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Third post-Newtonian dynamics of compact binaries: equations of motion in the centre-of-mass frame

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Abstract

The equations of motion of compact binary systems and their associated Lagrangian formulation have been derived in previous works at the third post-Newtonian (3PN) approximation of general relativity in harmonic coordinates. In the present work, we investigate the binary's relative dynamics in the centre-of-mass frame (centre of mass located at the origin of the coordinates). We obtain the 3PN-accurate expressions of the centre-of-mass positions and equations of the relative binary motion. We show that the equations derive from a Lagrangian (neglecting the radiation reaction), from which we deduce the conserved centre-of-mass energy and angular momentum at the 3PN order. The harmonic-coordinates centre-of-mass Lagrangian is equivalent, via a contact transformation of the particles' variables, to the centre-of-mass Hamiltonian in ADM coordinates that is known from the post-Newtonian ADM-Hamiltonian formalism. As an application we investigate the dynamical stability of circular binary orbits at the 3PN order.

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1. Introduction

The problem of the dynamics of two compact bodies is part of a larger programme aimed at unravelling the information contained in the gravitational-wave signals emitted by inspiralling and/or coalescing compact binaries (see [1, 2] for reviews). The current treatment of the problem is post-Newtonian (expansion when the speed of light $c \rightarrow +\infty$; following the standard practice we say that a term of order $1/c^{2n}$ relative to the Newtonian force belongs

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to the n PN approximation). The first breakthrough in the problem of dynamics has been the completion of the equations of motion of two point-like particles up to the 2PN order [3–10]. In recent years the (quite involved) equations of motion at the next 3PN order have also been successfully derived [11–20].

Up to the order 3.5PN there is a clean separation between the conservative part of the dynamics, made up of the ‘even’ Newtonian, 1PN, 2PN and 3PN approximations—with the 2PN and especially the 3PN ones being very difficult to obtain, and the part associated with the radiation reaction, and consisting of the ‘odd’ 2.5PN and 3.5PN orders (which are comparatively much simpler to control than 2PN and 3PN). In principle, the conservative part of the equations yields a point of dynamical general-relativistic instability at which there is (presumably) a transition from the adiabatic inspiral to the final plunge and coalescence. On the other hand, the non-conservative terms—i.e. 2.5PN computed in [3–8], 3.5PN and 4.5PN computed in [21–23, 10]—are determined by the boundary conditions imposed on the gravitational field at infinity.

One should not confuse the latter nomenclature for post-Newtonian orders with a different one applied to the gravitational field at future null infinity. There the ‘Newtonian’ order, which has a quadrupolar wave pattern, corresponds to the dominant odd term in the local equations of motion, i.e. 2.5PN, while the 1PN order, which is both quadrupolar and octupolar, corresponds to 3.5PN in the local equations. And so on. Because of the presence of tails at the 1.5PN order in the wave field at infinity there is a contribution at the 4PN order in the equations of motion that is ‘odd’ in the sense of being associated with radiation-reaction effects [24]. Similarly one expects that the known tails-of-tails [25] arising at the 3PN order in the wave field will correspond to an odd contribution at the 5.5PN order in the equations of motion⁴.

Two different methods, relying on two independent frameworks, have been applied to the equations of motion at the 3PN order. Jaranowski and Schäfer [11, 12], working within the ADM-Hamiltonian formalism of general relativity, derived the Hamiltonian describing the motion of two compact bodies, in ADM coordinates, and in the centre-of-mass frame. The Hamiltonian was later generalized to an arbitrary frame in [13]. Blanchet and Faye [16–19] (following the method proposed in [8]) performed a direct iteration of the equations of motion in harmonic coordinates, and in a general frame. The Lagrangian of the motion was then deduced from the equations of motion [20]. The end results provided by these two methods—ADM-Hamiltonian and harmonic-coordinates—have been shown to be physically equivalent [14, 20]: there exists a unique ‘contact’ transformation of the binary’s dynamical variables that transforms the harmonic-coordinates Lagrangian [20] into a different Lagrangian, whose Legendre transform agrees with the ADM-coordinates Hamiltonian [13].

In the works [11–14] and [16–20] the compact objects are modelled by structureless (non-spinning) point particles. Such a modelling is quite efficient and physically sound when describing the inspiral of compact binaries, but the shortcoming is the necessity of a regularization for removing the infinite self-field of each of the point particles. The regularization of Hadamard (or, more precisely, a refined form of it proposed in [18, 19] and implemented in the harmonic-coordinates approach) has been applied but turned out to be incomplete in the sense that one (and only one) numerical coefficient remains undetermined at the 3PN order: ω_s in the ADM-Hamiltonian formalism [11, 12], λ in the harmonic-coordinates approach [16, 17]. This coefficient has been computed in [15] with the help of a dimensional regularization instead of the Hadamard one, within the ADM-Hamiltonian formalism, with the result $\omega_s = 0$ or equivalently $\lambda = -\frac{1987}{3080}$ (below we shall keep the value of λ unspecified).

⁴ This difference by 2.5PN orders explains why the equations of motion are insufficient as regards the radiative aspects of the problem. For analysing the waves emitted by inspiralling compact binaries one needs not only the solution of the problem of motion but also the (equally crucial) solution of the problem of gravitational-wave generation [2].

The present paper's lineage is the harmonic-coordinates approach [16–20]. Its goal is the completion of the 3PN dynamics of compact binaries—equations of motion, Lagrangian, conserved integrals—in the frame where the centre-of-mass is located at the origin of the coordinates. Our motivation is that the centre-of-mass equations of motion constitute the needed starting point in many applications such as the one in [26]. In section 2, we recall the expression derived in [20] for the position of the centre-of-mass in an arbitrary harmonic-coordinates frame. In section 3, the individual positions of the particles in the centre-of-mass frame are obtained as functions of the relative separation and velocity. We then compute the 3PN-accurate centre-of-mass equations of motion. These equations are substantially simpler than in a general frame—though they are still quite lengthy (that is unavoidable at such a high post-Newtonian order). In particular, we recover the centre-of-mass equations of motion at the 2.5PN order derived by Lincoln and Will [27] on the basis of the general-frame 2.5PN equations of Damour and Deruelle [3–5]. In section 4, the 3PN relative Lagrangian (in harmonic coordinates), describing the conservative part of the dynamics, is obtained. Note that the centre-of-mass relative Lagrangian does not straightforwardly follow from the general-frame Lagrangian of de Andrade *et al* [20], because one is not *a priori* allowed to use in a Lagrangian some expressions which are consequences of the equations of motion derived from that Lagrangian. We found it convenient to derive the centre-of-mass Lagrangian *ab initio* using some guess work (i.e. adjusting a set of coefficients in order to reproduce after Lagrangian variation the correct equations of motion). From that centre-of-mass Lagrangian we then obtain in a standard way the Noetherian conserved energy and angular momentum, thereby completing our harmonic-coordinates approach.

Further investigations are proposed. Section 5 deals with the connection between the centre-of-mass Lagrangian and the centre-of-mass ADM-Hamiltonian. We check that the centre-of-mass reduction of the contact transformation worked out in [20] between the harmonic-coordinates Lagrangian and the ADM-coordinates Hamiltonian is identical—as it must surely be—to the contact transformation connecting the centre-of-mass versions of these Lagrangian and Hamiltonian. In the process we recover the 3PN Hamiltonian for the relative motion as computed by Jaranowski and Schäfer [11, 12]. Finally in section 6 we consider the problem of the stability against linear perturbations of circular orbits. We undertake the problem by perturbing both the equations of motion in harmonic coordinates, and the Hamiltonian equations in ADM coordinates (the two methods give equivalent results). We obtain a gauge-invariant criterion for the stability of circular orbits up to the 3PN order.

2. The centre-of-mass vector position

Our study starts with the expression, derived in [20], for the position G^i of the binary's centre of mass. In this section we briefly review the construction of G^i . Note that the centre-of-mass position can also be interpreted as the gravitational mass-type dipole moment. Actually, using a slight abuse of language, by centre-of-mass position G^i we really mean the gravitational dipole (its dimension is that of a mass times a length). In a future work [28], we shall show that the gravitational mass-type dipole moment which follows from a 3PN wave-generation formalism (instead of being inferred from the 3PN equations of motion) is in complete agreement with the present centre-of-mass vector G^i .

By equations of binary motion (in a given coordinate system) we mean the acceleration $a_A^i(t) = dv_A^i/dt$ of body A , where $A = 1, 2$ and the spatial index $i = 1, 2, 3$, as a function of the positions y_B^i and coordinate velocities $v_B^i(t) = dy_B^i/dt$. Equation (7.16) in [17] gives the 3PN equations of motion in the harmonic coordinate system. The 3PN Lagrangian in

harmonic coordinates (considering only the conservative part of the dynamics) is given by equation (4.1) in [20]; it takes the form

$$L = L_N[\mathbf{y}_A, \mathbf{v}_A] + \frac{1}{c^2} L_{1PN}[\mathbf{y}_A, \mathbf{v}_A] + \frac{1}{c^4} L_{2PN}[\mathbf{y}_A, \mathbf{v}_A, \mathbf{a}_A] + \frac{1}{c^6} L_{3PN}[\mathbf{y}_A, \mathbf{v}_A, \mathbf{a}_A]. \quad (2.1)$$

The successive post-Newtonian orders depend on the positions and velocities, and also, starting from the 2PN order, on the accelerations. The fact that a harmonic-coordinates Lagrangian necessarily becomes a ‘generalized’ one (depending on accelerations) at the 2PN approximation has been proved by Damour and Deruelle [4]⁵. At the 3PN order we found [20] that the Lagrangian also depends on accelerations, but it is notable that these accelerations are sufficient (i.e. there is no need to include derivatives of accelerations). Furthermore, the dependence upon the accelerations at both the 2PN and 3PN orders is linear. Indeed, one can always eliminate from a generalized Lagrangian, taking the form of a perturbative post-Newtonian expansion, a non-linear—for instance quadratic—term in the accelerations by adding a ‘double-zero’ counter term, whose Lagrangian variation is zero on-shell and therefore which does not contribute to the dynamics (we refer to [6] for a general discussion on acceleration-dependent terms in a post-Newtonian Lagrangian). The conservative part of the equations of motion of body A (neglecting the 2.5PN and 3.5PN radiation damping terms) can be written with the help of the variational derivative of the Lagrangian as

$$\frac{\delta L}{\delta y_A^i} \equiv \frac{\partial L}{\partial y_A^i} - \frac{d}{dt} \left(\frac{\partial L}{\partial v_A^i} \right) + \frac{d^2}{dt^2} \left(\frac{\partial L}{\partial a_A^i} \right) = \mathcal{O} \left(\frac{1}{c^8} \right). \quad (2.2)$$

It is important to remember that here the equations of motion are supposed to have been ‘order-reduced’ using the equations themselves at lower post-Newtonian order (i.e. any a_A^i occurring in a post-Newtonian term must be replaced by its expression in terms of the y_B^i and v_B^i following the equations of motion).

The existence of a centre-of-mass integral for the 3PN dynamics is a consequence of its invariance under Lorentz transformations or boosts. The Lorentz invariance of the (harmonic-coordinates) equations of motion was established in [17]. Technically, this means that a specific variant, defined in [19], of the Hadamard regularization that respects the Lorentz invariance, is to be implemented. Consider an infinitesimal deformation of the path of the two particles, say $\delta y_A^i(t) \equiv y_A^i(t) - y_A^i(t)$. Then the corresponding perturbation of the Lagrangian, i.e. $\delta L = L[\mathbf{y}'_A, \mathbf{v}'_A, \mathbf{a}'_A] - L[\mathbf{y}_A, \mathbf{v}_A, \mathbf{a}_A]$, reads, to the linearized order,

$$\delta L = \frac{dQ}{dt} + \sum_A \frac{\delta L}{\delta y_A^i} \delta y_A^i + \mathcal{O}(\delta \mathbf{y}_A^2). \quad (2.3)$$

It involves the total time derivative of the function

$$Q = \sum_A (p_A^i \delta y_A^i + q_A^i \delta v_A^i), \quad (2.4)$$

which is defined in terms of the momenta conjugate to the velocities and accelerations,

$$p_A^i = \frac{\delta L}{\delta v_A^i} \equiv \frac{\partial L}{\partial v_A^i} - \frac{d}{dt} \left(\frac{\partial L}{\partial a_A^i} \right), \quad (2.5)$$

$$q_A^i = \frac{\delta L}{\delta a_A^i} \equiv \frac{\partial L}{\partial a_A^i}. \quad (2.6)$$

⁵ This is consistent with a general argument of Martin and Sanz [29] that in coordinate systems which preserve the Lorentz invariance (as the harmonic coordinates do) the equations of motion at 2PN and higher orders cannot be derived from an ordinary Lagrangian.

Equations (2.3), (2.5) and (2.6) are nothing other than the Noetherian equations in the case of a generalized Lagrangian.

In the case of a Lorentz transformation, the change in the position of particle A , to linear order in the (constant) boost velocity V^i , is

$$\delta y_A^i = -V^i t + \frac{1}{c^2} V^j y_A^j v_A^i + \mathcal{O}(\mathbf{V}^2). \quad (2.7)$$

Because the 3PN dynamics is invariant under Lorentz boosts, the change in the Lagrangian given by equation (2.3) must take the form of a total time derivative on-shell, i.e. when the equations of motion (2.2) are satisfied. Hence there should exist a certain functional Z^i of the positions, velocities and accelerations such that (on-shell)

$$\delta L = V^i \frac{dZ^i}{dt} + \mathcal{O}(\mathbf{V}^2). \quad (2.8)$$

Using this, together with the particular form of the transformation law (2.7), into equation (2.3) we readily obtain the conservation (on-shell) of the Noetherian integral $K^i = G^i - tP^i$, where P^i , the total linear momentum, and G^i , the centre-of-mass position, are given by

$$P^i = \sum_A p_A^i, \quad (2.9)$$

$$G^i = -Z^i + \sum_A \left(-q_A^i + \frac{1}{c^2} [y_A^i p_A^j v_A^j + y_A^i q_A^j a_A^j + v_A^i q_A^j v_A^j] \right). \quad (2.10)$$

Since P^i is itself constant (indeed, apply equations (2.3), (2.4) to the case of a constant spatial translation: $\delta y_A^i = \epsilon^i$),

$$\frac{dP^i}{dt} = 0, \quad (2.11)$$

we find that the conservation of K^i implies that

$$\frac{dG^i}{dt} = P^i. \quad (2.12)$$

(We neglect terms of order $\mathcal{O}(c^{-8})$.) The centre-of-mass vector G^i is conserved in the rest frame where $P^i = 0$; it will be zero, by definition, in the centre-of-mass frame.

Applying these considerations to the 3PN equations of motion and Lagrangian in harmonic coordinates, we found [20] that indeed the variation of the Lagrangian uniquely defines some function Z^i (this is a confirmation of the boost symmetry of the equations of motion), and from it we explicitly determined the centre-of-mass vector position in an arbitrary harmonic-coordinates frame⁶:

$$\begin{aligned} G^i = & m_1 y_1^i + \frac{m_1}{c^2} \left\{ \left(-\frac{m_2}{2r_{12}} + \frac{v_1^2}{2} \right) y_1^i \right\} + \frac{m_1}{c^4} \left\{ m_2 \left(-\frac{7}{4}(n_{12}v_1) - \frac{7}{4}(n_{12}v_2) \right) v_1^i \right. \\ & + \left[-\frac{5}{4} \frac{m_1 m_2}{r_{12}^2} + \frac{7}{4} \frac{m_2^2}{r_{12}^2} + \frac{3v_1^4}{8} + \frac{m_2}{r_{12}} \left(-\frac{1}{8}(n_{12}v_1)^2 - \frac{1}{4}(n_{12}v_1)(n_{12}v_2) \right. \right. \\ & \left. \left. + \frac{1}{8}(n_{12}v_2)^2 + \frac{19}{8}v_1^2 - \frac{7}{4}(v_1v_2) - \frac{7}{8}v_2^2 \right) \right] y_1^i \left\} + \frac{m_1}{c^6} \left\{ \left[\frac{235}{24} \frac{m_1 m_2}{r_{12}} (n_{12}v_{12}) \right. \right. \right. \\ & \left. \left. - \frac{235}{24} \frac{m_2^2}{r_{12}} (n_{12}v_{12}) + m_2 \left(\frac{5}{12}(n_{12}v_1)^3 + \frac{3}{8}(n_{12}v_1)^2 (n_{12}v_2) + \frac{3}{8}(n_{12}v_1)(n_{12}v_2)^2 \right) \right. \right. \end{aligned}$$

⁶ Throughout this paper the gravitational constant is set to $G = 1$.

$$\begin{aligned}
& + \frac{5}{12}(n_{12}v_2)^3 - \frac{15}{8}(n_{12}v_1)v_1^2 - (n_{12}v_2)v_1^2 + \frac{1}{4}(n_{12}v_1)(v_1v_2) + \frac{1}{4}(n_{12}v_2)(v_1v_2) \\
& - (n_{12}v_1)v_2^2 - \frac{15}{8}(n_{12}v_2)v_2^2 \Big] v_1^i + \left[\frac{5v_1^6}{16} + \frac{m_2}{r_{12}} \left(\frac{1}{16}(n_{12}v_1)^4 \right. \right. \\
& + \frac{1}{8}(n_{12}v_1)^3(n_{12}v_2) + \frac{3}{16}(n_{12}v_1)^2(n_{12}v_2)^2 + \frac{1}{4}(n_{12}v_1)(n_{12}v_2)^3 - \frac{1}{16}(n_{12}v_2)^4 \\
& - \frac{5}{16}(n_{12}v_1)^2v_1^2 - \frac{1}{2}(n_{12}v_1)(n_{12}v_2)v_1^2 - \frac{11}{8}(n_{12}v_2)^2v_1^2 + \frac{53}{16}v_1^4 \\
& + \frac{3}{8}(n_{12}v_1)^2(v_1v_2) + \frac{3}{4}(n_{12}v_1)(n_{12}v_2)(v_1v_2) + \frac{5}{4}(n_{12}v_2)^2(v_1v_2) - 5v_1^2(v_1v_2) \\
& + \frac{17}{8}(v_1v_2)^2 - \frac{1}{4}(n_{12}v_1)^2v_2^2 - \frac{5}{8}(n_{12}v_1)(n_{12}v_2)v_2^2 + \frac{5}{16}(n_{12}v_2)^2v_2^2 \\
& + \left. \frac{31}{16}v_1^2v_2^2 - \frac{15}{8}(v_1v_2)v_2^2 - \frac{11}{16}v_2^4 \right) + \frac{m_1m_2}{r_{12}^2} \left(\frac{79}{12}(n_{12}v_1)^2 - \frac{17}{3}(n_{12}v_1)(n_{12}v_2) \right. \\
& + \frac{17}{6}(n_{12}v_2)^2 - \frac{175}{24}v_1^2 + \frac{40}{3}(v_1v_2) - \frac{20}{3}v_2^2 \Big) + \frac{m_2^2}{r_{12}^2} \left(-\frac{7}{3}(n_{12}v_1)^2 \right. \\
& + \frac{29}{12}(n_{12}v_1)(n_{12}v_2) + \frac{2}{3}(n_{12}v_2)^2 + \frac{101}{12}v_1^2 - \frac{40}{3}(v_1v_2) + \frac{139}{24}v_2^2 \Big) - \frac{19}{8} \frac{m_1m_2^2}{r_{12}^3} \\
& + \left. \frac{m_2^2m_2}{r_{12}^3} \left(\frac{13721}{1260} - \frac{22}{3} \ln \left(\frac{r_{12}}{r_1'} \right) \right) + \frac{m_2^3}{r_{12}^3} \left(-\frac{14351}{1260} + \frac{22}{3} \ln \left(\frac{r_{12}}{r_2'} \right) \right) \right] y_1^i \Big\} \\
& + 1 \leftrightarrow 2 + \mathcal{O} \left(\frac{1}{c^8} \right). \tag{2.13}
\end{aligned}$$

To the terms given above we must add those corresponding to the relabelling $1 \leftrightarrow 2$. We denote by $r_{12} = |\mathbf{y}_1 - \mathbf{y}_2|$, $n_{12}^i = (y_1^i - y_2^i)/r_{12}$ and $v_{12}^i = v_1^i - v_2^i$ the relative particles' separation, unit direction and velocity.

Expression (2.13) has been systematically order-reduced using the equations of motion and therefore depends only on the positions and velocities (no accelerations). Note the appearance at the 3PN order of some logarithmic terms, containing two constants r_1' and r_2' (one for each body) having the dimension of a length. It was proved in [17, 20] that these logarithms, and the r_A' therein, can be removed by an infinitesimal gauge transformation at the 3PN order. Thus we can refer to the r_A' as some 'gauge constants', since they are merely associated with a choice of coordinate system, and thereby do not carry any physical meaning: they will always cancel when deriving some physical, gauge-invariant results. On the other hand, we note that G^i is free of the physical regularization ambiguity λ present in the equations of motion and Lagrangian.

The previous derivation of the centre of mass neglected the effect of radiation reaction. To take into account this effect we introduce some appropriate modifications at the 2.5PN order of the linear momentum and centre of mass position:

$$\tilde{P}^i = P^i + \left(\frac{4m_1^2m_2}{5c^5r_{12}^2} n_{12}^i \left[v_{12}^2 - \frac{2m_1}{r_{12}} \right] + 1 \leftrightarrow 2 \right), \tag{2.14}$$

$$\tilde{G}^i = G^i + \left(\frac{4m_1m_2}{5c^5} v_1^i \left[v_{12}^2 - \frac{2(m_1+m_2)}{r_{12}} \right] + 1 \leftrightarrow 2 \right). \tag{2.15}$$

With these definitions, we find that the conservation laws (2.11) and (2.12), but now when taking into account the radiation-reaction effect, take in fact exactly the same form

$$\frac{d\tilde{P}^i}{dt} = \mathcal{O}\left(\frac{1}{c^7}\right), \quad (2.16)$$

$$\frac{d\tilde{G}^i}{dt} = \tilde{P}^i + \mathcal{O}\left(\frac{1}{c^7}\right). \quad (2.17)$$

This finding is quite normal: recall that the total linear momentum of an isolated system is conserved up to the 3PN order included. Indeed, the integral over the system of the local radiation-reaction forces is a total time derivative at the 2.5PN order, and therefore it does not contribute to any change in the total linear momentum. The modification of the linear momentum by radiation reaction, or net radiation ‘recoil’ of the source, is a smaller effect, of order 3.5PN—negligible in equations (2.16) and (2.17).

3. Equations of motion in the centre-of-mass frame

The positions and velocities of the two particles in the centre-of-mass frame at the 3PN order are obtained by solving the equation

$$\tilde{G}^i[\mathbf{y}_A, \mathbf{v}_A] = \mathcal{O}\left(\frac{1}{c^7}\right), \quad (3.1)$$

where \tilde{G}^i is defined in the previous section. Obviously, the solution must be determined iteratively, in a post-Newtonian perturbative sense, with systematic order reduction of the equations. At the Newtonian order we get

$$y_1^i = X_2 x^i + \mathcal{O}\left(\frac{1}{c^2}\right), \quad (3.2)$$

$$y_2^i = -X_1 x^i + \mathcal{O}\left(\frac{1}{c^2}\right). \quad (3.3)$$

In this paper we employ the following notation. The binary’s relative separation is

$$x^i = y_1^i - y_2^i, \quad (3.4)$$

$$r = |\mathbf{x}| \quad \text{and} \quad n^i = \frac{x^i}{r}. \quad (3.5)$$

For the relative velocity and acceleration we pose

$$v^i = \frac{dx^i}{dt} = v_1^i - v_2^i \quad \text{and} \quad \dot{r} = \mathbf{n} \cdot \mathbf{v}, \quad (3.6)$$

$$a^i = \frac{dv^i}{dt}, \quad (3.7)$$

(r and v^i were formerly denoted by r_{12} and v_{12}^i in equation (2.13)). Concerning the mass parameters we denote

$$X_1 = \frac{m_1}{m} \quad \text{and} \quad X_2 = \frac{m_2}{m}, \quad (3.8)$$

$$m = m_1 + m_2, \quad (3.9)$$

$$v = \frac{m_1 m_2}{m^2} = X_1 X_2 \quad \text{and} \quad \mu = m v. \quad (3.10)$$

All the expressions that are written in the centre-of-mass frame are conveniently parametrized by m and the very useful mass ratio ν (such that $\nu = \frac{1}{4}$ in the equal-mass case and $\nu \rightarrow 0$ in the test-mass limit for one of the bodies). Often it is convenient to consider reduced quantities, i.e. quantities divided by the reduced mass μ .

The Newtonian solution (3.2), (3.3) is inserted into the 1PN terms of equation (3.1) and we then obtain an equation for the 1PN corrections in y_1^i and y_2^i . Solving that equation we plug the result back into the 1PN and 2PN terms of equation (3.1) and obtain the 2PN corrections in the same way. The process continues at the next order and this finally results in the 3PN-accurate relationship between the individual centre-of-mass positions y_1^i and y_2^i and the relative position x^i and velocity v^i . In the course of the computation we use for the order reduction the centre-of-mass equations of relative motion at the 2PN order—that is, at one post-Newtonian order before the 3PN order we want to reach. Since we give below the result for the 3PN equations, we do not detail this step and simply present the final expressions. They are in the form

$$y_1^i = [X_2 + \nu(X_1 - X_2)\mathcal{P}]x^i + \nu(X_1 - X_2)\mathcal{Q}v^i + \mathcal{O}\left(\frac{1}{c^7}\right), \quad (3.11)$$

$$y_2^i = [-X_1 + \nu(X_1 - X_2)\mathcal{P}]x^i + \nu(X_1 - X_2)\mathcal{Q}v^i + \mathcal{O}\left(\frac{1}{c^7}\right), \quad (3.12)$$

where all the post-Newtonian corrections beyond the Newtonian result (3.2), (3.3) are proportional to the mass ratio ν and the mass difference $X_1 - X_2$. The two dimensionless coefficients \mathcal{P} and \mathcal{Q} depend on the mass parameters m , ν , the distance r , the relative velocity $v^2 = \mathbf{v}^2$ and the radial velocity $\dot{r} = \mathbf{n} \cdot \mathbf{v}$:

$$\begin{aligned} \mathcal{P} = & \frac{1}{c^2} \left[\frac{v^2}{2} - \frac{m}{2r} \right] + \frac{1}{c^4} \left[\frac{3v^4}{8} - \frac{3\nu v^4}{2} + \frac{m}{r} \left(-\frac{\dot{r}^2}{8} + \frac{3\dot{r}^2\nu}{4} + \frac{19v^2}{8} + \frac{3\nu v^2}{2} \right) \right. \\ & + \frac{m^2}{r^2} \left(\frac{7}{4} - \frac{\nu}{2} \right) \left. \right] + \frac{1}{c^6} \left[\frac{5v^6}{16} - \frac{11\nu v^6}{4} + 6v^2 v^6 + \frac{m}{r} \left(\frac{\dot{r}^4}{16} - \frac{5\dot{r}^4\nu}{8} \right) \right. \\ & + \frac{21\dot{r}^4 v^2}{16} - \frac{5\dot{r}^2 v^2}{16} + \frac{21\dot{r}^2 \nu v^2}{16} - \frac{11\dot{r}^2 v^2 v^2}{2} + \frac{53v^4}{16} - 7\nu v^4 - \frac{15v^2 v^4}{2} \left. \right) \\ & + \frac{m^2}{r^2} \left(-\frac{7\dot{r}^2}{3} + \frac{73\dot{r}^2\nu}{8} + 4\dot{r}^2 v^2 + \frac{101v^2}{12} - \frac{33\nu v^2}{8} + 3v^2 v^2 \right) \\ & \left. + \frac{m^3}{r^3} \left(-\frac{14351}{1260} + \frac{\nu}{8} - \frac{v^2}{2} + \frac{22}{3} \ln\left(\frac{r}{r_0''}\right) \right) \right], \quad (3.13) \end{aligned}$$

$$\begin{aligned} \mathcal{Q} = & \frac{1}{c^4} \left[-\frac{7m\dot{r}}{4} \right] + \frac{1}{c^5} \left[\frac{4mv^2}{5} - \frac{8m^2}{5r} \right] + \frac{1}{c^6} \left[m\dot{r} \left(\frac{5\dot{r}^2}{12} - \frac{19\dot{r}^2\nu}{24} - \frac{15v^2}{8} + \frac{21\nu v^2}{4} \right) \right. \\ & \left. + \frac{m^2\dot{r}}{r} \left(-\frac{235}{24} - \frac{21\nu}{4} \right) \right]. \quad (3.14) \end{aligned}$$

Up to the 2.5PN order we find agreement with the circular-orbit limit of equations (6.4) in [30] (note that the 2.5PN radiation-reaction term itself is proportional to the velocity and so it enters only the coefficient \mathcal{Q}).

In equation (3.13) we find that the logarithms appear at the 3PN order and only in the coefficient \mathcal{P} . They contain a particular combination r_0'' of the two gauge constants r_1' and r_2' that is defined by

$$(X_1 - X_2) \ln r_0'' = X_1^2 \ln r_1' - X_2^2 \ln r_2'. \quad (3.15)$$

This constant r_0'' happens to be different from a similar constant r_0' which will have to be introduced to the 3PN equations of relative motion and Lagrangian (see equation (3.19)).

The 3PN centre-of-mass equations of motion are obtained in a straightforward way by replacing in the general-frame 3PN equations derived in [17] (see equation (7.16) there) the positions by equations (3.11)–(3.13), and the velocities by the derivatives of equations (3.11)–(3.13) (applying as usual the order reduction of all accelerations where necessary). Actually for this purpose we do not need the equations (3.11)–(3.13) with the full 3PN precision; the 2PN-accurate ones are sufficient. We write the relative acceleration in the centre-of-mass frame in the form

$$\frac{dv^i}{dt} = -\frac{m}{r^2}[(1 + \mathcal{A})n^i + \mathcal{B}v^i] + \mathcal{O}\left(\frac{1}{c^7}\right), \quad (3.16)$$

and find that the coefficients \mathcal{A} and \mathcal{B} are

$$\begin{aligned} \mathcal{A} = \frac{1}{c^2} & \left\{ -\frac{3\dot{r}^2 v}{2} + v^2 + 3vv^2 - \frac{m}{r}(4 + 2v) \right\} + \frac{1}{c^4} \left\{ \frac{15\dot{r}^4 v}{8} - \frac{45\dot{r}^4 v^2}{8} - \frac{9\dot{r}^2 v v^2}{2} \right. \\ & + 6\dot{r}^2 v^2 v^2 + 3vv^4 - 4v^2 v^4 + \frac{m}{r} \left(-2\dot{r}^2 - 25\dot{r}^2 v - 2\dot{r}^2 v^2 - \frac{13vv^2}{2} + 2v^2 v^2 \right) \\ & + \frac{m^2}{r^2} \left(9 + \frac{87v}{4} \right) \left. \right\} + \frac{1}{c^5} \left\{ -\frac{24\dot{r} v v^2 m}{5 r} - \frac{136\dot{r} v m^2}{15 r^2} \right\} \\ & + \frac{1}{c^6} \left\{ -\frac{35\dot{r}^6 v}{16} + \frac{175\dot{r}^6 v^2}{16} - \frac{175\dot{r}^6 v^3}{16} + \frac{15\dot{r}^4 v v^2}{2} - \frac{135\dot{r}^4 v^2 v^2}{4} + \frac{255\dot{r}^4 v^3 v^2}{8} \right. \\ & - \frac{15\dot{r}^2 v v^4}{2} + \frac{237\dot{r}^2 v^2 v^4}{8} - \frac{45\dot{r}^2 v^3 v^4}{2} + \frac{11vv^6}{4} - \frac{49v^2 v^6}{4} + 13v^3 v^6 \\ & + \frac{m}{r} \left(79\dot{r}^4 v - \frac{69\dot{r}^4 v^2}{2} - 30\dot{r}^4 v^3 - 121\dot{r}^2 v v^2 + 16\dot{r}^2 v^2 v^2 + 20\dot{r}^2 v^3 v^2 \right. \\ & + \frac{75vv^4}{4} + 8v^2 v^4 - 10v^3 v^4 \left. \right) + \frac{m^2}{r^2} \left(\dot{r}^2 + \frac{32573\dot{r}^2 v}{168} + \frac{11\dot{r}^2 v^2}{8} - 7\dot{r}^2 v^3 \right. \\ & + \frac{615\dot{r}^2 v \pi^2}{64} - \frac{26987vv^2}{840} + v^3 v^2 - \frac{123v\pi^2 v^2}{64} - 110\dot{r}^2 v \ln\left(\frac{r}{r_0'}\right) \\ & \left. + 22vv^2 \ln\left(\frac{r}{r_0'}\right) \right) + \frac{m^3}{r^3} \left(-16 - \frac{41911v}{420} + \frac{44\lambda v}{3} - \frac{71v^2}{2} + \frac{41v\pi^2}{16} \right) \left. \right\}, \quad (3.17) \end{aligned}$$

$$\begin{aligned} \mathcal{B} = \frac{1}{c^2} & \left\{ -4\dot{r} + 2\dot{r}v \right\} + \frac{1}{c^4} \left\{ \frac{9\dot{r}^3 v}{2} + 3\dot{r}^3 v^2 - \frac{15\dot{r} v v^2}{2} - 2\dot{r} v^2 v^2 + \frac{m}{r} \left(2\dot{r} + \frac{41\dot{r}v}{2} + 4\dot{r}v^2 \right) \right\} \\ & + \frac{1}{c^5} \left\{ \frac{8vv^2 m}{5 r} + \frac{24v m^2}{5 r^2} \right\} + \frac{1}{c^6} \left\{ -\frac{45\dot{r}^5 v}{8} + 15\dot{r}^5 v^2 + \frac{15\dot{r}^5 v^3}{4} + 12\dot{r}^3 v v^2 \right. \\ & - \frac{111\dot{r}^3 v^2 v^2}{4} - 12\dot{r}^3 v^3 v^2 - \frac{65\dot{r} v v^4}{8} + 19\dot{r} v^2 v^4 + 6\dot{r} v^3 v^4 \\ & + \frac{m}{r} \left(\frac{329\dot{r}^3 v}{6} + \frac{59\dot{r}^3 v^2}{2} + 18\dot{r}^3 v^3 - 15\dot{r} v v^2 - 27\dot{r} v^2 v^2 - 10\dot{r} v^3 v^2 \right) \\ & \left. + \frac{m^2}{r^2} \left(-4\dot{r} - \frac{18169\dot{r}v}{840} + 25\dot{r}v^2 + 8\dot{r}v^3 - \frac{123\dot{r}v\pi^2}{32} + 44\dot{r}v \ln\left(\frac{r}{r_0'}\right) \right) \right\}. \quad (3.18) \end{aligned}$$

Up to the 2.5PN order the result agrees with the one given by Lincoln and Will [27]. At the 3PN order we have some gauge-dependent logarithms containing a constant r'_0 —distinct from r''_0 introduced in equation (3.15)⁷—which is the ‘logarithmic barycentre’ of the two constants r'_1 and r'_2 :

$$\ln r'_0 = X_1 \ln r'_1 + X_2 \ln r'_2. \quad (3.19)$$

In addition, there is the physical ambiguity λ due to the Hadamard self-field regularization (λ cannot be removed by any coordinate transformation); it appears at the 3PN order in the \mathcal{A} -coefficient.

4. Lagrangian and Noetherian conserved integrals

The Lagrangian for the relative centre-of-mass motion is obtained from the 3PN centre-of-mass equations of motion (3.16)–(3.18) in which one ignores for a moment the radiation-reaction 2.5PN term. The Lagrangian in harmonic coordinates will necessarily be a generalized one, depending on accelerations, from the 2PN order. At the 3PN order, further acceleration terms are necessary but we do not need to include derivatives of accelerations. Furthermore we can always restrict ourselves to a Lagrangian that is linear in the accelerations. Hence, our centre-of-mass Lagrangian, denoted with a calligraphic letter \mathcal{L} to distinguish it from the general-frame Lagrangian L , is of the form

$$\mathcal{L} = \mathcal{L}_N[\mathbf{x}, \mathbf{v}] + \frac{1}{c^2} \mathcal{L}_{1\text{PN}}[\mathbf{x}, \mathbf{v}] + \frac{1}{c^4} \mathcal{L}_{2\text{PN}}[\mathbf{x}, \mathbf{v}, \mathbf{a}] + \frac{1}{c^6} \mathcal{L}_{3\text{PN}}[\mathbf{x}, \mathbf{v}, \mathbf{a}]. \quad (4.1)$$

We recall that there is a large freedom for choosing a Lagrangian because we can always add to it the total time derivative of an arbitrary function. As a matter of convenience, we shall choose below a particular centre-of-mass Lagrangian in such a way that it is ‘close’ (in the sense that many coefficients are identical) to some ‘fictitious’ Lagrangian that is obtained from the general-frame one given in [20] by the mere Newtonian replacements $y_1^i \rightarrow X_2 x^i$, $y_2^i \rightarrow -X_1 x^i$. We immediately point out that such a fictitious Lagrangian is *not* the correct Lagrangian for describing the centre-of-mass relative motion. Indeed, the actual centre-of-mass variables involve many post-Newtonian corrections given by equations (3.11)–(3.14), so the actual centre-of-mass Lagrangian must contain some extra terms in addition to those of the previous fictitious Lagrangian. However, we find that these extra terms arise only at the 2PN order and not before. We did not try to find a general method for obtaining systematically the centre-of-mass Lagrangian given the general-frame one. Though such a method might exist it was in fact simpler to proceed by guess-work, i.e. to introduce some unknown coefficients in front of all possible types of terms, and to adjust these coefficients so that the Lagrangian reproduces the correct equations of motion. Our result (when divided by the reduced mass $\mu = mv$) is then

$$\begin{aligned} \frac{\mathcal{L}}{\mu} = & \frac{v^2}{2} + \frac{m}{r} + \frac{1}{c^2} \left\{ \frac{v^4}{8} - \frac{3vv^4}{8} + \frac{m}{r} \left(\frac{\dot{r}^2 v}{2} + \frac{3v^2}{2} + \frac{vv^2}{2} \right) - \frac{m^2}{2r^2} \right\} \\ & + \frac{1}{c^4} \left\{ \frac{v^6}{16} - \frac{7vv^6}{16} + \frac{13v^2v^6}{16} + \frac{m}{r} \left(\frac{3\dot{r}^4 v^2}{8} - \frac{\dot{r}^2 a_n v r}{8} + \frac{\dot{r}^2 v v^2}{4} \right. \right. \\ & \left. \left. - \frac{5\dot{r}^2 v^2 v^2}{4} + \frac{7a_n v r v^2}{8} + \frac{7v^4}{8} - \frac{5vv^4}{4} - \frac{9v^2 v^4}{8} - \frac{7\dot{r} v r a_v}{4} \right) \right\} \end{aligned}$$

⁷ They are related by

$$(X_1 - X_2) \ln \left(\frac{r''_0}{r'_0} \right) = X_1 X_2 \ln \left(\frac{r'_1}{r'_2} \right).$$

$$\begin{aligned}
& + \frac{m^2}{r^2} \left(\frac{\dot{r}^2}{2} + \frac{41\dot{r}^2 v}{8} + \frac{3\dot{r}^2 v^2}{2} + \frac{7v^2}{4} - \frac{27vv^2}{8} + \frac{v^2 v^2}{2} \right) + \frac{m^3}{r^3} \left(\frac{1}{2} + \frac{15v}{4} \right) \Bigg\} \\
& + \frac{1}{c^6} \left\{ \frac{5v^8}{128} - \frac{59vv^8}{128} + \frac{119v^2 v^8}{64} - \frac{323v^3 v^8}{128} + \frac{m}{r} \left(\frac{5\dot{r}^6 v^3}{16} + \frac{\dot{r}^4 a_n vr}{16} \right. \right. \\
& - \frac{5\dot{r}^4 a_n v^2 r}{16} - \frac{3\dot{r}^4 v v^2}{16} + \frac{7\dot{r}^4 v^2 v^2}{4} - \frac{33\dot{r}^4 v^3 v^2}{16} - \frac{3\dot{r}^2 a_n vr v^2}{16} - \frac{\dot{r}^2 a_n v^2 r v^2}{16} \\
& + \frac{5\dot{r}^2 v v^4}{8} - \frac{3\dot{r}^2 v^2 v^4}{16} + \frac{75\dot{r}^2 v^3 v^4}{16} + \frac{7a_n vr v^4}{8} - \frac{7a_n v^2 r v^4}{2} + \frac{11v^6}{16} - \frac{55vv^6}{16} \\
& \left. + \frac{5v^2 v^6}{2} + \frac{65v^3 v^6}{16} + \frac{5\dot{r}^3 vr a_v}{12} - \frac{13\dot{r}^3 v^2 r a_v}{8} - \frac{37\dot{r} vr v^2 a_v}{8} + \frac{35\dot{r} v^2 r v^2 a_v}{4} \right) \\
& + \frac{m^2}{r^2} \left(-\frac{109\dot{r}^4 v}{144} - \frac{259\dot{r}^4 v^2}{36} + 2\dot{r}^4 v^3 - \frac{17\dot{r}^2 a_n vr}{6} + \frac{97\dot{r}^2 a_n v^2 r}{12} + \frac{\dot{r}^2 v^2}{4} \right. \\
& - \frac{41\dot{r}^2 v v^2}{6} - \frac{2287\dot{r}^2 v^2 v^2}{48} - \frac{27\dot{r}^2 v^3 v^2}{4} + \frac{203a_n vr v^2}{12} + \frac{149a_n v^2 r v^2}{6} \\
& \left. + \frac{45v^4}{16} + \frac{53vv^4}{24} + \frac{617v^2 v^4}{24} - \frac{9v^3 v^4}{4} - \frac{235\dot{r} vr a_v}{24} + \frac{235\dot{r} v^2 r a_v}{6} \right) \\
& + \frac{m^3}{r^3} \left(\frac{3\dot{r}^2}{2} - \frac{12041\dot{r}^2 v}{420} + \frac{37\dot{r}^2 v^2}{4} + \frac{7\dot{r}^2 v^3}{2} - \frac{123\dot{r}^2 v \pi^2}{64} \right. \\
& + \frac{5v^2}{4} + \frac{387vv^2}{70} - \frac{7v^2 v^2}{4} + \frac{v^3 v^2}{2} + \frac{41v\pi^2 v^2}{64} + 22\dot{r}^2 v \ln \left(\frac{r}{r'_0} \right) \\
& \left. - \frac{22vv^2}{3} \ln \left(\frac{r}{r'_0} \right) \right) + \frac{m^4}{r^4} \left(-\frac{3}{8} - \frac{2747v}{140} + \frac{11\lambda v}{3} + \frac{22v}{3} \ln \left(\frac{r}{r'_0} \right) \right) \Bigg\}. \quad (4.2)
\end{aligned}$$

Witness the acceleration terms present at the 2PN and 3PN orders: our notation is $a_n \equiv \mathbf{a} \cdot \mathbf{n}$ and $a_v \equiv \mathbf{a} \cdot \mathbf{v}$ for the scalar products between $a^i = dv^i/dt$ and the direction n^i and velocity v^i . The conservative part of the equations of motion is then identical (after order reduction of the accelerations) to

$$\frac{\delta \mathcal{L}}{\delta x^i} \equiv \frac{\partial \mathcal{L}}{\partial x^i} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial v^i} \right) + \frac{d^2}{dt^2} \left(\frac{\partial \mathcal{L}}{\partial a^i} \right) = \mathcal{O} \left(\frac{1}{c^8} \right). \quad (4.3)$$

From the Lagrangian one deduces the conserved energy and angular momentum using the generalized formulae (neglecting $\mathcal{O}(c^{-8})$)

$$E = v^i p^i + a^i q^i - \mathcal{L}, \quad (4.4)$$

$$J^i = \varepsilon_{ijk} (x^j p^k + v^j q^k), \quad (4.5)$$

(the first one being a generalized version of the Legendre transform), where the conjugate momenta read

$$p^i = \frac{\delta \mathcal{L}}{\delta v^i} \equiv \frac{\partial \mathcal{L}}{\partial v^i} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial a^i} \right), \quad (4.6)$$

$$q^i = \frac{\delta \mathcal{L}}{\delta a^i} \equiv \frac{\partial \mathcal{L}}{\partial a^i}. \quad (4.7)$$

Alternatively one can compute the centre-of-mass energy and angular momentum directly from the general-frame quantities E and J^i given by equations (4.2) and (4.4) in [20] by replacing all variables by their centre-of-mass expressions given by equations (3.11)–(3.14);

the result is the same. For the energy we obtain

$$\begin{aligned}
\frac{E}{\mu} = & \frac{v^2}{2} - \frac{m}{r} + \frac{1}{c^2} \left\{ \frac{3v^4}{8} - \frac{9vv^4}{8} + \frac{m}{r} \left(\frac{\dot{r}^2 v}{2} + \frac{3v^2}{2} + \frac{vv^2}{2} \right) + \frac{m^2}{2r^2} \right\} \\
& + \frac{1}{c^4} \left\{ \frac{5v^6}{16} - \frac{35vv^6}{16} + \frac{65v^2v^6}{16} + \frac{m}{r} \left(-\frac{3\dot{r}^4 v}{8} + \frac{9\dot{r}^4 v^2}{8} + \frac{\dot{r}^2 vv^2}{4} \right. \right. \\
& \quad \left. \left. - \frac{15\dot{r}^2 v^2 v^2}{4} + \frac{21v^4}{8} - \frac{23vv^4}{8} - \frac{27v^2 v^4}{8} \right) + \frac{m^2}{r^2} \left(\frac{\dot{r}^2}{2} + \frac{69\dot{r}^2 v}{8} + \frac{3\dot{r}^2 v^2}{2} \right. \right. \\
& \quad \left. \left. + \frac{7v^2}{4} - \frac{55vv^2}{8} + \frac{v^2 v^2}{2} \right) + \frac{m^3}{r^3} \left(-\frac{1}{2} - \frac{15v}{4} \right) \right\} + \frac{1}{c^6} \left\{ \frac{35v^8}{128} - \frac{413vv^8}{128} \right. \\
& \quad + \frac{833v^2 v^8}{64} - \frac{2261v^3 v^8}{128} + \frac{m}{r} \left(\frac{5\dot{r}^6 v}{16} - \frac{25\dot{r}^6 v^2}{16} + \frac{25\dot{r}^6 v^3}{16} - \frac{9\dot{r}^4 vv^2}{16} \right. \\
& \quad \left. + \frac{21\dot{r}^4 v^2 v^2}{4} - \frac{165\dot{r}^4 v^3 v^2}{16} - \frac{21\dot{r}^2 vv^4}{16} - \frac{75\dot{r}^2 v^2 v^4}{16} + \frac{375\dot{r}^2 v^3 v^4}{16} + \frac{55v^6}{16} \right. \\
& \quad \left. - \frac{215vv^6}{16} + \frac{29v^2 v^6}{4} + \frac{325v^3 v^6}{16} \right) + \frac{m^2}{r^2} \left(-\frac{731\dot{r}^4 v}{48} + \frac{41\dot{r}^4 v^2}{4} + 6\dot{r}^4 v^3 \right. \\
& \quad \left. + \frac{3\dot{r}^2 v^2}{4} + \frac{31\dot{r}^2 vv^2}{2} - \frac{815\dot{r}^2 v^2 v^2}{16} - \frac{81\dot{r}^2 v^3 v^2}{4} + \frac{135v^4}{16} - \frac{97vv^4}{8} + \frac{203v^2 v^4}{8} \right. \\
& \quad \left. - \frac{27v^3 v^4}{4} \right) + \frac{m^3}{r^3} \left(\frac{3\dot{r}^2}{2} + \frac{803\dot{r}^2 v}{840} + \frac{51\dot{r}^2 v^2}{4} + \frac{7\dot{r}^2 v^3}{2} - \frac{123\dot{r}^2 vv^2}{64} \right. \\
& \quad \left. + \frac{5v^2}{4} - \frac{6747vv^2}{280} - \frac{21v^2 v^2}{4} + \frac{v^3 v^2}{2} + \frac{41v\pi^2 v^2}{64} + 22\dot{r}^2 v \ln \left(\frac{r}{r'_0} \right) \right. \\
& \quad \left. - \frac{22vv^2}{3} \ln \left(\frac{r}{r'_0} \right) \right) + \frac{m^4}{r^4} \left(\frac{3}{8} + \frac{2747v}{140} - \frac{11\lambda v}{3} - \frac{22v}{3} \ln \left(\frac{r}{r'_0} \right) \right) \left. \right\}. \quad (4.8)
\end{aligned}$$

As for the centre-of-mass angular momentum we get

$$\begin{aligned}
\frac{J^i}{\mu} = & \varepsilon_{ijk} x^j v^k \left[1 + \frac{1}{c^2} \left\{ (1-3v) \frac{v^2}{2} + \frac{m}{r} (3+v) \right\} + \frac{1}{c^4} \left\{ \frac{3v^4}{8} - \frac{21vv^4}{8} + \frac{39v^2 v^4}{8} \right. \right. \\
& \quad \left. \left. + \frac{m}{r} \left(-\dot{r}^2 v - \frac{5\dot{r}^2 v^2}{2} + \frac{7v^2}{2} - 5vv^2 - \frac{9v^2 v^2}{2} \right) + \frac{m^2}{r^2} \left(\frac{7}{2} - \frac{41v}{4} + v^2 \right) \right\} \right. \\
& \quad \left. + \frac{1}{c^6} \left\{ \frac{5v^6}{16} - \frac{59vv^6}{16} + \frac{119v^2 v^6}{8} - \frac{323v^3 v^6}{16} + \frac{m}{r} \left(\frac{3\dot{r}^4 v}{4} - \frac{3\dot{r}^4 v^2}{4} - \frac{33\dot{r}^4 v^3}{8} \right. \right. \right. \\
& \quad \left. \left. - 3\dot{r}^2 vv^2 + \frac{7\dot{r}^2 v^2 v^2}{4} + \frac{75\dot{r}^2 v^3 v^2}{4} + \frac{33v^4}{8} - \frac{71vv^4}{4} + \frac{53v^2 v^4}{4} + \frac{195v^3 v^4}{8} \right) \right. \\
& \quad \left. + \frac{m^2}{r^2} \left(\frac{\dot{r}^2}{2} - \frac{287\dot{r}^2 v}{24} - \frac{317\dot{r}^2 v^2}{8} - \frac{27\dot{r}^2 v^3}{2} + \frac{45v^2}{4} - \frac{161vv^2}{6} + \frac{105v^2 v^2}{4} \right. \right. \\
& \quad \left. \left. - 9v^3 v^2 \right) + \frac{m^3}{r^3} \left(\frac{5}{2} - \frac{5199v}{280} - 7v^2 + v^3 + \frac{41v\pi^2}{32} - \frac{44v}{3} \ln \left(\frac{r}{r'_0} \right) \right) \right] \left. \right\}. \quad (4.9)
\end{aligned}$$

(The energy involves the regularization ambiguity λ , while the angular momentum is free of any physical ambiguity.) These quantities are conserved in the sense that their time variation equals the radiation-reaction effect. One can therefore modify them with terms of ‘odd order’ to take into account the radiation reaction due to gravitational wave emission. For instance,

in the leading 2.5PN radiation reaction one conventionally chooses that the right-hand sides of the balance equations for energy and angular momentum take the standard form appropriate to the quadrupolar approximation. We then pose

$$\tilde{E} = E + \frac{8m^3 \dot{r} v^2}{5c^5 r^2}, \quad (4.10)$$

$$\tilde{J}^i = J^i - \frac{8m^3 \dot{r} v^2}{5c^5 r^2} \varepsilon_{ijk} x^j v^k. \quad (4.11)$$

This choice is in agreement with the results of [21–23]. Then we can easily check that

$$\frac{d\tilde{E}}{dt} = -\frac{1}{5c^5} \ddot{Q}_{ij} \ddot{Q}_{ij} + \mathcal{O}\left(\frac{1}{c^7}\right), \quad (4.12)$$

$$\frac{d\tilde{J}^i}{dt} = -\frac{2}{5c^5} \varepsilon_{ijk} \ddot{Q}_{jl} \ddot{Q}_{kl} + \mathcal{O}\left(\frac{1}{c^7}\right), \quad (4.13)$$

where the Newtonian trace-free quadrupole moment is $Q_{ij} = \mu(x^i x^j - \frac{1}{3} \delta^{ij} r^2)$.

5. Lagrangian and Hamiltonian in ADM-coordinates

In [20] (see also [14]) we determined the ‘contact’ transformation between the particles’ variables in harmonic coordinates and those in ADM (or rather ADM-type⁸) coordinates. By contact transformation we mean the relation between the particles’ trajectories in harmonic coordinates, $y_A^i(t)$, and the corresponding ones in ADM coordinates, say $Y_A^i(t)$. We recall that in the contact transformation, i.e.

$$\delta y_A^i(t) = Y_A^i(t) - y_A^i(t), \quad (5.1)$$

the time coordinate t is to be viewed as a ‘dummy’ variable⁹. There is a unique transformation (5.1) such that the 3PN harmonic-coordinates Lagrangian of de Andrade *et al* [20] (in a general frame) is changed into another Lagrangian whose Legendre transform coincides with the 3PN ADM-coordinates Hamiltonian derived by Damour *et al* [13]. The explicit expression of this general-frame contact transformation can be found in section 4.2 of [20].

Now we are in a position to obtain the relation between the relative separation vector $x^i \equiv y_1^i - y_2^i$ in harmonic coordinates and the one $X^i \equiv Y_1^i - Y_2^i$ in ADM coordinates (do not confuse the relative distances x^i and X^i between the two particles with the spatial position vector of some field event in these coordinates). Namely,

$$\delta x^i \equiv X^i - x^i = \delta y_1^i - \delta y_2^i, \quad (5.2)$$

where δy_1^i and δy_2^i are given by equations (4.8)–(4.10) in [20]. (We shall always view such equalities as (5.2) as functional equalities, i.e. valid for any dummy time variable t .) One replaces in equation (5.2) all the variables by their centre-of-mass counterparts following equations (3.11)–(3.14). Actually, since the contact transformation is already of relative order 2PN, the calculation is quite immediate and requires only the equations (3.11)–(3.14) to 1PN

⁸ Strictly speaking, the ADM coordinates we are considering differ from the actual ADM coordinates at the 3PN order by a shift of phase-space coordinates that is given in [13].

⁹ The contact transformation is not a coordinate transformation between the spatial vectors in both coordinates, but takes also into account the fact that the time coordinate changes as well, i.e. $\delta y_A^i = \xi^i(y_A) - \xi^0(y_A) v_A^i / c$, where $\xi^\mu(y_A)$ denotes the four-dimensional change between the harmonic and ADM coordinates, when evaluated at the position $y_A = (t, y_A^i)$.

order. As a result the spatial separation vectors X^i and x^i in both coordinates (each one living within the spatial slice of constant time appropriate to each of the coordinate systems) are related to each other by

$$\begin{aligned} \frac{\delta x^i}{m} = \frac{1}{c^4} \left\{ \left[\frac{\dot{r}^2 v}{8} - \frac{5v v^2}{8} + \frac{m}{r} \left(-\frac{1}{4} - 3v \right) \right] n^i + \frac{9\dot{r}v}{4} v^i \right\} \\ + \frac{1}{c^6} \left\{ \left[-\frac{\dot{r}^4 v}{16} + \frac{5\dot{r}^4 v^2}{16} + \frac{5\dot{r}^2 v v^2}{16} - \frac{15\dot{r}^2 v^2 v^2}{16} - \frac{v v^4}{2} + \frac{11v^2 v^4}{8} \right. \right. \\ + \frac{m}{r} \left(\frac{161\dot{r}^2 v}{48} - \frac{5\dot{r}^2 v^2}{2} - \frac{451v v^2}{48} - \frac{3v^2 v^2}{8} \right) \\ + \frac{m^2}{r^2} \left(\frac{2773v}{280} + \frac{21v\pi^2}{32} - \frac{22v}{3} \ln \left(\frac{r}{r_0'} \right) \right) \left. \right] n^i \\ + \left[-\frac{5\dot{r}^3 v}{12} + \frac{29\dot{r}^3 v^2}{24} + \frac{17\dot{r}v v^2}{8} - \frac{21\dot{r}v^2 v^2}{4} + \frac{m}{r} \left(\frac{43\dot{r}v}{3} + 5\dot{r}v^2 \right) \right] v^i \left. \right\}. \end{aligned} \quad (5.3)$$

Below we shall deduce from this formula the radius of the circular orbit in ADM coordinates, say R_0 , versus the one in harmonic coordinates, i.e. r_0 (see equation (6.39)).

Now we look for the centre-of-mass Lagrangian in ADM coordinates. Since in ADM coordinates the Lagrangian is ‘ordinary’ (no accelerations) the contact transformation must be such that it removes the acceleration terms present in harmonic coordinates—more precisely, it must make them in the form of a total time derivative which is irrelevant to the dynamics. Following an investigation similar to the one in section 3.2 of [20] (see notably equation (3.18) there) we know that \mathcal{L}^{ADM} differs from \mathcal{L} by two terms: (1) the functional variation of \mathcal{L} induced at the linearized order by the contact transformation of the relative path as given by equations (5.2), (5.3); (2) the total time derivative of a function \mathcal{F} of the relative position and velocity. We can limit our consideration to the linearized order because δx^i is at least of order 2PN, so the non-linear terms do not contribute before the 4PN order and are negligible here. Hence we necessarily have the following *functional* equality (by which we mean the equality between functions of the same *dummy* variables $\mathbf{x}, \mathbf{v}, \mathbf{a}$):

$$\mathcal{L}^{\text{ADM}}[\mathbf{x}, \mathbf{v}] = \mathcal{L}[\mathbf{x}, \mathbf{v}, \mathbf{a}] - \frac{\delta \mathcal{L}}{\delta x^i} \delta x^i + \frac{d\mathcal{F}}{dt}, \quad (5.4)$$

in which δx^i is explicitly given by equation (5.3), and where the function \mathcal{F} is for the moment unknown¹⁰. We insist that in the present calculation the contact transformation δx^i is known so that the only freedom left is the choice of \mathcal{F} . This contrasts with our earlier study in [20] where both the contact transformation of the individual paths, δy_1^i and δy_2^i , and some arbitrary function, say F , had to be varied and determined. The reason, of course, is that once δy_1^i and δy_2^i are known from [20] we have no choice for δx^i which must be equal to the centre-of-mass reduction of the difference $\delta y_1^i - \delta y_2^i$ (see equation (5.2)). Thus, despite the smaller freedom that we presently have in the adjustment of parameters, the calculation *must* work with that δx^i and not with another one.

The function $\mathcal{F} = \mathcal{F}[\mathbf{x}, \mathbf{v}]$ is not difficult to determine in order to match perfectly the ADM Hamiltonian. Note that after adding the total time derivative of that function, not

¹⁰ Note that the minus sign in front of the second term in equation (5.4) differs from the one in equation (3.18) of [20]. The reason is because we have corrected a sign inconsistency in [20]: namely the equation (3.12) there, together with the adopted definition $\delta y_A^i = y_A^i - y_A^i$, is inconsistent with (3.13) and (3.18); but this does not change any of the results of [20].

only has one been able to remove all the accelerations, but also one has gauged away all the logarithms which were present in the harmonic-coordinates Lagrangian. We get

$$\begin{aligned} \frac{\mathcal{F}}{m\dot{r}} = \frac{1}{c^4} & \left[-\frac{\nu v^2}{4} + \frac{m}{r} \left(\frac{1}{4} + 3\nu \right) \right] + \frac{1}{c^6} \left[-\frac{\nu \dot{r}^2 v^2}{16} + \frac{19\nu^2 \dot{r}^2 v^2}{48} - \frac{\nu v^4}{16} + \frac{19\nu^2 v^4}{16} \right. \\ & + \frac{m}{r} \left(-\frac{43\nu \dot{r}^2}{144} - \frac{97\nu^2 \dot{r}^2}{36} + \frac{v^2}{8} - \frac{217\nu v^2}{48} - \frac{665\nu^2 v^2}{24} \right) \\ & \left. + \frac{m^2}{r^2} \left(\frac{3}{4} - \frac{113\nu}{280} + 6\nu^2 - \frac{21\nu\pi^2}{32} + \frac{22\nu}{3} \ln \left(\frac{r}{r'_0} \right) \right) \right]. \end{aligned} \quad (5.5)$$

Next, equation (5.4) together with the explicit expressions (5.3) and (5.5) gives the ADM centre-of-mass Lagrangian. Once the calculation is done we have to express it using the names appropriate to the ADM variables: $X^i = x^i + \delta x^i$, which means the separation distance R , the relative square velocity V^2 and the radial velocity $\dot{R} = \mathbf{N} \cdot \mathbf{V}$. The formula is

$$\begin{aligned} \frac{\mathcal{L}^{\text{ADM}}}{\mu} = \frac{m}{R} + \frac{V^2}{2} + \frac{1}{c^2} & \left\{ \frac{V^4}{8} - \frac{3\nu V^4}{8} + \frac{m}{R} \left(\frac{\nu \dot{R}^2}{2} + \frac{3V^2}{2} + \frac{\nu V^2}{2} \right) - \frac{m^2}{2R^2} \right\} \\ & + \frac{1}{c^4} \left\{ \frac{V^6}{16} - \frac{7\nu V^6}{16} + \frac{13\nu^2 V^6}{16} + \frac{m}{R} \left(\frac{3\nu^2 \dot{R}^4}{8} + \frac{\nu \dot{R}^2 V^2}{2} - \frac{5\nu^2 \dot{R}^2 V^2}{4} \right. \right. \\ & \left. \left. + \frac{7V^4}{8} - \frac{3\nu V^4}{2} - \frac{9\nu^2 V^4}{8} \right) + \frac{m^2}{R^2} \left(\frac{3\nu \dot{R}^2}{2} + \frac{3\nu^2 \dot{R}^2}{2} + 2V^2 - \nu V^2 + \frac{\nu^2 V^2}{2} \right) \right. \\ & \left. + \frac{m^3}{R^3} \left(\frac{1}{4} + \frac{3\nu}{4} \right) \right\} + \frac{1}{c^6} \left\{ \frac{5V^8}{128} - \frac{59\nu V^8}{128} + \frac{119\nu^2 V^8}{64} - \frac{323\nu^3 V^8}{128} \right. \\ & + \frac{m}{R} \left(\frac{5\nu^3 \dot{R}^6}{16} + \frac{9\nu^2 \dot{R}^4 V^2}{16} - \frac{33\nu^3 \dot{R}^4 V^2}{16} + \frac{\nu \dot{R}^2 V^4}{2} - 3\nu^2 \dot{R}^2 V^4 \right. \\ & \left. + \frac{75\nu^3 \dot{R}^2 V^4}{16} + \frac{11V^6}{16} - \frac{7\nu V^6}{2} + \frac{59\nu^2 V^6}{16} + \frac{65\nu^3 V^6}{16} \right) \\ & + \frac{m^2}{R^2} \left(-\frac{5\nu \dot{R}^4}{12} + \frac{17\nu^2 \dot{R}^4}{12} + 2\nu^3 \dot{R}^4 + \frac{39\nu \dot{R}^2 V^2}{16} - \frac{29\nu^2 \dot{R}^2 V^2}{8} \right. \\ & \left. - \frac{27\nu^3 \dot{R}^2 V^2}{4} + \frac{47V^4}{16} - \frac{15\nu V^4}{4} - \frac{25\nu^2 V^4}{16} - \frac{9\nu^3 V^4}{4} \right) \\ & \left. + \frac{m^3}{R^3} \left(\frac{77\nu \dot{R}^2}{16} + \frac{5\nu^2 \dot{R}^2}{4} + \frac{7\nu^3 \dot{R}^2}{2} + \frac{3\nu \dot{R}^2 \pi^2}{64} + \frac{13V^2}{8} - \frac{409\nu V^2}{48} - \frac{5\nu^2 V^2}{8} \right. \right. \\ & \left. \left. + \frac{\nu^3 V^2}{2} - \frac{\nu \pi^2 V^2}{64} \right) + \frac{m^4}{R^4} \left(-\frac{1}{8} - \frac{1881\nu}{280} + \frac{11\nu\lambda\nu}{3} + \frac{21\nu\pi^2}{32} \right) \right\}. \end{aligned} \quad (5.6)$$

This is an ordinary Lagrangian and we apply the ordinary Legendre transform to obtain the Hamiltonian, which is a function of the conjugate momentum

$$P^i = \frac{\partial \mathcal{L}^{\text{ADM}}}{\partial V^i}, \quad (5.7)$$

$$P^2 \equiv \mathbf{P}^2 \quad \text{and} \quad P_R \equiv \mathbf{N} \cdot \mathbf{P}. \quad (5.8)$$

We find perfect agreement with the centre-of-mass Hamiltonian derived in [11–13]:

$$\begin{aligned}
\frac{\mathcal{H}^{\text{ADM}}}{\mu} = & \frac{P^2}{2} - \frac{m}{R} + \frac{1}{c^2} \left\{ -\frac{P^4}{8} + \frac{3\nu P^4}{8} + \frac{m}{R} \left(-\frac{P_R^2 \nu}{2} - \frac{3P^2}{2} - \frac{\nu P^2}{2} \right) + \frac{m^2}{2R^2} \right\} \\
& + \frac{1}{c^4} \left\{ \frac{P^6}{16} - \frac{5\nu P^6}{16} + \frac{5\nu^2 P^6}{16} + \frac{m}{R} \left(-\frac{3P_R^4 \nu^2}{8} - \frac{P_R^2 P^2 \nu^2}{4} \right. \right. \\
& \left. \left. + \frac{5P^4}{8} - \frac{5\nu P^4}{2} - \frac{3\nu^2 P^4}{8} \right) + \frac{m^2}{R^2} \left(\frac{3P_R^2 \nu}{2} + \frac{5P^2}{2} + 4\nu P^2 \right) \right. \\
& \left. + \frac{m^3}{R^3} \left(-\frac{1}{4} - \frac{3\nu}{4} \right) \right\} + \frac{1}{c^6} \left\{ -\frac{5P^8}{128} + \frac{35\nu P^8}{128} - \frac{35\nu^2 P^8}{64} + \frac{35\nu^3 P^8}{128} \right. \\
& \left. + \frac{m}{R} \left(-\frac{5P_R^6 \nu^3}{16} + \frac{3P_R^4 P^2 \nu^2}{16} - \frac{3P_R^4 P^2 \nu^3}{16} + \frac{P_R^2 P^4 \nu^2}{8} \right. \right. \\
& \left. \left. - \frac{3P_R^2 P^4 \nu^3}{16} - \frac{7P^6}{16} + \frac{21\nu P^6}{8} - \frac{53\nu^2 P^6}{16} - \frac{5\nu^3 P^6}{16} \right) \right. \\
& \left. + \frac{m^2}{R^2} \left(\frac{5P_R^4 \nu}{12} + \frac{43P_R^4 \nu^2}{12} + \frac{17P_R^2 P^2 \nu}{16} + \frac{15P_R^2 P^2 \nu^2}{8} - \frac{27P^4}{16} + \frac{17\nu P^4}{2} \right. \right. \\
& \left. \left. + \frac{109\nu^2 P^4}{16} \right) + \frac{m^3}{R^3} \left(-\frac{85P_R^2 \nu}{16} - \frac{7P_R^2 \nu^2}{4} - \frac{25P^2}{8} - \frac{335\nu P^2}{48} - \frac{23\nu^2 P^2}{8} \right. \right. \\
& \left. \left. - \frac{3P_R^2 \nu \pi^2}{64} + \frac{\nu P^2 \pi^2}{64} \right) + \frac{m^4}{R^4} \left(\frac{1}{8} + \frac{1881\nu}{280} - \frac{11\lambda\nu}{3} - \frac{21\nu\pi^2}{32} \right) \right\}. \quad (5.9)
\end{aligned}$$

Recall that λ is related to the so-called ‘static’ regularization-ambiguity constant ω_s of [11, 12] by $\lambda = -\frac{3}{11}\omega_s - \frac{1987}{3080}$. We have $\omega_s = 0$ according to [15]. On the other hand, the ‘kinetic’ ambiguity constant ω_k of [11, 12] has been fixed to the value $\omega_k = \frac{41}{24}$ by the explicit Lorentz invariance of the equations of motion in harmonic coordinates [16, 17], and, equivalently, by the requirement of existence of generators for the Poincaré algebra in the ADM-Hamiltonian formalism [13].

Finally let us present, for completeness, the formulae for the centre-of-mass positions which are analogous to equations (3.11)–(3.14) but in ADM coordinates. We have

$$Y_1^i = [X_2 + \nu(X_1 - X_2)\hat{\mathcal{P}}]X^i + \nu(X_1 - X_2)\hat{\mathcal{Q}}V^i + \mathcal{O}\left(\frac{1}{c^7}\right), \quad (5.10)$$

$$Y_2^i = [-X_1 + \nu(X_1 - X_2)\hat{\mathcal{P}}]X^i + \nu(X_1 - X_2)\hat{\mathcal{Q}}V^i + \mathcal{O}\left(\frac{1}{c^7}\right), \quad (5.11)$$

where the post-Newtonian coefficients $\hat{\mathcal{P}}$ and $\hat{\mathcal{Q}}$ are given by

$$\begin{aligned}
\hat{\mathcal{P}} = & \frac{1}{c^2} \left[\frac{V^2}{2} - \frac{m}{2R} \right] + \frac{1}{c^4} \left[\frac{3V^4}{8} - \frac{3\nu V^4}{2} + \frac{m}{R} \left(\frac{3\nu \dot{R}^2}{4} + \frac{7V^2}{4} + \frac{3\nu V^2}{2} \right) + \frac{m^2}{R^2} \left(\frac{1}{4} - \frac{\nu}{2} \right) \right] \\
& + \frac{1}{c^6} \left[\frac{5V^6}{16} - \frac{11\nu V^6}{4} + 6\nu^2 V^6 + \frac{m}{R} \left(\frac{21\dot{R}^4 \nu^2}{16} + \frac{7\dot{R}^2 \nu V^2}{4} - \frac{11\dot{R}^2 \nu^2 V^2}{2} \right. \right. \\
& \left. \left. + \frac{45V^4}{16} - \frac{109\nu V^4}{16} - \frac{15\nu^2 V^4}{2} \right) + \frac{m^2}{R^2} \left(\frac{9\dot{R}^2 \nu}{4} + 4\dot{R}^2 \nu^2 + \frac{23V^2}{8} + \frac{29\nu V^2}{16} \right. \right. \\
& \left. \left. + 3\nu^2 V^2 \right) + \frac{m^3}{R^3} \left(-\frac{1}{8} + \frac{\nu}{8} - \frac{\nu^2}{2} \right) \right], \quad (5.12)
\end{aligned}$$

$$\hat{\mathcal{Q}} = \frac{1}{c^5} \left[\frac{4mV^2}{5} - \frac{8m^2}{5R} \right] + \frac{1}{c^6} \left[\frac{m^2 \nu \dot{R}}{4R} \right]. \quad (5.13)$$

At the 2PN order the result is identical with the one given by Wex in his appendix A [31]¹¹. By differentiating equations (5.10)–(5.11) with respect to time we obtain the centre-of-mass velocities V_1^i and V_2^i in terms of the relative position X^i and velocity V^i . We have checked that by replacing into the obtained relations the velocities V_1^i and V_2^i by their expressions depending on the conjugate momenta P_1^i and P_2^i as deduced from the variation of the general-frame Lagrangian, and by expressing the velocity V^i in terms of P^i following the variation of the centre-of-mass Lagrangian (equation (5.7)), with both replacements being made with the full 3PN accuracy, one ends up with the simple equations

$$P_1^i = P^i = -P_2^i, \quad (5.14)$$

which are indeed the ones appropriate to a linear momentum that is conserved.

6. On the dynamical stability of circular orbits

As an application of the previous formalism let us investigate the problem of the stability, against dynamical perturbations, of circular orbits at the 3PN order. We propose to use two different methods, one based on a perturbation at the level of the equations of motion (3.16)–(3.18) in harmonic coordinates, the other one consisting of perturbing the Hamiltonian equations in ADM coordinates for the Hamiltonian (5.9). We shall find a criterion for the stability of orbits and shall present it in an invariant way (the same in different coordinate systems). We shall check that our two methods agree on the result.

We deal first with the perturbation of the equations of motion, following Kidder *et al* [32] (see their section III.A). We introduce polar coordinates (r, φ) in the orbital plane and pose $u \equiv \dot{r}$ and $\omega \equiv \dot{\varphi}$ (beware that in this paper $u = \dot{r}$, and *not* another standard notation in central force problems, $u = 1/r$). Then equation (3.16) yields the system of equations

$$\dot{r} = u, \quad (6.1)$$

$$\dot{u} = -\frac{m}{r^2}[1 + \mathcal{A} + \mathcal{B}u] + r\omega^2, \quad (6.2)$$

$$\dot{\omega} = -\omega \left[\frac{m}{r^2} \mathcal{B} + \frac{2u}{r} \right], \quad (6.3)$$

where \mathcal{A} and \mathcal{B} are given by equations (3.17), (3.18) as functions of r, u and ω (through $v^2 = u^2 + r^2\omega^2$). In the case of an orbit which is circular apart from the adiabatic inspiral at the 2.5PN order (we neglect the 2.5PN radiation-reaction effect), we have $\dot{r} = \dot{u} = \dot{\omega} = 0$ hence $u = 0$. Equation (6.2) gives thereby the angular velocity ω_0 of the circular orbit as

$$\omega_0^2 = \frac{m}{r_0^3}(1 + \mathcal{A}_0). \quad (6.4)$$

Solving iteratively this relation at the 3PN order using the equations of motion (3.16)–(3.18) we obtain ω_0 as a function of the circular-orbit radius r_0 in harmonic coordinates (the result agrees with the one of [16, 17]):

$$\omega_0^2 = \frac{m}{r_0^3} \left\{ 1 + \frac{m}{r_0 c^2}(-3 + \nu) + \frac{m^2}{r_0^2 c^4} \left(6 + \frac{41}{4}\nu + \nu^2 \right) + \frac{m^3}{r_0^3 c^6} \left(-10 + \left[-\frac{67759}{840} + \frac{41}{64}\pi^2 \right. \right. \right. \\ \left. \left. \left. + 22 \ln \left(\frac{r_0}{r_0'} \right) + \frac{44}{3}\lambda \right] \nu + \frac{19}{2}\nu^2 + \nu^3 \right) + \mathcal{O} \left(\frac{1}{c^8} \right) \right\}. \quad (6.5)$$

¹¹ Since the contact transformation we consider relates together the *conservative* parts of the dynamics in harmonic and ADM-type coordinates (and affects only the 2PN and 3PN orders), the radiation-reaction damping term at the 2.5PN order in equation (5.13) is the same as in harmonic coordinates. This is merely a definition of a particular ADM-type dynamics (*a priori* different from the one in ADM coordinates *stricto-sensu*), in which the ‘odd’ terms, associated with radiation reaction, are the same as in harmonic coordinates.

(Please do not confuse the circular-orbit radius r_0 with the constant r'_0 entering the logarithm at the 3PN order and which is defined by equation (3.19).)

Now we investigate the equations of linear perturbations around the circular orbit defined by the constants $r_0, u_0 = 0$ (actually, if we were to include the radiation-reaction damping, $u_0 = \mathcal{O}(c^{-5})$) and ω_0 . We pose

$$r = r_0 + \delta r, \quad (6.6)$$

$$u = \delta u, \quad (6.7)$$

$$\omega = \omega_0 + \delta \omega, \quad (6.8)$$

where $\delta r, \delta u$ and $\delta \omega$ denote some perturbations of the circular orbit. Then a system of linear equations follows:

$$\dot{\delta r} = \delta u, \quad (6.9)$$

$$\dot{\delta u} = \alpha_0 \delta r + \beta_0 \delta \omega, \quad (6.10)$$

$$\dot{\delta \omega} = \gamma_0 \delta u, \quad (6.11)$$

where the coefficients, which solely depend on the unperturbed circular orbit, read [32]

$$\alpha_0 = 3\omega_0^2 - \frac{m}{r_0^2} \left(\frac{\partial \mathcal{A}}{\partial r} \right)_0, \quad (6.12)$$

$$\beta_0 = 2r_0\omega_0 - \frac{m}{r_0^2} \left(\frac{\partial \mathcal{A}}{\partial \omega} \right)_0, \quad (6.13)$$

$$\gamma_0 = -\omega_0 \left[\frac{2}{r_0} + \frac{m}{r_0^2} \left(\frac{\partial \mathcal{B}}{\partial u} \right)_0 \right]. \quad (6.14)$$

In obtaining equations (6.12)–(6.14) we use the fact that \mathcal{A} is a function of the square u^2 through $v^2 = u^2 + r^2\omega^2$, so that $\partial \mathcal{A}/\partial u$ is proportional to u and thus vanishes in the unperturbed configuration (because $u = \delta u$). On the other hand, since the radiation reaction is neglected, \mathcal{B} also is proportional to u (see equation (3.18)), so only $\partial \mathcal{B}/\partial u$ can contribute at the zeroth perturbative order. Now by examining the fate of perturbations that are proportional to some $e^{i\sigma t}$, we arrive at the condition for the frequency σ of the perturbation to be real, and hence for stable circular orbits to exist, as being [32]

$$\hat{C}_0 \equiv -\alpha_0 - \beta_0\gamma_0 > 0. \quad (6.15)$$

Substituting into this \mathcal{A} and \mathcal{B} at the 3PN order we then arrive at the orbital-stability criterion

$$\hat{C}_0 = \frac{m}{r_0^3} \left\{ 1 + \frac{m}{r_0 c^2} (-9 + \nu) + \frac{m^2}{r_0^2 c^4} \left(30 + \frac{65}{4} \nu + \nu^2 \right) + \frac{m^3}{r_0^3 c^6} \left(-70 + \left[-\frac{45\,823}{840} - \frac{451}{64} \pi^2 \right. \right. \right. \\ \left. \left. \left. + 22 \ln \left(\frac{r_0}{r'_0} \right) - \frac{88}{3} \lambda \right] \nu + \frac{19}{2} \nu^2 + \nu^3 \right) + \mathcal{O} \left(\frac{1}{c^8} \right) \right\}, \quad (6.16)$$

where we recall that r_0 is the radius of the orbit in harmonic coordinates.

Our second method is to use the Hamiltonian equations based on the 3PN Hamiltonian in ADM coordinates given by equation (5.9). We introduce the polar coordinates (R, Ψ) in the orbital plane—we assume that the orbital plane is equatorial, given by $\Theta = \frac{\pi}{2}$ in the spherical coordinate system (R, Θ, Ψ) —and make the substitution

$$P^2 = P_R^2 + \frac{P_\Psi^2}{R^2}, \quad (6.17)$$

into the Hamiltonian. This yields a ‘reduced’ Hamiltonian that is a function of R, P_R and P_Ψ : $\mathcal{H} = \mathcal{H}[R, P_R, P_\Psi]$, and describes the motion in polar coordinates in the orbital plane (henceforth we denote $\mathcal{H} = \mathcal{H}^{\text{ADM}}/\mu$). The Hamiltonian equations then read

$$\frac{dR}{dt} = \frac{\partial \mathcal{H}}{\partial P_R}, \quad (6.18)$$

$$\frac{d\Psi}{dt} = \frac{\partial \mathcal{H}}{\partial P_\Psi}, \quad (6.19)$$

$$\frac{dP_R}{dt} = -\frac{\partial \mathcal{H}}{\partial R}, \quad (6.20)$$

$$\frac{dP_\Psi}{dt} = 0. \quad (6.21)$$

Evidently the constant P_Ψ is nothing but the conserved angular-momentum integral. For circular orbits we have $R = R_0$ (a constant) and $P_R = 0$, so

$$\frac{\partial \mathcal{H}}{\partial R} [R_0, 0, P_\Psi^0] = 0, \quad (6.22)$$

which gives the angular momentum P_Ψ^0 of the circular orbit as a function of R_0 , and

$$\omega_0 \equiv \left(\frac{d\Psi}{dt} \right)_0 = \frac{\partial \mathcal{H}}{\partial P_\Psi} [R_0, 0, P_\Psi^0], \quad (6.23)$$

which yields the angular frequency of the circular orbit ω_0 —the same as in equation (6.5)—in terms of R_0 ¹²:

$$\omega_0^2 = \frac{m}{R_0^3} \left\{ 1 + \frac{m}{R_0 c^2} (-3 + \nu) + \frac{m^2}{R_0^2 c^4} \left(\frac{21}{4} - \frac{5}{8} \nu + \nu^2 \right) + \frac{m^3}{R_0^3 c^6} \left(-7 + \left[-\frac{54\,629}{1680} + \frac{167}{64} \pi^2 + \frac{44}{3} \lambda \right] \nu - \frac{31}{8} \nu^2 + \nu^3 \right) + \mathcal{O} \left(\frac{1}{c^8} \right) \right\}. \quad (6.24)$$

We consider now a perturbation of the circular orbit defined by

$$P_R = \delta P_R, \quad (6.25)$$

$$P_\Psi = P_\Psi^0 + \delta P_\Psi, \quad (6.26)$$

$$R = R_0 + \delta R, \quad (6.27)$$

$$\omega = \omega_0 + \delta \omega. \quad (6.28)$$

It is easy to verify that the Hamiltonian equations (6.18)–(6.21), when worked out at the linearized order, read as

$$\delta \dot{P}_R = -\pi_0 \delta R - \rho_0 \delta P_\Psi, \quad (6.29)$$

$$\delta \dot{P}_\Psi = 0, \quad (6.30)$$

$$\delta \dot{R} = \sigma_0 \delta P_R, \quad (6.31)$$

$$\delta \dot{\omega} = \rho_0 \delta R + \tau_0 \delta P_\Psi, \quad (6.32)$$

where the coefficients, which depend on the unperturbed orbit, are given by

$$\pi_0 = \frac{\partial^2 \mathcal{H}}{\partial R^2} [R_0, 0, P_\Psi^0], \quad (6.33)$$

¹² The last equation,

$$\frac{\partial \mathcal{H}}{\partial P_R} [R_0, 0, P_\Psi^0] = 0,$$

which is equivalent to $R = \text{const} = R_0$, is automatically verified because \mathcal{H} is a quadratic function of P_R and hence $\partial \mathcal{H} / \partial P_R$ is zero for circular orbits.

$$\rho_0 = \frac{\partial^2 \mathcal{H}}{\partial R \partial P_\Psi} [R_0, 0, P_\Psi^0], \quad (6.34)$$

$$\sigma_0 = \frac{\partial^2 \mathcal{H}}{\partial P_R^2} [R_0, 0, P_\Psi^0], \quad (6.35)$$

$$\tau_0 = \frac{\partial^2 \mathcal{H}}{\partial P_\Psi^2} [R_0, 0, P_\Psi^0]. \quad (6.36)$$

By looking to solutions proportional to some $e^{i\sigma t}$ one obtains some real frequencies, and therefore one finds stable circular orbits, if and only if

$$\hat{C}_0 \equiv \pi_0 \sigma_0 > 0. \quad (6.37)$$

Using the Hamiltonian (5.9) we readily obtain

$$\begin{aligned} \hat{C}_0 = \frac{m}{R_0^3} \left\{ 1 + \frac{m}{R_0 c^2} (-9 + \nu) + \frac{m^2}{R_0^2 c^4} \left(\frac{117}{4} + \frac{43}{8} \nu + \nu^2 \right) \right. \\ \left. + \frac{m^3}{R_0^3 c^6} \left(-61 + \left[\frac{135403}{1680} - \frac{325}{64} \pi^2 - \frac{88}{3} \lambda \right] \nu - \frac{31}{8} \nu^2 + \nu^3 \right) + \mathcal{O} \left(\frac{1}{c^8} \right) \right\}. \end{aligned} \quad (6.38)$$

This result does not look the same as our previous result (6.16), but this is simply due to the fact that it depends on the ADM radial separation R_0 instead of the harmonic one r_0 . Fortunately, we have derived in section 5 all the material needed to connect R_0 to r_0 with the 3PN accuracy. Indeed, with equations (5.2), (5.3) we have the relation valid for general orbits between the separation vectors in both coordinate systems. Specializing that relation to circular orbits we readily find

$$\begin{aligned} R_0 = r_0 \left\{ 1 + \frac{m^2}{r_0^2 c^4} \left(-\frac{1}{4} - \frac{29}{8} \nu \right) + \frac{m^3}{r_0^3 c^6} \left(\left[\frac{3163}{1680} + \frac{21}{32} \pi^2 \right. \right. \right. \\ \left. \left. - \frac{22}{3} \ln \left(\frac{r_0}{r_0'} \right) \right] \nu + \frac{3}{8} \nu^2 \right) + \mathcal{O} \left(\frac{1}{c^8} \right) \right\}. \end{aligned} \quad (6.39)$$

The difference between R_0 and r_0 is made out of 2PN and 3PN terms only. Inserting equation (6.39) into equation (6.38) and re-expanding to 3PN order we find that indeed our basic stability-criterion function \hat{C}_0 comes out the same with our two methods.

Finally, let us give to the function \hat{C}_0 an invariant meaning by expressing it with the help of the orbital frequency ω_0 of the circular orbit, or, more conveniently, of the frequency-related parameter¹³

$$x_0 \equiv \left(\frac{m \omega_0}{c^3} \right)^{2/3}. \quad (6.40)$$

¹³ From the inverse of equation (6.5) we obtain r_0 as a function of x_0 . For completeness we give the relations linking both r_0 and R_0 to the x_0 -parameter:

$$\begin{aligned} \frac{m}{r_0 c^2} = x_0 \left\{ 1 + \left(1 - \frac{\nu}{3} \right) x_0 + \left(1 - \frac{65}{12} \nu \right) x_0^2 + \left(1 + \left[-\frac{10151}{2520} - \frac{41}{192} \pi^2 - \frac{22}{3} \ln \left(\frac{r_0}{r_0'} \right) - \frac{44}{9} \lambda \right] \nu \right. \right. \\ \left. \left. + \frac{229}{36} \nu^2 + \frac{\nu^3}{81} \right) x_0^3 + \mathcal{O}(x_0^4) \right\}, \end{aligned}$$

$$\begin{aligned} \frac{m}{R_0 c^2} = x_0 \left\{ 1 + \left(1 - \frac{\nu}{3} \right) x_0 + \left(\frac{5}{4} - \frac{43}{24} \nu \right) x_0^2 \right. \\ \left. + \left(\frac{7}{4} + \left[\frac{23759}{5040} - \frac{167}{192} \pi^2 - \frac{44}{9} \lambda \right] \nu + \frac{85}{36} \nu^2 + \frac{\nu^3}{81} \right) x_0^3 + \mathcal{O}(x_0^4) \right\}. \end{aligned}$$

This allows us to write the criterion for stability as $C_0 > 0$, where $C_0 = \frac{m^2}{c^6 x_0^3} \hat{C}_0$ admits the gauge-invariant form (the same in all coordinate systems)

$$C_0 = 1 - 6x_0 + 14\nu x_0^2 + \left(\left[\frac{5954}{35} - \frac{123}{16} \pi^2 - 44\lambda \right] \nu - 14\nu^2 \right) x_0^3 + \mathcal{O}(x_0^4). \quad (6.41)$$

This form is more interesting than the coordinate-dependent expressions (6.16) or (6.38), not only because of its invariant form, but also because as we see the 1PN term yields exactly the Schwarzschild result that the innermost stable circular orbit or ISCO of a test particle (i.e. in the limit $\nu \rightarrow 0$) is located at $x_{\text{ISCO}} = 1/6$. Thus we find that, at the 1PN order, but for *any* mass ratio ν ,

$$x_{\text{ISCO}}^{\text{1PN}} = \frac{1}{6}. \quad (6.42)$$

One could have expected that some deviations of the order of ν already occur at the 1PN order, but it turns out that only from the 2PN order does one find the occurrence of some non-Schwarzschildian corrections proportional to ν . At the 2PN order we obtain

$$x_{\text{ISCO}}^{\text{2PN}} = \frac{3}{14\nu} \left(1 - \sqrt{1 - \frac{14\nu}{9}} \right). \quad (6.43)$$

For equal masses this gives $x_{\text{ISCO}}^{\text{2PN}} \simeq 0.187$. Note also that the effect of the finite mass corrections is to increase the frequency of the ISCO with respect to the Schwarzschild result (i.e. to make it more inward): $x_{\text{ISCO}}^{\text{2PN}} = \frac{1}{6} [1 + \frac{7}{18} \nu + \mathcal{O}(\nu^2)]$. Finally, at the 3PN order, for equal masses $\nu = \frac{1}{4}$ and for the value of the ambiguity parameter $\lambda = -\frac{1987}{3080}$ (equivalent to $\omega_s = 0$), we find that according to our criterion *all* the circular orbits are stable. More generally, we find that at the 3PN order *all* orbits are stable when the mass ratio is $\nu > \nu_c$ where $\nu_c \simeq 0.183$.

Note that the above stability criterion C_0 gives an innermost stable circular orbit, when it exists, that is not necessarily the same as—and actually differs from—the innermost circular orbit or ICO, which is defined by the point at which the centre-of-mass binding energy of the binary for circular orbits reaches its minimum [33]. In this respect the present formalism, which is based on systematic post-Newtonian expansions (without using post-Newtonian resummation techniques such as Padé approximants [34]), differs from some ‘Schwarzschild-like’ methods such as the effective-one-body approach [35] in which the ICO happens to be also an innermost stable circular orbit or ISCO.

As a final comment, let us note that the use of a *truncated* post-Newtonian series such as equation (6.41) to determine the ISCO is *a priori* meaningful only if we are able to bound the neglected error terms. Furthermore, since we are dealing with a stability criterion, it is not completely clear that the higher-order post-Newtonian correction terms, even if they are numerically small, will not change qualitatively the response of the orbit to the dynamical perturbation. This is indeed a problem, which cannot be answered rigorously with the present formalism. However, in the regime of the ISCO (when it exists), we have seen that x_0 is rather small: $x_0 \simeq 0.2$ (this is also approximately the value for the ICO computed in [33]), which indicates that the neglected terms in the truncated series (6.41) should not contribute very much, because they involve at least a factor $x_0^4 \simeq 0.002$. On the other hand, we pointed out that in the limit $\nu \rightarrow 0$ the criterion C_0 gives back the correct *exact* result, $x_{\text{ISCO}}^{\nu \rightarrow 0} = \frac{1}{6}$. This contrasts with the gauge-dependent power series (6.16) or (6.38) which give only some approximate results. Based on these observations, we feel that it is reasonable to expect that the gauge-invariant stability criterion defined by equation (6.41) is physically meaningful.

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References

- [1] Will C M 1984 *Proc. 8th Nishinomiya-Yukawa Symposium on Relativistic Cosmology* ed M Sasaki (Universal Academy)
- [2] Blanchet L 2002 *Living Rev. Rel.* **5** 3
- [3] Damour T and Deruelle N 1981 *Phys. Lett. A* **87** 81
- [4] Damour T and Deruelle N 1981 *C. R. Acad. Sci., Paris II* **293** 537
- [5] Damour T 1982 *C. R. Acad. Sci., Paris II* **294** 1355
- [6] Damour T and Schäfer G 1985 *Gen. Rel. Grav.* **17** 879
- [7] Kopejkin S M 1985 *Astron. Zh.* **62** 889
- [8] Blanchet L, Faye G and Ponsot B 1998 *Phys. Rev. D* **58** 124002
- [9] Itoh Y, Futamase T and Asada H 2001 *Phys. Rev. D* **63** 064038
- [10] Pati M E and Will C M 2002 *Phys. Rev. D* **65** 104008
- [11] Jaranowski P and Schäfer G 1998 *Phys. Rev. D* **57** 7274
- [12] Jaranowski P and Schäfer G 1999 *Phys. Rev. D* **60** 124003
- [13] Damour T, Jaranowski P and Schäfer G 2000 *Phys. Rev. D* **62** 021501(R)
- [14] Damour T, Jaranowski P and Schäfer G 2001 *Phys. Rev. D* **63** 044021
- [15] Damour T, Jaranowski P and Schäfer G 2001 *Phys. Lett. B* **513** 147
- [16] Blanchet L and Faye G 2000 *Phys. Lett. A* **271** 58
- [17] Blanchet L and Faye G 2001 *Phys. Rev. D* **63** 062005
- [18] Blanchet L and Faye G 2000 *J. Math. Phys.* **41** 7675
- [19] Blanchet L and Faye G 2001 *J. Math. Phys.* **42** 4391
- [20] de Andrade V C, Blanchet L and Faye G 2001 *Class. Quantum Grav.* **18** 753
- [21] Iyer B R and Will C M 1993 *Phys. Rev. Lett.* **70** 113
- [22] Iyer B R and Will C M 1995 *Phys. Rev. D* **52** 6882
- [23] Gopakumar A, Iyer B R and Iyer S 1997 *Phys. Rev. D* **55** 6030
Gopakumar A, Iyer B R and Iyer S 1998 *Phys. Rev. D* **57** 6562 (errata)
- [24] Blanchet L and Damour T 1988 *Phys. Rev. D* **37** 1411
- [25] Blanchet L 1998 *Class. Quantum Grav.* **15** 113
- [26] Mora T and Will C M 2002 *Phys. Rev. D* **66** 101501 (Preprint gr-qc 0208089)
- [27] Lincoln C W and Will C M 1990 *Phys. Rev. D* **42** 1123
- [28] Blanchet L and Iyer B R in preparation
- [29] Martin J and Sanz J L 1979 *J. Math. Phys.* **20** 26
- [30] Blanchet L 1996 *Phys. Rev. D* **54** 1417
- [31] Wex N 1995 *Class. Quantum Grav.* **12** 983
- [32] Kidder L E, Will C M and Wiseman A G 1993 *Phys. Rev. D* **47** 3281
- [33] Blanchet L 2002 *Phys. Rev. D* **65** 124009
Blanchet L 2002 *Proc. 25th Johns Hopkins Workshop I* Ciufolini and L Lusanna (Singapore: World Scientific) (Preprint gr-qc 0209089) at press
- [34] Damour T, Iyer B R and Sathyaprakash B S 1998 *Phys. Rev. D* **57** 885
- [35] Buonanno A and Damour T 1999 *Phys. Rev. D* **59** 084006