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THIRD ORDER NONDEGENERATE HOMOTOPIES OF SPACE CURVES

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1. Introduction

A space curve given by a C^3 immersion $X: S^1 \to E^3$ of the unit circle S^1 into a Euclidean 3-space E^3 is said to be third order nondegenerate if it has nonvanishing torsion, and of course nonvanishing curvature. Examples of such curves are coiled springs which are joined into closed curves; some telephone cords are made this way. Two such space curves are third order nondegenerately homotopic if they are connected by a 1-parameter family of such curves. (See Feldman [1] for the general definition of a pth order nondegenerate map between manifolds of arbitrary dimensions.)

Theorem 1. *There are four third order nondegenerate homotopy classes of space curves.*

Let e_1 be the unit tangent vector of the curve X, respecting the orientation when that is prescribed. The spherical image or tangential indicatrix is given by the map $e_1: S^1 \to S^2$, where S^2 is the unit 2-sphere in E^3. It is easy to check that because the torsion of the curve X never vanishes the geodesic curvature of its spherical image is never zero. Furthermore, because the curve is closed, the spherical image must cross every great circle, or what is the same, contain the center of S^2 in the interior of its convex hull; this observation about closed space curves is due to C. Loewner (see Fenchel [3]). Thus a 1-parameter family of space curves each with nonzero torsion gives rise to a 1-parameter family of spherical curves, each of which has nonzero geodesic curvature and which contains the center of S^2 in its convex hull.

In [5] we have classified second order nondegenerate homotopy classes of curves on the unit 2-sphere; a second order nondegenerate curve on S^2 being one such that the geodesic curvature is not zero. Two nondegenerate curves are nondegenerately homotopic if and only if they are regularly homotopic, their geodesic curvatures have the same sign and they are either both simple or both have double points. Nondegenerate simple curves must lie in a hemisphere (see Fenchel [3]), but among the classes of curves with double points it is possible to find nondegenerate curves which cross every great circle. Representatives of the four second order nondegenerate homotopy classes of

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curves with double points are pictured in Fig. 1. The pictures show both hemispheres as seen under parallel projection from above. All four curves follow along on two intersecting small circles except where they make loops. The four curves have the property that they cross every great circle as may be verified by noting that each curve crosses its antipodal image. Curves 1 and 2 are regularly homotopic with geodesic curvature of opposite sign. Curves 3 and 4 represent the other regular homotopy class again with geodesic curvature of both signs.

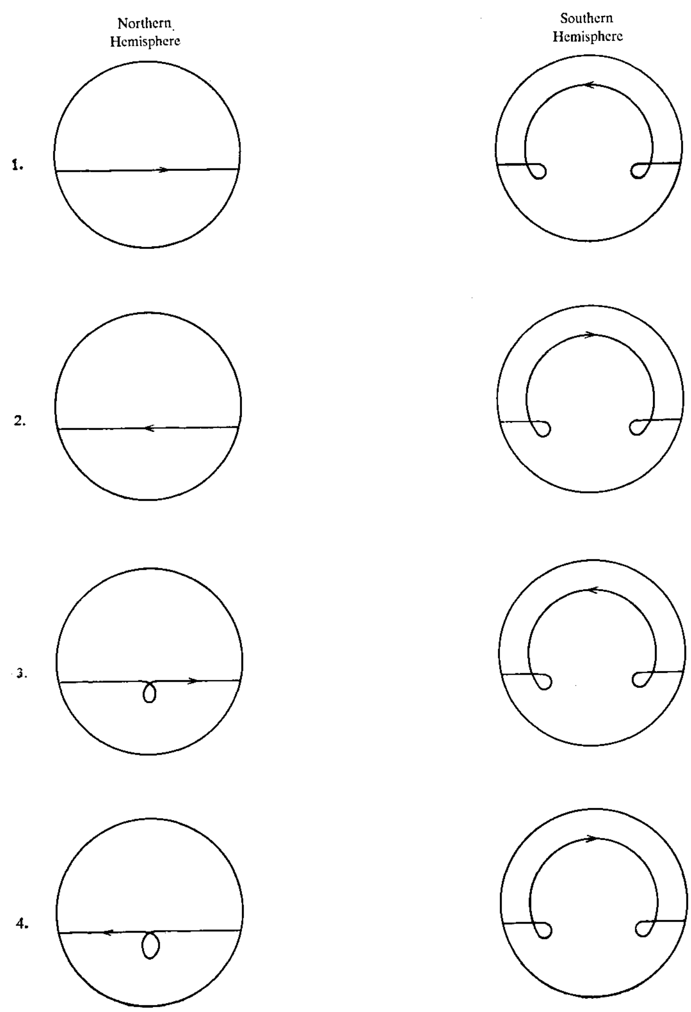

Fig. 1

The remark of Loewner that the spherical image of a closed space curve crosses every great circle has a converse due to Fenchel [3], namely, every spherical curve which crosses every great circle is the spherical image of some closed space curve. Thus there exist space curves whose spherical images are the four curves in Fig. 1. These space curves have nonzero torsion because their spherical images have nonzero geodesic curvature, and they are in distinct third order nondegenerate homotopy classes because their spherical images are in distinct second order nondegenerate homotopy classes.

The proof of Theorem 1 relies on an integration theorem of E. Feldman [2]. Suppose $X_0, X_1\colon S^1 \to E^3$ are two closed space curves whose respective spherical images are e_0, e_1. If e_0, e_1 are connected by a homotopy e_t, which for each t crosses every great circle, then there exists a homotopy of closed space curves X_t joining X_0 and X_1 such that the spherical image of X_t is e_t. Feldman uses this theorem to classify space curves under second order nondegerate homotopy, and shows that two space curves with nonzero curvature are second order nondegenerately homotopic if and only if their spherical images are regularly homotopic. There are two regular homotopy classes of curves on S^2 (see Smale [6]), and thus there are two second order nondegenerate homotopy classes of space curves.

Theorem 2. *Two curves on S^2 with nonzero geodesic curvature which cross every great circle are homotopic through such curves if and only if they are regularly homotopic and their geodesic curvatures have the same sign. In particular, any such curve is homotopic through such curves to one of the curves of Fig.* 1.

Theorem 1 is now a consequence of Theorem 2, the integration theorem of Feldman, and the previous discussion.

Corollary 3. *Two space curves with nonzero torsion, which are second order nondegenerately homotopic and whose torsion have the same sign, are third order nondegenerately homotopic.*

Proof. Let X_0, X_1 be the two curves, and e_0, e_1 their spherical images. Then e_0, e_1 cross every great circle and have nonzero geodesic curvature. Furthermore, since X_0, X_1 are second order nondegenerately homotopic, e_0, e_1 are regularly homotopic. Finally, since X_0, X_1 have torsion of the same sign, e_0, e_1 have geodesic curvature of the same sign. By Theorem 2, e_0 and e_1 are each homotopic by a homotopy of spherical curves which cross every great circle and have nonzero geodesic curvature. Thus by Feldman's integration theorem the curves X_0, X_1 are third order nondegenerately homotopic.

We may use the langage of screws to give representatives of the four third order nondegenerate homotopy classes. Bend a machine screw (which does not taper) around in a circle and join it to itself. Then the four classes are determined by the right or left handedness of the screw and whether there are an even or odd number of threads.

2. Proof of Theorem 2

We begin by recalling some techniques used in [5].

Lemma 4. *Let X_0, X_1: $[0, 1] \to E^2$ be two oriented arcs with positive curvature. Suppose that the arcs are identical in neighborhoods of their endpoints and that the total turning of the tangent for both arcs is the same. Then the two arcs are nondegenerately homotopic by a homotopy which leaves the arcs fixed on neighborhoods of their endpoints.*

This is contained in Theorem 5 of [5]. The basic idea is to choose parametrizations of X_0, X_1 so that the tangents are parallel at corresponding points, and then to use a linear homotopy.

Lemma 5. *Let X: $[0, 1] \to S^2$ be a nondegenerate arc on the unit 2-sphere. Suppose that $X(0), X(1)$ are in the southern hemisphere and that X meets the equator transversally at two points. Then X is nondegenerately homotopic to an arc Y, which agrees with X in neighborhoods of the end points and lies entirely in the lower hemisphere. Furthermore, the neighborhoods of the endpoints of X are never moved during the homotopy.*

This lemma enables us to "pull arcs out of a hemisphere". The proof is contained in [5] in several parts. There are two cases to be considered. Either the arc X is "troublesome" or it is not. If the arc is not troublesome, then the proof is straightforward enough (see Lemma 8 of [5]). If the arc is troublesome, then the arc must first be pushed up over the top of the hemisphere (see Lemma 9 of [5]) forming two simple arcs. These are then individually pulled into the lower hemisphere. The main tool for doing second order nondegenerate homotopies on the unit 2-sphere in [5] is central projection of a hemispherical arc and then an application of Lemma 4. This tool we use here also.

We shall need to glue together arcs in such a way that the curvature is continuous. We may do this by using spiral arcs (see Guggenheimer [4, pp. 48–52]), and shall not spell out the details but speak merely of "turning a corner by adding a loop" or of "flattening a curve locally" without much ado.

To continue with the proof of Theorem 2 let X: $S^1 \to S^2$ be an immersion with nonzero geodesic curvature which crosses every great circle. Using rather standard arguments we may assume that X has only finitely many transversal double points and no triple points. Let us fix an orientation of S^1. Then $X(t_0, t_1)$ will mean the image of the open arc from t_0 to t_1 in the sense of this orientation. By an oriented pair of points we mean the pair of points (t_0, t_1) and the arc from t_0 to t_1. We may choose a pair of double points (t_0, t_1) so that the arc $X(t_0, t_1)$ is simple; this is done by induction and the fact that there are only finitely many double points. Since a simple arc $X(t_0, t_1)$ lies in a hemisphere (see Fenchel [3]), it may be projected centrally to a simple nondegenerate arc in the plane, which either bounds a convex body or else a heart shaped region. The next two paragraphs show that we may assume the plane arc bounds a convex body rather than a heart shaped region.

Consider the tangent plane of S^2 at a point of the curve $X(t)$. The tangent line at the point of $X(t)$ divides the tangent plane into two half spaces. Since the curve is nondegenerate, the projection of X'' onto the tangent plane will be independent of $X'(t)$ and therefore determines a half plane which we call the preferred side of the tangent. Oriented pairs of double points (t_0, t_1) are then of two kinds. If the tangent vector $X'(t_1)$ lies to the preferred side of the tangent line at t_0, we say (t_0, t_1) is a double point of the first kind, otherwise of the second kind. The preferred side of a nondegenerate plane curve is analogously defined, and central projection respects the preference of sides. If $X(t_0, t_1)$ is of the second kind it projects into the boundary of a convex body, and if $X(t_0, t_1)$ is of the first kind it projects into the boundary of a heart shaped region. Thus we need only to show that we may always find a simple arc of the second kind.

Suppose the simple arc $X(t_0, t_1)$ is of the first kind. Let $X(\alpha, \beta)$, $\alpha < t_0 < t_1 < \beta$, be contained in the hemisphere of projection, $X(\alpha)$ and $X(\beta)$ lying on the boundary. Since central projection of this hemisphere maps the simple arc $X(t_0, t_1)$ to a simple closed curve which contains $X(t_1 + \varepsilon)$ for small ε, there must be a first point $t_2, t_1 < t_2 < \beta$, such that $X(t_2)$ lies on the simple arc $X(t_0, t_1)$, and therefore there is a point $t_3, t_0 < t_3 < t_1$, such that $X(t_3) = X(t_2)$. If $X(t_3, t_2)$ is simple, then it is a simple arc of the second kind. If $X(t_3, t_2)$ is not simple, let $t_4, t_3 < t_4 < t_2$, be a point such that $X(t_3, t_4)$ is simple and $X(t_3, t_4 + \varepsilon)$ is not. Thus there is a point $t_3 \leq t_5 < t_4$ such that $X(t_5, t_4)$ is simple and $X(t_5) = X(t_4)$. Suppose $X(t) = X(t')$ for $t_0 < t < t' < t_4$. Since $t' < t_4 < t_2$, we must then have $t_1 < t$, so that $t_3 < t < t' < t_4$. But $X(t_3, t_4)$ is simple; so no such t, t' exist. Hence we see that $X(t_0 + \varepsilon, t_4)$ is simple for any $\varepsilon > 0$. If $X(t_5, t_4)$ is of the first kind, then $X(t_5, t_4)$ would contain $X(t_5 - \varepsilon)$ in its interior and $X(t_0 + \varepsilon)$ outside; this would contradict the fact that $X(t_0 + \varepsilon, t_4)$ is simple. Thus $X(t_5, t_4)$ must be a simple arc of the second kind. Hence in every case we have found a simple arc of the second kind; call this arc $X(t_0, t_1), X(t_0) = X(t_1)$.

In the next several paragraphs we show that we may deform the arc $X(t_0, t_1)$ to a simple arc, which projects to a convex curve and furthermore has the property that for an appropriately small $\varepsilon > 0, X(t_0 - \varepsilon, t_1 + \varepsilon)$ crosses every great circle. We do this by deforming $X(t_0, t_1)$ so that it travels along on its osculating small circles B_0, B_1 at $X(t_0) = X(t_1)$ except of course where it leaves the small circles to turn the corner at the further end. We begin this construction in the next paragraph by "preparing" the double point, namely, we deform the curve X so that in a neighborhood of $X(t_0) = X(t_1)$ the osculating circles are constant of arbitrarily small geodesic curvature.

Since $X(t_0, t_1)$ projects centrally to the boundary of a convex figure, it lies to one side of its tangent great circles. Let C_0, C_1 be the tangent great circles of X at t_0, t_1 respectively, and B_0, B_1 be the small circles which are translates of C_0, C_1

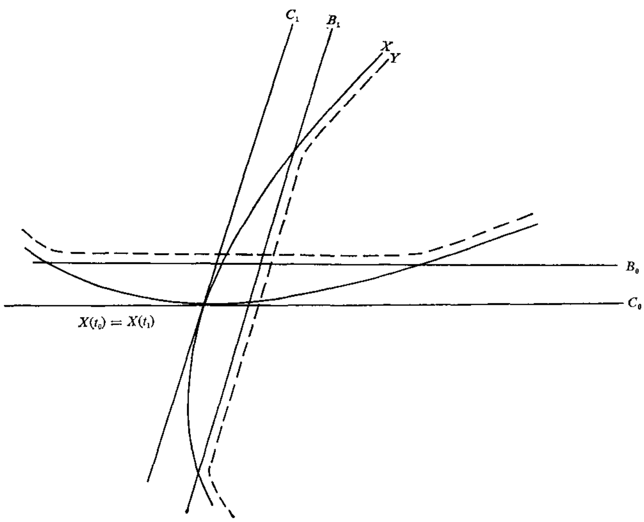

Fig. 2

a distance δ each in the preferred direction of the respective tangent lines at $X(t_0) = X(t_1)$. δ is chosen so small that the arcs of X through the double point cross transversally B_0, B_1 as shown in Fig. 2. Define Y to be equal to X except that on the arcs through the double point the curve follows along the small circles B_0, B_1 as indicated by Fig. 2. Here we must use spiral arcs to make the geodesic curvature continuous at the intersections of X and B_0, B_1. Fig. 3 shows the central projection of a neighborhood of the double point $X(t_0) = X(t_1)$. By use of Lemma 4 we may nondegenerately deform X to Y by a deformation which does not move X except near the double point. Furthermore the maximum distance any point is moved tends to 0 as $\delta \to 0$. Thus by choosing δ sufficiently small we may assure that at each time during the homotopy the curves cross every great circle. Let us now rename the curve Y by the letter X. Then we may assume that X has a simple arc $X(t_0, t_1)$ with double point $X(t_0) = X(t_1)$ which projects to the boundary of a convex body such that at the intersection the curve is following on small circles of arbitrarily small geodesic curvature.

Choose $\varepsilon > 0$ so that the spherical triangle Δ bounded by C_0, C_1 and the shorter arc of the great circle from $X(t_0 - \varepsilon)$ to $X(t_1 + \varepsilon)$ has the property that any great circle meeting an interior point meets either the arc $X(t_0 - \varepsilon, t_0)$ or the arc $X(t_1, t_1 + \varepsilon)$. Choose $p^* \in \Delta$, and let p be the antipodal point of p^*.

Let W be the wedge bounded by C_0, C_1 which contains $X(t_0, t_1)$, and V be the portion of W bounded by B_0, B_1, the osculating circles at the double point. Now $p \in W$. Thus using the previously described flattening process we may

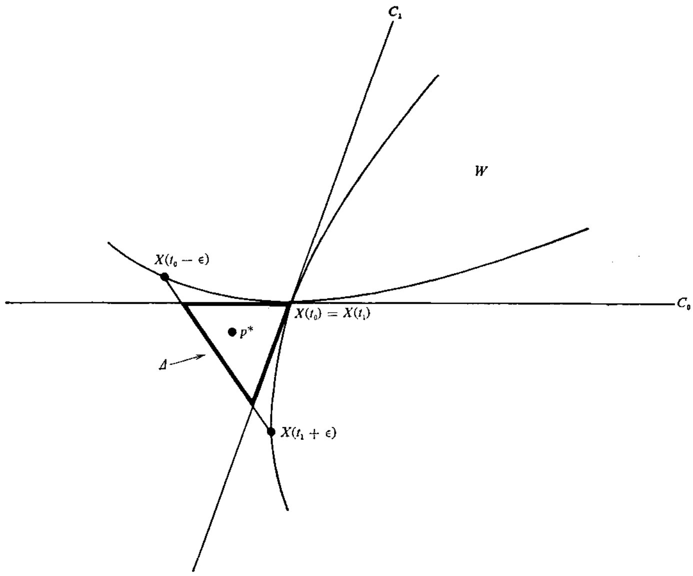

Fig. 3

assume that B_0, B_1 are arbitrarily near C_0, C_1 and hence that $p \in V$. Let Y be the curve equal to X outside the arc $X(t_0, t_1)$ and from t_0 to t_1 it traverses B_0 and returns along B_1 making a turn at the farther end where B_0, B_1 meet. The arc $Y(t_0, t_1)$ lies in W and therefore in a hemisphere. When projected centrally $Y(t_0, t_1)$ contains $X(t_0, t_1)$ and also p in its convex hull. This is because $Y(t_0, t_1)$ traverses the boundary of V except where it makes a small turn, and V contains $X(t_0, t_1)$ and p.

Now $Y(t_0, t_1), X(t_0, t_1)$ and p are all contained in a hemisphere H. Let $h\colon H \to R^2$ be central projection, and define a homotopy Y_s from X to Y as follows: $Y_s(t) = X(t)$ if t is not on the oriented arc (t_0, t_1). $Y_s(t_0) = Y_s(t_1) = X(t_0)$, and $Y_s(t) = h^{-1}(\varphi_s(t))$ for t on the oriented arc (t_0, t_1), where φ_s is a homotopy from $h \circ X$ to $h \circ Y$ given by Lemma 4, and is the linear homotopy between $h \circ X$ and $h \circ Y$ when they are parametrized to make the tangents parallel at corresponding points.

Y_s is a homotopy from X to Y through curves whose geodesic curvature never vanishes. Since $h \circ X(t_0, t_1)$ is inside the convex hull of $\varphi_s(t_0, t_1)$ for any s, any line which crosses $h \circ X(t_0, t_1)$ must cross $\varphi_s(t_0, t_1)$, and therefore any great circle which crosses $X(t_0, t_1)$ must cross $Y_s(t_0, t_1)$ for any s. In order to see that Y_s crosses every great circle for each s suppose that for some s_0, Y_{s_0} failed to cross some great circle C. Since X crosses C, and $X(t) = Y_{s_0}(t)$ for $t \notin (t_0, t_1)$, X must cross C at points in (t_0, t_1). Thus C crosses the arc $X(t_0, t_1)$; this implies that C crosses $Y_s(t_0, t_1)$ and hence Y_s.

Let K be the great circle through $X(t_0)$, p, p^*, and C be any other great circle. C meets K say at a point y on the half circle of K joining p to p^* containing $X(t_0)$. If y is on the shorter arc of K from $X(t_0)$ to p, then y is inside of the simple loop $Y(t_0, t_1)$ and so C crosses the arc $Y(t_0, t_1)$. If y is on the shorter arc of K from p^* to $X(t_0)$, then y is in the spherical triangle Δ. Hence C must cross either the arc $Y(t_0 - \varepsilon, t_0)$ or the arc $Y(t_1, t_1 + \varepsilon)$, so that every great circle crosses the arc $Y(t_0 - \varepsilon, t_1 + \varepsilon)$.

Rename the curve Y by the letter X.

The remainder of the proof in outline is as follows. We choose hemispheres H_1, H_2 with $\partial H_1 = \partial H_2$ and pull all the arcs cut from X by H_2 into H_1, except we do not move the arc $X(t_0 - \varepsilon, t_1 + \varepsilon)$. This requires Lemma 5. The curve crosses every great circle at each time because $X(t_0 - \varepsilon, t_1 + \varepsilon)$ was not moved and that arc itself crosses every great circle. The curve is now in the position of Fig. 4a), and the remaining deformations are illustrated in Figs. 4b), 5 and 6.

Let L be the axis of the great circle K, and H be the hemisphere containing the wedge W such that ∂H contains $X(t_0)$ and the poles of K. Since L lies in the plane of ∂H, rotate H about L through a small amount to obtain a new hemisphere H_1, which contains $X(t_0)$ and p^* in the interior and cuts off a simple arc A of $X(t_0, t_1)$ near p. We may even assume that the spherical triangle $\triangle$ lies in H_1.

Now by a slight homotopy of X we may assume that X meets ∂H_1 only finitely many times. Let H_2 be the other hemisphere with $\partial H_2 = \partial H_1$. Recall $A = H_2 \cap X(t_0, t_1)$, so X meets H_2 in finitely many arcs $A, A_1, \cdots, A_n$. Suppose the arc $A_i = X(t_i, t_{i+1})$. Because $X(t_0-\varepsilon, t_1+\varepsilon) \cap H_2 = A \neq A_i$, note first that (t_i, t_{i+1}) is disjoint from $(t_0 - \varepsilon, t_1 + \varepsilon)$. Furthermore, we may choose $\varepsilon_1 > 0$ so that $(t_i - \varepsilon_1, t_{i+1} + \varepsilon_1)$ is disjoint from $(t_0 - \varepsilon, t_1 + \varepsilon)$. Let h_t be the homotopy given by Lemma 5, which pulls the arc A_i out of H_2 and into H_1. Since h_t is a second order nondegenerate homotopy which is constant outside of $(t_i - \varepsilon_1, t_{i+1} + \varepsilon_1)$, h_t does not move $X(t_0 - \varepsilon, t_1 + \varepsilon)$ and hence has the property that for each t it crosses every great circle. In this way by finitely many applications of Lemma 5 we may assume that only the arc A lies in the hemisphere H_2.

The arc $X(t_0 - \varepsilon, t_1 + \varepsilon)$ meets H_1 along two intersecting small circles B_0 and B_1, and the point p^* lies in the spherical triangle $\triangle$. Let B_2 be a small circle through $X(t_0 - \varepsilon)$ and $X(t_1 + \varepsilon)$ whose plane is parallel to the line of intersection of the planes of B_0 and B_1. Thus B_0, B_1, B_2 possess a common parallel line l, and B_2 is not a great circle because it may be checked that p and p^* lie to one side of B_2. Fig. 4a) is the projection of H_1 centrally onto the plane. Fig. 4b) is an arc which is identical with X along B_0 and B_1 and then goes along B_2 as indicated. The corners are turned as indicated, and a little loop is attached and turned enough times so that the total turning of the tangent in 4a) and 4b) is the same. By Lemma 4 there is a nondegenerate homotopy, leaving portions of the arc along B_0 and B_1 fixed, which moves the

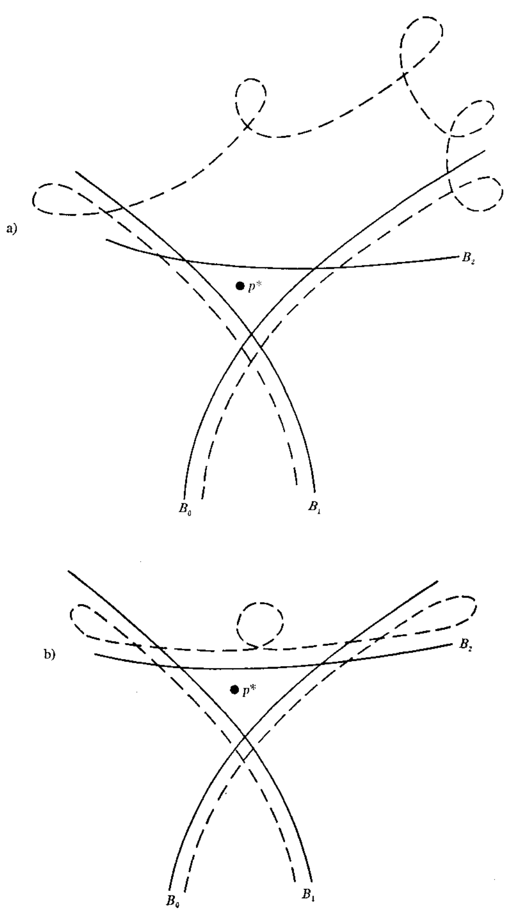

Fig. 4

one arc to the other. On the sphere we therefore have a homotopy which leaves $X(t_0 - \varepsilon, t_1 + \varepsilon)$ fixed, and hence each intermediate curve crosses every great circle. The curve now looks like γ_1 of Fig. 5. We have added in the figure a small circle B_3, whose plane is perpendicular to the common parallel line to B_0, B_1, B_2, as shown in Fig. 5.

Construct curves $\gamma_2, \gamma_3, \gamma_4$ of Fig. 5 by following along the small circles B_0, B_1, B_2, B_3 as indicated. The arrow indicates that the loop is traversed many times. We construct γ_2 so that the indicated loop is traversed as many times as γ_1. In γ_3 and γ_4 the indicated loop is traversed one more time than for γ_1. The view of S^2 is the same for each of the curves γ_i, namely, we view S^2 along the line l parallel to the planes of B_0, B_1, B_2 under parallel projection. Let H_i be the great circles parallel to the small circles B_i, H_i^+ the hemisphere bounded

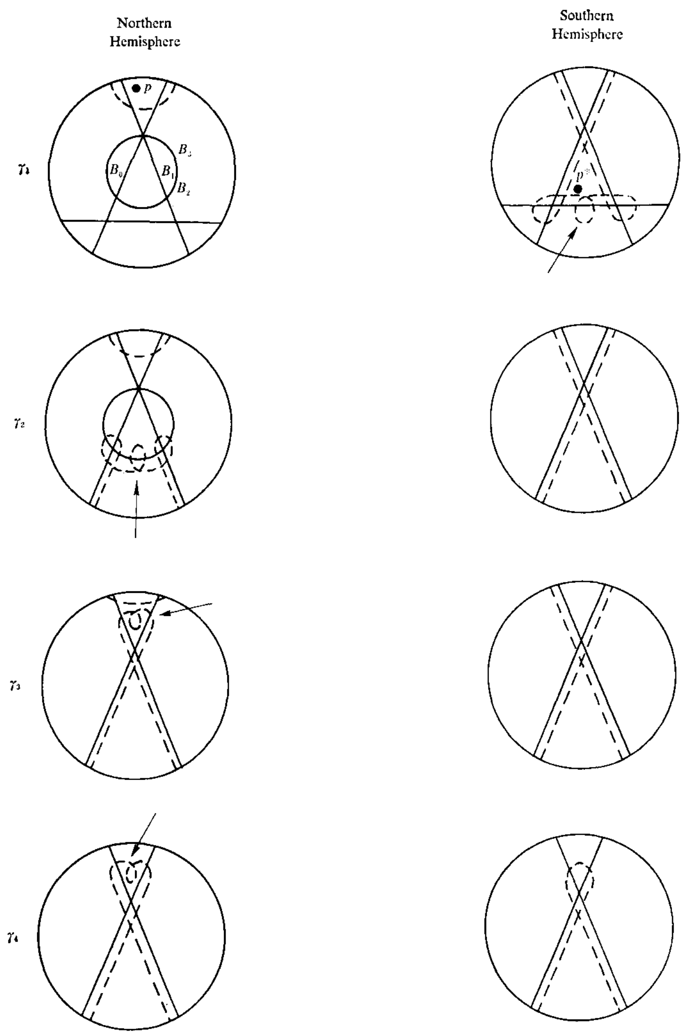

Fig. 5

by H_i which contains B_i, and H_i^- the other hemisphere, $i = 0, 1, 2, 3$. To deform from γ_1 to γ_4 we successively centrally project from the hemispheres H_2^+, H_3^+, H_2^- and use Lemma 4. Note that going from γ_1 to γ_3 the arc $X(t_0 - \varepsilon, t_1 + \varepsilon)$ never moves. But this arc itself crosses every great circle so that the homotopy from γ_1 to γ_2 and γ_2 to γ_3 are of the required type. In going from γ_3 to γ_4 note that the complimentary arc to $X(t_0, t_1)$, which we may call $X(t_1, t_0)$, is never moved. But this arc also crosses every great circle as may be seen by checking that the convex hull of $X(t_1, t_0)$ contains the center of S^2. *Consequently any* *spherical curve which crosses every great circle and has nonzero geodesic curvature may be deformed through such curves to a curve lying on two small circles with loops attached, as in* γ_4 *of Fig.* 5.

It remains now to show how to cancel the extra loops as indicated by the arrow in Fig. 5. To describe the remaining homotopies we will construct curves γ_i as shown in Fig. 6, $i = 1, \cdots, 6$. For this purpose let C_3 be a great circle, C_3^+ and C_3^- its hemispheres. Let C_0, C_1, C_2 be three great circles which meet at the poles of C_3 in equal angles, and B_0, B_1, B_2 be three small circles parallel to C_0, C_1, C_2 chosen to form equilateral spherical triangles in both C_3^+ and C_3^- with the poles of C_3 as center. Let C_i^+ be the hemisphere bounded by C_i which contains B_i, $i = 0, 1, 2$, and B_3, B_4 be small circles parallel to C_3, $B_3 \subset C_3^+$, $B_4 \subset C_3^-$, chosen so that B_3, B_4 pass through the vertices of the equilateral triangles. The curves γ_i are constructed by following along the small circles B_i, $i = 0, \cdots, 4$, turning corners as indicated in Fig. 6. To pass from γ_4 of Fig. 5 to γ_1 of Fig. 6 we may have to open the planes of the small circles B_0, B_1 of Fig. 5, but this is surely a homotopy of the required type. Let us next assume that the loop indicated by the arrow in γ_4 of Fig. 5 is traversed twice. We may therefore show the loop as two distinct loops each traversed once as in γ_1 of Fig. 6. If the loop indicated by the arrow in γ_4 of Fig. 5 is traversed more than twice, we may attach an extra loop to γ_1 of Fig. 6 and allow it to be carried along through out the homotopy from γ_1 to γ_6. Each of the curves γ_i has nonzero geodesic curvature, and they all intersect their antipodal image transversally and hence cross every great circle. To perform the homotopies we project centrally from a hemisphere and apply Lemma 4. In going from γ_i to γ_{i+1}, $i = 1, \cdots, 5$, we use respectively hemispheres $C_3^+, C_1^+, C_2^+, C_3^-, C_3^-$. It is possible to go from γ_4 to γ_6 by a simple application of Lemma 4. We have included γ_5 just to make it easier to show that all the intermediate curves cross every great circle. To show that each intermediate curve of the homotopy from γ_i to γ_{i+1} crosses every great circle we will display a point $*$, which is unmoved during the homotopy and such that the intermediate curves cross their antipodal images at that point. For the homotopies from γ_1 to γ_4 we may let $*$ be the midpoint of the arc $B_1 \cap C_3^-$, for the homotopy from γ_4 to γ_5 let $*$ be the midpoint of the arc $B_2 \cap C_3^-$, and for the homotopy from γ_5 to γ_6 let $*$ be the midpoint of the arc $B_4 \cap C_2^+$. Then one verifies that in each case $*$ has the above properties so that all the intermediate curves cross every great circle. We see that γ_6 follows along on 2-small circles turning corners with small loops traversed once. By opening the planes of the small circles B_1 and B_4 so that they make an angle of $60°$ and then rotating S^2 we may deform γ_6 into γ_1 without the two loops. Thus we have cancelled two of the loops of γ_1.

By continuing the circuit of going from γ_1 to γ_6 and back to γ_1 we kill off two loops at a time achieving eventually one of four curves, namely, curves, like γ_1 of Fig. 6, which follow along on two intersecting small circles with either no loop or one loop attached and with either orientation. One checks that these

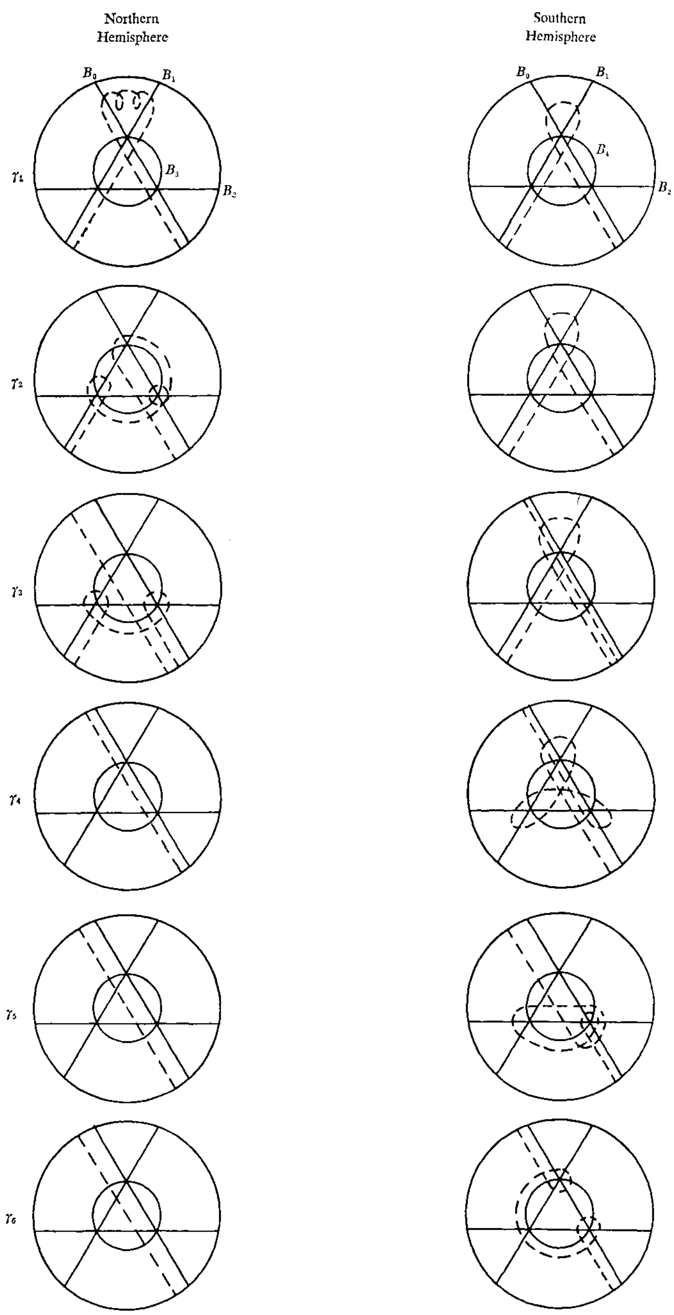
Northern Hemisphere
Southern Hemisphere
B_0
B_1
B_2
B_3
B_4
γ_1
γ_2
γ_3
γ_4
γ_5
γ_6

Fig. 6

four curves may be distinguished by the sign of their geodsic curvature and their regular homotopy class. Thus two curves, which cross every great circle, have geodesic curvature of the same sign and lie in the same regular homotopy class, may be brought to the same curve by a nondegenerate homotopy with intermediate curves crossing every great circle. Consequently, the two curves themselves are homotopic by such a homotopy. This completes the proof of Theorem 2.

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