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Research Article

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Third-order differential equations with three-point boundary conditions

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Abstract: In this paper, a third-order ordinary differential equation coupled to three-point boundary conditions is considered. The related Green's function changes its sign on the square of definition. Despite this, we are able to deduce the existence of positive and increasing functions on the whole interval of definition, which are convex in a given subinterval. The nonlinear considered problem consists on the product of a positive real parameter, a nonnegative function that depends on the spatial variable and a time dependent function, with negative sign on the first part of the interval and positive on the second one. The results hold by means of fixed point theorems on suitable cones.

Keywords: third-order equations, three-point boundary conditions, Green's function, degree theory

MSC 2020: 34B05, 34B08, 34B10, 34B15, 34B18, 34B27, 34L30

1 Introduction

Third-order three-point boundary value problems arise in several areas of applied mathematics and physics. Some particular models of deflection of a curved beam with constant or varying cross sections, three-layer beams, electromagnetic waves, study of the equilibrium states of a hated bar and others can be found in [1]. Jiang and Agarwal [2] proved that the singular third-order boundary value problem

$$\begin{gathered} y'''(x) = (1-y)^\lambda g(y), \quad 0 < x < \infty, \quad \lambda > 0, \\ y(0) = 0, \quad \lim_{x\to\infty} y(x) = 1, \quad \lim_{x\to\infty} y'(x) = \lim_{x\to\infty} y''(x) = 0, \end{gathered}$$

where $g(y)$ is positive and continuous on $(0, 1]$, has a unique solution. This kind of problem arises in the study of draining and coating flows.

Later, using Krasnoselskii's fixed-point theorem, Sun [3] proved the existence of infinite positive solutions of the BVP

$$\begin{gathered} u'''(t) = \lambda a(t) f(t, u(t)), \quad 0 < t < 1, \\ u(0) = u'(\eta) = u''(1) = 0, \quad \eta \in \left(\frac{1}{2}, 1\right), \end{gathered}$$

assuming that f is sublinear or superlinear with respect to the second variable.

Li [4] studied the same problems with two-point boundary conditions

$$u(0) = u'(0) = u''(1) = 0,$$

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while Liu et al. in [5,6] studied the aforementioned problem with three-point boundary conditions

$$u(a) = u(b) = u''(b) = 0 \quad \text{and} \quad u(a) = u'(b) = u''(a) = 0,$$

respectively. In [7], the authors considered the problem with conditions as follows:

$$u(0) = u'(0) = 0, \quad u'(1) = \alpha u'(\eta), \quad 0 < \eta < 1, \quad 1 < \alpha < \frac{1}{\eta}.$$

In all these papers, the existence of positive solution follows from the fact that the corresponding Green's function is strictly positive. In [8], Palamides and Veloni studied the singular BVP

$$u'''(t) = -a(t)f(t, u(t)), \quad 0 < t < 1,$$
$$u(0) = u'(1) = u''(\eta) = 0, \quad \eta \in [0, 1/2].$$

The corresponding Green's function $G(t, s)$ for this problem is not a definite sign function for $(t, s) \in [0, 1] \times [0, 1]$. The solution $u(t) = \int_0^1 G(t, s)a(s)f(s, u(s))ds$ may still be positive, i.e., if its initial values $u'(0)$ and $u''(0)$ are positive. This observation is based on an analysis of the corresponding vector field on the phase-plane (u', u''), proposed in [9] and in some references therein. It is worth noticing that a positive and increasing solution was obtained in [8], where the proof is based on the classical Krasnoselskii's fixed point theorem in cones.

The aim of this paper is to study the existence, nonexistence and multiplicity of solutions of the third-order nonlinear differential equation

$$u'''(t) = -\lambda p(t)f(u(t)), \quad \text{a.e. } t \in [0, 1] \equiv I, \tag{1}$$

coupled with more general three-point boundary value conditions, namely

$$u(0) = 0, \quad u''(\eta) = \alpha u'(1), \quad u'(1) = \beta u(1), \tag{2}$$

with $0 \le \alpha \le 1$, $0 \le \beta < \frac{2}{2-\alpha}$ and $0 \le \eta \le \frac{1}{3}$.

The following assumptions on the nonlinear part are assumed:

(F) $\lambda > 0$ is a parameter, $p \in L^\infty(I)$ is such that $p < 0$ a.e. on $[0, \eta]$ and $p > 0$ a.e. on $[\eta, 1]$ and $f : [0, \infty) \to [0, \infty)$ is a continuous function.

Moreover, it is worth noticing that the properties of the corresponding sign-changing Green's function make it necessary to construct a different kind of cone, similar to the one recently used in [10]. Using this cone we will impose some conditions in order to assure the existence of positive and increasing solutions of the considered problem, which will also be convex in a certain subset of its interval of definition.

The paper is organized as follows: in Section 2 we study the linear problem and we deduce the exact expression of the corresponding Green's function and some of its properties as well as some properties of its first- and second-order derivative. Using these properties, in Section 3, we impose some sufficient conditions on the nonlinearity that allow us to deduce the existence of at least one positive solution of problem (1)–(2). The results are based on the fixed point index theory. In Section 4, we give some conditions under which there is no solution for the considered problem. Finally, in Section 5 we illustrate the given results with some examples.

2 Linear problem

Consider, for any $y \in C([0, 1])$, the following three-point linear boundary value problem

$$u'''(t) = -y(t), \quad 0 \le t \le 1, \tag{3}$$

$$u(0) = 0, \quad u''(\eta) = \alpha u'(1), \quad u'(1) = \beta u(1), \tag{4}$$

with $0 \le \alpha \le 1$, $0 \le \beta < \frac{2}{2-\alpha}$ and $0 \le \eta \le \frac{1}{2}$

First, we prove the following result in which the sign properties of the related Green's function is deduced.

Lemma 2.1. *Let* $0 \le \alpha \le 1$, $0 \le \beta < \frac{2}{2-\alpha}$ *and* $0 \le \eta \le \frac{1}{2}$. *The Green's function G, related to problem* (3)–(4), *has the following sign properties*:

$$G(t,s) \le 0 \quad \text{and} \quad \frac{\partial}{\partial t} G(t,s) \le 0 \quad \text{for } 0 \le s \le \eta,$$
$$G(t,s) \ge 0 \quad \text{and} \quad \frac{\partial}{\partial t} G(t,s) \ge 0 \quad \text{for } \eta \le s \le 1.$$

Proof. Integrating three times the linear problem gives us that

$$u''(t) = \int_0^t -y(s)\,\mathrm{d}s + A,$$
$$u'(t) = \int_0^t (s-t)y(s)\,\mathrm{d}s + At + B,$$
$$u(t) = \int_0^t \left(st - \frac{s^2}{2} - \frac{t^2}{2}\right) y(s)\,\mathrm{d}s + A\frac{t^2}{2} + Bt + C.$$

The first condition $u(0) = 0$ implies that $C = 0$.

Next, $u''(\eta) = \alpha u'(1)$ is rewritten as

$$\int_0^\eta -y(s)\,\mathrm{d}s + A = \alpha\left(\int_0^1 (s-1)y(s)\,\mathrm{d}s + A + B\right),$$

whence

$$B = \frac{1-\alpha}{\alpha}A - \frac{1}{\alpha}\int_0^\eta y(s)\,\mathrm{d}s - \int_0^1 (s-1)y(s)\,\mathrm{d}s.$$

Then, using $u'(1) = \beta u(1)$, we obtain that

$$\int_0^1 (s-1)y(s)\,\mathrm{d}s + A + B = \beta\left(\int_0^1 \left(s - \frac{s^2}{2} - \frac{1}{2}\right) y(s)\,\mathrm{d}s + \frac{A}{2} + B\right).$$

If we substitute B with the expression from above and simplify, we get that

$$A = \frac{\alpha\beta}{2+\alpha\beta-2\beta}\int_0^1 (1-s^2)y(s)\,\mathrm{d}s + \frac{2(1-\beta)}{2+\alpha\beta-2\beta}\int_0^\eta y(s)\,\mathrm{d}s.$$

Thus,

$$\begin{aligned} B &= \frac{1-\alpha}{\alpha}A - \frac{1}{\alpha}\int_0^\eta y(s)\,\mathrm{d}s - \int_0^1 (s-1)y(s)\,\mathrm{d}s \\ &= \frac{(1-\alpha)\beta}{2+\alpha\beta-2\beta}\int_0^1 (1-s^2)y(s)\,\mathrm{d}s + \int_0^1 (1-s)y(s)\,\mathrm{d}s + \frac{2(1-\alpha)(1-\beta)}{\alpha(2+\alpha\beta-2\beta)}\int_0^\eta y(s)\,\mathrm{d}s - \frac{1}{\alpha}\int_0^\eta y(s)\,\mathrm{d}s. \end{aligned}$$

Finally, we obtain that

$$\begin{aligned} u(t) &= \int_0^t \left(st - \frac{s^2}{2} - \frac{t^2}{2}\right) y(s)\,\mathrm{d}s + A\frac{t^2}{2} + Bt \\ &= \int_0^t \left(st - \frac{s^2}{2} - \frac{t^2}{2}\right) y(s)\,\mathrm{d}s + \int_0^1 \frac{\alpha\beta(1-s^2)}{2(2+\alpha\beta-2\beta)} t^2 y(s)\,\mathrm{d}s + \int_0^\eta \frac{(1-\beta)}{2+\alpha\beta-2\beta} t^2 y(s)\,\mathrm{d}s \\ &\quad + \int_0^1 \left(\frac{2-\beta}{2+\alpha\beta-2\beta} - s - \frac{(1-\alpha)\beta}{2+\alpha\beta-2\beta} s^2\right) t\, y(s)\,\mathrm{d}s + \int_0^\eta \frac{\beta-2}{2+\alpha\beta-2\beta} t\, y(s)\,\mathrm{d}s. \end{aligned}$$

As a result, we have that

(1) if $s \geq \eta$, then

$$G(t,s) = \begin{cases} \dfrac{\alpha\beta(1-s^2)}{2(2+\alpha\beta-2\beta)} t^2 + \left(\dfrac{2-\beta}{2+\alpha\beta-2\beta} - s - \dfrac{(1-\alpha)\beta}{2+\alpha\beta-2\beta} s^2\right) t, & s \geq t, \\ \dfrac{2\beta - 2 - \alpha\beta s^2}{2(2+\alpha\beta-2\beta)} t^2 + \left(\dfrac{2-\beta}{2+\alpha\beta-2\beta} - \dfrac{(1-\alpha)\beta}{2+\alpha\beta-2\beta} s^2\right) t - \dfrac{s^2}{2}, & s \leq t, \end{cases}$$

and

(2) if $s \leq \eta$, then

$$G(t,s) = \begin{cases} \dfrac{2+\alpha\beta-2\beta-\alpha\beta s^2}{2(2+\alpha\beta-2\beta)} t^2 - \left(s + \dfrac{(1-\alpha)\beta s^2}{2+\alpha\beta-2\beta}\right) t, & s \geq t, \\ \dfrac{-\alpha\beta s^2}{2(2+\alpha\beta-2\beta)} t^2 - \dfrac{(1-\alpha)\beta s^2}{2+\alpha\beta-2\beta} t - \dfrac{s^2}{2}, & s \leq t. \end{cases}$$

Now, we will study the sign properties of function G.

First, suppose that $s \geq \eta$ and $s \geq t$. In this case, we have that

$$G(t,s) = \frac{\alpha\beta(1-s^2)}{2(2+\alpha\beta-2\beta)} t^2 + \left(\frac{2-\beta}{2+\alpha\beta-2\beta} - s - \frac{(1-\alpha)\beta}{2+\alpha\beta-2\beta} s^2\right) t,$$

which gives us that $G(0,s) = 0$ and

$$\frac{\partial}{\partial t} G(t,s) = \frac{\alpha\beta(1-s^2)}{2+\alpha\beta-2\beta} t + \frac{2-\beta}{2+\alpha\beta-2\beta} - s - \frac{(1-\alpha)\beta}{2+\alpha\beta-2\beta} s^2.$$

Thus, $\frac{\partial}{\partial t} G(t,s) \geq 0$ is equivalent to

$$\alpha\beta(1-s^2)t + 2 - \beta \geq s(2+\alpha\beta-2\beta) + (1-\alpha)\beta s^2,$$

which is the same as

$$2 - 2s \geq \beta(1 + s(\alpha-2) + (1-\alpha)s^2 - \alpha(1-s^2)t).$$

It is enough to show that

$$2 - 2s \geq \frac{2}{2-\alpha}(1 + s(\alpha-2) + (1-\alpha)s^2 - \alpha(1-s^2)t),$$

which is equivalent to

$$(1-s^2)(1+\alpha t - \alpha) \geq 0$$

and the last one clearly holds since $\alpha \leq 1$ and $s \leq 1$.

As a result, in the case when $s \geq \eta$ and $s \geq t$, we have that $\frac{\partial}{\partial t} G(t,s) \geq 0$ and $G(0,s) = 0$.

Second, suppose that $s \geq \eta$ and $s \leq t$. In this case, it is fulfilled that

$$G(t,s) = \frac{2\beta - 2 - \alpha\beta s^2}{2(2 + \alpha\beta - 2\beta)} t^2 + \left(\frac{2-\beta}{2+\alpha\beta-2\beta} - \frac{(1-\alpha)\beta}{2+\alpha\beta-2\beta} s^2 \right) t - \frac{s^2}{2},$$

which gives us that

$$\frac{\partial}{\partial t} G(t,s) = \frac{2\beta - 2 - \alpha\beta s^2}{2 + \alpha\beta - 2\beta} t + \frac{2-\beta}{2+\alpha\beta-2\beta} - \frac{(1-\alpha)\beta}{2+\alpha\beta-2\beta} s^2.$$

Thus, $\frac{\partial}{\partial t} G(t,s) \geq 0$ is equivalent to

$$(2\beta - 2 - \alpha\beta s^2)t + 2 - \beta \geq (1-\alpha)\beta s^2,$$

which is the same as

$$2 - 2t \geq \beta(1 + s^2 + \alpha s^2 t - \alpha s^2 - 2t).$$

Since $\beta < \frac{2}{2-\alpha}$, it is enough to show that

$$2 - 2t \geq \frac{2}{2-\alpha}(1 + s^2 + \alpha s^2 t - \alpha s^2 - 2t),$$

which is equivalent to

$$(1 - s^2)(1 + \alpha t - \alpha) \geq 0$$

and the last one clearly holds since $\alpha \leq 1$ and $s \leq 1$.

Moreover,

$$G(s,s) = \frac{2\beta - 2 - \alpha\beta s^2}{2(2 + \alpha\beta - 2\beta)} s^2 + \left(\frac{2-\beta}{2+\alpha\beta-2\beta} - \frac{(1-\alpha)\beta}{2+\alpha\beta-2\beta} s^2 \right) s - \frac{s^2}{2}.$$

Thus, $G(s,s) \geq 0$ is equivalent to

$$(2\beta - 2 - \alpha\beta s^2)s^2 + 2(2-\beta)s - 2(1-\alpha)\beta s^3 \geq s^2(2 + \alpha\beta - 2\beta),$$

which is the same as

$$4s(1-s) \geq \beta s(2 + \alpha s)(1-s)^2.$$

It is enough to show that

$$4 \geq \frac{2}{2-\alpha}(2 + \alpha s)(1 - s),$$

or, which is the same,

$$2 + 2s + \alpha s^2 \geq 2\alpha + \alpha s$$

and the last one holds since $2 \geq 2\alpha$ and $2s \geq \alpha s$.

As a result, in the case when $s \geq \eta$ and $s \leq t$, we have that $\frac{\partial}{\partial t} G(t,s) \geq 0$ and $G(s,s) \geq 0$.

Now, suppose that $s \leq \eta$ and $s \geq t$. In this case, we have that

$$G(t,s) = \frac{2 + \alpha\beta - 2\beta - \alpha\beta s^2}{2(2 + \alpha\beta - 2\beta)} t^2 - \left(\frac{(1-\alpha)\beta s^2}{2+\alpha\beta-2\beta} + s \right) t,$$

which gives us that $G(0,s) = 0$ and

$$\frac{\partial}{\partial t} G(t,s) = \frac{2 + \alpha\beta - 2\beta - \alpha\beta s^2}{2 + \alpha\beta - 2\beta} t - \frac{(1-\alpha)\beta s^2}{2+\alpha\beta-2\beta} - s.$$

Thus, $\frac{\partial}{\partial t} G(t,s) \leq 0$ is equivalent to

$$(2 + \alpha\beta - 2\beta - \alpha\beta s^2)t \leq (1-\alpha)\beta s^2 + s(2 + \alpha\beta - 2\beta),$$

which is the same as

$$(2+\alpha\beta-2\beta)(s-t)+\beta s^2(1+\alpha t-\alpha)\geq 0$$

and the last inequality clearly holds as $s \geq t$ and $\alpha \leq 1$.

As a result, in the case when $s \leq \eta$ and $s \geq t$, we have that $\frac{\partial}{\partial t}G(t,s) \leq 0$ and $G(0,s)=0$.

Finally, suppose that $s \leq \eta$ and $s \leq t$. In this case, we have that

$$G(t,s)=\frac{-\alpha\beta s^2}{2(2+\alpha\beta-2\beta)}t^2-\frac{(1-\alpha)\beta s^2}{2+\alpha\beta-2\beta}t-\frac{s^2}{2}.$$

Obviously, since $\alpha \leq 1$, we have that $G(t,s) \leq 0$. Moreover,

$$\frac{\partial}{\partial t}G(t,s)=\frac{-\alpha\beta s^2}{2+\alpha\beta-2\beta}t-\frac{(1-\alpha)\beta s^2}{2+\alpha\beta-2\beta}\leq 0.$$

As a result, in the case when $s \leq \eta$ and $s \leq t$, we have that $\frac{\partial}{\partial t}G(t,s) \leq 0$ and $G(s,s) \leq 0$ and the result is proved. □

As a direct consequence of previous result, for $0 \leq s \leq \eta$, we obtain that

$$\max_{t\in I}|G(t,s)|=-\min\{G(t,s)\}=-G(1,s)=\frac{s^2}{2+\alpha\beta-2\beta},$$

while, for $\eta \leq s \leq 1$

$$\max_{t\in I}|G(t,s)|=G(1,s)=\frac{1-s^2}{2+\alpha\beta-2\beta}.$$

Consequently, since $\eta \in (0,1/2]$,

$$\max_{t,s\in I}|G(t,s)|=\frac{\max\{\eta^2,1-\eta^2\}}{2+\alpha\beta-2\beta}=\frac{1-\eta^2}{2+\alpha\beta-2\beta}. \tag{5}$$

Remark 2.2. We point out that, under the conditions for α and β, we have that $2+\alpha\beta-2\beta>0$. Moreover, if $\alpha=\beta=0$, we obtain the expression of Green's function given in [8].

Lemma 2.3. *Let $G(t,s)$ be Green's function defined in* Lemma 2.1. *Then,*

$$\max_{t,s\in I}\left|\frac{\partial}{\partial t}G(t,s)\right|\leq\frac{2+\alpha\beta-\beta}{2+\alpha\beta-2\beta}. \tag{6}$$

Proof. A direct computation gives us that if $s \geq \eta$, then

$$\left|\frac{\partial}{\partial t}G(t,s)\right|=\begin{cases}\dfrac{\alpha\beta t-2s-\alpha\beta s+2\beta s+2-\beta-\beta s^2(1+\alpha t-\alpha)}{2+\alpha\beta-2\beta}, & s\geq t,\\[2mm] \dfrac{2\beta t-2t+2-\beta-\beta s^2(1+\alpha t-\alpha)}{2+\alpha\beta-2\beta}, & s\leq t\end{cases}$$

and

(2) if $s \leq \eta$, then

$$\left|\frac{\partial}{\partial t}G(t,s)\right|=\begin{cases}\dfrac{(2+\alpha\beta-2\beta)(s-t)+\beta s^2(1+\alpha t-\alpha)}{2+\alpha\beta-2\beta}, & s\geq t,\\[2mm] \dfrac{\beta s^2(1+\alpha t-\alpha)}{2+\alpha\beta-2\beta}, & s\leq t.\end{cases}$$

Thus, if $s \geq \eta$ and $s \geq t$, then (6) is equivalent to

$$\beta s^2(1+\alpha t-\alpha)+\alpha\beta(1+s-t)+2s(1-\beta)\ge 0,$$

which holds.

If $s \ge \eta$ and $s \le t$, then (6) is the same as

$$\alpha\beta+\beta s^2(1+\alpha t-\alpha)+2t(1-\beta)\ge 0,$$

which is true.

If $s \le \eta$ and $s \ge t$, then we have

$$\left|\frac{\partial}{\partial t}G(t,s)\right| = \frac{(2+\alpha\beta-2\beta)(s-t)+\beta s^2(1+\alpha t-\alpha)}{2+\alpha\beta-2\beta} \le \frac{2+\alpha\beta-2\beta+\beta}{2+\alpha\beta-2\beta} = \frac{2+\alpha\beta-\beta}{2+\alpha\beta-2\beta}.$$

Finally, if $s \le \eta$ and $s \le t$, then

$$\left|\frac{\partial}{\partial t}G(t,s)\right| = \frac{\beta s^2(1+\alpha t-\alpha)}{2+\alpha\beta-2\beta} \le \frac{\beta}{2+\alpha\beta-2\beta} \le \frac{2+\alpha\beta-\beta}{2+\alpha\beta-2\beta}.$$ □

On the other hand, we have that, for $s \ge \eta$,

$$\frac{\partial^2}{\partial t^2}G(t,s) = \begin{cases} \dfrac{\alpha\beta(1-s^2)}{2+\alpha\beta-2\beta}, & s>t, \\ \dfrac{2\beta-2-\alpha\beta s^2}{2+\alpha\beta-2\beta}, & s<t \end{cases}$$

and for $s \le \eta$,

$$\frac{\partial^2}{\partial t^2}G(t,s) = \begin{cases} \dfrac{2+\alpha\beta-2\beta-\alpha\beta s^2}{2+\alpha\beta-2\beta}, & s>t, \\ \dfrac{-\alpha\beta s^2}{2+\alpha\beta-2\beta}, & s<t. \end{cases}$$

So, as a direct consequence, we deduce the following result

Lemma 2.4. *Let* $0 \le \alpha \le 1$, $0 \le \beta < \frac{2}{2-\alpha}$ *and* $0 \le \eta \le \frac{1}{2}$. *Then, Green's function* G, *related to problem* (3)–(4), *satisfies*

$$\frac{\partial^2}{\partial t^2}G(t,s) \le 0 \quad \textit{for all } s<t$$

and

$$\frac{\partial^2}{\partial t^2}G(t,s) \ge 0 \quad \textit{for all } s>t.$$

Now, define the cone

$$K := \{y \in C^1(I) :\ y(t) \ge 0,\ \ y'(t) \ge 0,\ \ t \in I\}. \tag{7}$$

Lemma 2.5. *Let* $0 \le \alpha \le 1$, $0 \le \beta < \frac{2}{2-\alpha}$, $0 \le \eta \le \frac{1}{2}$ *and* G *be the related Green's function to problem* (3)–(4). *Let* $y \in K$. *Then the unique solution of the linear boundary value problem* (3)–(4) *is such that*

$$u(t) = \int_0^1 G(t,s)y(s)\,ds \in K.$$

Moreover, $u \in C^2(I)$ *and* $u''(t) \ge 0$ *for all* $t \in [0,\eta]$.

Proof. First, we will show that $u(t) \geq 0$ on I.

If $t \leq \eta$, using that $G(t,s) \leq 0$ for $0 \leq s \leq \eta$ and $G(t,s) \geq 0$ for $\eta \leq s \leq 1$, we have

$$
\begin{aligned}
u(t) &= \int_0^1 G(t,s)y(s)\mathrm{d}s = \int_0^t G(t,s)y(s)\mathrm{d}s + \int_t^\eta G(t,s)y(s)\mathrm{d}s + \int_\eta^1 G(t,s)y(s)\mathrm{d}s \\
&= \int_0^t \left(\frac{-\alpha\beta s^2}{2(2+\alpha\beta-2\beta)}t^2 - \frac{(1-\alpha)\beta s^2}{2+\alpha\beta-2\beta}t - \frac{s^2}{2}\right)y(s)\mathrm{d}s \\
&\quad + \int_t^\eta \left(\frac{2+\alpha\beta-2\beta-\alpha\beta s^2}{2(2+\alpha\beta-2\beta)}t^2 - \left(s + \frac{(1-\alpha)\beta s^2}{2+\alpha\beta-2\beta}\right)t\right)y(s)\mathrm{d}s \\
&\quad + \int_\eta^1 \left(\frac{\alpha\beta(1-s^2)}{2(2+\alpha\beta-2\beta)}t^2 + \left(\frac{2-\beta}{2+\alpha\beta-2\beta} - s - \frac{(1-\alpha)\beta}{2+\alpha\beta-2\beta}s^2\right)t\right)y(s)\mathrm{d}s \\
&\geq \max_{0\leq s\leq t} y(s) \int_0^t \left(\frac{-\alpha\beta s^2}{2(2+\alpha\beta-2\beta)}t^2 - \frac{(1-\alpha)\beta s^2}{2+\alpha\beta-2\beta}t - \frac{s^2}{2}\right)\mathrm{d}s \\
&\quad + \max_{t\leq s\leq \eta} y(s) \int_t^\eta \left(\frac{2+\alpha\beta-2\beta-\alpha\beta s^2}{2(2+\alpha\beta-2\beta)}t^2 - \left(s + \frac{(1-\alpha)\beta s^2}{2+\alpha\beta-2\beta}\right)t\right)\mathrm{d}s \\
&\quad + \min_{\eta\leq s\leq 1} y(s) \int_\eta^1 \left(\frac{\alpha\beta(1-s^2)}{2(2+\alpha\beta-2\beta)}t^2 + \left(\frac{2-\beta}{2+\alpha\beta-2\beta} - s - \frac{(1-\alpha)\beta}{2+\alpha\beta-2\beta}s^2\right)t\right)\mathrm{d}s \\
&= y(t) \int_0^t \left(\frac{-\alpha\beta s^2}{2(2+\alpha\beta-2\beta)}t^2 - \frac{(1-\alpha)\beta s^2}{2+\alpha\beta-2\beta}t - \frac{s^2}{2}\right)\mathrm{d}s \\
&\quad + y(\eta) \int_t^\eta \left(\frac{2+\alpha\beta-2\beta-\alpha\beta s^2}{2(2+\alpha\beta-2\beta)}t^2 - \left(s + \frac{(1-\alpha)\beta s^2}{2+\alpha\beta-2\beta}\right)t\right)\mathrm{d}s \\
&\quad + y(\eta) \int_\eta^1 \left(\frac{\alpha\beta(1-s^2)}{2(2+\alpha\beta-2\beta)}t^2 + \left(\frac{2-\beta}{2+\alpha\beta-2\beta} - s - \frac{(1-\alpha)\beta}{2+\alpha\beta-2\beta}s^2\right)t\right)\mathrm{d}s \\
&\geq y(\eta)\left(-\frac{1}{6}t\frac{-6+\alpha\beta-6\beta\eta+6\eta\beta t+2\beta-2\beta t\alpha-6t\eta+12\eta+\alpha\beta t^2+2t^2-2\beta t^2}{2+\alpha\beta-2\beta}\right) \geq 0.
\end{aligned}
$$

The last one holds since

$$-6+\alpha\beta-6\beta\eta+6\eta\beta t+2\beta-2\beta t\alpha-6t\eta+12\eta+\alpha\beta t^2+2t^2-2\beta t^2 \leq 0$$

is equivalent to

$$6-12\eta-2t^2+6t\eta \geq \beta(\alpha-6\eta+6\eta t+2-2t\alpha+\alpha t^2-2t^2).$$

Using that $\beta < \frac{2}{2-\alpha}$, we only need to show that

$$(3-6\eta-t^2+3t\eta)(2-\alpha) = 6-3\alpha-12\eta+6\alpha\eta-2t^2+\alpha t^2+6\eta t-3\eta t\alpha \geq \alpha-6\eta+6\eta t+2-2t\alpha+\alpha t^2-2t^2,$$

which is the same as

$$(2-3\eta)(2(1-\alpha)+t\alpha) \geq 0$$

and this clearly holds since $\eta \leq \frac{1}{2}$ and $\alpha \leq 1$.

Now, if $t \geq \eta$, using again that $G(t,s) \leq 0$ for $0 \leq s \leq \eta$ and $G(t,s) \geq 0$ for $\eta \leq s \leq 1$, we have

$$\begin{aligned}
u(t) &= \int_0^1 G(t,s)\,\mathrm{d}s = \int_0^\eta G(t,s)y(s)\,\mathrm{d}s + \int_\eta^t G(t,s)y(s)\,\mathrm{d}s + \int_t^1 G(t,s)y(s)\,\mathrm{d}s \\
&= \int_0^\eta \left(\frac{-\alpha\beta s^2}{2(2+\alpha\beta-2\beta)}t^2 - \frac{(1-\alpha)\beta s^2}{2+\alpha\beta-2\beta}t - \frac{s^2}{2}\right)y(s)\,\mathrm{d}s \\
&\quad + \int_\eta^t \left(\frac{2\beta-2-\alpha\beta s^2}{2(2+\alpha\beta-2\beta)}t^2 + \left(\frac{2-\beta}{2+\alpha\beta-2\beta} - \frac{(1-\alpha)\beta}{2+\alpha\beta-2\beta}s^2\right)t - \frac{s^2}{2}\right)y(s)\,\mathrm{d}s \\
&\quad + \int_t^1 \left(\frac{\alpha\beta(1-s^2)}{2(2+\alpha\beta-2\beta)}t^2 + \left(\frac{2-\beta}{2+\alpha\beta-2\beta} - s - \frac{(1-\alpha)\beta}{2+\alpha\beta-2\beta}s^2\right)t\right)y(s)\,\mathrm{d}s \\
&\geq \max_{0\leq s\leq\eta} y(s) \int_0^\eta \left(\frac{-\alpha\beta s^2}{2(2+\alpha\beta-2\beta)}t^2 - \frac{(1-\alpha)\beta s^2}{2+\alpha\beta-2\beta}t - \frac{s^2}{2}\right)\mathrm{d}s \\
&\quad + \min_{\eta\leq s\leq t} y(s) \int_\eta^t \left(\frac{2\beta-2-\alpha\beta s^2}{2(2+\alpha\beta-2\beta)}t^2 + \left(\frac{2-\beta}{2+\alpha\beta-2\beta} - \frac{(1-\alpha)\beta}{2+\alpha\beta-2\beta}s^2\right)t - \frac{s^2}{2}\right)\mathrm{d}s \\
&\quad + \min_{t\leq s\leq 1} y(s) \int_t^1 \left(\frac{\alpha\beta(1-s^2)}{2(2+\alpha\beta-2\beta)}t^2 + \left(\frac{2-\beta}{2+\alpha\beta-2\beta} - s - \frac{(1-\alpha)\beta}{2+\alpha\beta-2\beta}s^2\right)t\right)\mathrm{d}s \\
&= y(\eta) \int_0^\eta \left(\frac{-\alpha\beta s^2}{2(2+\alpha\beta-2\beta)}t^2 - \frac{(1-\alpha)\beta s^2}{2+\alpha\beta-2\beta}t - \frac{s^2}{2}\right)\mathrm{d}s \\
&\quad + y(\eta) \int_\eta^t \left(\frac{2\beta-2-\alpha\beta s^2}{2(2+\alpha\beta-2\beta)}t^2 + \left(\frac{2-\beta}{2+\alpha\beta-2\beta} - \frac{(1-\alpha)\beta}{2+\alpha\beta-2\beta}s^2\right)t - \frac{s^2}{2}\right)\mathrm{d}s \\
&\quad + y(t) \int_t^1 \left(\frac{\alpha\beta(1-s^2)}{2(2+\alpha\beta-2\beta)}t^2 + \left(\frac{2-\beta}{2+\alpha\beta-2\beta} - s - \frac{(1-\alpha)\beta}{2+\alpha\beta-2\beta}s^2\right)t\right)\mathrm{d}s \\
&\geq y(\eta)\left(-\frac{1}{6}t\frac{-6+\alpha\beta-6\beta\eta+6\eta\beta t+2\beta-2\beta t\alpha-6t\eta+12\eta+\alpha\beta t^2+2t^2-2\beta t^2}{2+\alpha\beta-2\beta}\right) \geq 0.
\end{aligned}$$

Next, we will show that $u'(t) \geq 0$ on I.
If $t \leq \eta$, since $\frac{\partial}{\partial t}G(t,s) \leq 0$ for $0 \leq s \leq \eta$ and $\frac{\partial}{\partial t}G(t,s) \geq 0$ for $\eta \leq s \leq 1$, we have

$$\begin{aligned}
u'(t) &= \int_0^t -\frac{\beta(1+\alpha t-\alpha)}{2+\alpha\beta-2\beta}s^2 y(s)\,\mathrm{d}s + \int_t^\eta \left(t - s - \frac{\beta(1+\alpha t-\alpha)}{2+\alpha\beta-2\beta}s^2\right)y(s)\,\mathrm{d}s \\
&\quad + \int_\eta^1 \left(\frac{2+\alpha\beta t-\beta}{2+\alpha\beta-2\beta} - s - \frac{\beta(1+\alpha t-\alpha)}{2+\alpha\beta-2\beta}s^2\right)y(s)\,\mathrm{d}s \\
&\geq \max_{0\leq s\leq t} y(s) \int_0^t -\frac{\beta(1+\alpha t-\alpha)}{2+\alpha\beta-2\beta}s^2\,\mathrm{d}s + \max_{t\leq s\leq\eta} y(s) \int_t^\eta \left(t - s - \frac{\beta(1+\alpha t-\alpha)}{2+\alpha\beta-2\beta}s^2\right)\mathrm{d}s \\
&\quad + \min_{\eta\leq s\leq 1} y(s) \int_\eta^1 \left(\frac{2+\alpha\beta t-\beta}{2+\alpha\beta-2\beta} - s - \frac{\beta(1+\alpha t-\alpha)}{2+\alpha\beta-2\beta}s^2\right)\mathrm{d}s
\end{aligned}$$

$$= y(t)\int_0^t -\frac{\beta(1+\alpha t-\alpha)}{2+\alpha\beta-2\beta}s^2\,\mathrm{d}s + y(\eta)\int_t^\eta\left(t-s-\frac{\beta(1+\alpha t-\alpha)}{2+\alpha\beta-2\beta}s^2\right)\mathrm{d}s$$
$$+ y(\eta)\int_\eta^1\left(\frac{2+\alpha\beta t-\beta}{2+\alpha\beta-2\beta}-s-\frac{\beta(1+\alpha t-\alpha)}{2+\alpha\beta-2\beta}s^2\right)\mathrm{d}s$$
$$\geq y(\eta)\left(t\eta-\frac{t^2}{2}+\frac{2+\alpha\beta t-\beta}{2+\alpha\beta-2\beta}(1-\eta)-\frac{1}{2}-\frac{\beta(1+\alpha t-\alpha)}{2+\alpha\beta-2\beta}\frac{1}{3}\right)\geq 0.$$

The last inequality holds since it is equivalent to

$$6(1-t)(1+t-2\eta)\geq\beta(2+\alpha+12t\eta+3\alpha t^2-4\alpha t-6\eta-6t^2),$$

which is satisfied since $\beta\leq\frac{2}{2-\alpha}$ and

$$3(1-t)(1+t-2\eta)(2-\alpha)\geq 2+\alpha+12t\eta+3\alpha t^2-4\alpha t-6\eta-6t^2$$

is equivalent to

$$(1+\alpha t-\alpha)(2-3\eta)\geq 0.$$

If $\eta\leq t$, since $\frac{\partial}{\partial t}G(t,s)\leq 0$ for $0\leq s\leq\eta$ and $\frac{\partial}{\partial t}G(t,s)\geq 0$ for $\eta\leq s\leq 1$ we have

$$u'(t)=\int_0^\eta -\frac{\beta(1+\alpha t-\alpha)}{2+\alpha\beta-2\beta}s^2y(s)\mathrm{d}s+\int_\eta^t\left(\frac{2+2\beta t-2t-\beta}{2+\alpha\beta-2\beta}-\frac{\beta(1+\alpha t-\alpha)}{2+\alpha\beta-2\beta}s^2\right)y(s)\mathrm{d}s$$
$$+\int_t^1\left(\frac{2+\alpha\beta t-\beta}{2+\alpha\beta-2\beta}-s-\frac{\beta(1+\alpha t-\alpha)}{2+\alpha\beta-2\beta}s^2\right)y(s)\mathrm{d}s$$
$$\geq\max_{0\leq s\leq\eta}y(s)\int_0^\eta -\frac{\beta(1+\alpha t-\alpha)}{2+\alpha\beta-2\beta}s^2\,\mathrm{d}s+\min_{\eta\leq s\leq t}y(s)\int_\eta^t\left(\frac{2+2\beta t-2t-\beta}{2+\alpha\beta-2\beta}-\frac{\beta(1+\alpha t-\alpha)}{2+\alpha\beta-2\beta}s^2\right)\mathrm{d}s$$
$$+\min_{t\leq s\leq 1}y(s)\int_t^1\left(\frac{2+\alpha\beta t-\beta}{2+\alpha\beta-2\beta}-s-\frac{\beta(1+\alpha t-\alpha)}{2+\alpha\beta-2\beta}s^2\right)\mathrm{d}s$$
$$=y(\eta)\int_0^\eta -\frac{\beta(1+\alpha t-\alpha)}{2+\alpha\beta-2\beta}s^2\mathrm{d}s+y(\eta)\int_\eta^t\left(\frac{2+2\beta t-2t-\beta}{2+\alpha\beta-2\beta}-\frac{\beta(1+\alpha t-\alpha)}{2+\alpha\beta-2\beta}s^2\right)\mathrm{d}s$$
$$+y(t)\int_t^1\left(\frac{2+\alpha\beta t-\beta}{2+\alpha\beta-2\beta}-s-\frac{\beta(1+\alpha t-\alpha)}{2+\alpha\beta-2\beta}s^2\right)\mathrm{d}s$$
$$\geq y(\eta)\left(\frac{2+2\beta t-2t-\beta}{2+\alpha\beta-2\beta}(t-\eta)+\frac{2+\alpha\beta t-\beta}{2+\alpha\beta-2\beta}(1-t)-\frac{1}{2}+\frac{t^2}{2}-\frac{\beta(1+\alpha t-\alpha)}{2+\alpha\beta-2\beta}\frac{1}{3}\right)\geq 0.$$

The last inequality holds since it is equivalent to

$$6(1-t)(1+t-2\eta)\geq\beta(2+\alpha+12t\eta+3\alpha t^2-4\alpha t-6\eta-6t^2)$$

and we already showed above that it is true.

As a result, we have that $u(0)=0$, $u(t)\geq 0$ and $u'(t)\geq 0$ for all $t\in I$.

Similarly as above, for $0\leq t\leq\eta$, since

$$2(\eta-t)\geq\frac{2}{2-\alpha}\left(2((\eta-t))+\alpha t-\frac{2}{3}\alpha\right)\geq\beta\left(2((\eta-t))+\alpha t-\frac{2}{3}\alpha\right),$$

we conclude that

$$
\begin{aligned}
u''(t) &= \int_0^1 \frac{\partial^2}{\partial t^2} G(t,s) y(s)\,\mathrm{d}s \\
&= \int_0^t \frac{-\alpha\beta s^2}{2+\alpha\beta-2\beta} y(s)\,\mathrm{d}s + \int_t^\eta \frac{2+\alpha\beta-2\beta-\alpha\beta s^2}{2+\alpha\beta-2\beta} y(s)\,\mathrm{d}s + \int_\eta^1 \frac{\alpha\beta(1-s^2)}{2+\alpha\beta-2\beta} y(s)\,\mathrm{d}s \\
&\geq \max_{0\leq s\leq t} y(s) \int_0^t \frac{-\alpha\beta s^2}{2+\alpha\beta-2\beta}\mathrm{d}s + \min_{t\leq s\leq \eta} \int_t^\eta \frac{2+\alpha\beta-2\beta-\alpha\beta s^2}{2+\alpha\beta-2\beta}\mathrm{d}s + \min_{\eta\leq s\leq 1} \int_\eta^1 \frac{\alpha\beta(1-s^2)}{2+\alpha\beta-2\beta}\mathrm{d}s \\
&= y(t) \int_0^t \frac{-\alpha\beta s^2}{2+\alpha\beta-2\beta}\mathrm{d}s + y(t) \int_t^\eta \frac{2+\alpha\beta-2\beta-\alpha\beta s^2}{2+\alpha\beta-2\beta}\mathrm{d}s + y(\eta) \int_\eta^1 \frac{\alpha\beta(1-s^2)}{2+\alpha\beta-2\beta}\mathrm{d}s \\
&\geq y(t)\left(\int_0^t \frac{-\alpha\beta s^2}{2+\alpha\beta-2\beta}\mathrm{d}s + \int_t^\eta \frac{2+\alpha\beta-2\beta-\alpha\beta s^2}{2+\alpha\beta-2\beta}\mathrm{d}s + \int_\eta^1 \frac{\alpha\beta(1-s^2)}{2+\alpha\beta-2\beta}\mathrm{d}s \right) \\
&= \frac{y(t)}{2+\alpha\beta-2\beta}\left(2\eta - 2t - 2\beta\eta + 2\beta t - \alpha\beta t + \frac{2}{3}\alpha\beta\right) \geq 0.
\end{aligned}
$$

□

In order to deduce existence results for the nonlinear problem, we will work in more restrictive cones. To this end, from the fact that

$$
\frac{\partial}{\partial s}\left(\frac{G(t,s)}{G(1,s)}\right) = \begin{cases} -\dfrac{t((\alpha-2)\beta+2)}{(s+1)^2} & \text{if } s>t \text{ and } s>\eta, \\ -\dfrac{t((\alpha-2)\beta+2)(s-t)}{s^3} & \text{if } s>t \text{ and } s<\eta, \\ -\dfrac{s(t-1)^2((\alpha-2)\beta+2)}{(s^2-1)^2} & \text{if } s<t \text{ and } s>\eta, \\ 0 & \text{if } s<t \text{ and } s<\eta, \end{cases}
$$

we deduce that $\frac{\partial}{\partial s}\left(\frac{G(t,s)}{G(1,s)}\right) \leq 0$ for all $t, s \in I$, $0 \leq \alpha \leq 1$, $0 \leq \beta < \frac{2}{2-\alpha}$ and $0 \leq \eta \leq \frac{1}{2}$.

As a direct consequence,

Lemma 2.6. *Let* $0 \leq \alpha \leq 1$, $0 \leq \beta < \frac{2}{2-\alpha}$ *and* $0 \leq \eta < \frac{1}{2}$ *and* G *be the related Green's function to problem* (3)–(4). Then, for all $(t,s) \in (0,1] \times (0,1)$ *the following inequalities are fulfilled*:

$$
\frac{G(t,s)}{G(1,s)} \leq \lim_{s\to 0^+} \frac{G(t,s)}{G(1,s)} \leq \frac{1}{2}\beta(t-1)(\alpha(t-1)+2)+1 \leq 1 \tag{8}
$$

and

$$
\frac{G(t,s)}{G(1,s)} \geq \lim_{s\to 1^-} \frac{G(t,s)}{G(1,s)} = \frac{1}{2}\alpha\beta(t-1)t + t. \tag{9}
$$

Now, by defining $g(t) := \frac{1}{2}\alpha\beta(t-1)t + t$ and $H(t,s) := G(t,s) - g(t)\,G(1,s)$, we deduce, from Lemmas 2.1 and 2.6 the following direct consequence.

Corollary 2.7. *Let* $0 \leq \alpha \leq 1$, $0 \leq \beta < \frac{2}{2-\alpha}$ *and* $0 \leq \eta \leq \frac{1}{2}$. *Then*

$$
\begin{aligned}
G(t,s) &\leq g(t)G(1,s) \quad \textit{for } 0 \leq s \leq \eta, \\
G(t,s) &\geq g(t)G(1,s) \quad \textit{for } \eta \leq s \leq 1.
\end{aligned}
$$

Thus, by denoting for any $y \in C(I)$

$$\|y\|_\infty = \max\{|y(t)|,\ t \in I\},$$

we are in a position to introduce a more restrictive cone than K, defined in (7), as follows:

$$K_0 := \{y \in C^1(I) : y \in K,\ y(t) \geq g(t)\|y\|_\infty,\ t \in I\}. \tag{10}$$

So, we assume that the solution of problem (3)–(4) belongs to the previous cone, when η is in the more restrictive interval $[0, 1/3]$. The result is the following.

Lemma 2.8. *Let* $0 \leq \alpha \leq 1$, $0 \leq \beta < \frac{2}{2-\alpha}$, $0 \leq \eta \leq \frac{1}{3}$ *and G be the related Green's function to problem* (3)–(4). *Let* $y \in K_0$. *Then the unique solution of the linear boundary value problem* (3)–(4) *is such that*

$$u(t) = \int_0^1 G(t,s)y(s)\,ds \in K_0.$$

Proof. As it has been proved in Lemma 2.5, we know that $u \in K$. Let us see that $u(t) \geq g(t)\|u\|_\infty$ for all $t \in I$.

To this end, we must use the expression

$$H(t,s) := G(t,s) - g(t)G(1,s) = \begin{cases} \frac{1}{2}s^2(t-1), & s < t \text{ and } s < \eta, \\ \frac{1}{2}(s-1)^2 t, & s \geq t \text{ and } s \geq \eta, \\ \frac{1}{2}(t-1)(s^2-t), & s < t \text{ and } s \geq \eta, \\ \frac{1}{2}t((s-2)s+t), & s \geq t \text{ and } s < \eta. \end{cases}$$

If $t \leq \eta$, using Corollary 2.7 and arguing as in the proof of Lemma 2.5, we have that

$$\begin{aligned} u(t) - g(t)\|u\|_\infty &= u(t) - g(t)u(1) = \int_0^1 H(t,s)y(s)\,ds \\ &= \int_0^t H(t,s)y(s)\,ds + \int_t^\eta H(t,s)y(s)\,ds + \int_\eta^1 H(t,s)y(s)\,ds \\ &= \int_0^t \frac{1}{2}s^2(t-1)y(s)\,ds + \int_t^\eta \frac{1}{2}t((s-2)s+t)y(s)\,ds + \int_\eta^1 \frac{1}{2}(s-1)^2 t y(s)\,ds \\ &\geq y(t)\int_0^t \frac{1}{2}s^2(t-1)\,ds + y(\eta)\int_t^\eta \frac{1}{2}t((s-2)s+t)\,ds + y(\eta)\int_\eta^1 \frac{1}{2}(s-1)^2 t\,ds \\ &\geq y(\eta)\frac{1}{6}(1-t)t(t+1-3\eta), \end{aligned}$$

which is nonnegative on $[0, \eta]$ if and only if $\eta \leq 1/3$.

Analogously, when $t \geq \eta$, using Corollary 2.7 and arguing as in the proof of Lemma 2.5, we have that

$$\begin{aligned} u(t) - g(t)\|u\|_\infty &= \int_0^\eta H(t,s)y(s)\,ds + \int_\eta^t H(t,s)y(s)\,ds + \int_t^1 H(t,s)y(s)\,ds \\ &= \int_0^\eta \frac{1}{2}s^2(t-1)y(s)\,ds + \int_\eta^t \frac{1}{2}(t-1)(s^2-t)y(s)\,ds + \int_t^1 \frac{1}{2}(s-1)^2 t y(s)\,ds \end{aligned}$$

$$\geq y(\eta)\int_0^{\eta}\frac{1}{2}s^2(t-1)\,\mathrm{d}s + y(\eta)\int_{\eta}^{t}\frac{1}{2}t((s-2)s+t)\,\mathrm{d}s + y(t)\int_t^1\frac{1}{2}(s-1)^2 t\,\mathrm{d}s$$
$$\geq y(\eta)\frac{1}{6}(1-t)t(t+1-3\eta),$$

which is nonnegative on $[\eta, 1]$ if and only if $\eta \leq 1/3$. □

Remark 2.9. Notice that Lemmas 2.5 and 2.8 remain valid for any $y \in C(I)$, which is nonnegative and monotone nondecreasing in I.

Now, define $h(t) := 1 + \alpha(t-1)$ and

$$H_2(t,s) := \frac{\partial}{\partial t}G(t,s) - h(t)\frac{\partial}{\partial t}G(1,s) = \begin{cases} 0, & s < t \text{ and } s < \eta, \\ 1-s, & s \geq t \text{ and } s \geq \eta, \\ 1-t, & s < t \text{ and } s \geq \eta, \\ t-s, & s \geq t \text{ and } s < \eta. \end{cases}$$

So, it is obvious that

Corollary 2.10. *Let* $0 \leq \alpha \leq 1$, $0 \leq \beta < \frac{2}{2-\alpha}$ *and* $0 \leq \eta \leq \frac{1}{2}$. *Then*

$$\frac{\partial}{\partial t}G(t,s) \leq h(t)\frac{\partial}{\partial t}G(1,s) \quad \textit{for } 0 \leq s \leq \eta,$$
$$\frac{\partial}{\partial t}G(t,s) \geq h(t)\frac{\partial}{\partial t}G(1,s) \quad \textit{for } \eta \leq s \leq 1.$$

Let us see that u, the unique solution of problem (3)–(4), is such that $u'(t) \geq h(t)u'(1)$ for all $t \in I$.

If $t \leq \eta$, using Corollary 2.10 and arguing as in the proof of Lemma 2.5, we have that

$$\begin{aligned} u'(t) - h(t)u'(1) &= \int_0^1 H_2(t,s)y(s)\,\mathrm{d}s \\ &= \int_0^t H_2(t,s)y(s)\,\mathrm{d}s + \int_t^{\eta} H_2(t,s)y(s)\,\mathrm{d}s + \int_{\eta}^1 H_2(t,s)y(s)\,\mathrm{d}s \\ &= \int_t^{\eta}(t-s)y(s)\,\mathrm{d}s + \int_{\eta}^1(1-s)y(s)\,\mathrm{d}s \\ &\geq y(\eta)\int_t^{\eta}(t-s)\,\mathrm{d}s + y(\eta)\int_{\eta}^1(1-s)\,\mathrm{d}s \\ &= y(\eta)\left(\frac{1}{2}(1-t)(-2\eta+t+1)\right), \end{aligned}$$

which is nonnegative on $[0, \eta]$ if and only $0 < \eta \leq 1/2$.

Analogously, when $t \geq \eta$, using Corollary 2.10 and arguing as in the proof of Lemma 2.5, we have that

$$\begin{aligned} u'(t) - h(t)u'(1) &= \int_0^{\eta} H_2(t,s)y(s)\,\mathrm{d}s + \int_{\eta}^{t} H_2(t,s)y(s)\,\mathrm{d}s + \int_t^1 H_2(t,s)y(s)\,\mathrm{d}s \\ &= \int_{\eta}^{t}(1-t)y(s)\,\mathrm{d}s + \int_t^1(1-s)y(s)\,\mathrm{d}s \end{aligned}$$

$$\geq y(\eta)\int_{\eta}^{t}(1-t)\,\mathrm{d}s + y(t)\int_{t}^{1}(1-s)\,\mathrm{d}s$$
$$\geq y(\eta)\left(\frac{1}{2}(1-t)(-2\eta+t+1)\right),$$

which is nonnegative on $[\eta, 1]$ if and only $0 < \eta \leq 1/2$.

3 Existence results

In this section, using the fixed point index theory, we will give some sufficient conditions that will ensure the existence of positive solutions of problem (1)–(2). We follow some ideas developed on [10].

Let us consider the Banach space $E = C^1(I)$ equipped with the norm

$$\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty\}.$$

Taking into account the properties satisfied by Green's function and its derivatives, we define the cone K_1 in E as follows:

$$K_1 := \{y \in C^1(I) : y \in K_0,\ \ y'(t) \geq h(t)y'(1),\ \ t \in I\}, \tag{11}$$

with K_0 defined in (10).

Moreover, it is well known that the solutions of problem (1)–(2) correspond with the fixed points of the integral operator

$$Tu(t) = \lambda\int_0^1 G(t,s)p(s)f(u(s))\,\mathrm{d}s, \quad t \in I. \tag{12}$$

Lemma 3.1. T: $K_1 \to K_1$ *is a completely continuous operator.*

Proof. Let $u \in K_1$. From the sign conditions on function p assumed in (F) and Lemma 2.1 we obtain that $Tu(t) \geq 0$, $(Tu)'(t) \geq 0$ for all $t \in I$.

Moreover, using Lemma 2.6 and condition (F), we have that

$$g(t)p(s)G(1,s) \leq p(s)G(t,s) \leq p(s)G(1,s), \quad \text{for a.e. } t, s \in I. \tag{13}$$

Thus, for $t \in I$, we have that

$$\begin{aligned} Tu(t) &= \lambda\int_0^1 G(t,s)p(s)f(u(s))\,\mathrm{d}s \\ &\geq \lambda\int_0^1 g(t)p(s)G(1,s)f(u(s))\,\mathrm{d}s \\ &\geq \lambda\int_0^1 g(t)\sup_{t\in I}\{p(s)G(t,s)\}f(u(s))\,\mathrm{d}s \\ &\geq \lambda g(t)\sup_{t\in I}\left\{\int_0^1 p(s)G(t,s)f(u(s))\,\mathrm{d}s\right\} = g(t)\|Tu\|_\infty. \end{aligned}$$

Using Corollary 2.10 and condition (F), we have that

$$h(t)p(s)\frac{\partial}{\partial t}G(1,s) \leq p(s)\frac{\partial}{\partial t}G(t,s), \quad \text{for a.e. } t, s \in I. \tag{14}$$

Thus, for $t \in I$, we have that

$$(Tu)'(t) = \lambda \int_0^1 \frac{\partial}{\partial t} G(t,s)p(s)f(u(s))\,\mathrm{d}s \geq \lambda \int_0^1 h(t)p(s)\frac{\partial}{\partial t}G(1,s)f(u(s))\,\mathrm{d}s = h(t)(Tu)'(1).$$

As a result we obtain that $Tu \in K_1$.

Now, we will show that T is a compact operator. Let us consider a bounded set $B \subset \{u \in E : \|u\| \leq r\}$. First, we will prove that $T(B)$ is uniformly bounded in $C^1(I)$.

First, notice that, since p is a L^∞-function, there is a constant $D_1 > 0$ such that $0 \leq |p(t)|f(u(t)) \leq D_1$ for a.e. $t \in I$ and all $u \in B$.

Thus, from (5), for $u \in B$, we have that

$$\|Tu\|_\infty = \sup_{t\in[0,1]} \left| \lambda \int_0^1 G(t,s)p(s)f(u(s))\,\mathrm{d}s \right| \leq \lambda \int_0^1 \max_{t,s\in I}|G(t,s)||p(s)|f(u(s))\,\mathrm{d}s \leq \lambda \frac{1-\eta^2}{2+\alpha\beta-2\beta} D_1 := C_1,$$

and from 6 it follows that

$$\|(Tu)'\|_\infty = \sup_{t\in[0,1]} \left| \lambda \int_0^1 \frac{\partial}{\partial t}G(t,s)p(s)f(u(s))\,\mathrm{d}s \right| \leq \lambda \int_0^1 \max_{t,s\in I} \left| \frac{\partial}{\partial t}G(t,s) \right| |p(s)|f(u(s))\,\mathrm{d}s \leq \lambda \frac{2+\alpha\beta-\beta}{2+\alpha\beta-2\beta} D_1 := C_2.$$

Thus, $\|Tu\| \leq \max\{C_1, C_2\}$ for all $u \in B$.

Now, we will prove that $T(B)$ is equicontinuous in $C^1(I)$. Let $t_1, t_2 \in I$. Without loss of generality, suppose that $t_1 < t_2$. Then,

$$\begin{aligned} |Tu(t_1) - Tu(t_2)| &\leq \lambda \int_0^1 |G(t_1,s) - G(t_2,s)||p(s)|f(u(s))\,\mathrm{d}s \\ &\leq \lambda D_1 \int_0^1 |G(t_1,s) - G(t_2,s)|\,\mathrm{d}s \end{aligned}$$

and since $G(\cdot, s)$ is continuous, we have that for all $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ (independent of $u \in B$) such that if $|t_1 - t_2| < \delta$, then $|Tu(t_1) - Tu(t_2)| < \varepsilon$ for all $u \in B$.

Using the same arguments, we have

$$|(Tu)'(t_1) - (Tu)'(t_2)| \leq \lambda D_1 \int_0^1 \left| \frac{\partial}{\partial t}G(t_1,s) - \frac{\partial}{\partial t}G(t_2,s) \right| \mathrm{d}s,$$

whence $|(Tu)'(t_1) - (Tu)'(t_2)| < \varepsilon$ for all $u \in B$.

Therefore, $T(B)$ is equicontinuous in $C^1(I)$.

As a consequence, by the Ascoli-Arzela theorem, we obtain that $T(B)$ is relatively compact in $C^1(I)$, which gives us that T is a completely continuous operator. □

Define $\Lambda = \int_0^1 G(1,s)p(s)g(s)\,\mathrm{d}s > 0$, $p^* = \max_{s\in I}|p(s)|$ and denote, assuming that both limits exist,

$$f_0 = \lim_{x\to 0^+} \frac{f(x)}{x} \quad \text{and} \quad f^\infty = \lim_{x\to +\infty} \frac{f(x)}{x}.$$

Theorem 3.2. *Assume that* $0 \leq \alpha \leq 1$, $0 \leq \beta < \frac{2}{2-\alpha}$ *and* $0 \leq \eta \leq \frac{1}{3}$. *If* $\frac{2+\alpha\beta-\beta}{2+\alpha\beta-2\beta} f^\infty p^* < \Lambda f_0$, *then for all* $\lambda \in \left(\frac{1}{\Lambda f_0}, \frac{2+\alpha\beta-2\beta}{(2+\alpha\beta-\beta)f^\infty p^*}\right)$, *problem* (1)–(2) *have at least one positive solution that belongs to the cone* K_1.

Proof. Assume, at first, that $f_0 \in (0, +\infty)$, let $\lambda \in \left(\frac{1}{\Lambda f_0}, \frac{2+\alpha\beta-2\beta}{(2+\alpha\beta-\beta)f^\infty p^*}\right)$ and choose $\varepsilon \in (0, f_0)$ such that

$$\frac{1}{\Lambda(f_0-\varepsilon)} < \lambda < \frac{2+\alpha\beta-2\beta}{(2+\alpha\beta-\beta)(f^\infty+\varepsilon)p^*}.$$

From the definition of f_0, it follows that there exists $\delta_1 > 0$ such that when $0 \le u(t) \le \delta_1$, for all $t \in I$, we have

$$f(u(t)) > (f_0-\varepsilon)u(t) \text{ for all } t \in I.$$

Let $\Omega_{\delta_1} = \{u \in K_1 : \|u\| < \delta_1\}$ and choose $u \in \partial\Omega_{\delta_1}$. We will prove that $Tu \not\preccurlyeq u$, being $\preccurlyeq$ the order induced in the cone K_1. So, since $p(s)G(1, s) \ge 0$ for all $s \in I$ and $u \in K_1$, we have

$$\begin{aligned} Tu(1) &= \lambda \int_0^1 G(1,s)p(s)f(u(s))\,ds \\ &\ge \lambda(f_0-\varepsilon)\int_0^1 p(s)G(1,s)u(s)\,ds \\ &\ge \lambda(f_0-\varepsilon)\|u\|_\infty \int_0^1 p(s)G(1,s)g(s)\,ds \\ &= \lambda(f_0-\varepsilon)u(1)\int_0^1 p(s)G(1,s)g(s)\,ds > u(1). \end{aligned}$$

Thus, we have that $Tu(t) \le u(t)$ is not true for all $t \in I$, which is a necessary condition to have $u - Tu \in K \subset K_1$. As a consequence $Tu \not\preccurlyeq u$ and, from [11, Theorem 7.11, (iii)], we deduce that

$$i_{K_1}(T, \Omega_{\delta_1}) = 0.$$

Assuming now that $f_0 = +\infty$, let $\lambda > 0$ and $M > 0$ be such that $\lambda > 1/(M\Lambda)$. So, from the definition of f_0 we have that there exists $\delta_1 > 0$ such that when $0 < u(t) \le \delta_1$, for all $t \in I$, we have

$$f(u(t)) > Mu(t) \quad \text{for all } t \in I.$$

So, arguing as above we deduce again that

$$i_{K_1}(T, \Omega_{\delta_1}) = 0.$$

On the other hand, due to the definition of f^∞, we know that there exists $\delta_2 > 0$ such that when $\min_{t\in I}\{u(t)\} \ge \delta_2$,

$$f(u(t)) \le (f^\infty+\varepsilon)u(t) \le (f^\infty+\varepsilon)\|u\|_\infty \quad \text{for all } t \in I.$$

Define $\Omega_{\delta_2} = \{u \in K_1 : \min_{t\in I}|u(t)| < \delta_2\}$. We note that Ω_{δ_2} is an unbounded subset of the cone K_1. Because of this, the fixed point index of the operator T with respect to Ω_{δ_2}, $i_{K_1}(T, \Omega_{\delta_2})$ is only defined in the case that the set of fixed points of the operator T in Ω_{δ_2}, that is, $(I - T)^{-1}(\{0\}) \cap \Omega_{\delta_2}$, is compact (see [11] for the details). We will see that $i_{K_1}(T, \Omega_{\delta_2})$ can be defined in this case.

First of all, since $(I - T)$ is a continuous operator, it is obvious that $(I - T)^{-1}(\{0\}) \cap \Omega_{\delta_2}$ is closed. Moreover, we can assume that $(I - T)^{-1}(\{0\}) \cap \Omega_{\delta_2}$ is bounded. Indeed, on the contrary, we would have infinite fixed points of T in Ω_{δ_2} and it would be immediately deduced that problem (1)–(2) have an infinite number of positive solutions. Therefore, we may assume that there exists a constant $M > 0$ such that $\|u\| < M$ for all $u \in (I - T)^{-1}(\{0\}) \cap \Omega_{\delta_2}$.

Finally, using similar arguments as before, see also [10], since $(I - T)^{-1}(\{0\}) \cap \Omega_{\delta_2}$ is bounded, we deduce that $(I - T)^{-1}(\{0\}) \cap \Omega_{\delta_2}$ is equicontinuous.

Now, we will prove that $\|Tu\| \le \|u\|$ for all $u \in \partial\Omega_{\delta_2}$. Let $u \in \partial\Omega_{\delta_2}$. Then, for any $t \in I$, using (5), we have

$$|Tu(t)| \leq \lambda \int_0^1 |G(t,s)||p(s)|f(u(s))\,\mathrm{d}s$$
$$\leq \lambda \frac{1-\eta^2}{2+\alpha\beta-2\beta} p^* \int_0^1 (f^\infty + \varepsilon)\|u\|_\infty \mathrm{d}s$$
$$\leq \lambda \frac{1-\eta^2}{2+\alpha\beta-2\beta} p^* (f^\infty + \varepsilon)\|u\| < \|u\|.$$

We point out that the last inequality holds because of $1-\eta^2 < 1 < 2+\alpha\beta-\beta$.

Now, from (6), we deduce

$$|(Tu)'(t)| \leq \lambda \int_0^1 \left|\frac{\partial}{\partial t} G(t,s)\right| |p(s)|\, f(u(s))\,\mathrm{d}s$$
$$\leq \lambda \frac{2+\alpha\beta-\beta}{2+\alpha\beta-2\beta} p^* \int_0^1 (f^\infty + \varepsilon)\|u\|_\infty \mathrm{d}s$$
$$\leq \lambda \frac{2+\alpha\beta-\beta}{2+\alpha\beta-2\beta} p^* (f^\infty + \varepsilon)\|u\| < \|u\|,$$

whence, we deduce that $\|Tu\| \leq \|u\|$ and as a consequence, see [11, Theorem 7.3], we have that $i_{K_1}(T, \Omega_{\delta_2}) = 1$. Then, we conclude that T has a fixed point in $\Omega_{\delta_2} \backslash \overline{\Omega}_{\delta_1}$, which is a positive solution of problem (1)–(2). □

Consequently, we obtain the following results.

Corollary 3.3. *Assume that condition* (F) *holds. Then,*

(*i*) *If* $f_0 = \infty$ *and* $f^\infty = 0$, *then for all* $\lambda > 0$ *problem* (1)–(2) *has at least one positive solution.*

(*ii*) *If* $f_0 = \infty$ *and* $0 < f^\infty < \infty$, *then for all* $\lambda \in \left(0, \frac{2+\alpha\beta-2\beta}{(2+\alpha\beta-\beta)f^\infty p^*}\right)$ *problem* (1)–(2) *has at least one positive solution.*

(*iii*) *If* $0 < f_0 < \infty$ *and* $f^\infty = 0$, *then for all* $\lambda > \frac{1}{\Lambda f_0}$ *problem* (1)–(2) *has at least one positive solution.*

In the sequel, we will obtain some alternative results that ensure the existence of solutions of problem (1)–(2). First of all, we will recall some classical results regarding fixed point theory (see [12,13] for more details).

Lemma 3.4. *Let X be a Banach space and D be an open bounded set with $D_K = D \cap K \neq \emptyset$ and $\overline{D}_K \neq K$. Assume that $F : \overline{D}_K \to K$ is a compact map such that $x \neq Fx$ for $x \in \partial D_K$. Then the fixed point index $i_K(F, D_K)$ satisfies the following properties*:

(*1*) *If there exists $e \in K\backslash\{0\}$ such that $x \neq Fx + \alpha e$ for all $x \in \partial D_K$ and $\alpha > 0$, then $i_K(F, D_K) = 0$.*

(*2*) *If $\mu x \neq Fx$ for all $x \in \partial D_K$ and for every $\mu \geq 1$, then $i_K(F, D_K) = 1$.*

(*3*) *Let D^1 be open in X with $\overline{D}^1 \subset D_K$. If $i_K(F, D_K) = 1$ and $i_K(F, D_K^1) = 0$, then F has a fixed point in $D_K \backslash \overline{D}_K^1$. The same result holds if $i_K(F, D_K) = 0$ and $i_K(F, D_K^1) = 1$.*

We will consider the sets

$$K_\rho = \{u \in K_1 : \|u\| < \rho\}$$

and

$$V_\rho = \{u \in K_1 : \min_{t \in I} u(t) < \rho\}.$$

It is clear that $K_\rho \subset V_\rho$.

Lemma 3.5. *Denote*

$$f^{\rho} = \sup\left\{\frac{|p(t)|\, f(u)}{\rho}; \quad (t,u) \in I \times [0,\rho]\right\}.$$

If there exists $\rho > 0$ *such that* $\lambda f^{\rho} < \frac{2+\alpha\beta-2\beta}{2+\alpha\beta-\beta}$, *then* $i_{K_1}(T, K_\rho) = 1$.

Proof. We will prove that $Tu \neq \mu u$ for all $u \in \partial K_\rho$ and every $\mu \geq 1$. Suppose, on the contrary, that there exists some $u \in \partial K_\rho$ and $\mu \geq 1$ such that

$$\mu u(t) = \lambda \int_0^1 G(t,s)p(s)f(u(s))\,\mathrm{d}s.$$

Taking the supremum for $t \in I$, we obtain that

$$\mu\, \|u\|_\infty = \lambda \sup_{t\in I} \int_0^1 G(t,s)p(s)f(u(s))\,\mathrm{d}s \leq \lambda\rho f^{\rho}\frac{1-\eta^2}{2+\alpha\beta-2\beta} < \rho.$$

On the other hand, we have that

$$\mu u'(t) = \lambda \int_0^1 \frac{\partial}{\partial t}G(t,s)p(s)f(u(s))\,\mathrm{d}s$$

and

$$\mu\|u'\|_\infty = \lambda \sup_{t\in[0,1]} \int_0^1 \left|\frac{\partial}{\partial t}G(t,s)\right| |p(s)|\, f(u(s))\,\mathrm{d}s \leq \lambda\rho f^{\rho}\frac{2+\alpha\beta-\beta}{2+\alpha\beta-2\beta} < \rho.$$

As a consequence, it is deduced that $\mu\rho < \rho$, which is a contradiction with the assumption that $\mu \geq 1$. Therefore, the result is proved. □

Lemma 3.6. *Let*

$$M = \left(\int_0^1 |G(1,s)|\,\mathrm{d}s\right)^{-1}$$

and

$$f_{\rho} = \inf\left\{\frac{|p(t)|f(u)}{\rho}; \quad (t,u) \in I \times [0,\rho]\right\}.$$

If there exists $\rho > 0$ *such that* $\lambda\, f_\rho > M$, *then* $i_{K_1}(T, K_\rho) = 0$.

Proof. We will prove that there exists $e \in K_1 \setminus \{0\}$ such that $u \neq Tu + \alpha$ for all $u \in \partial V_\rho$ and all $\alpha > 0$.

Since $0 \leq g(t) \leq 1$ for all $t \in I$, we can take $e(t) = 1 \in K_1$ and suppose that there exists some $u \in \partial V_\rho$ and $\alpha > 0$ such that $u = Tu + \alpha$. Then,

$$u(1) = \lambda \int_0^1 |G(1,s)||p(s)|f(u(s))\,\mathrm{d}s + \alpha \geq \lambda\rho f_\rho \int_0^1 |G(1,s)|\,\mathrm{d}s > \rho,$$

which is a contradiction. Therefore, the result holds. □

From previous lemmas, it is possible to formulate the following theorem, in which we give some conditions under which problem (12) is solvable.

Theorem 3.7. *Assume* $0 < \eta < 1/3$. *Then problem* (1)–(2) *has at least one nontrivial solution in* K_1 *if one of the following conditions hold.*

(*C1*) *There exist* $\rho_1, \rho_2 \in (0, \infty)$, $\rho_1 < \rho_2$, *such that* $\lambda f_{\rho_1} > M$ *and* $\lambda f^{\rho_2} < \frac{2+\alpha\beta-2\beta}{2+\alpha\beta-\beta}$.

(*C2*) *There exist* $\rho_1, \rho_2 \in (0, \infty)$, $\rho_1 < \rho_2$, *such that* $\lambda f^{\rho_1} < \frac{2+\alpha\beta-2\beta}{2+\alpha\beta-\beta}$ *and* $\lambda f_{\rho_2} > M$.

4 Nonexistence results

In the following theorem, we give some conditions to ensure that the integral equation (12) has no nontrivial solution in K_1.

Theorem 4.1. *Let* $[a, b] \subset I$, *with* $a > 0$, *be given. If one of the following conditions holds*

(*i*) $f(x) < m^* x$ *for every* $x \geq 0$, *where*

$$m^* = \left(\lambda \sup_{t \in I} \int_0^1 G(t,s)p(s)\,\mathrm{d}s \right)^{-1}.$$

(*ii*) $f(x) > m_* x$ *for every* $x \geq 0$, *where*

$$m_* = \left(\lambda \inf_{t \in [a,b]} \int_a^b G(t,s)p(s)\,\mathrm{d}s \right)^{-1}.$$

Then problem (1)–(2) *has no nontrivial solution in* K_1.

Proof. (i) Suppose, on the contrary, that there exists $u \in K_1$ such that $u = Tu$. Let $t_0 \in I$ be such that $\|u\| = u(t_0)$. Then,

$$\begin{aligned} \|u\| &= \lambda \int_0^1 |G(t_0,s)||p(s)|f(u(s))\,\mathrm{d}s \\ &< \lambda m^* \int_0^1 |G(t_0,s)||p(s)|u(s)\,\mathrm{d}s \\ &\leq \lambda m^* \|u\| \int_0^1 |G(t_0,s)||p(s)|\,\mathrm{d}s \leq \|u\|, \end{aligned}$$

which is a contradiction.

(ii) Suppose, on the contrary, that there exists $u \in K_1$ such that $u = Tu$. Let $t_0 \in [a, b]$ be such that $u(t_0) = \min_{t \in [a,b]} u(t)$. Then, for $t \in [a, b]$ we have that

$$\begin{aligned} u(t_0) &= \lambda \int_0^1 G(t_0,s)p(s)f(u(s))\,\mathrm{d}s \\ &\geq \lambda \int_a^b G(t_0,s)p(s)f(u(s))\,\mathrm{d}s \end{aligned}$$

$$> \lambda m_* \int_a^b G(t_0, s)p(s)u(s)\,ds$$

$$\geq \lambda m_* u(t_0) \inf_{t\in[a,b]} \int_a^b G(t, s)p(s)\,ds \geq u(t_0),$$

which is a contradiction. □

5 Examples

In this section, we will illustrate our main results with examples. Moreover, we will show that the existence results obtained in Theorems 3.2 and 3.7 are not comparable.

Example 5.1. Let us consider the problem with $f(u) = u^\gamma$, $\gamma \in (0, 1)$ and $p(t) = q(t)\arctan(t - \eta)$, with $c_1 \geq q(t) \geq c_2 > 0$ for all $t \in I$ and let $0 \leq \alpha \leq 1$, $0 \leq \beta < \frac{2}{2-\alpha}$ and $0 \leq \eta \leq \frac{1}{3}$. That is,

$$u''' = -\lambda u^\gamma q(t)\arctan(t - \eta), \quad t \in I,$$
$$u(0) = 0, \quad u''(\eta) = \alpha u'(1), \quad u'(1) = \beta u(1).$$

In this case,

$$f_0 = +\infty \quad \text{and} \quad f^\infty = 0.$$

Then, Theorem 3.2 gives us that there exists at least one positive solution of the considered problem for all $\lambda > 0$.

On the other hand, for $\rho > 0$,

$$f_\rho = \inf\left\{\frac{q(t)\arctan|t - \eta|u^\gamma}{\rho}; \quad (t, u) \in I \times [0, \rho]\right\} = 0$$

and it is not possible to find a positive ρ, such that $\lambda f_\rho > M$, which means that Theorem 3.7 cannot be applied in this case.

Example 5.2. Now, consider the problem with $f(u) = uq(u)$, such that $D > q(u) \geq c > 0$ for all $t \in I$ and $p(t) = \arctan(t - \eta)$, where

$$D \equiv \frac{2 + \alpha\beta - 2\beta}{\lambda(1 - \eta^2)\left(\left(-\frac{1}{2}\ln(((\eta - 2)\eta + 2)(\eta^2 + 1)) - (\eta - 1)\arctan(1 - \eta) + \eta\arctan\eta\right)\right)}.$$

Namely,

$$u''' = -\lambda uq(u)\arctan(t - \eta), \quad t \in I,$$
$$u(0) = 0, \quad u''(\eta) = \alpha u'(1), \quad u'(1) = \beta u(1),$$

where $0 \leq \alpha \leq 1$, $0 \leq \beta < \frac{2}{2-\alpha}$ and $0 \leq \eta \leq \frac{1}{3}$ are arbitrarily chosen. Since,

$$\frac{1}{m^*} = \lambda \sup_{t\in I} \int_0^1 G(t, s)p(s)\,ds \leq \lambda \max_{t,s\in I}|G(t, s)| \int_0^1 |\arctan(s - \eta)|\,ds = \frac{1}{D},$$

then, $f(u) = uq(u) < uD = um^*$.

Thus, condition (i) in Theorem 4.1 holds, which shows that the considered problem has no nontrivial solutions in K_1.

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