\
&\quad+\frac{1}{(k+1)!}\int_0^\infty\cdots\int_0^\infty u^{k+1}_{t_{n-k}}\widetilde{\nabla}_{t_0}u_{t_1}\cdots\widetilde{\nabla}_{t_{n-k-1}}u_{t_{n-k}}dt_1\cdots dt_{n-k}\\
&= \sum_{i=2}^{k+1}\frac{1}{i!}(\widetilde{\nabla}^*u)^{n+1-i}D_{t_0}\int_0^\infty u^i_s ds+\frac{1}{(k+1)!}(\widetilde{\nabla}^*u)^{n-k}u^{k+1}_{t_0},
\end{aligned}$$

which shows by induction that (2.4) holds for $k=1,\ldots,n$. In particular, for $k=n$ we have

$$(\widetilde{\nabla}^*u)^n u_t = \frac{1}{(n+1)!}u^{n+1}_t+\sum_{i=2}^{n+1}\frac{1}{i!}(\widetilde{\nabla}^*u)^{n+1-i}D_t\int_0^\infty u^i_s ds, \qquad t\in\mathbb{R}_+,$$

which yields (2.3) by integration with respect to $t\in\mathbb{R}_+$ and duality. □

Cumulant operators

We recall the definition of the cumulant operators introduced in § 6 of [17] on the Poisson space. Given $k \geq 2$ and $u \in \widetilde{\mathbb{D}}_{2,1}(H)$ an $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-adapted process we define the cumulant operators

$$\Gamma_k^u : \mathbb{D}_{2,1} \longrightarrow L^2(\Omega), \qquad k \geq 2,$$

by

$$\Gamma_k^u F := F\langle (\widetilde{\nabla} u)^{k-2} u, u\rangle + \langle (\widetilde{\nabla} u)^{k-1} u, DF\rangle, \qquad k \geq 2.$$

As a consequence of Lemma 2.3 we have

$$\Gamma_k^u \mathbf{1} = \frac{1}{(k-1)!} \int_0^\infty u_t^k dt + \sum_{i=2}^{k-1} \frac{1}{i!} \left\langle (\widetilde{\nabla} u)^{k-1-i} u, D \int_0^\infty u_t^i dt \right\rangle,$$

and $|\Gamma_k^u \mathbf{1}| \leq \|u\|_{W_{2,1}(\mathbb{R}_+)}^k$, a.s., $k \geq 2$.

When $h \in W_{2,1}(\mathbb{R}_+)$ is a deterministic function we find

$$\Gamma_k^h \mathbf{1} = \frac{1}{(k-1)!} \int_0^\infty h^k(t) dt = \frac{1}{(k-1)!} \kappa_k(h), \qquad k \geq 2,$$

where $\kappa_k(h) = \int_0^\infty h^k(s) ds$ is the cumulant of order $k \geq 2$ of the compensated Poisson stochastic integral $\int_0^\infty h(t) d(N_t - t)$.

Poisson-Skorohod integral

In addition, D has a closable adjoint operator δ with domain $\mathrm{Dom}(\delta)$, that satisfies the duality relation

$$E\left[\int_0^\infty u_t D_t F dt\right] = E[F\delta(u)], \qquad F \in \mathbb{D}_{2,1}, \quad u \in \mathrm{Dom}(\delta), \tag{2.5}$$

and coincides with the compensated Poisson stochastic integral on square-integrable processes $(u_t)_{t\in\mathbb{R}_+}$ adapted to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ generated by $(N_t)_{t\in\mathbb{R}_+}$, i.e., we have

$$\delta(u) = \int_0^\infty u_t d(N_t - t).$$

In addition, the operators $\widetilde{\nabla}$, δ and D satisfy the commutation relation

$$D_t\delta(u) = u_t + \delta(\widetilde{\nabla}_t^* u), \tag{2.6}$$

for $u \in \widetilde{\mathbb{D}}_{2,1}(H)$ an $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-adapted process, cf. Lemma 4.5 in [17] and Relations (2.16), (2.19) and Lemma 2.4 in [19].

Recall that when $(u_t)_{t\in\mathbb{R}_+}$ is an $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-adapted process, $\delta(u)$ coincides with the Itô integral of u.

3 Edgeworth type expansions

Classical Edgeworth expansions are used in particular as asymptotic expansions around the Gaussian cumulative distribution function $\Phi(x)$ for the cumulative distribution function $P(F \leq x)$ of a centered random variable F with $E[F^2] = 1$, as

$$\Phi(x) + c_1\phi(x)H_1(x) + \cdots + c_m\phi(x)H_m(x) + \cdots ,$$

where $\phi(x)$, $x \in \mathbb{R}$, is the standard Gaussian density, $H_k(x)$ is the Hermite polynomial of degree $k \geq 1$, and c_k is a coefficient depending on the sequence of cumulants $(\kappa_n)_{n\geq 1}$ of a random variable F, cf. Chapter 5 of [8] and § A.4 of [12].

Edgeworth type expansions of the form

$$E[Fg(F)] = \sum_{l=1}^{n} \frac{\kappa_{l+1}}{l!} E[g^{(l)}(F)] + E[g^{(n+1)}(F)\Gamma_{n+1}F], \quad n \geq 1,$$

have been obtained by the Malliavin calculus in [9], [2], [4], written here for F a centered random variable, where Γ_{n+1} is a cumulant type operator on the Wiener space such that $n!E[\Gamma_n F]$ coincides with the cumulant κ_{n+1} of order $n+1$, $n \in \mathbb{N}$, cf. [11], extending results of [1].

An infinite Edgeworth type expansion for the compensated Poisson stochastic integral of a deterministic function $f \in \bigcap_{p=2}^{\infty} L^p(\mathbb{R}_+)$ can be written as

$$E\big[\delta(f)g(\delta(f))\big] \;=\; E\left[\int_0^\infty f(s)\big(g(\delta(f)+f(s)) - g(\delta(f))\big)ds\right] \tag{3.1}$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty} \frac{1}{k!} \int_0^{\infty} f^{k+1}(s) ds E\big[g^{(k)}(\delta(f))\big] \\
&= \sum_{k=1}^{\infty} \frac{1}{k!} \kappa_{k+1}(f) E\big[g^{(k)}(\delta(f))\big], \qquad g \in \mathcal{C}_b^{\infty}(\mathbb{R}),
\end{aligned}
$$

using a standard integration by parts for finite difference operators on the Poisson space.

In this section we establish an Edgeworth type expansion of any finite order with an explicit remainder term for the compensated Poisson stochastic integral $\delta(u)$ of an $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-adapted process u.

Before proceeding to the statement of general expansions in Proposition 3.1, we derive an expansion of order one for a deterministic integrand $f \in W_{2,1}(\mathbb{R}_+)$. In the sequel we let $\mathcal{C}_b^n(\mathbb{R}_+)$ denote the space of $\mathcal{C}_b^n$ functions bounded together with their derivatives on $\mathbb{R}_+$, $n \geq 1$.

By the duality relation (2.5) between D and δ, the chain rule of derivation for D and the commutation relation (2.6) we get, for $g \in \mathcal{C}_b^2(\mathbb{R})$,

$$
\begin{aligned}
&E\big[\delta(f) g(\delta(f))\big] = E\big[g'(\delta(f))\langle f, D\delta(f)\rangle\big] \\
&= E\big[g'(\delta(f))\langle f, f\rangle\big] + E\big[g'(\delta(f))\langle f, \delta(\widetilde{\nabla} * f)\rangle\big] \\
&= E\big[g'(\delta(f))\langle f, f\rangle\big] + E\big[\langle \widetilde{\nabla}^* f, D(g'(\delta(f)) f)\rangle\big] \\
&= E\big[g'(\delta(f))\langle f, f\rangle\big] + E\big[g''(\delta(f))\langle (\widetilde{\nabla} f) f, D\delta(f)\rangle\big] \\
&= E\big[g'(\delta(f))\langle f, f\rangle\big] + \frac{1}{2}\int_0^{\infty} f^3(t) dt E\big[g''(\delta(f))\big] + E\big[g''(\delta(f))\langle (\widetilde{\nabla} f) f, \delta(\widetilde{\nabla}^* f)\rangle\big] \\
&= \kappa_2(f) E\big[g'(\delta(f))\big] + \frac{1}{2}\kappa_3(f) E\big[g''(\delta(f))\big] + E\big[g''(\delta(f))\langle (\widetilde{\nabla} f) f, \delta(\widetilde{\nabla}^* f)\rangle\big],
\end{aligned}
$$

since by Lemma 2.3 we have

$$
\langle (\widetilde{\nabla} f) f, f\rangle = -\int_0^{\infty} f(t) f'(t) \int_0^t f(s) ds dt = \frac{1}{2}\int_0^{\infty} f^3(t) dt = \frac{1}{2}\kappa_3^f.
$$

In the next proposition we derive general Edgeworth type expansions for adapted integrands processes $(u_t)_{t\in\mathbb{R}_+}$.

Proposition 3.1 *Let $n \geq 0$ and assume that $u \in \widetilde{\mathbb{D}}_{n+1,1}(H)$ is an $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-adapted process. Then for all $g \in \mathcal{C}_b^{n+1}(\mathbb{R})$ and bounded $G \in \mathbb{D}_{2,1}$ we have*

$$E\big[G\delta(u)g(\delta(u))\big] = E\big[g(\delta(u))\langle u, DG\rangle\big] + \sum_{k=1}^{n} E\big[g^{(k)}(\delta(u))\Gamma_{k+1}^u G\big]$$
$$+ E\left[Gg^{(n+1)}(\delta(u))\left(\int_0^\infty \frac{u_s^{n+2}}{(n+1)!}ds + \sum_{k=2}^{n+1}\frac{1}{k!}\left\langle(\widetilde{\nabla}u)^{n+1-k}u, D\int_0^\infty u_t^k dt\right\rangle + \langle(\widetilde{\nabla}u)^n u, \delta(\widetilde{\nabla}^* u)\rangle\right)\right].$$

Proof. By the duality relation (2.5) between D and δ, the chain rule of derivation for D and the commutation relation (2.6), we get

$$\begin{aligned}
&E\big[Gg(\delta(u))\langle(\widetilde{\nabla}u)^k u, D\delta(u)\rangle\big] - E\big[Gg'(\delta(u))\langle(\widetilde{\nabla}u)^{k+1}u, D\delta(u)\rangle\big]\\
&= E\big[Gg(\delta(u))\langle(\widetilde{\nabla}u)^k u, u\rangle\big] + E[Gg(\delta(u))\langle(\widetilde{\nabla}u)^k u, \delta(\widetilde{\nabla}^* u)\rangle] - E\big[Gg'(\delta(u))\langle(\widetilde{\nabla}u)^{k+1}u, D\delta(u)\rangle\big]\\
&= E\big[Gg(\delta(u))\langle(\widetilde{\nabla}u)^k u, u\rangle\big] + E\big[\langle\widetilde{\nabla}^* u, D(Gg(\delta(u))(\widetilde{\nabla}u)^k u)\rangle\big] - E\big[Gg'(\delta(u))\langle(\widetilde{\nabla}u)^{k+1}u, D\delta(u)\rangle\big]\\
&= E\big[Gg(\delta(u))\langle(\widetilde{\nabla}u)^k u, u\rangle\big] + E\big[g(\delta(u))\langle(\widetilde{\nabla}u)^{k+1}u, DG\rangle\big] + E\big[Gg(\delta(u))\langle\widetilde{\nabla}^* u, D((\widetilde{\nabla}u)^k u)\rangle\big]\\
&= E\big[g(\delta(u))\Gamma_{k+2}^u G\big],
\end{aligned}$$

since $\langle\widetilde{\nabla}^* u, D((\widetilde{\nabla}u)^k u)\rangle = 0$ as u is an $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-adapted process, see Lemma 4.4 of [16]. Therefore, we have

$$\begin{aligned}
&E\big[G\delta(u)g(\delta(u))\big] = E\big[Gg'(\delta(u))\langle u, D\delta(u)\rangle\big] + E\big[g(\delta(u))\langle u, DG\rangle\big]\\
&= E\big[g(\delta(u))\langle u, DG\rangle\big] + E\big[Gg^{(n+1)}(\delta(u))\langle(\widetilde{\nabla}u)^n u, D\delta(u)\rangle\big]\\
&\quad + \sum_{k=0}^{n-1}\Big(E\big[Gg^{(k+1)}(\delta(u))\langle(\widetilde{\nabla}u)^k u, D\delta(u)\rangle\big] - E\big[Gg^{(k+2)}(\delta(u))\langle(\widetilde{\nabla}u)^{k+1}u, D\delta(u)\rangle\big]\Big)\\
&= E\big[g(\delta(u))\langle u, DG\rangle\big] + \sum_{k=1}^{n} E\big[g^{(k)}(\delta(u))\Gamma_{k+1}^u G\big] + E\big[Gg^{(n+1)}(\delta(u))\langle(\widetilde{\nabla}u)^n u, D\delta(u)\rangle\big]\\
&= E\big[g(\delta(u))\langle u, DG\rangle\big] + \sum_{k=1}^{n} E\big[g^{(k)}(\delta(u))\Gamma_{k+1}^u G\big]\\
&\quad + E\big[Gg^{(n+1)}(\delta(u))\langle(\widetilde{\nabla}u)^n u, u\rangle\big] + E\big[Gg^{(n+1)}(\delta(u))\langle(\widetilde{\nabla}u)^n u, \delta(\widetilde{\nabla}^* u)\rangle\big],
\end{aligned}$$

and we conclude by Lemma 2.3. □

When $G = 1$, Proposition 3.1 shows that

$$E\big[\delta(u)g(\delta(u))\big] = \sum_{k=1}^{n+1}\frac{1}{k!}E\left[\int_0^\infty u_s^{k+1}ds\, g^{(k)}(\delta(u))\right]$$

$$+\sum_{k=2}^{n+1}\sum_{i=2}^{k}\frac{1}{i!}E\left[\left\langle(\widetilde{\nabla}u)^{k-i}u, D\int_0^\infty u_t^i dt\right\rangle g^{(k)}(\delta(u))\right]$$
$$+E\big[g^{(n+1)}(\delta(u))\langle(\widetilde{\nabla}u)^n u, \delta(\widetilde{\nabla}^* u)\rangle\big], \qquad n \geq 0.$$

On the other hand, when $f \in W_{2,1}(\mathbb{R}_+)$ is a deterministic function and $g \in \mathcal{C}_b^\infty(\mathbb{R})$ we find

$$\begin{aligned} E\big[\delta(f)g(\delta(f))\big] &= \sum_{k=1}^{n+1}\frac{1}{k!}\int_0^\infty f^{k+1}(s)dsE\big[g^{(k)}(\delta(f))\big] + E\big[g^{(n+1)}(\delta(f))\langle(\widetilde{\nabla}f)^n f, \delta(\widetilde{\nabla}^* f)\rangle\big] \\ &= \sum_{k=1}^{n+1}\frac{1}{k!}\kappa_{k+1}(f)E\big[g^{(k)}(\delta(f))\big] + E\big[g^{(n+1)}(\delta(f))\langle(\widetilde{\nabla}f)^n f, \delta(\widetilde{\nabla}^* f)\rangle\big], \end{aligned}$$

$n \geq 0$, showing, as n tends to $+\infty$, that

$$E\big[\delta(f)g(\delta(f))\big] = \sum_{k=1}^{\infty}\int_0^\infty \frac{f^{k+1}(s)}{k!}dsE\big[g^{(k)}(\delta(f))\big] = E\left[\int_0^\infty f(s)\big(g(\delta(f)+f(s)) - g(\delta(f))\big)ds\right],$$

which recovers (3.1) and the standard integration by parts identity for finite difference operators on the Poisson space.

4 Stein approximation

We let $\mathcal{N} \simeq \mathcal{N}(0,1)$ denote a standard Gaussian random variable. In comparison with the results of [2], our bounds apply to a different stochastic integral representation.

In the case $n = 0$, Proposition 3.1 reads

$$E\big[g'(\delta(u))\langle u, u\rangle - \delta(u)g(\delta(u))\big] = -E\big[g'(\delta(u))\langle u, \delta(\widetilde{\nabla}^* u)\rangle\big],$$

for $u \in \widetilde{\mathbb{D}}_{2,1}(H)$ and $g \in \mathcal{C}_b^1(\mathbb{R})$. Applying this relation to the solution g_x of the Stein equation

$$\mathbf{1}_{(-\infty,x]}(z) - \Phi(z) = g_x'(z) - zg_x(z), \qquad z \in \mathbb{R},$$

which satisfies $\|g_x\|_\infty \leq \sqrt{2\pi}/4$ and $\|g_x'\|_\infty \leq 1$, cf. Lemma 2.2-(v) of [6], yields the expansion

$$P(\delta(u) \leq x) - \Phi(x) = E\big[(1 - \langle u, u\rangle)g_x'(\delta(u))\big] - E\big[\langle u, \delta(\widetilde{\nabla}u)\rangle g_x'(\delta(u))\big], \quad x \in \mathbb{R},$$

around the Gaussian cumulative distribution function $\Phi(x)$, with $u \in \widetilde{\mathbb{D}}_{2,1}(H)$.

On the other hand, given $h : \mathbb{R} \to \mathbb{R}$ an absolutely continuous function with bounded derivative, the functional equation

$$h(z) - \mathrm{E}[h(\mathcal{N})] = g'(z) - zg(z), \qquad z \in \mathbb{R}, \tag{4.1}$$

has a solution $g_h \in \mathcal{C}_b^1(\mathbb{R})$ which is twice differentiable and satisfies the bounds

$$\|g_h'\|_\infty \leq \|h'\|_\infty \quad \text{and} \quad \|g_h''\|_\infty \leq 2\|h'\|_\infty, \qquad x \in \mathbb{R},$$

cf. Lemma 1.2-(v) of [10] and references therein.

Proposition 4.1 *Let $u \in \widetilde{\mathbb{D}}_{2,1}(H)$ be adapted with respect to the Poisson filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. We have*

$$\begin{aligned} d(\delta(u), \mathcal{N}) \ &\leq \ E\big[|1 - \langle u, u\rangle|\big] + E\left[\left|\int_0^\infty u_s^3 ds + \left\langle u, D\int_0^\infty u_t^2 dt\right\rangle\right|\right] \\ &\quad +2E\big[|\langle(\widetilde{\nabla} u)u, \delta(\widetilde{\nabla}^* u)\rangle|\big]. \end{aligned} \tag{4.2}$$

Proof. For $n = 1$ and $G = 1$, Proposition 3.1 shows that

$$\begin{aligned} E[\delta(u)g(\delta(u))] \ &= \ E[g'(\delta(u))\langle u, u\rangle] + \frac{1}{2}E\left[g''(\delta(u))\left(\int_0^\infty u_s^3 ds + \left\langle u, D\int_0^\infty u_t^2 dt\right\rangle\right)\right] \\ &\quad +E\big[g''(\delta(u))\langle(\widetilde{\nabla} u)u, \delta(\widetilde{\nabla} u)\rangle\big], \end{aligned}$$

hence for any absolutely continuous function $h : \mathbb{R} \to \mathbb{R}$ with bounded derivative, denoting by g_h the solution to (4.1) we have

$$\begin{aligned} &E[h(\delta(u))] - E[h(\mathcal{N})] = E[\delta(u)g_h(\delta(u)) - g_h'(\delta(u))] \\ &\quad = \ E[g_h'(\delta(u))(\langle u, u\rangle - 1)] + \frac{1}{2}E\left[g''(\delta(u))\left(\int_0^\infty u_s^3 ds + \left\langle u, D\int_0^\infty u_t^2 dt\right\rangle\right)\right] \\ &\qquad +2E\big[g_h''(\delta(u))\langle(\widetilde{\nabla} u)u, \delta(\widetilde{\nabla}^* u)\rangle\big], \end{aligned}$$

hence

$$\begin{aligned} |E[\delta(u)h(\delta(u))] - E[h(\mathcal{N})]| \ &\leq \ \|h'\|_\infty E\big[|1 - \langle u, u\rangle|\big] \\ &\quad +\|h'\|_\infty E\left[\left|\int_0^\infty u_s^3 ds + \left\langle u, D\int_0^\infty u_t^2 dt\right\rangle\right|\right] \\ &\quad +2\|h'\|_\infty E\big[|\langle(\widetilde{\nabla} u)u, \delta(\widetilde{\nabla}^* u)\rangle|\big], \end{aligned}$$

which yields (4.2). □

As a consequence of Proposition 4.1 and the Itô isometry we have the following corollary.

Corollary 4.2 *Let $u \in \widetilde{\mathbb{D}}_{2,1}(H)$ be adapted with respect to the Poisson filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. We have*

$$d(\delta(u),\mathcal{N}) \leq |1-\mathrm{Var}[\delta(u)]| + \sqrt{\mathrm{Var}\big[\|u\|_H^2\big]} + E\left[\left|\int_0^\infty u_s^3 ds + \left\langle u, D\int_0^\infty u_t^2 dt\right\rangle\right|\right] + 2E\big[|\langle(\widetilde{\nabla}u)u, \delta(\widetilde{\nabla}^* u)\rangle|\big]. \tag{4.3}$$

Proof. By the Itô isometry we have

$$\mathrm{Var}[\delta(u)] = E\left[\left(\int_0^\infty u_t d(N_t - t)\right)^2\right] = E[\langle u,u\rangle],$$

hence

$$\begin{aligned} E\big[|1-\langle u,u\rangle|\big] &\leq E\big[|1-E[\langle u,u\rangle]|\big] + E\big[|\langle u,u\rangle - E[\langle u,u\rangle]|\big] \\ &= |1-\mathrm{Var}[\delta(u)]| + \sqrt{E[(\langle u,u\rangle - E[\langle u,u\rangle])^2]} \\ &= |1-\mathrm{Var}[\delta(u)]| + \sqrt{\mathrm{Var}\big[\|u\|_H^2\big]}. \end{aligned}$$

□

In particular, when $\mathrm{Var}[\delta(u)] = 1$, (4.3) shows that

$$d(\delta(u),\mathcal{N}) \leq \sqrt{\mathrm{Var}\big[\|u\|_H^2\big]} + E\left[\left|\int_0^\infty u_s^3 ds + \left\langle u, D\int_0^\infty u_t^2 dt\right\rangle\right|\right] + 2E\big[|\langle(\widetilde{\nabla}u)u, \delta(\widetilde{\nabla}^* u)\rangle|\big].$$

5 Examples

Adapted integrands

Although the present approach does not apply directly to the classical normalized sequence

$$F_k := \frac{T_k - k}{\sqrt{k}} = -\frac{1}{\sqrt{k}}\int_0^\infty \mathbf{1}_{[0,T_k]}(t) d(N_t - t), \quad k \geq 1,$$

due to the lack of time differentiability of the adapted integrand $t \mapsto \mathbf{1}_{[0,T_k]}(t)$, examples of this form can be treated via smoothed processes u_k as in (5.1) below, see also Corollary 5.2.

Proposition 5.1 *Let*

$$u_k(t) := g(t)\mathbf{1}_{[0,T_k]}(t) + g(T_k)\mathbf{1}_{(T_k,\infty)}(t)f(t-T_k), \qquad t \in \mathbb{R}_+, \tag{5.1}$$

$k \in \mathbb{N}$, where g is a Lipschitz function on $\mathbb{R}_+$ and $f \in W_{2,1}(\mathbb{R}_+)$ satisfies $f(0) = 1$. Then for every $k \in \mathbb{N}$ we have

$$\begin{aligned}
& d(\delta(u_k), \mathcal{N}) \qquad\qquad (5.2)\\
&\le E\left[\left|1 - \int_0^{T_k} g^2(t)dt - \|f\|^2_{L^2(\mathbb{R}_+)} g^2(T_k)\right|\right] + E\left[\left|\int_0^{T_k} (g^2(s) - g^2(T_k))g(s)ds\right|\right]\\
&\quad + E\left[\,|g^3(T_k)|\,\right]\left|\int_0^\infty f^3(t)dt\right| + 2\|f\|^2_{L^2(\mathbb{R}_+)} E\left[\left|g(T_k)g'(T_k)\int_0^{T_k} g(t)dt\right|\right]\\
&\quad + 2E\left[\left|\int_0^{T_k} g'(t)\int_0^t g(s)ds \int_t^{T_k} g'(s)d(N_s - s)dt\right|\right]\\
&\quad + 2\|f\|_{L^2(\mathbb{R}_+)} E\left[\left|g'(T_k)\int_0^{T_k} g'(t)\int_0^t g(s)dsdt\right|\right]\\
&\quad + 2\|f\|_{L^2(\mathbb{R}_+)} E\left[\left|g(T_k)g'(T_k)\int_0^{T_k} g(s)ds\right|\right]\sqrt{\int_0^\infty t|f'(t)|^2 dt}\\
&\quad + 2\|f\|_{L^2(\mathbb{R}_+)} E\left[|g^3(T_k)|\right]\int_0^\infty t|f'(t)|^2 dt.
\end{aligned}$$

Proof. We have

$$\delta(u_k) = \int_0^\infty u_k(t)d(N_t - t) = \int_0^{T_k} g(t)d(N_t - t) + g(T_k)\int_{T_k}^\infty f(t - T_k)d(N_t - t),$$

and

$$\mathrm{Var}[\delta(u_k)] = E\left[\int_0^{T_k} g^2(t)dt\right] + E[g^2(T_k)]\int_0^\infty f^2(t)dt.$$

On the other hand, we have

$$\begin{aligned}
&\widetilde{\nabla}_s u_k(t) = D_s u_k(t) - \big(g'(t)\mathbf{1}_{[0,T_k]}(t) + g(T_k)f'(t-T_k)\mathbf{1}_{(T_k,\infty)}(t)\big)\mathbf{1}_{[0,t]}(s)\\
&= -\big(g'(T_k)\mathbf{1}_{(T_k,\infty)}(t)f(t-T_k) - g(T_k)\mathbf{1}_{(T_k,\infty)}(t)f'(t-T_k)\big)\mathbf{1}_{[0,T_k]}(s)\\
&\quad - \big(g'(t)\mathbf{1}_{[0,T_k]}(t) + g(T_k)f'(t-T_k)\mathbf{1}_{(T_k,\infty)}(t)\big)\mathbf{1}_{[0,t]}(s)\\
&= -\big(g'(t)\mathbf{1}_{[0,T_k]}(t) + g'(T_k)\mathbf{1}_{(T_k,\infty)}(t)f(t-T_k)\big)\mathbf{1}_{[0,t]}(s)\mathbf{1}_{[0,T_k]}(s) - g(T_k)\mathbf{1}_{\{T_k<s<t\}}f'(t-T_k),
\end{aligned}$$

and

$$(\widetilde{\nabla}u_k)u_k(t) \;=\; -\int_0^\infty g(s)\big(g'(t)\mathbf{1}_{[0,T_k]}(t) + g'(T_k)\mathbf{1}_{(T_k,\infty)}(t)f(t-T_k)\big)\mathbf{1}_{[0,t]}(s)\mathbf{1}_{[0,T_k]}(s)ds$$

$$
\begin{aligned}
& -g^2(T_k)f'(t-T_k)\int_0^\infty f(s-T_k)\mathbf{1}_{\{T_k<s<t\}}ds \\
= \quad & -\mathbf{1}_{[0,T_k]}(t)g'(t)\int_0^t g(s)ds - \mathbf{1}_{(T_k,\infty)}(t)g'(T_k)f(t-T_k)\int_0^{T_k} g(s)ds \\
& +\mathbf{1}_{(T_k,\infty)}(t)g^2(T_k)f'(t-T_k)\int_{T_k}^t f(s-T_k)ds.
\end{aligned}
$$

Given that

$$
\begin{aligned}
\delta(\widetilde{\nabla}_s^* u_k) \quad = \quad & -\left(\int_s^{T_k} g'(t)d(N_t-t) + g'(T_k)\int_{T_k}^\infty f(t-T_k)d(N_t-t)\right)\mathbf{1}_{[0,T_k]}(s) \\
& +\mathbf{1}_{\{T_k<s\}}g(T_k)\int_s^\infty f'(t-T_k)d(N_t-t),
\end{aligned}
$$

this yields

$$
\begin{aligned}
& \langle(\widetilde{\nabla}u_k)u_k, \delta(\widetilde{\nabla}^* u_k)\rangle \\
& = \int_0^{T_k}\left(\int_t^{T_k} g'(s)d(N_s-s) + g'(T_k)\int_{T_k}^\infty f(s-T_k)d(N_s-s)\right)g'(t)\int_0^t g(s)dsdt \\
& \quad - g(T_k)\int_{T_k}^\infty\int_t^\infty f'(s-T_k)d(N_s-s)g'(T_k)f(t-T_k)\int_0^{T_k} g(s)dsdt \\
& \quad + g^3(T_k)\int_{T_k}^\infty\int_t^\infty f'(s-T_k)d(N_s-s)f'(t-T_k)\int_{T_k}^t f(s-T_k)dsdt \\
& \simeq \int_0^{T_k} g'(t)\int_t^{T_k} g'(s)d(\hat{N}_s-s)\int_0^t g(s)dsdt \\
& \quad + g'(T_k)\int_0^\infty f(s)d(\hat{N}_s-s)\int_0^{T_k} g'(t)\int_0^t g(s)dsdt \\
& \quad - g(T_k)g'(T_k)\int_0^{T_k} g(s)ds\int_0^\infty f(t)\int_t^\infty f'(s)d(\hat{N}_s-s)dt \\
& \quad + g^3(T_k)\int_0^\infty f'(t)\int_t^\infty f'(s)d(\hat{N}_s-s)\int_0^t f(s)dsdt,
\end{aligned}
$$

where "$\simeq$" denotes equality in distribution and $(\hat{N}_t)_{t\in\mathbb{R}_+}$ is a standard Poisson process independent of $(T_k)_{k\geq 1}$. We also have

$$
\begin{aligned}
D_s\int_0^\infty u_k^2(t)dt \quad & = \quad D_s\left(\int_0^{T_k} g^2(t)dt + g^2(T_k)\int_{T_k}^\infty f^2(t-T_k)dt\right) \\
& = \quad D_s\left(\int_0^{T_k} g^2(t)dt + g^2(T_k)\int_0^\infty f^2(t)dt\right)
\end{aligned}
$$

$$
\begin{aligned}
&= g^2(T_k)D_sT_k + 2g(T_k)g'(T_k)\int_0^\infty f^2(t)dtD_sT_k \\
&= -\left(g^2(T_k) + 2g(T_k)g'(T_k)\int_0^\infty f^2(t)dt\right)\mathbf{1}_{[0,T_k]}(s),
\end{aligned}
$$

consequently we have

$$
\left\langle u_k, D\int_0^\infty u_k^2(t)dt\right\rangle = -\left(g^2(T_k) + 2g(T_k)g'(T_k)\int_0^\infty f^2(t)dt\right)\int_0^{T_k} g(t)dt,
$$

and

$$
\begin{aligned}
&\int_0^\infty u_k^3(s)ds + \left\langle u_k, D\int_0^\infty u_k^2(t)dt\right\rangle \\
&= \int_0^{T_k}(g^2(s) - g^2(T_k))g(s)ds + g^3(T_k)\int_0^\infty f^3(t)dt - 2g(T_k)g'(T_k)\int_0^\infty f^2(t)dt\int_0^{T_k} g(t)dt.
\end{aligned}
$$

Hence by Corollary 4.2 we find

$$
\begin{aligned}
&d(\delta(u_k), \mathcal{N}) \\
&\leq |1 - \mathrm{Var}[\delta(u_k)]| + \sqrt{\mathrm{Var}\left[\|u_k\|_H^2\right]} + E\left[\left|\int_0^\infty u_k^3(s)ds + \left\langle u_k, D\int_0^\infty u_k^2(t)dt\right\rangle\right|\right] \\
&\quad +2E\left[|\langle(\widetilde{\nabla}u_k)u_k, \delta(\widetilde{\nabla}^*u_k)\rangle|\right] \\
&\leq E\left[\left|1 - \int_0^{T_k} g^2(t)dt - g^2(T_k)\int_0^\infty f^2(t)dt\right|\right] \\
&\quad +E\left[\left|\int_0^{T_k}(g^2(s) - g^2(T_k))g(s)ds\right|\right] \\
&\quad +E\left[\left|g^3(T_k)\right|\right]\left|\int_0^\infty f^3(t)dt\right| + 2E\left[\left|g(T_k)g'(T_k)\int_0^{T_k} g(t)dt\right|\right]\int_0^\infty f^2(t)dt \\
&\quad +2E\left[\left|\int_0^{T_k}\int_t^{T_k} g'(s)d(N_s - s)g'(t)\int_0^t g(s)dsdt\right|\right] \\
&\quad +2E\left[\left|g'(T_k)\int_0^{T_k} g'(t)\int_0^t g(s)dsdt\right|\right]E\left[\left|\int_0^\infty f(s)d(N_s - s)\right|\right] \\
&\quad +2E\left[\left|g(T_k)g'(T_k)\int_0^{T_k} g(s)ds\right|\right]E\left[\left|\int_0^\infty f(t)\int_t^\infty f'(s)d(N_s - s)dt\right|\right] \\
&\quad +2E\left[\left|g^3(T_k)\right|\right]E\left[\left|\int_0^\infty f'(t)\int_t^\infty f'(s)d(N_s - s)\int_0^t f(s)dsdt\right|\right].
\end{aligned}
$$

Finally, we note that

$$
E\left[\left|\int_0^\infty f(t)\int_t^\infty f'(s)d(N_s - s)dt\right|\right] = E[|\langle f, \delta(\widetilde{\nabla}^*f)\rangle|]
$$

$$
\begin{aligned}
&\leq \|f\|_{L^2(\mathbb{R}_+)} \sqrt{E[\|\delta(\widetilde{\nabla}^* f)\|^2_{L^2(\mathbb{R}_+)}]} \\
&\leq \|f\|_{L^2(\mathbb{R}_+)} \sqrt{E\left[\int_0^\infty |\delta(\widetilde{\nabla}^*_t f)|^2 dt\right]} \\
&= \|f\|_{L^2(\mathbb{R}_+)} \sqrt{\int_0^\infty \int_0^\infty |\widetilde{\nabla}_t f(s)|^2 ds dt} \\
&= \|f\|_{L^2(\mathbb{R}_+)} \sqrt{\int_0^\infty \int_0^\infty |f'(s)|^2 \mathbf{1}_{[0,s]}(t) ds dt} \\
&= \|f\|_{L^2(\mathbb{R}_+)} \sqrt{\int_0^\infty s|f'(s)|^2 ds},
\end{aligned}
$$

and

$$
\begin{aligned}
E\left[\left|\int_0^\infty f'(t) \int_t^\infty f'(s) d(N_s - s) \int_0^t f(s) ds dt\right|\right] &= E\big[|\langle(\widetilde{\nabla} f)f, \delta(\widetilde{\nabla}^* f)\rangle|\big] \\
&\leq \|(\widetilde{\nabla} f)f\|_{L^2(\mathbb{R}_+)} \sqrt{E\left[\int_0^\infty |\delta(\widetilde{\nabla}^*_t f)|^2 dt\right]} \\
&\leq \|f\|_{L^2(\mathbb{R}_+)} \|\widetilde{\nabla} f\|^2_{L^2(\mathbb{R}^2_+)} = \|f\|_{L^2(\mathbb{R}_+)} \int_0^\infty s|f'(s)|^2 ds.
\end{aligned}
$$

The above bound can also be obtained as

$$
\begin{aligned}
E\big[|\langle(\widetilde{\nabla} f)f, \delta(\widetilde{\nabla}^* f)\rangle|\big] &= E\big[|\delta((\widetilde{\nabla} f)^2 f)|\big] \leq \sqrt{E\big[|\delta((\widetilde{\nabla} f)^2 f)|^2\big]} \\
&= \|(\widetilde{\nabla} f)^2 f\|_{L^2(\mathbb{R}_+)} \leq \|f\|_{L^2(\mathbb{R}_+)} \|\widetilde{\nabla} f\|^2_{L^2(\mathbb{R}^2_+)} \\
&= \|f\|_{L^2(\mathbb{R}_+)} \int_0^\infty s|f'(s)|^2 ds.
\end{aligned}
$$

□

When u_k is an adapted process of the form

$$
u_k(t) := g_k(t)\mathbf{1}_{[0,T_k]}(t), \qquad t \in \mathbb{R}_+,
$$

where g_k is Lipschitz, and e.g. $f(t) := e^{-nt}$, Proposition 5.2 shows, by letting n tend to infinity, that

$$
d\left(\int_0^{T_k} g_k(t) d(N_t - t), \mathcal{N}\right) \leq E\left[\left|1 - \int_0^{T_k} g_k^2(t) dt\right|\right] + E\left[\left|\int_0^{T_k} (g_k^2(s) - g_k^2(T_k)) g_k(s) ds\right|\right]
$$

$$+2E\left[\left|\int_0^{T_k} g_k'(t)\int_t^{T_k} g_k'(s)d(N_s-s)\int_0^t g_k(s)dsdt\right|\right].$$

As a consequence, similarly to Corollary 4.2 we have the following result.

Corollary 5.2 *Let g_k be a Lipschitz function on $\mathbb{R}_+$ for $k \geq 0$. We have*

$$\begin{aligned} d\left(\int_0^{T_k} g_k(t)d(N_t-t),\mathcal{N}\right) &\leq E\left[\left|1-E\left[\int_0^{T_k} g_k^2(t)dt\right]\right|\right]+\sqrt{\mathrm{Var}\left[\int_0^{T_k} g_k^2(t)dt\right]} \\ &\quad +E\left[\left|\int_0^{T_k}(g_k^2(s)-g_k^2(T_k))g_k(s)ds\right|\right] \\ &\quad +2E\left[\left|\int_0^{T_k} g_k'(t)\int_t^{T_k} g_k'(s)d(N_s-s)\int_0^t g_k(s)dsdt\right|\right]. \end{aligned}$$

For example, when $g_k(t)=1/\sqrt{k}$ is constant, Corollary 5.2 recovers the standard Berry-Esseen rate

$$d\left(\frac{T_k-k}{\sqrt{k}},\mathcal{N}\right)\leq\frac{\sqrt{\mathrm{Var}[T_k]}}{k}=\frac{1}{\sqrt{k}},\qquad k\geq 1.$$

Deterministic integrands

Taking $k=0$ and $g_0(0)=1$ in Proposition 5.1, we also find the following corollary.

Corollary 5.3 *Let $f\in W_{2,1}(\mathbb{R}_+)$ be a deterministic function. We have*

$$d\left(\int_0^\infty f(t)d(N_t-t),\mathcal{N}\right)\leq\left|1-\int_0^\infty f^2(t)dt\right|+\left|\int_0^\infty f^3(t)dt\right|+2\|f\|_{L^2(\mathbb{R}_+)}\int_0^\infty t|f'(t)|^2dt.$$

Considering for example $f_k(t)$ of the form

$$f_k(t):=\frac{1}{\sqrt{k}}g(t/k),\qquad k\geq 1,$$

where $g\in\mathcal{C}^1(\mathbb{R})$ is such that $\|g\|_{L^2(\mathbb{R}_+)}=\|f_k\|_{L^2(\mathbb{R}_+)}=1$, we have

$$\begin{aligned} d\left(\int_0^\infty f_k(t)d(N_t-t),\mathcal{N}\right) &\leq \left|\int_0^\infty f_k^3(s)ds\right|+2\int_0^\infty t|f_k'(t)|^2dt \\ &= \frac{1}{\sqrt{k}}\left|\int_0^\infty g^3(t)dt\right|+\frac{2}{k}\int_0^\infty t|g'(t)|^2dt, \qquad (5.3) \end{aligned}$$

$k \geq 1$. When g has constant sign, e.g. $g(x) := \sqrt{2b}e^{-bt}$ with $b > 0$ and $f_k(t) = \sqrt{2b/k}e^{-bt/k}$, (5.3) does not improve on the standard Berry-Esseen convergence rate

$$d\left(\int_0^\infty g_k(t)d(N_t - t), \mathcal{N}\right) \leq \frac{2}{3}\sqrt{\frac{2b}{k}}, \quad \text{and} \quad d\left(\int_0^\infty g_k(t)d(N_t - t), \mathcal{N}\right) \leq \sqrt{\frac{b}{3k}},$$

respectively obtained from (1.4) and (1.5), see § 4.2 of [20]. On the other hand, choosing g such that

$$\int_0^\infty g^3(t)dt = 0,$$

we find the bound

$$d\left(\int_0^\infty f_k(t)d(N_t - t), \mathcal{N}\right) \leq \frac{2}{k}\int_0^\infty t|g'(t)|^2 dt, \qquad k \geq 1, \tag{5.4}$$

while Corollary 4.4 of [20] and Corollary 3.4 of [14] only yield the Berry-Esseen rates

$$\begin{aligned} d\left(\int_0^\infty f_k(t)d(N_t - t), \mathcal{N}\right) &\leq |1 - \|f_k\|^2_{L^2(\mathbb{R}_+)}| + \|f_k\|_{L^2(\mathbb{R}_+)}\sqrt{\int_0^\infty t|f_k'(t)|^2 dt} \\ &= \frac{1}{\sqrt{k}}\sqrt{\int_0^\infty t|g'(t)|^2 dt}, \qquad k \geq 1, \end{aligned} \tag{5.5}$$

by (1.5), and

$$\begin{aligned} d\left(\int_0^\infty f_k(t)d(N_t - t), \mathcal{N}\right) &\leq \left|1 - \|f_k\|^2_{L^2(\mathbb{R}_+)}\right| + \int_0^\infty |f_k^3(s)|ds \\ &= \frac{1}{\sqrt{k}}\int_0^\infty |g(t)|^3 dt, \end{aligned} \tag{5.6}$$

by (1.4). For another example, choosing

$$f_k(t) := \frac{1}{C\sqrt{k}}\sum_{i=1}^m a_i e^{-b_i t/k}, \qquad k \geq 1,$$

with $\|f_k\|_{L^2(\mathbb{R}_+)} = 1$, for $b_i > 0$, $a_i \in \mathbb{R}$, $i = 1, \ldots, m$, such that

$$\sum_{1 \leq i,j,l \leq m} \frac{a_i a_j a_l}{b_i + b_j + b_l} = 0,$$

(5.4) yields the bound

$$d\left(\int_0^\infty f_k(t)d(N_t - t), \mathcal{N}\right) \leq \frac{1}{C^2 k}\sum_{1 \leq i,j \leq m} \frac{a_i a_j b_i b_j}{(b_i + b_j)^2}, \qquad k \geq 1.$$

In particular, when $f_k(t) := k^{-1/2}(e^{-t/k} - ae^{-bt/k})/C$ with $a, b > 0$ and

$$C^2 := \frac{1}{2} - \frac{2a}{1+b} + \frac{a^2}{2b} = \frac{b(b+1) - 4ab + a^2(1+b)}{2b(b+1)} > 0,$$

for any $b > 0$ we can choose $a > 0$ satisfying the equation

$$\int_0^\infty f_k^3(t)dt = \frac{1}{C^3\sqrt{k}}\left(\frac{1}{3} - \frac{3a}{2+b} + \frac{3a^2}{1+2b} - \frac{a^3}{3b}\right) = 0,$$

which yields the bound

$$d\left(\int_0^\infty f_k(t)d(N_t - t), \mathcal{N}\right) \;\leq\; \frac{c(a,b)}{k}, \qquad k \geq 1,$$

where $c(a, b)$ depends only on $a, b > 0$.

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