

# Thirty Years of Inventory Routing

Leandro C. Coelho

CIRRELT and Faculté des sciences de l'administration, Laval University, Quebec G1V 0A6, Canada,  
leandro.coelho@cirrelt.ca

Jean-François Cordeau, Gilbert Laporte

CIRRELT and HEC Montréal, Montréal H3T 2A7, Canada  
{jean-francois.cordeau@cirrelt.ca, gilbert.laporte@cirrelt.ca}

The inventory-routing problem (IRP) dates back 30 years. It can be described as the combination of vehicle-routing and inventory management problems, in which a supplier has to deliver products to a number of geographically dispersed customers, subject to side constraints. It provides integrated logistics solutions by simultaneously optimizing inventory management, vehicle routing, and delivery scheduling. Some exact algorithms and several powerful metaheuristic and matheuristic approaches have been developed for this class of problems, especially in recent years. The purpose of this article is to provide a comprehensive review of this literature, based on a new classification of the problem. We categorize IRPs with respect to their structural variants and the availability of information on customer demand.

*Key words:* inventory routing; survey; literature review; history

*History:* Received: September 2012; revision received: January 2013; accepted: March 2013. Published online in *Articles in Advance*.

## 1. Introduction

The inventory-routing problem (IRP) integrates inventory management, vehicle routing, and delivery-scheduling decisions. Its study is rooted in the seminal paper of Bell et al. (1983), published 30 years ago. The IRP arises in the context of vendor-managed inventory (VMI), a business practice aimed at reducing logistics costs and adding business value. In VMI, a supplier makes the replenishment decisions for products delivered to customers, based on specific inventory and supply chain policies (Angulo, Nachtmann, and Waller 2004; Lee and Seungjin 2008; Simchi-Levi, Chen, and Bramel 2005). This practice is often described as a win-win situation: vendors save on distribution and production costs because they can coordinate shipments made to different customers, and buyers also benefit by not allocating efforts to inventory control. In such contexts, the supplier has to make three simultaneous decisions: (1) when to serve a given customer, (2) how much to deliver to this customer when it is served, and (3) how to combine customers into vehicle routes.

### 1.1. Origins of the Inventory-Routing Problem

The first studies published on the IRP were mostly variations of models designed for the vehicle-routing problem (VRP) and heuristics developed to take inventory costs into consideration. Bell et al. (1983) dealt with the case where only transportation costs

are included, demand is stochastic, and customer inventory levels must be met. This was followed by a number of variants of the problem defined by the same authors. Some other early papers on the IRP are worthy of mention: Federgruen and Zipkin (1984) have modified the VRP heuristic of Fisher and Jaikumar (1981) to accommodate inventory and shortage costs in a random demand environment; Blumenfeld et al. (1985) have considered distribution, inventory, and production setup costs; Burns et al. (1985) have analyzed trade-offs between transportation and inventory costs, using an approximation of travel costs; Dror, Ball, and Golden (1985) have studied short-term solutions. The latter study was extended to stochastic demand by Dror and Ball (1987). The paper of Dror and Levy (1986) adapts earlier VRP heuristics to the solution of a weekly IRP, whereas Anily and Federgruen (1990) have proposed the first clustering algorithm for the IRP. Most of these papers assume that the consumption rate at the customer locations is known and deterministic. Despite the large number of contributions on distribution and inventory problems before this period, the integration of these two features proved difficult to handle, not only because of limited computing power, but also because the available algorithms could not easily handle large and complex combinatorial problems, such as those combining routing and inventory management decisions.

## 1.2. Typologies of the Problem

We classify IRPs according to two schemes: the first one refers to the structural variants present in IRPs, whereas the second one is related to the availability of information on the demand. This classification scheme is different from the one proposed in Andersson et al. (2010) as we separate the structure of the problem from the availability of information, whereas Andersson et al. combine both. Our motivation is to better distinguish models from algorithms. We also include more structural criteria than Andersson et al. (2010), e.g., the inventory policy employed.

Many variants of the IRP have been described over the past 30 years. There does not really exist a standard version of the problem. We will therefore refer to “basic versions” of the IRP, on which most of the research effort has concentrated, and to “extensions of the basic versions,” which are more elaborate. The basic versions are presented in Table 1. They can be classified according to seven criteria, namely, time horizon, structure, routing, inventory policy, inventory decisions, fleet composition, and fleet size.

In Table 1, time refers to the horizon taken into account by the IRP model. It can either be finite or infinite. The number of suppliers and customers may vary, and therefore the structure can be one-to-one when there is only one supplier serving one customer, one-to-many in the most common case with one supplier and several customers, or less frequently, many-to-many with several suppliers and several customers. Routing can be direct when there is only one customer per route, multiple when there are several customers in the same route, or continuous when there is no central depot, like in several maritime applications. Inventory policies define preestablished rules to replenish customers. The two most common are the maximum-level (ML) policy and the order-up-to level (OU) policy. Under an ML inventory policy, the replenishment level is flexible, but bounded by the capacity available at each customer. Under an OU policy, whenever a customer is visited, the quantity delivered is that to fill its inventory capacity. Inventory decisions determine how inventory management

is modeled. If the inventory is allowed to become negative, then back-ordering occurs and the corresponding demand will be served at a later stage; if there are no back orders, then the extra demand is considered as lost sales. In both cases there may exist a penalty for the stockout. In deterministic contexts, one can also restrict the inventory to be nonnegative. Finally, the last two criteria refer to fleet composition and size. The fleet can be either homogeneous or heterogeneous, and the number of vehicles available may be fixed at one, fixed at many, or unconstrained.

The second classification refers to the time when information on demand becomes known. If it is fully available to the decision maker at the beginning of the planning horizon, the problem is then deterministic; if its probability distribution is known, it is stochastic, which yields the stochastic inventory-routing problem (SIRP). Dynamic IRPs arise when demand is not fully known in advance, but is gradually revealed over time, as opposed to what happens in a static context. In this case, one can still exploit its statistical distribution in the solution process, yielding a dynamic and stochastic inventory-routing problem (DSIRP).

## 1.3. Applications

Several applications of the IRP have been documented. Most arise in maritime logistics, namely in ship routing and inventory management. Literature reviews are provided in Ronen (1993); Christiansen, Fagerholt, and Ronen (2004); Christiansen et al. (2007); Christiansen et al. (2013). The problems described in these surveys involve a many-to-many structure with continuous routes (Christiansen 1999; Christiansen and Nygreen 1998a, b), direct deliveries (Stålhanne et al. 2012), several products (Bausch, Brown, and Ronen 1998; Persson and Göthe-Lundgren 2005; Ronen 2002), and stochastic demand (Qu, Bookbinder, and Iyogun 1999). More complex configurations involve the presence of time windows and the typical cost structure of the maritime environment (i.e., demurrage and overage rates) (Song and Furman 2013), and soft data-derived time windows to help gain robustness (Christiansen and Nygreen 2005). Problems in which storage capacities, production, and consumption rates are variable have been studied by Engineer et al. (2012); Grønhaug et al. (2010); Uggen, Fodstad, and Nørstebø (2013). Problems arising in the chemical components industry (Dauzère-Péres et al. 2007; Miller 1987) and in the oil and gas industries (Al-Khayyal and Hwang 2007; Grønhaug et al. 2010; Persson and Göthe-Lundgren 2005; Rakke et al. 2011; Shen, Chu, and Chen 2011; Song and Furman 2013) are also a frequent source of applications in a maritime environment.

Nonmaritime applications of the IRP arise in a large variety of industries, including the distribution of gas-using tanker trucks (Bard et al. 1998; Bell

**Table 1** Structural Variants of the IRP

Criteria	Possible options		
Time horizon	Finite	Infinite	
Structure	One-to-one	One-to-many	Many-to-many
Routing	Direct	Multiple	Continuous
Inventory policy	Maximum level (ML)	Order-up-to level (OU)	
Inventory decisions	Lost sales	Back-order	Nonnegative
Fleet composition	Homogeneous	Heterogeneous	
Fleet size	Single	Multiple	Unconstrained

Source. Adapted from Andersson et al. (2010).

et al. 1983; Campbell and Savelsbergh 2004; Golden, Assad, and Dahl 1984; Trudeau and Dror 1992), road-based distribution of automobile components (Alegre, Laguna, and Pacheco 2007; Blumenfeld et al. 1985, 1987; Stacey, Natarajathinam, and Sox 2007) and of perishable items (Federgruen and Zipkin 1984; Federgruen, Prastacos, and Zipkin 1986).

Other applications include the transportation of groceries (Custódio and Oliveira 2006; Gaur and Fisher 2004; Mercer and Tao 1996), cement (Christiansen et al. 2011), fuel (Popović, Vidović, and Radivojević 2012), blood (Hemmelmayer et al. 2009), livestock (Oppen, Løkketangen, and Desrosiers 2010), and waste organic oil (Aksen et al. 2012).

Note that not all of these papers deal with the IRP as described. Some of them optimize vehicle routes or dispatching of vehicles only, without considering inventory costs. However, inventory concerns appear as constraints, ensuring that demand is satisfied and that customer and transportation capacities are respected (Bard et al. 1998; Bausch, Brown, and Ronen 1998; Oppen, Løkketangen, and Desrosiers 2010).

#### 1.4. Aim and Organization of the Paper

The aim of this paper is to present a comprehensive literature review of the IRP, including its main variants, models, and algorithms. It complements the survey of Andersson et al. (2010), which puts more emphasis on industrial applications. In contrast, our contribution focuses on the methodological aspects of the problem. Other less recent reviews are those of Cordeau et al. (2007) and Bertazzi, Savelsbergh, and Speranza (2008). Our aim is to provide an up-to-date and in-depth presentation and classification of the research conducted in this area.

The remainder of the paper is organized as follows. In §2 we describe the basic versions of the IRP as well as its models and solutions procedures. A number of meaningful extensions of the problem are then presented in §3. This is followed by the description of the stochastic version of the problem in §4 and by the dynamic and stochastic IRP in §5. Benchmark instances are described in §6, and our conclusions follow in §7.

## 2. Basic Versions of the Inventory-Routing Problem

The basic IRP is defined on a graph  $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ , where  $\mathcal{V} = \{0, \dots, n\}$  is the vertex set and  $\mathcal{A} = \{(i, j): i, j \in \mathcal{V}, i \neq j\}$  is the arc set. Vertex 0 represents the supplier, and the vertices of  $\mathcal{V}' = \mathcal{V} \setminus \{0\}$  represent customers. Both the supplier and customers incur unit inventory-holding costs  $h_i$  per period ( $i \in \mathcal{V}$ ), and each customer has an inventory-holding capacity  $C_i$ . The length of the planning horizon is  $p$ , and at each

time period  $t \in \mathcal{T} = \{1, \dots, p\}$  the quantity of product made available at the supplier is  $r^t$ . We assume the supplier has sufficient inventory to meet all of the demand during the planning horizon and that inventories are not allowed to be negative. The variables  $I_0^t$  and  $I_i^t$  are defined as the inventory levels at the end of period  $t$ , respectively, at the supplier and at customer  $i$ . At the beginning of the planning horizon the decision maker knows the current inventory level of the supplier and of all customers ( $I_0^0$  and  $I_i^0$  for  $i \in \mathcal{V}'$ ), and has full knowledge of the demand  $d_i^t$  of each customer  $i$  for each time period  $t$ . A set  $\mathcal{K} = \{1, \dots, K\}$  of vehicles with capacity  $Q_k$  are available. Each vehicle is able to perform one route per time period to deliver products from the supplier to a subset of customers. A routing cost  $c_{ij}$  is associated with arc  $(i, j) \in \mathcal{A}$ .

The objective of the problem is to minimize the total inventory distribution cost while meeting the demand of each customer. The replenishment plan is subject to the following constraints:

- the inventory level at each customer can never exceed its maximum capacity;
- inventory levels are not allowed to be negative;
- the supplier's vehicles can perform at most one route per time period, each starting and ending at the supplier;
- vehicle capacities cannot be exceeded.

The solution to the problem determines which customers to serve in each time period, which of the supplier's vehicles to use, how much to deliver to each visited customer, and the delivery routes. Clearly, the IRP just defined is deterministic and static because consumption rates are fixed and known beforehand.

The basic IRP is NP-hard because it subsumes the classical VRP. As a result, most papers propose heuristics for its solution, but a number of exact algorithms are also available. In Table 2 we present the papers mentioned in this section on the deterministic IRP. These will be further described when we present exact algorithms in §2.1 and heuristics in §2.2.

### 2.1. Exact Algorithms

All models presented in this section were developed assuming the cost matrix is symmetric. In such cases, it is natural to define the problem on an undirected graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ , where  $\mathcal{E} = \{(i, j): i, j \in \mathcal{V}, i < j\}$ , and to use routing variables associated with the edges, which is computationally more efficient. It is straightforward to extend edge-based formulations to the directed case.

Archetti et al. (2007) have developed the first branch-and-cut algorithm for a single-vehicle IRP. This algorithm is able to solve both the OU and the ML versions, differing by a single constraint. Subtour elimination constraints are added dynamically as cuts in the search tree whenever an incumbent solution

Table 2 Classification of the Papers on the Basic Versions of the IRP

Reference	Structure			Routing			Inventory policy		Inventory decisions			Fleet composition		Fleet size		
	Time horizon	One- to-one	Many-to-many	Direct	Multiple	Continuous	Maximum level (ML)	Order-up-to level (OU)	Lost sales	Backlogging	Nonnegative	Homogeneous	Heterogeneous	Single	Multiple	Unconstrained
Bell et al. (1983)	✓	✓			✓		✓		✓				✓		✓	
Burns et al. (1985)	✓	✓			✓		✓		✓			✓				
Dror, Ball, and Golden (1985)	✓	✓			✓			✓	✓			✓			✓	
Dror and Levy (1986)	✓	✓			✓		✓				✓	✓			✓	
Dror and Ball (1987)	✓	✓			✓		✓					✓			✓	
Chien, Balakrishnan, and Wong (1989)	✓	✓			✓		✓			✓			✓		✓	
Anily and Federgruen (1990)		✓			✓		✓		✓			✓				✓
Gallego and Simchi-Levi (1990)		✓			✓		✓		✓			✓				✓
Campbell et al. (1998)	✓	✓			✓		✓					✓				✓
Christiansen (1999)	✓		✓			✓	✓		✓				✓		✓	
Bertazzi, Paletta, and Speranza (2002)	✓	✓			✓			✓			✓	✓		✓		
Ronen (2002)	✓						✓		✓			✓				✓
Ribeiro and Lourenço (2003)	✓	✓			✓		✓		✓			✓				✓
Abdelmaguid (2004)	✓	✓			✓		✓					✓			✓	
Campbell and Savelsbergh (2004)	✓	✓			✓		✓		✓			✓			✓	
Abdelmaguid and Dessouky (2006)	✓	✓			✓		✓			✓		✓			✓	
Aghezzaf, Raa, and van Landeghem (2006)		✓			✓		✓		✓			✓			✓	

Table 2 (Continued)

Reference	Structure			Routing			Inventory policy		Inventory decisions			Fleet composition			Fleet size		
	Time horizon	One- to-one	Many- to-many	Direct	Multiple	Continuous	Maximum level (ML)	Order-up-to level (OU)	Inventory decisions			Homogeneous	Heterogeneous	Single	Multiple	Unconstrained	
									Lost sales	Backlogging	Nonnegative						
Archetti et al. (2007)	✓	✓			✓		✓	✓		✓		✓		✓			
Raa and Aghezzaf (2008)	✓	✓			✓		✓	✓		✓		✓			✓		
Savelsbergh and Song (2008)	✓		✓			✓	✓		✓			✓			✓		
Zhao, Chen, and Zang (2008)	✓	✓			✓		✓	✓				✓					✓
Abdelmaguid, Dessouky, and Ordóñez (2009)	✓	✓			✓		✓		✓				✓		✓		
Boudia and Prins (2009)	✓	✓			✓		✓		✓			✓			✓		
Raa and Aghezzaf (2009)	✓	✓			✓		✓		✓			✓			✓		
Geiger and Sevaux (2011a)	✓	✓			✓		✓		✓			✓					✓
Soljani and Süral (2011)	✓	✓			✓			✓				✓		✓			
Adulyasak, Cordeau, and Jans (2013)	✓	✓			✓		✓	✓		✓		✓			✓		
Archetti et al. (2012)	✓	✓			✓		✓	✓		✓		✓		✓			
Coelho, Cordeau, and Laporte (2012a)	✓	✓			✓		✓	✓		✓		✓		✓			
Coelho, Cordeau, and Laporte (2012b)	✓	✓			✓		✓	✓		✓		✓			✓		
Michel and Vanderbeck (2012)	✓	✓			✓			✓		✓		✓			✓		
Coelho and Laporte (2013a)	✓	✓			✓		✓	✓		✓		✓	✓		✓		
Coelho and Laporte (2013b)	✓	✓			✓		✓	✓		✓		✓			✓		
Hewitt et al. (2013)	✓		✓			✓	✓		✓				✓				✓

violates them. Archetti et al. have also derived some valid inequalities to strengthen the model and were able to solve instances with up to 50 customers in a three-period horizon, and 30 customers in a six-period horizon within two hours of computing time. Despite considering only one vehicle, the Archetti et al. (2007) model is somewhat more general than others because it incorporates not only inventory holding costs at the customers, but also at the supplier. It was later improved by Solyalı and Süral (2011), who used a stronger formulation with shortest-path networks representing customer replenishments, as well as a heuristic to provide an initial upper bound to the branch-and-cut algorithm. Solyalı and Süral considered only the OU policy and solved larger instances with up to 15 customers and 12 periods, 25 customers and nine periods, and 60 customers in a three-period horizon.

Recently, algorithms capable of solving exact multivehicle versions of the IRP have been introduced. Coelho and Laporte (2013b) and Adulyasak, Cordeau, and Jans (2013) have proposed an extension of the Archetti et al. (2007) formulation under the OU and ML policies to account for multiple vehicles, and have solved it in a branch-and-cut fashion. Assuming again that the transportation cost matrix is symmetric, their proposed model is undirected to reduce the number of variables. Thus, their model uses variables  $x_{ij}^{kt}$  equal to the number of times edge  $(i, j)$  is used on the route of vehicle  $k$  in period  $t$ . It also uses variables  $y_i^{kt}$  equal to one if and only if vertex  $i$  (the supplier or a customer) is visited by vehicle  $k$  in period  $t$ . Let  $I_i^t$  denote the inventory level at vertex  $i \in \mathcal{V}$  at the end of period  $t \in \mathcal{T}$ , and  $q_i^{kt}$  denote the quantity of product delivered from the supplier to customer  $i$  using vehicle  $k$  in time period  $t$ . Assuming that the OU inventory policy applies, the problem can then be formulated as

$$\text{minimize } \left\{ \sum_{i \in \mathcal{V}} \sum_{t \in \mathcal{T}} h_i I_i^t + \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{V}, i < j} \sum_{k \in \mathcal{K}} \sum_{t \in \mathcal{T}} c_{ij} x_{ij}^{kt} \right\} \quad (1)$$

subject to

$$I_0^t = I_0^{t-1} + r^t - \sum_{k \in \mathcal{K}} \sum_{i \in \mathcal{V}'} q_i^{kt} \quad t \in \mathcal{T}, \quad (2)$$

$$I_0^t \geq 0 \quad t \in \mathcal{T}, \quad (3)$$

$$I_i^t = I_i^{t-1} + \sum_{k \in \mathcal{K}} q_i^{kt} - d_i^t \quad i \in \mathcal{V}' \quad t \in \mathcal{T}, \quad (4)$$

$$I_i^t \geq 0 \quad i \in \mathcal{V} \quad t \in \mathcal{T}, \quad (5)$$

$$I_i^t \leq C_i \quad i \in \mathcal{V} \quad t \in \mathcal{T}, \quad (6)$$

$$\sum_{k \in \mathcal{K}} q_i^{kt} \leq C_i - I_i^{t-1} \quad i \in \mathcal{V}' \quad t \in \mathcal{T}, \quad (7)$$

$$q_i^{kt} \geq C_i y_i^{kt} - I_i^{t-1} \quad i \in \mathcal{V}' \quad k \in \mathcal{K} \quad t \in \mathcal{T}, \quad (8)$$

$$q_i^{kt} \leq C_i y_i^{kt} \quad i \in \mathcal{V}' \quad k \in \mathcal{K} \quad t \in \mathcal{T}, \quad (9)$$

$$\sum_{i \in \mathcal{V}'} q_i^{kt} \leq Q_k y_0^{kt} \quad k \in \mathcal{K} \quad t \in \mathcal{T}, \quad (10)$$

$$\sum_{j \in \mathcal{V}, i < j} x_{ij}^{kt} + \sum_{j \in \mathcal{V}, i > j} x_{ji}^{kt} = 2y_i^{kt} \quad i \in \mathcal{V} \quad k \in \mathcal{K} \quad t \in \mathcal{T}, \quad (11)$$

$$\sum_{i \in \mathcal{S}} \sum_{j \in \mathcal{S}, i < j} x_{ij}^{kt} \leq \sum_{i \in \mathcal{S}} y_i^{kt} - y_m^{kt} \quad \mathcal{S} \subseteq \mathcal{V}' \quad k \in \mathcal{K} \quad t \in \mathcal{T} \quad m \in \mathcal{S}, \quad (12)$$

$$q_i^{kt} \geq 0 \quad i \in \mathcal{V}' \quad k \in \mathcal{K} \quad t \in \mathcal{T}, \quad (13)$$

$$x_{i0}^{kt} \in \{0, 1, 2\} \quad i \in \mathcal{V}' \quad k \in \mathcal{K} \quad t \in \mathcal{T}, \quad (14)$$

$$x_{ij}^{kt} \in \{0, 1\} \quad i, j \in \mathcal{V}' \quad k \in \mathcal{K} \quad t \in \mathcal{T}, \quad (15)$$

$$y_i^{kt} \in \{0, 1\} \quad i \in \mathcal{V} \quad k \in \mathcal{K} \quad t \in \mathcal{T}. \quad (16)$$

Constraints (2) define the inventory at the supplier, whereas constraints (3) prevent stockouts at the supplier; constraints (4) and (5) are similar and apply to the customers. Constraints (6) impose maximal inventory level at the customers. Note that these constraints assume that the inventory at the end of the period cannot exceed the maximum available holding capacity, which means that during the period, before all demand has happened, the inventory capacity could be temporarily exceeded. This is a usual assumption in IRP models. Constraints (7)–(9) link the quantities delivered to the routing variables. In particular, they only allow a vehicle to deliver products to a customer if the customer is visited by this vehicle, and enforce the OU policy. Constraints (10) ensure that vehicle capacities are respected, whereas constraints (11) and (12) are degree constraints and subtour elimination constraints, respectively. The latter are relaxed and added as cuts whenever they are violated in the search tree. Constraints (13)–(16) enforce integrality and non-negativity conditions on the variables.

This formulation can be solved using branch and cut by making use of the capabilities of modern MIP solvers. Instances with up to 45 customers, three periods, and three vehicles have been solved to optimality with CPLEX. Adulyasak, Cordeau, and Jans (2013) have compared this model with a two-index formulation that yielded better lower bounds on larger instances that could not be solved exactly with the three-index formulation. This formulation has been recently extended to solve a multiproduct version of the IRP by Coelho and Laporte (2013a), for which details are presented in §3.2

## 2.2. Heuristic Algorithms

Most of the early papers on the IRP have applied simple heuristics to simplified versions of the problem. These explore the solution space through the use of simple neighborhood structures such as interchanges,

and typically decompose the IRP into hierarchical subproblems, where the solution to one subproblem is used in the next step. Examples include an assignment heuristic (Dror, Ball, and Golden 1985), interchange algorithm (Dror and Levy 1986), trade-offs based on approximate routing costs (Burns et al. 1985), and a clustering heuristic (Anily and Federgruen 1990).

Current heuristic algorithms are rather involved and are able to obtain high-quality solutions to difficult optimization problems. They rely on the concept of *metaheuristics*, which apply local search procedures and a strategy to avoid local optima, and perform a thorough evaluation of the search space (Gendreau and Potvin 2010). New developments in this area include the hybridization of different metaheuristic concepts to create more powerful algorithms (Raidl, Puchinger, and Blum 2010) and also the hybridization of a heuristic and a mathematical programming algorithm, yielding so-called *matheuristic* algorithms (Maniezzo, Stützle, and Voß 2009). Recent IRP papers using some of these techniques include iterated local search (Ribeiro and Lourenço 2003), variable neighborhood search (Zhao, Chen, and Zang 2008), greedy randomized adaptive search (Campbell and Savelsbergh 2004), memetic algorithms (Boudia and Prins 2009), tabu search (Archetti et al. 2012), and adaptive large neighborhood search (Coelho, Cordeau, and Laporte 2012c).

Bell et al. (1983) analyzed the case where only transportation costs are included, but inventory levels must be met at the customers. A short term solution is presented in Dror and Ball (1987) and Dror, Ball, and Golden (1985), based on the assignment of customers to optimal replenishment periods, and on the computation of the expected increase in cost when the customer is visited in another period. Dror, Ball, and Golden (1985) offered the first algorithmic comparison for the IRP with two major simplifications: (1) an OU policy applies and (2) customers are only visited once during the planning period. Dror and Ball (1987) also applied the OU policy, and has been widely used by many researchers.

Building on the idea of adapting previous VRP algorithms and heuristics, Dror and Levy (1986) proposed a vertex interchange algorithm for a weekly IRP. They generated an initial solution to a VRP by keeping track of vehicle capacities and customer inventories, thus improving the initial solution scheme presented in Dror, Ball, and Golden (1985). Burns et al. (1985) developed formulas based on the trade-offs between transportation and inventory costs using an approximation of traveling costs. They showed that under direct shipping, the optimal delivery size is the economic order quantity.

Clustering heuristics were proposed by Anily and Federgruen (1990) and Campbell and Savelsbergh (2004). Direct deliveries were studied by Gallego and Simchi-Levi (1990), who evaluated their long-term effectiveness. Aghezzaf, Raa, and van Landeghem (2006) allowed vehicles to perform more than one route per period and modified the approach employed by Anily and Federgruen (1990) by using heuristic column generation. Their work was later extended by Raa and Aghezzaf (2009) who have added driving time constraints. Construction and improvement heuristics were proposed by Chien, Balakrishnan, and Wong (1989) for a version of the problem with a heterogeneous fleet. Considering backlogging, a construction heuristic was put forward by Abdelmaguid (2004) and was later outperformed by the genetic algorithm of Abdelmaguid and Dessouky (2006). Heuristics for the IRP with backlogging were later reviewed by Abdelmaguid, Dessouky, and Ordóñez (2009).

Savelsbergh and Song (2008) solved a problem in which a single producer cannot usually meet the demand of its customers because they are too far away. This leads to the formulation of a problem with several suppliers and trips lasting longer than one period. This problem is called the IRP with continuous moves and is solved through a local search algorithm applied on an initial solution generated by a randomized greedy heuristic.

Considering a cyclic planning approach where a long-term distribution pattern can be derived, Raa and Aghezzaf (2008) developed an algorithm allowing vehicles to perform multiple tours. Initially, customers are partitioned over vehicles using a column generation algorithm. Then, for each vehicle, the set of customers assigned to it is partitioned over different tours for which frequencies are then determined. For each partition of customers over tours and each combination of tour frequencies, a delivery schedule is then made to check feasibility.

With the aim of identifying Pareto-optimal solutions, Geiger and Sevaux (2011a) compared different solutions with respect to the two opposing terms in the objective function. When a customer is visited very frequently, its inventory cost is low but routing becomes expensive, and vice versa. This is important when considering changes in some of the parameters, for example, when fuel prices increase or when focusing on the computation of “green” solutions.

A heuristic column generation algorithm is used to solve a tactical IRP in Michel and Vanderbeck (2012), where customer demands are deterministic and are clustered to be served by different vehicles; routing costs are approximated. This heuristic yields solutions that deviate by approximately 6% from the optimum and improve upon industrial practice by 10%

with respect to travel distances and the number of vehicles used.

A two-phase heuristic based on a linear programming model was proposed by Campbell et al. (1998). In the first phase, the exact visiting period and quantity to be delivered to each customer are calculated. Then, in the second phase, customers are sequenced into vehicle routes. This model is difficult to solve because of the high number of possible routes, and also because of the length of the planning horizon. Considering a small set of routes and aggregating periods toward the end of the horizon makes the model more tractable. The output of this first phase specifies how much to deliver to each customer in each period of the planning horizon. This information then becomes the input of a standard algorithm for the VRP with time windows, which is solved for each period in the second phase. Since decisions are taken separately in the two phases, the second phase can only be optimal with respect to the solution obtained from the first phase. Besides, this model considers time constraints explicitly, but does not include any consideration for the inventory holding costs.

Bertazzi, Paletta, and Speranza (2002) have proposed a fast local search algorithm for the single-vehicle case in which an OU inventory policy is applied. This policy decreases the flexibility of the decision maker by restricting the set of possible solutions to the problem—the simplified problem is solved heuristically. A first step creates a feasible solution, and a second one is applied as long as a given minimum improvement is made to the total cost function. This is achieved by removing all possible customer pairs and computing a series of shortest paths to determine the periods in which the customers should be reinserted. Specifically, shortest paths are computed on acyclic networks  $\mathcal{N}_i$ , one for each customer  $i$ . Each node of  $\mathcal{N}_i$  corresponds to a discrete time instant between 0 and  $p + 1$ , and an arc  $(t, t')$  is defined if no stockout occurs at customer  $i$  whenever it is not visited in the interval  $[t, t']$ ; the quantity delivered to  $i$  at each time period will be that to fill the customer capacity, and each arc cost is the sum of the inventory and routing costs associated with visiting customer  $i$  in the interval  $[t, t']$ . Bertazzi, Paletta, and Speranza (2002) consider both inventory and transportation costs, and it is relevant to note that the supplier also incurs inventory costs, which was not generally the case in previous papers. Computational experiments have shown that this heuristic works extremely fast, but the optimality gap is sometimes larger than 5%.

Archetti et al. (2012) have designed a more involved heuristic combining tabu search with the exact solution of mixed-integer linear programs (MILPs) used

to approximate routing decisions. It operates with a combination of a tabu search heuristic embedded within four neighborhood search operators and two MILPs to further refine the solutions. Starting from a feasible solution, the algorithm explores the neighborhood of the current solution and performs occasional jumps to new regions of the search space. Infeasible solutions are temporarily accepted, namely, because of a stockout at the supplier or exceeded vehicle capacity. Results show that the heuristic performs remarkably well on benchmark instances, with an optimality gap usually below 0.1%.

Coelho, Cordeau, and Laporte (2012c) have developed an adaptive large neighborhood search (ALNS) matheuristic that can solve the IRP as a special case of a broader problem including transshipments. This algorithm works in two phases, first creating vehicle routes by means of the ALNS operators and then determining delivery quantities through the use of an exact minimum-cost network flow algorithm. When no transshipments are considered, this matheuristic performs slightly worse than the algorithm of Archetti et al. (2012). Finally, Coelho, Cordeau, and Laporte (2012a) have proposed an extension of the previous algorithm to the multivehicle version of the IRP. In this problem, the ALNS creates vehicle routes, and delivery quantities are again optimized by means of a min-cost network flow algorithm. Better solutions are obtained by approximating the costs of inserting or removing customers from existing solutions through the exact solution of a MILP, as in Archetti et al. (2012).

Fast primal solutions are obtained by a branch-and-price guided search in Hewitt et al. (2013). The problem at hand is a maritime IRP dealing with a single product, many-to-many structure, distributed by a heterogeneous fleet of vessels over a finite horizon. Experiments show that it performs significantly faster than solving the MILP using a state-of-the-art solver, and it is able to obtain solutions that are comparable in terms of cost.

### 3. Extensions of the Basic Versions

Almost every combination of the criteria presented in Table 1 has been studied at some point over the past 30 years. Specific versions of the IRP include the IRP with a single customer (Bertazzi and Speranza 2002; Dror and Ball 1987; Speranza and Ukovich 1996; Solyalı and Süral 2008), the IRP with multiple customers (Archetti et al. 2007; Bell et al. 1983; Chien, Balakrishnan, and Wong 1989; Coelho and Laporte 2013b; Coelho, Cordeau, and Laporte 2012c), the IRP with direct deliveries (Bertazzi 2008; Gallego and Simchi-Levi 1990, 1994; Hall 1992; Kleywegt, Nori, and Savelsbergh 2002; Mishra and Raghunathan 2004), the multi-item



IRP (Bausch, Brown, and Ronen 1998; Qu, Bookbinder, and Iyogun 1999; Sindhuchao et al. 2005; Speranza and Ukovich 1994), the IRP with several suppliers and customers (Benoist et al. 2011), and the IRP with heterogeneous fleet (Chien, Balakrishnan, and Wong 1989; Christiansen 1999; Coelho and Laporte 2013b; Persson and Göthe-Lundgren 2005), among others. However, some common criteria are more relevant and have received more attention. Table 3 presents the papers cited in this section, which covers deterministic extensions of the IRP.

### 3.1. The Production-Routing Problem

Because VMI provides advantages to both the supplier and the customers, it is natural to think that integrating one more element of the supply chain may lead to an even better performance. This extra element may be external (the supplier of the supplier) or may include other activities of the supplier, such as production planning. This leads to the production-inventory-routing problem, also called the production-routing problem (PRP). The PRP integrates inventory and lot-sizing decisions over a given planning horizon with the design of vehicle routes to perform the deliveries. Thus, it integrates the lot-sizing problem and the vehicle-routing problem. With respect to the IRP, the PRP is more general in that it integrates production and distribution decisions.

Chandra (1993) and Chandra and Fisher (1994) were among the first to integrate production decisions within the IRP. They were followed by Chandra and Fisher (1994); Herer and Roundy (1997); Fumero and Vercellis (1999); Bertazzi, Paletta, and Speranza (2005); Bard and Nananukul (2009, 2010). More recent works in this direction include those of Archetti et al. (2011) and Adulyasak, Cordeau, and Jans (2012).

In the same vein, other levels of integration have been proposed. For instance, Blumenfeld et al. (1985) considered distribution, inventory, and production setup costs. Ahmadi-Javid and Azad (2010) proposed a broader mechanism that simultaneously optimizes location, allocation, capacity, inventory, and routing decisions in supply chain design under stochastic demand.

### 3.2. The IRP with Multiple Products

In some versions of the IRP, several products are handled at once. Speranza and Ukovich (1994, 1996) studied the case with predetermined frequencies for a multiproduct flow for a single customer. Bertazzi, Speranza, and Ukovich (1997) later extended these studies to handle multiple customers. Carter et al. (1996) have also proposed a two-phase heuristic to solve the multiproduct version of the IRP. A particular case of the multi-item IRP was analyzed by Popović, Vidović, and Radivojević (2012), in which different

types of fuel are delivered to a set of customers by vehicles with compartments. The problem was solved by means of a variable neighborhood search heuristic because the proposed MILP could only handle the smallest instance from a practical application. Moin, Salhi, and Aziz (2011) analyzed variation of the multi-product version that also considers multiple suppliers but only one customer (many-to-one structure). The authors derived lower and upper bounds after solving a linear mathematical formulation with a commercial solver and then compute better upper bounds by means of a genetic algorithm. Most of these results were improved by Mjirda et al. (2012) who developed a variable neighborhood search heuristic. Building up on the previous structure, Ramkumar et al. (2012) studied the many-to-many case and proposed a MILP formulation for a multi-item multidepot IRP. However, their computational results show the limitations of the method since several small instances with only two vehicles, two products, two suppliers, three customers, and three periods could not be solved to optimality in eight hours of computing time.

An exact MILP was proposed by Coelho and Laporte (2013a) to solve a multivehicle multiproduct version of the problem. It deals with shared inventory capacity and shared vehicle capacity for all products. Their implementation is able to solve instances with up to five products, five vehicles, three periods, and 30 customers. Note that a multiproduct formulation for a deterministic maritime problem was proposed by Ronen (2002), but only very small instances could be solved. For this reason, the author developed a heuristic to solve the problem, which was a simplification of a stochastic one based on forecasts and predetermined safety stocks levels.

### 3.3. The IRP with Direct Deliveries and Transshipment

Another variation of the IRP deals with direct deliveries, as the one studied by Kleywegt, Nori, and Savelsbergh (2002) and Bertazzi (2008). Making exclusive use of direct deliveries simplifies the problem because it removes the routing dimension from it. Direct deliveries are shown to be effective when economic order quantities for the customers are close to the vehicle capacities (Gallego and Simchi-Levi 1990, 1994). Li, Chen, and Chu (2010) developed an analytic method for performance evaluation of this delivery strategy, and the effectiveness can be represented as a function of system parameters.

A number of replenishment policies have been proposed in this context. Power-of-two policies were analyzed by Herer and Roundy (1997), a fixed partition policy combined with a tabu search heuristic was studied by Zhao, Wang, and Lai (2007), and a stationary nested joint replenishment policy was developed

Table 3 Classification of the Papers on Extensions of the Basic Versions of the IRP

Reference	Time horizon		Products		Structure			Routing		Inventory policy		Inventory decisions		Fleet composition		Fleet size	
	Finite	Infinite	Single	Many	One-to-one	One-to-many	Many-to-many	Routing		Maximum level (ML)	Order-up-to level (OU)	Lost sales	Backlogging	Homogeneous	Heterogeneous	Single	Multiple
								Direct	Multiple								
Blumenfeld et al. (1985)	✓			✓	✓	✓	✓	✓		✓		✓		✓			✓
Roundy (1985)	✓		✓		✓	✓			✓	✓		✓		✓			✓
Gallego and Simchi-Levi (1990)	✓		✓		✓	✓		✓		✓		✓		✓			✓
Hall (1992)	✓		✓		✓	✓		✓		✓		✓		✓			✓
Gallego and Simchi-Levi (1994)	✓		✓		✓	✓		✓		✓		✓		✓			✓
Speranza and Ukovich (1994)	✓			✓	✓	✓		✓		✓		✓		✓			✓
Carter et al. (1996)	✓			✓	✓	✓			✓	✓			✓				
Speranza and Ukovich (1996)				✓	✓	✓			✓	✓		✓		✓		✓	✓
Bertazzi, Speranza, and Ukovich (1997)	✓			✓	✓	✓			✓	✓		✓		✓			✓
Herer and Roundy (1997)	✓		✓		✓	✓		✓	✓	✓		✓		✓		✓	✓
Viswanathan and Mathur (1997)	✓			✓	✓	✓			✓	✓		✓		✓			
Bausch, Brown, and Ronen (1998)	✓			✓			✓		✓	✓			✓				
Bertazzi and Speranza (2002)	✓			✓	✓			✓		✓		✓		✓			✓
Persson and Göthe-Lundgren (2005)	✓			✓			✓		✓	✓		✓			✓		✓
Sindhuchao et al. (2005)				✓	✓			✓	✓	✓		✓		✓		✓	✓
Stacey, Natarajathinam, and Sox (2007)	✓			✓	✓			✓	✓	✓		✓		✓		✓	✓
Zhao, Wang, and Lai (2007)	✓		✓		✓	✓		✓		✓		✓		✓			✓
Bertazzi (2008)	✓			✓	✓	✓		✓		✓		✓		✓			✓
Solyali and Süral (2008)	✓		✓		✓			✓		✓		✓		✓			✓
Li, Chen, and Chu (2010)	✓		✓		✓			✓		✓		✓		✓		✓	
Benoist et al. (2011)	✓		✓		✓		✓		✓	✓		✓		✓			
Moin, Salhi, and Aziz (2011)	✓		✓		✓			✓	✓	✓		✓		✓			
Coelho, Cordeau, and Laporte (2012a)	✓		✓		✓	✓		✓	✓	✓		✓		✓			
Coelho, Cordeau, and Laporte (2012b)	✓				✓	✓		✓		✓		✓		✓			
Mjirda et al. (2012)	✓			✓	✓			✓	✓	✓		✓		✓		✓	✓
Popović, Vidović, and Radivojević (2012)	✓			✓	✓			✓	✓	✓		✓		✓		✓	✓
Ramkumar et al. (2012)	✓						✓		✓	✓		✓				✓	✓
Coelho and Laporte (2013a)	✓		✓		✓			✓	✓	✓		✓		✓		✓	✓
Coelho and Laporte (2013b)	✓			✓	✓			✓	✓	✓		✓		✓		✓	✓

by Viswanathan and Mathur (1997) for a multiproduct case. Most of the IRP literature considers continuous decision variables for the delivery times. Under this assumption, the optimal replenishment time may be noninteger, which can constitute an inconvenience for some suppliers. Roundy (1985) studied the case with multiple customers receiving direct deliveries at discrete times, and defined frequency-based policies proven to be within 2% of the optimum in the worst case. In this model, inventory holding costs are linear, but there are fixed ordering and delivery costs.

Direct deliveries from the supplier and lateral transshipments between customers have also been used in conjunction with multicustomer routes to increase the flexibility of the system. Transshipments were formally introduced within the IRP framework by Coelho, Cordeau, and Laporte (2012c). They included planned transshipment decisions within a deterministic framework as a way of reducing distribution costs. Coelho, Cordeau, and Laporte (2012b) later used transshipments within a DSIRP framework as a means of mitigating stockouts when demand exceeded the available inventory. Emergency transshipments proved to be a valuable option for decreasing average stockouts while significantly reducing distribution costs.

### 3.4. The Consistent IRP

Some authors have noted that a cost-optimal solution may sometimes result in inconveniences both to the supplier and to the customers. This is the case, for example, when very small deliveries take place on consecutive days, followed by a very large delivery, after which the customer is not visited for a long period. Another example, this time undesirable for the supplier, is that it could be optimal to dispatch a mix of almost full and almost empty vehicles, which does not yield a proper load balancing and may irritate some drivers. It is possible to alleviate some of these problems by introducing some consistency features into the basic IRP, which has already been done in the context of the VRP. Thus, some authors have included workforce management within the periodic VRP for assigning territories to drivers as in Christofides and Beasley (1984); Beasley (1984); Barlett and Ghoshal (2002); Zhong, Hall, and Dessouky (2007). This is an indirect way of enforcing driver consistency, which was formally put forward by Groër, Golden, and Wasil (2009). Smilowitz, Nowak, and Jiang (2013) analyzed potential trade-offs between workforce management and travel distance goals in a multiobjective PVRP. Another example of consistency is the spacing of deliveries to customers, which ensures smoother operations (see, e.g., Ohlmann, Fry, and Thomas 2008). This type of requirement is often modeled as constraints in the

context of the periodic VRP (Christofides and Beasley 1984; Francis, Smilowitz, and Tzur 2008). Finally, the quantities delivered to customers were also controlled to avoid large variations over time, which are negatively perceived by customers (Beamon 1999).

Quality-of-service features were incorporated in IRP solutions by Coelho, Cordeau, and Laporte (2012a). This was achieved by ensuring consistent solutions from three different aspects: quantities delivered, frequency of the deliveries, and workforce management. These authors have shown through extensive computational experiments on benchmark instances that ensuring consistent solutions over time increases the cost of the solution between 1% and 8% on average.

## 4. Stochastic Inventory Routing

In the SIRP, the supplier knows customer demand only in a probabilistic sense. Demand stochasticity means that shortages may occur. To discourage them, a penalty is imposed whenever a customer runs out of stock, and this penalty is usually modeled as a proportion of the unsatisfied demand. Unsatisfied demand is typically considered to be lost, that is, there is no backlogging. The objective of the SIRP remains the same as in the deterministic case, but is written to accommodate the stochastic and unknown future parameters: the supplier must determine a distribution policy that maximizes its expected discounted value (revenue minus costs) over the planning horizon, which can be finite or infinite. Typical problems dealing with SIRP applications arise in the oil and gas industry (Bard et al. 1998; Federgruen and Zipkin 1984; Trudeau and Dror 1992). Table 4 lists the papers cited in this section.

### 4.1. Finite Horizon

Several heuristic algorithms exist for the SIRP with a finite horizon. Federgruen and Zipkin (1984) have modified the VRP heuristic of Fisher and Jaikumar (1981) to accommodate inventory and shortage costs in a random demand environment. Federgruen, Prastacos, and Zipkin (1986) extended the work of Federgruen and Zipkin (1984) to allow multiple products, in their case, perishable items. Using a different approach, Golden, Assad, and Dahl (1984) determined which customers to visit based on their degree of urgency, before solving the routing problem heuristically by means of the Clarke and Wright (1964) algorithm.

Jaillet et al. (2002) solved a short-term problem using the rolling horizon framework of Bard et al. (1998), which is later approximated as a periodic solution over a long-term horizon. This problem includes satellite facilities where trucks can be replenished during their route, and direct deliveries for emergency deliveries when customers run out of stock.

Table 4 Classification of the Papers on the Stochastic IRP

Reference	Structure			Inventory policy				Fleet composition			Fleet size				
	Time horizon	One-to-one	One-to-many	Routing		Maximum level (ML)	Order-up-to level (OU)	Inventory decisions		Fleet composition		Fleet size			
				Direct	Multiple			Continuous	Lost sales	Backlogging	Homogeneous	Heterogeneous	Single	Multiple	Unconstrained
Federgruen and Zipkin (1984)	✓		✓		✓	✓		✓			✓		✓		
Golden, Assad, and Dahl (1984)	✓		✓		✓	✓		✓				✓			
Federgruen, Prastacos, and Zipkin (1986)	✓		✓		✓	✓		✓				✓			
Trudeau and Dror (1992)	✓			✓	✓	✓		✓				✓			
Minkoff (1993)		✓		✓	✓	✓		✓				✓			
Bard et al. (1998)	✓		✓		✓	✓		✓				✓			
Campbell et al. (1998)	✓		✓		✓	✓		✓				✓			
Qu, Bookbinder, and Iyogun (1999)			✓		✓	✓		✓					✓		
Berman and Larson (2001)			✓		✓	✓		✓				✓			
Jaillet et al. (2002)	✓		✓		✓	✓		✓					✓		
Kleywegt, Nori, and Savelsbergh (2002)					✓	✓		✓					✓		
Adelman (2004)	✓	✓		✓		✓		✓					✓		
Kleywegt, Nori, and Savelsbergh (2004)	✓	✓		✓		✓		✓					✓		
Aghezzaf (2008)	✓	✓		✓		✓		✓					✓		
Hvattum and Løkketangen (2009)	✓	✓		✓		✓		✓					✓		
Hvattum, Løkketangen, and Laporte (2009)						✓									
Huang and Lin (2010)	✓	✓		✓			✓	✓					✓		
Geiger and Sevaux (2011a)	✓	✓		✓		✓		✓					✓		
Liu and Lee (2011)	✓	✓		✓		✓		✓					✓		
Solyali, Cordeau, and Laporte (2012)	✓	✓		✓		✓			✓				✓		

Geiger and Sevaux (2011b) studied a problem with unknown demand varying within 10% of a mean value. They proposed several policies based on delivery frequencies for each customer. They provide the Pareto front approximation of such policies when moving from a total routing-optimized solution to an inventory-optimized one. To solve the problem for many periods, they apply the record-to-record travel heuristic of Li, Golden, and Wasil (2007).

The classical road-based IRP with time windows was solved by Liu and Lee (2011). Their algorithm uses a combination of variable neighborhood search and tabu search. However, the effectiveness of the algorithm cannot be completely assessed because the computational comparison is made against three algorithms designed for the VRP with time windows.

#### 4.2. Infinite Horizon

Given the size and the complexity of the SIRP, Minkoff (1993) proposed a heuristic approach based on a Markov decision model to a problem somewhat similar to the IRP, called the delivery dispatching problem. He simplified the objective function, making it a sum of smaller and simpler objective functions, one for each customer, and solved the problem heuristically. This model is one of the few to work with an unconstrained fleet. Also working with a variant of the IRP, Berman and Larson (2001) used dynamic programming to solve the case where the demand probability distributions are known, adjusting the amount of goods delivered to each customer, to minimize the expected sum of penalties associated with early or late deliveries.

Campbell et al. (1998) introduced a dynamic programming model for the SIRP in which only transportation and stockout costs are taken into account. To simplify the model, no inventory holding costs are incurred. At the beginning of each period, the supplier knows the inventory level at each of the customers and decides which customers to visit, how much to deliver to each, how to combine them into routes, and which routes to assign to each of the available vehicles. The components of their Markov decision process are the following (Campbell et al. 1998):

- The state  $x$  is the current inventory at each customer and the state space  $\mathcal{X}$  is  $[0, C_1] \times [0, C_2] \times \cdots \times [0, C_n]$ . Let  $X_t \in \mathcal{X}$  denote the state at time  $t$ .
- The action space  $\mathcal{A}(x)$  for each state  $x$  is the set of all itineraries satisfying constraints such as vehicle capacities and customer inventory capacities. Let  $\mathcal{A} \equiv \bigcup_{x \in \mathcal{X}} \mathcal{A}(x)$  denote the set of all possible itineraries and  $A_t \in \mathcal{A}(X_t)$  denote the itinerary chosen at time  $t$ .
- The Markov transition function  $R$  obtained from the known demand probability distribution. For any state  $x \in \mathcal{X}$ , any itinerary  $a \in \mathcal{A}(x)$ , and any

(measurable) subset  $B \subseteq \mathcal{X}$ , the transition follows

$$P[X_{t+1} \in B \mid X_t = x, A_t = a] = \int_B R[dy \mid x, a]. \quad (17)$$

- The only costs taken into account are transportation costs, which depend on the vehicle tours, and a stockout penalty cost. Let  $c(x, a)$  denote the expected daily cost if the process is in state  $x$  and itinerary  $a \in \mathcal{A}(x)$  is chosen.

- Let  $\alpha \in [0, 1)$  denote the discount factor. The objective is to minimize the expected total discounted cost over an infinite horizon. Let  $V^*(x)$  denote the optimal expected cost given that the initial state is  $x$ , i.e.,

$$V^*(x) \equiv \inf_{\{A_t\}_{t=0}^{\infty}} E \left[ \sum_{t=0}^{\infty} \alpha^t c(X_t, A_t) \mid X_0 = x \right]. \quad (18)$$

The actions are restricted in the sense that  $A_t$  depends only on the history of the system; when one decides which itinerary to choose, one does not know what the future holds. Under certain usual conditions, equation (18) can be written as

$$V^*(x) \equiv \inf_{a \in \mathcal{A}(x)} \left\{ c(x, a) + \alpha \int_{\mathcal{X}} V^*(y) R[dy \mid x, a] \right\}. \quad (19)$$

Equation (19) can only be solved using classical dynamic programming algorithms if the state space  $\mathcal{X}$  is small, which is not the case for practical instances of the SIRP. Campbell et al. (1998) state that it is possible to solve the problem by approximating the value function  $V^*(x)$  with a function  $\hat{V}(x, \beta)$  that depends on a vector of parameters  $\beta$ . This is the approach followed by Kleywegt, Nori, and Savelsbergh (2002, 2004) who, as in Campbell et al. (1998), use a Markov decision process to formulate the SIRP. Here, a set of customers must be served from a warehouse by means of a fleet of homogeneous capacitated vehicles. Each customer has an inventory capacity, and the problem is modeled in discrete time. Inventory at each customer at any given time is known to the supplier. Customer demands are stochastic and independent from each other, and the supplier knows the joint probability distribution of their demands, which does not change over time. The supplier must decide which customers to visit, how much to deliver to them, how to combine customers into routes, and which routes to assign to each vehicle. The set of admissible decisions is constrained by vehicle and customer capacities, driver working hours, possible time windows at the customers, and by any other constraint imposed by the system or the application. Although demands are stochastic, the cost of each decision is known to the supplier. Thus, Kleywegt, Nori, and Savelsbergh (2002, 2004) consider traveling costs, shortages that

are proportional to the amount of unsatisfied and lost demand and holding costs. These models consider a revenue proportional to the quantities delivered. The problem is formulated to maximize the expected discounted value over an infinite horizon as a discrete time Markov decision process.

Kleywegt, Nori, and Savelsbergh (2002) studied the case with direct deliveries only, whereas Kleywegt, Nori, and Savelsbergh (2004) limited the routing to at most three customers per route. In the paper by Adelman (2004), there is no limit on the number of customers to be served in a route, except for the limits resulting from maximal route duration and vehicle capacity. The approach taken by Adelman is a little different and works as follows. Using a value function not made up of individual customer values, but of marginal transportation costs, he compares stockout costs with replenishment policies, choosing the one that maximizes the value. A linear program is derived from the value function, and its optimal dual prices are used to calculate the optimal policy of the semi-Markov decision process. In the direct deliveries study of Kleywegt, Nori, and Savelsbergh (2002), optimal solutions were obtained on instances with up to 60 customers and up to 16 vehicles, whereas in Kleywegt, Nori, and Savelsbergh (2004) instances with up to 15 customers and five vehicles were solved.

There are few exceptions to the dynamic programming approach. Qu, Bookbinder, and Iyogun (1999) develop a periodic policy for a multi-item IRP and Huang and Lin (2010) solve it by means of an ant colony optimization algorithm. Hvattum and Løkketangen (2009) and Hvattum, Løkketangen, and Laporte (2009) solve the IRP by capturing the stochastic information over a short horizon. In Hvattum and Løkketangen (2009), the problem is solved using a GRASP, which successively increases the volume delivered to customers. Hvattum, Løkketangen, and Laporte (2009) state that it is sufficient to capture the stochastics of the SIRP over a finite horizon, which is achieved through truncated scenario trees, both breadthwise and depthwise.

#### 4.3. Robust Optimization

A different way to model and solve the SIRP is through the use of robust optimization. This solution framework is appropriate to deal with uncertainty where no information is available on the parameter probability distributions. This is achieved by optimizing the problem while ensuring feasibility for all possible realizations of the bounded uncertain parameters, also called a minimax solution. Usually, studies on the SIRP assume that one knows the probability distribution of demand, which is generally not the case in practice. Aghezzaf (2008) considers the case of normally distributed customer demands and travel times

with constant averages and bounded standard deviations. He uses robust optimization to determine the distribution plan through a nonlinear mixed-integer programming formulation, which is feasible for all possible realizations of the random variables. Monte Carlo simulation is used to improve the plan's critical parameters (replenishment cycle times and safety stock levels). Solyali, Cordeau, and Laporte (2012) proposed such an exact approach based on robust optimization, which we describe as follows.

In their model, a supplier distributes a single product to  $n$  customers, using a vehicle of capacity  $Q$ , over a finite discrete time horizon  $p$ . The dynamic uncertain demand at each customer  $i \in \mathcal{V} = \{1, \dots, n\}$  in period  $t \in \mathcal{T} = \{1, \dots, p\}$  is  $d_i^t$ . The probability distribution of the demand is unknown, but one knows that it can take any value in the interval  $[\bar{d}_i^t - \hat{d}_i^t, \bar{d}_i^t + \hat{d}_i^t]$ , where  $\bar{d}_i^t$  is the nominal value (point estimate), and  $\hat{d}_i^t$  is the maximum deviation for the demand of  $i$  in period  $t$ . An inventory holding cost equal to  $h_i^t$  per unit at customer  $i$  in period  $t$  is incurred at the customers. Backlogging is allowed, and each unit backlogged in period  $t$  at customer  $i$  costs  $g_i^t$ , where  $g_i^t > h_i^t$ . There is a fixed-vehicle dispatching cost  $f_t$  for using the vehicle in period  $t$ . If the vehicle leaves customer  $i \in \mathcal{V}' = \mathcal{V} \cup \{0\}$  heading to customer  $j$ , it incurs a cost  $c_{ij}$ , and transportation costs are assumed to be symmetric.

The problem is formulated as follows. Let  $q_{itk}$  be the total inventory cost of replenishing customer  $i$  in period  $t \in \mathcal{T}$  to satisfy its demand in period  $k \in \mathcal{T}$ ;  $q_{i,T+1,k}$  is the total inventory cost of not meeting the demand of customer  $i$  in period  $k \in \mathcal{T}$ ;  $w_{itk}$  is the fraction of the demand of customer  $i$  in period  $k \in \mathcal{T}$  delivered in period  $t \in \mathcal{T}$ ; and  $w_{i,T+1,k}$  is the fraction of the unsatisfied demand of customer  $i$  in period  $k \in \mathcal{T}$ . Additionally, let  $y_{it}$  be 1 if the inventory of customer  $i$  is replenished in period  $t \in \mathcal{T}$  and 0 otherwise; let  $y_{0t}$  be 1 if the vehicle is used in period  $t \in \mathcal{T}$  and 0 otherwise; and let  $x_{ij}^t$  be the number of times the edge  $(i, j)$  is traversed in period  $t \in \mathcal{T}$ . The derivation of the robust formulation is rather involved, and the reader is referred to Solyali, Cordeau, and Laporte (2012) for details. Their final robust formulation ensuring feasibility for any  $d_i^k \in [\bar{d}_i^k - \hat{d}_i^k, \bar{d}_i^k + \hat{d}_i^k]$  is

$$\begin{aligned} \text{minimize } & \left\{ \sum_{t \in \mathcal{T}} f_t y_{0t} + \sum_{i \in \mathcal{V}'} \sum_{j \in \mathcal{V}', i < j} \sum_{t \in \mathcal{T}} c_{ij} x_{ij}^t \right. \\ & \left. + \sum_{i \in \mathcal{V}'} \sum_{t=1}^{p+1} \sum_{k=1}^p q_{itk} w'_{itk} \right\} \end{aligned} \quad (20)$$

$$\text{subject to } \sum_{i \in \mathcal{V}'} \sum_{k=1}^p w'_{itk} \leq Q y_{0t} \quad t \in \mathcal{T}, \quad (21)$$

$$w'_{itk} \geq 0 \quad i \in \mathcal{V} \quad k \in \mathcal{T} \quad 1 \leq t \leq p+1, \quad (22)$$

$$\sum_{t=1}^{p+1} w'_{itk} \geq \bar{d}_i^t + \hat{d}_i^t \quad i \in \mathcal{V} \quad k \in \mathcal{T}, \quad (23)$$

$$w'_{itk} \leq (\bar{d}_i^t + \hat{d}_i^t) y_{it} \quad i \in \mathcal{V} \quad t \in \mathcal{T} \quad k \in \mathcal{T}, \quad (24)$$

$$\sum_{j \in \mathcal{V}', i < j} x_{ij}^t + \sum_{j \in \mathcal{V}', i > j} x_{ji}^t = 2y_{it} \quad i \in \mathcal{V}' \quad t \in \mathcal{T}, \quad (25)$$

$$\sum_{i \in \mathcal{P}} \sum_{j \in \mathcal{P}, i < j} x_{ij}^t \leq \sum_{i \in \mathcal{P}} y_{it} - y_{mt} \quad \mathcal{P} \subseteq \mathcal{V} \quad t \in \mathcal{T} \quad m \in \mathcal{P}, \quad (26)$$

$$y_{it} \leq y_{0t} \quad i \in \mathcal{V} \quad t \in \mathcal{T}, \quad (27)$$

$$x_{ij}^t \in \{0, 1\} \quad i, j \in \mathcal{V}, i < j \quad t \in \mathcal{T}, \quad (28)$$

$$x_{i0}^t \in \{0, 1, 2\} \quad i \in \mathcal{V} \quad t \in \mathcal{T}, \quad (29)$$

$$y_{it} \in \{0, 1\} \quad i \in \mathcal{V}' \quad t \in \mathcal{T}, \quad (30)$$

$$w_{itk} \geq 0 \quad i \in \mathcal{V} \quad k \in \mathcal{T} \quad 1 \leq t \leq p+1, \quad (31)$$

where  $q_{itk} = \sum_{l=1}^{k-1} h_i^l$  if  $t \leq k$ ,  $q_{itk} = \sum_{l=k}^{t-1} g_i^l$  if  $t > k$ , and  $w'_{itk} = d_i^k w_{itk}$ .

The objective function (20) is the sum of the fixed vehicle dispatching, transportation, inventory holding, and shortage costs. Constraints (21) ensure that the vehicle capacity is not exceeded. Constraints (22) and (23) guarantee that quantities delivered are non-negative and ensure robustness for any realization of the demand. Constraints (24) link variables  $w'_{itk}$  and  $y_{it}$ , forcing the  $y$  variables to take value 1 if the  $w'$  variables are positive. Thus,  $(\bar{d}_i^t + \hat{d}_i^t)$  is used as a big-M value. Constraints (25) are degree constraints, guaranteeing that if  $i$  is visited in period  $t$ , then there are two edges incident to it. Constraints (26) are sub-tour elimination constraints. Constraints (27) ensure that the vehicle starts its tour from the supplier and are used to strengthen the formulation. Constraints (28)–(30) and (31) are integrality constraints and non-negativity constraints, respectively.

Using this formulation, Solyali, Cordeau, and Laporte (2012) have solved instances with up to seven periods and 30 customers within a reasonable computing time.

## 5. Dynamic and Stochastic Inventory-Routing Problem

In the dynamic IRP, customer demand is gradually revealed over time, e.g., at the end of each period, and one must solve the problem repeatedly with the available information. We are aware of two studies on this problem, both making use of probabilistic knowledge of the demand. In dynamic and stochastic inventory-routing problems (DSIRP), customer demand is known in a probabilistic sense and revealed over time, thus yielding a dynamic and stochastic problem.

Solving a dynamic problem consists of proposing a *solution policy* as opposed to computing a static output (Berbeglia, Cordeau, and Laporte 2010). A possible policy is to optimize a static instance whenever new information becomes available. The drawback of such a method is that it is often very time consuming to solve a large number of instances. A more common policy is to apply the static algorithm only once and then reoptimize the problem through a heuristic whenever new information is made available. A third policy, which can be combined with either of the first two, is to take advantage of the probabilistic knowledge of future information and make use of forecasts. For more information on the solution of dynamic problems, see Psaraftis (1998); Ghiani et al. (2003); Berbeglia, Cordeau, and Laporte (2010).

Recently, Bertazzi et al. (2013) and Coelho, Cordeau, and Laporte (2012b) introduced solution methodologies that can handle DSIRPs, with a goal of minimizing the total inventory, distribution, and shortage costs. The paper by Bertazzi et al. proposes a heuristic rollout algorithm that uses a sampling approach to generate demand scenarios for the current period and considers the average demand for future ones. Decisions are made by solving a MIP by branch and cut in each period. Bertazzi et al. considered the OU policy, and tested their algorithm on instances with up to 35 customers and three periods, 15 customers and six periods, and 10 customers and nine periods. Coelho, Cordeau, and Laporte (2012b) introduced a different methodology that can make use of historical data in the form of forecasts to take future unknown demands into account, thus being able to efficiently solve instances in which the demand presents a trend or seasonality. The problem at each time period is solved by ALNS and the heuristic was implemented in a rolling horizon framework. Both the OU and ML policies were considered and results were reported on instances with up to 200 customers and 20 periods.

## 6. Benchmark Instances

Benchmark instance sets are now available to the research community and allow for a better assessment and comparison of algorithms. We have aggregated these instances into a single website to make their access easier and to encourage other researchers to use them; they are all available at <http://www.leandro-coelho.com/instances>. The first set was proposed by Archetti et al. (2007) and comprises 160 instances ranging from five to 50 customers, with three and six periods, respectively. These were used to evaluate the algorithms of Bertazzi, Paletta, and Speranza (2002); Archetti et al. (2007); Solyali and Süral (2011); Archetti et al. (2012); Coelho, Cordeau, and Laporte (2012a, c); Coelho and Laporte (2013b).

A newer, larger, and more challenging data set proposed by Archetti et al. (2012) contains 60 instances with six periods and up to 200 customers. This set has been used to evaluate the algorithms of Archetti et al. (2012); Coelho, Cordeau, and Laporte (2012a); and Coelho and Laporte (2013b). A large set for the problem with multiple vehicles and multiple products with 675 instances with varying number of vehicles, periods, customers, and products has been proposed by Coelho and Laporte (2013a). Finally, Coelho, Cordeau, and Laporte (2012b) have proposed a large test bed for the DSIRP, containing 450 instances.

## 7. Conclusions

The IRP was introduced 30 years ago by Bell et al. (1983) and has since evolved into a rich research area. Several versions of the problem have been studied, and applications are encountered in many settings, primarily in maritime transportation. Our survey provides a classification of the IRP literature under two dimensions: the structure of the problem and the time at which information becomes available. Because IRPs are typically very hard to solve, most algorithms are heuristics. These have gradually evolved from simple interchange schemes to more sophisticated meta-heuristics, sometimes combined with exact methods. In recent years, we have also witnessed the emergence of exact branch-and-cut algorithms that can be implemented within the framework of general-purpose solvers. Over the years, part of the research effort has shifted toward the study of rich extensions of the basic IRP model. These include the production-routing problem, the IRP with multiple products, the IRP with direct deliveries and transshipment, and the consistent IRP. Finally, several authors have moved away from the deterministic and static version of the IRP and have proposed models and algorithms capable of handling its stochastic and dynamic versions. We believe this paper has helped unify the rapidly expanding body of knowledge on the IRP and will stimulate other researchers to pursue the study of this fascinating field.

## Acknowledgments

The authors thank the associate editor as well as two anonymous referees for their valuable comments that helped improve this paper. This work was partly supported by the Canadian Natural Sciences and Engineering Research Council [Grants 227837-09 and 39682-10]. This support is gratefully acknowledged.

## References

Abdelmaguid TF (2004) Heuristic approaches for the integrated inventory distribution problem. Ph.D. dissertation, University of Southern California, Los Angeles.

- Abdelmaguid TF, Dessouky MM (2006) A genetic algorithm approach to the integrated inventory-distribution problem. *Internat. J. Production Res.* 44(21):4445–4464.
- Abdelmaguid TF, Dessouky MM, Ordóñez F (2009) Heuristic approaches for the inventory-routing problem with backloging. *Comput. Indust. Engrg* 56(4):1519–1534.
- Adelman D (2004) A price-directed approach to stochastic inventory/routing. *Oper. Res.* 52(4):499–514.
- Adulyasak Y, Cordeau J-F, Jans R (2012) Optimization-based adaptive large neighborhood search for the production routing problem. *Transportation Sci.*, ePub ahead of print November 28, <http://dx.doi.org/10.1287/trsc.1120.0443>.
- Adulyasak Y, Cordeau J-F, Jans R (2013) Formulations and branch-and-cut algorithms for multivehicle production and inventory routing problems. *INFORMS J. Comput.*, ePub ahead of print June 14, <http://dx.doi.org/10.1287/ijoc.2013.0550>.
- Aghezzaf E-H (2008) Robust distribution planning for the supplier-managed inventory agreements when demand rates and travel times are stationary. *J. Oper. Res. Soc.* 59(8):1055–1065.
- Aghezzaf E-H, Raa B, van Landeghem H (2006) Modeling inventory routing problems in supply chains of high consumption products. *Eur. J. Oper. Res.* 169(3):1048–1063.
- Ahmadi-Javid A, Azad N (2010) Incorporating location, routing and inventory decisions in supply chain network design. *Transportation Res. Part E: Logist. Transportation Rev.* 46(5):582–597.
- Aksen D, Kaya O, Salman F, Akça Y (2012) Selective and periodic inventory routing problem for waste vegetable oil collection. *Optim. Lett.* 6(6):1063–1080.
- Al-Khayyal F, Hwang S-J (2007) Inventory constrained maritime routing and scheduling for multi-commodity liquid bulk, part I: Applications and model. *Eur. J. Oper. Res.* 176(1):106–130.
- Alegre J, Laguna M, Pacheco J (2007) Optimizing the periodic pick-up of raw materials for a manufacturer of auto parts. *Eur. J. Oper. Res.* 179(3):736–746.
- Andersson H, Hoff A, Christiansen M, Hasle G, Løkketangen A (2010) Industrial aspects and literature survey: Combined inventory management and routing. *Comput. Oper. Res.* 37(9):1515–1536.
- Angulo A, Nachtmann H, Waller MA (2004) Supply chain information sharing in a vendor managed inventory partnership. *J. Bus. Logist.* 25(1):101–120.
- Anily S, Federgruen A (1990) One warehouse multiple retailer systems with vehicle routing costs. *Management Sci.* 36(1):92–114.
- Archetti C, Bertazzi L, Hertz A, Speranza MG (2012) A hybrid heuristic for an inventory routing problem. *INFORMS J. Comput.* 24(1):101–116.
- Archetti C, Bertazzi L, Laporte G, Speranza MG (2007) A branch-and-cut algorithm for a vendor-managed inventory-routing problem. *Transportation Sci.* 41(3):382–391.
- Archetti C, Bertazzi L, Paletta G, Speranza MG (2011) Analysis of the maximum level policy in a production-distribution system. *Comput. Oper. Res.* 38(12):1731–1746.
- Bard JF, Nananukul N (2009) Heuristics for a multiperiod inventory routing problem with production decisions. *Comput. Indust. Engrg* 57(3):713–723.
- Bard JF, Nananukul N (2010) A branch-and-price algorithm for an integrated production and inventory routing problem. *Comput. Oper. Res.* 37(12):2202–2217.
- Bard JF, Huang L, Jaillet P, Dror M (1998) A decomposition approach to the inventory routing problem with satellite facilities. *Transportation Sci.* 32(2):189–203.
- Barlett CA, Ghoshal S (2002) Building competitive advantage through people. *MIT Sloan Management Rev.* 43(2):34–41.
- Bausch DO, Brown GG, Ronen D (1998) Scheduling short-term marine transport of bulk products. *Maritime Policy Management* 25(4):335–348.
- Beamom BM (1999) Measuring supply chain performance. *Internat. J. Oper. Production Management* 19(3):275–292.
- Beasley JE (1984) Fixed routes. *J. Oper. Res. Soc.* 35(1):49–55.



- Bell WJ, Dalberto LM, Fisher ML, Greenfield AJ, Jaikumar R, Kedia P, Mack RG, Prutzman PJ (1983) Improving the distribution of industrial gases with an on-line computerized routing and scheduling optimizer. *Interfaces* 13(6):4–23.
- Benoist T, Gardi F, Jeanjean A, Estellon B (2011) Randomized local search for real-life inventory routing. *Transportation Sci.* 45(3):381–398.
- Berbeglia G, Cordeau J-F, Laporte G (2010) Dynamic pickup and delivery problems. *Eur. J. Oper. Res.* 202(1):8–15.
- Berman O, Larson RC (2001) Deliveries in an inventory/routing problem using stochastic dynamic programming. *Transportation Sci.* 35(2):192–213.
- Bertazzi L (2008) Analysis of direct shipping policies in an inventory-routing problem with discrete shipping times. *Management Sci.* 54(4):748–762.
- Bertazzi L, Speranza MG (2002) Continuous and discrete shipping strategies for the single link problem. *Transportation Sci.* 36(3):314–325.
- Bertazzi L, Paletta G, Speranza MG (2002) Deterministic order-up-to level policies in an inventory routing problem. *Transportation Sci.* 36(1):119–132.
- Bertazzi L, Paletta G, Speranza MG (2005) Minimizing the total cost in an integrated vendor-managed inventory system. *J. Heuristics* 11(5–6):393–419.
- Bertazzi L, Savelsbergh M, Speranza MG (2008) Inventory routing. Golden BL, Raghavan S, Wasil EA, eds. *The Vehicle Routing Problem: Latest Advances and New Challenges* (Springer, New York), 49–72.
- Bertazzi L, Speranza MG, Ukovich W (1997) Minimization of logistic costs with given frequencies. *Transportation Res. Part B: Methodological* 31(4):327–340.
- Bertazzi L, Bosco A, Guerriero F, Laganà D (2013) A stochastic inventory routing problem with stock-out. *Transportation Res. Part C: Emerging Tech.* 27(1):89–107.
- Blumenfeld DE, Burns LD, Diltz JD, Daganzo CF (1985) Analyzing trade-offs between transportation, inventory and production costs on freight networks. *Transportation Res. Part B: Methodological* 19(5):361–380.
- Blumenfeld DE, Burns LD, Daganzo CF, Frick MC, Hall RW (1987) Reducing logistic costs at General Motors. *Interfaces* 17(1):26–47.
- Boudia M, Prins C (2009) A memetic algorithm with dynamic population management for an integrated production-distribution problem. *Eur. J. Oper. Res.* 195(3):703–715.
- Burns LD, Hall RW, Blumenfeld DE, Daganzo CF (1985) Distribution strategies that minimize transportation and inventory costs. *Oper. Res.* 33(3):469–490.
- Campbell AM, Savelsbergh MWP (2004) A decomposition approach for the inventory-routing problem. *Transportation Sci.* 38(4):488–502.
- Campbell AM, Clarke L, Kleywegt AJ, Savelsbergh MWP (1998) The inventory routing problem. Crainic TG, Laporte G, eds. *Fleet Management and Logistics* (Springer, Boston), 95–113.
- Carter MW, Farvolden JM, Laporte G, Xu J (1996) Solving an integrated logistics problem arising in grocery distribution. *INFOR* 34(4):290–306.
- Chandra P (1993) A dynamic distribution model with warehouse and customer replenishment requirements. *J. Oper. Res. Soc.* 44(7):681–692.
- Chandra P, Fisher ML (1994) Coordination of production and distribution planning. *Eur. J. Oper. Res.* 72(3):503–517.
- Chien TW, Balakrishnan A, Wong RT (1989) An integrated inventory allocation and vehicle routing problem. *Transportation Sci.* 23(2):67–76.
- Christiansen M (1999) A decomposition of a combined inventory and time constrained ship routing problem. *Transportation Sci.* 33(1):3–16.
- Christiansen M, Nygreen B (1998a) A method for solving ship routing problems with inventory constraints. *Ann. Oper. Res.* 81:357–378.
- Christiansen M, Nygreen B (1998b) Modelling path flows for a combined ship routing and inventory management problem. *Ann. Oper. Res.* 82:391–412.
- Christiansen M, Nygreen B (2005) Robust inventory ship routing by column generation. Desaulniers G, Desrosiers J, Solomon MM, eds. *Column Generation* (Springer, New York), 197–224.
- Christiansen M, Fagerholt K, Ronen D (2004) Ship routing and scheduling: Status and perspectives. *Transportation Sci.* 38(1):1–18.
- Christiansen M, Fagerholt K, Nygreen B, Ronen D (2007) Maritime transportation. Barnhart C, Laporte G, eds. *Transportation, Handbooks in Operations Research and Management Science*, Vol. 14 (North-Holland, Amsterdam), 189–284.
- Christiansen M, Fagerholt K, Nygreen B, Ronen D (2013) Ship routing and scheduling in the new millennium. *Eur. J. Oper. Res.* 228(3):467–483.
- Christiansen M, Fagerholt K, Flatberg T, Haugen Ø, Kloster O, Lund EH (2011) Maritime inventory routing with multiple products: A case study from the cement industry. *Eur. J. Oper. Res.* 208(1):86–94.
- Christofides N, Beasley JE (1984) The periodic routing problem. *Networks* 14(2):237–256.
- Clarke G, Wright JW (1964) Scheduling of vehicles from a central depot to a number of delivery points. *Oper. Res.* 12(4):568–581.
- Coelho LC, Laporte G (2013a) A branch-and-cut algorithm for the multi-product multi-vehicle inventory-routing problem. *Internat. J. Production Res.*, ePub ahead of print February 13, <http://dx.doi.org/10.1080/00207543.2012.757668>.
- Coelho LC, Laporte G (2013b) The exact solution of several classes of inventory-routing problems. *Comput. Oper. Res.* 40(2):558–565.
- Coelho LC, Cordeau J-F, Laporte G (2012a) Consistency in multi-vehicle inventory-routing. *Transportation Res. Part C: Emerging Tech.* 24(1):270–287.
- Coelho LC, Cordeau J-F, Laporte G (2012b) Dynamic and stochastic inventory-routing. Technical report, CIRRELT-2012-37, Montreal, Canada.
- Coelho LC, Cordeau J-F, Laporte G (2012c) The inventory-routing problem with transshipment. *Comput. Oper. Res.* 39(11):2537–2548.
- Cordeau J-F, Laporte G, Savelsbergh MWP, Vigo D (2007) Vehicle routing. Barnhart C, Laporte G, eds. *Transportation, Handbooks in Operations Research and Management Science*, Vol. 14 (North-Holland, Amsterdam), 367–428.
- Custódio A, Oliveira R (2006) Redesigning distribution operations: A case study on integrating inventory management and vehicle routes design. *Internat. J. Logist.* 9(2):169–187.
- Dauzère-Péres S, Nordli A, Olstad A, Haugen K, Koester U, Olav MP, Teistklub G, Reistad A (2007) Omya Hustadmarmor optimizes its supply chain for delivering calcium carbonate slurry to European paper manufacturers. *Interfaces* 37(1):39–51.
- Dror M, Ball MO (1987) Inventory/routing: Reduction from an annual to a short-period problem. *Naval Res. Logist.* 34(6):891–905.
- Dror M, Levy L (1986) A vehicle routing improvement algorithm comparison of a “greedy” and a matching implementation for inventory routing. *Comput. Oper. Res.* 13(1):33–45.
- Dror M, Ball MO, Golden BL (1985) A computational comparison of algorithms for the inventory routing problem. *Ann. Oper. Res.* 4:3–23.
- Engineer FG, Furman KC, Nemhauser GL, Savelsbergh MWP, Song J-H (2012) A branch-and-price-and-cut algorithm for single-product maritime inventory routing. *Oper. Res.* 60(1):106–122.
- Federgruen A, Zipkin P (1984) A combined vehicle routing and inventory allocation problem. *Oper. Res.* 32(5):1019–1037.
- Federgruen A, Prastacos G, Zipkin PH (1986) An allocation and distribution model for perishable products. *Oper. Res.* 34(1):75–82.
- Fisher ML, Jaikumar R (1981) A generalized assignment heuristic for vehicle-routing. *Networks* 11(2):109–124.

- Francis P, Smilowitz K, Tzur M (2008) The periodic vehicle routing problem and its extensions. Golden BL, Raghavan S, Wasil EA, eds. *The Vehicle Routing Problem: Latest Advances and New Challenges* (Springer, New York), 239–261.
- Fumero F, Vercellis C (1999) Synchronized development of production, inventory, and distribution schedules. *Transportation Sci.* 33(3):330–340.
- Gallego G, Simchi-Levi D (1990) On the effectiveness of direct shipping strategy for the one-warehouse multi-retailer  $r$ -systems. *Management Sci.* 36(2):240–243.
- Gallego G, Simchi-Levi D (1994) Rejoinder to “A note on bounds for direct shipping costs.” *Management Sci.* 40(10):1393.
- Gaur V, Fisher ML (2004) A periodic inventory routing problem at a supermarket chain. *Oper. Res.* 52(6):813–822.
- Geiger MJ, Sevaux M (2011a) On the use of reference points for the biobjective inventory routing problem. Ceschia S, Di Gaspero L, Loghi M, Schaerf A, Urli T, eds. *Proc. MIC 2011: The IX Metaheuristics Internat. Conf. Udine* (University of Udine, Udine, Italy), 141–149.
- Geiger MJ, Sevaux M (2011b) Practical inventory routing: A problem definition and an optimization method. Artner C, Doerner KF, Hartl RF, Tricoire F, eds. *Proc. EU/MEeting: Workshop on Client-Centered Logist. Internat. Aid Vienna* (University of Vienna, Vienna, Austria), 32–35.
- Gendreau M, Potvin J-Y (2010) *Handbook of Metaheuristics* International Series in Operations Research and Management Science, Vol. 146 (Springer, New York).
- Ghiani G, Guerriero F, Laporte G, Musmanno R (2003) Real-time vehicle routing: Solutions concepts, algorithms and parallel computing strategies. *Eur. J. Oper. Res.* 151(1):1–11.
- Golden BL, Assad AA, Dahl R (1984) Analysis of a large-scale vehicle-routing problem with an inventory component. *Large Scale Systems Inform. Decision Tech.* 7(2–3):181–190.
- Groër C, Golden BL, Wasil EA (2009) The consistent vehicle routing problem. *Manufacturing Service Oper. Management* 11(4):630–643.
- Grønhaug R, Christiansen M, Desaulniers G, Desrosiers J (2010) A branch-and-price method for a liquefied natural gas inventory routing problem. *Transportation Sci.* 44(3):400–415.
- Hall RW (1992) A note on bounds for direct shipping cost. *Management Sci.* 38(8):1212–1214.
- Hemmelmayr V, Doerner KF, Hartl RF, Savelsbergh MWP (2009) Delivery strategies for blood products supplies. *OR Spectrum* 31(4):707–725.
- Herer YT, Roundy R (1997) Heuristics for one-warehouse multi-retailer distribution problem with performance bounds. *Oper. Res.* 45(1):102–115.
- Hewitt M, Nemhauser GL, Savelsbergh MWP, Song J-H (2013) A branch-and-price guided search approach to maritime inventory routing. *Comput. Oper. Res.* 40(5):1410–1419.
- Huang S-H, Lin P-C (2010) A modified ant colony optimization algorithm for multiitem inventory routing problems with demand uncertainty. *Transportation Res. Part E: Logist. Transportation Rev.* 46(5):598–611.
- Hvattum LM, Løkketangen A (2009) Using scenario trees and progressive hedging for stochastic inventory routing problems. *J. Heuristics* 15(6):527–557.
- Hvattum LM, Løkketangen A, Laporte G (2009) Scenario tree-based heuristics for stochastic inventory-routing problems. *INFORMS J. Comput.* 21(2):268–285.
- Jaillet P, Bard JF, Huang L, Dror M (2002) Delivery cost approximations for inventory routing problems in a rolling horizon framework. *Transportation Sci.* 36(3):292–300.
- Kleywegt AJ, Nori VS, Savelsbergh MWP (2002) The stochastic inventory routing problem with direct deliveries. *Transportation Sci.* 36(1):94–118.
- Kleywegt AJ, Nori VS, Savelsbergh MWP (2004) Dynamic programming approximations for a stochastic inventory routing problem. *Transportation Sci.* 38(1):42–70.
- Lee HL, Seungjin W (2008) The whose, where, and how of inventory control design. *Supply Chain Management Rev.* 12(8):22–29.
- Li F, Golden BL, Wasil EA (2007) A record-to-record travel algorithm for solving the heterogeneous fleet vehicle routing problem. *Comput. Oper. Res.* 34(9):2734–2742.
- Li J, Chen H, Chu F (2010) Performance evaluation of distribution strategies for the inventory routing problem. *Eur. J. Oper. Res.* 202(2):412–419.
- Liu S-C, Lee W-T (2011) A heuristic method for the inventory routing problem with time windows. *Expert Systems Appl.* 38(10):13223–13231.
- Maniezzo V, Stützle T, Voß S (2009) *Matheuristics: Hybridizing Metaheuristics and Mathematical Programming*, Vol. 10 (Springer, New York).
- Mercer A, Tao X (1996) Alternative inventory and distribution policies of a food manufacturer. *J. Oper. Res. Soc.* 47(6):755–765.
- Michel S, Vanderbeck F (2012) A column-generation based tactical planning method for inventory routing. *Oper. Res.* 60(2):382–397.
- Miller, D. M. (1987) An interactive, computer-aided ship scheduling system. *Eur. J. Oper. Res.* 32(3):363–379.
- Minkoff AS (1993) A Markov decision model and decomposition heuristic for dynamic vehicle dispatching. *Oper. Res.* 41(1):77–90.
- Mishra BK, Raghunathan S (2004) Retailer- vs. vendor-managed inventory and brand competition. *Management Sci.* 50(4):445–457.
- Mjirda A, Jarboui B, Macedo R, Hanafi S (2012) A variable neighborhood search for the multi-product inventory routing problem. *Electronic Notes in Discrete Math.* 39(1):91–98.
- Moin NH, Salhi S, Aziz NAB (2011) An efficient hybrid genetic algorithm for the multi-product multi-period inventory routing problem. *Internat. J. Production Econom.* 133(1):334–343.
- Ohlmann JW, Fry MJ, Thomas BW (2008) Route design for lean production systems. *Transportation Sci.* 42(3):352–370.
- Oppen J, Løkketangen A, Desrosiers J (2010) Solving a rich vehicle routing and inventory problem using column generation. *Comput. Oper. Res.* 37(7):1308–1317.
- Persson JA, Göthe-Lundgren M (2005) Shipment planning at oil refineries using column generation and valid inequalities. *Eur. J. Oper. Res.* 163(3):631–652.
- Popović D, Vidović M, Radivojević G (2012) Variable neighborhood search heuristic for the inventory routing problem in fuel delivery. *Expert Systems Appl.* 39(18):13390–13398.
- Psaraftis HN (1998) Dynamic vehicle routing problems. Golden BL, Assad AA, eds. *Vehicle Routing: Methods and Studies* (North-Holland, Amsterdam), 223–248.
- Qu WW, Bookbinder JH, Iyogun P (1999) An integrated inventory-transportation system with modified periodic policy for multiple products. *Eur. J. Oper. Res.* 115(2):254–269.
- Raa B, Aghezzaf E-H (2008) Designing distribution patterns for long-term inventory routing with constant demand rates. *Internat. J. Production Econom.* 112(1):255–263.
- Raa B, Aghezzaf E-H (2009) A practical solution approach for the cyclic inventory routing problem. *Eur. J. Oper. Res.* 192(2):429–441.
- Raidl GR, Puchinger J, Blum C (2010) Metaheuristic hybrids. Gendreau M, Potvin J-Y, eds. *Handbook of Metaheuristics* (Springer, New York), 469–496.
- Rakke JG, Stålhane M, Moe CR, Christiansen M, Andersson H, Fagerholt K, Norstad I (2011) A rolling horizon heuristic for creating a liquefied natural gas annual delivery program. *Transportation Res. Part C: Emerging Tech.* 19(5):896–911.
- Ramkumar N, Subramanian P, Narendran T, Ganesh K (2012) Mixed integer linear programming model for multi-commodity multi-depot inventory routing problem. *OPSEARCH* 49(4):413–429.
- Ribeiro R, Lourenço HR (2003) Inventory-routing model for a multi-period problem with stochastic and deterministic demand. Technical report 275, Department of Economics and Business, Universitat Pompeu Fabra, Barcelona.
- Ronen D (1993) Ship scheduling: The last decade. *Eur. J. Oper. Res.* 71(3):325–333.
- Ronen D (2002) Marine inventory routing: Shipments planning. *J. Oper. Res. Soc.* 53(1):108–114.

- Roundy R (1985) 98%-effective integer-ratio lot-sizing for one-warehouse multi-retailer systems. *Management Sci.* 31(11):1416–1430.
- Savelsbergh MWP, Song JH (2008) An optimization algorithm for the inventory routing problem with continuous moves. *Comput. Oper. Res.* 35(7):2266–2282.
- Shen Q, Chu F, Chen H (2011) A Lagrangian relaxation approach for a multi-mode inventory routing problem with transshipment in crude oil transportation. *Comput. Chemical Engrg.* 35(10):2113–2123.
- Simchi-Levi D, Chen X, Bramel J (2005) *The Logic of Logistics: Theory, Algorithms, and Applications for Logistics and Supply Chain Management* (Springer-Verlag, New York).
- Sindhuchao S, Romeijn HE, Akçali E, Boondiskulchok R (2005) An integrated inventory-routing system for multi-item joint replenishment with limited vehicle capacity. *J. Global Optim.* 32(1):93–118.
- Smilowitz K, Nowak M, Jiang T (2013) Workforce management in periodic delivery operations. *Transportation Sci.* 47(2):214–230.
- Solyalı O, Süral H (2008) A single supplier-single retailer system with an order-up-to level inventory policy. *Oper. Res. Lett.* 36(5):543–546.
- Solyalı O, Süral H (2011) A branch-and-cut algorithm using a strong formulation and an a priori tour-based heuristic for an inventory-routing problem. *Transportation Sci.* 45(3):335–345.
- Solyalı O, Cordeau J-F, Laporte G (2012) Robust inventory routing under demand uncertainty. *Transportation Sci.* 46(3):327–340.
- Song JH, Furman KC (2013) A maritime inventory routing problem: Practical approach. *Comput. Oper. Res.* 40(3):657–665.
- Speranza MG, Ukovich W (1994) Minimizing transportation and inventory costs for several products on a single link. *Oper. Res.* 42(5):879–894.
- Speranza MG, Ukovich W (1996) An algorithm for optimal shipments with given frequencies. *Naval Res. Logist.* 43(5):655–671.
- Stacey J, Natarajathinam M, Sox C (2007) The storage constrained, inbound inventory routing problem. *Internat. J. Phys. Distribution Logist. Management* 37(6):484–500.
- Stålthane M, Rakke JG, Moe CR, Andersson H, Christiansen M, Fagerholt K (2012) A construction and improvement heuristic for a liquefied natural gas inventory routing problem. *Comput. Indust. Engrg.* 62(1):245–255.
- Trudeau P, Dror M (1992) Stochastic inventory routing: Route design with stockouts and route failures. *Transportation Sci.* 26(3):171–184.
- Uggen KT, Fodstad M, Nørstebø VS (2013) Using and extending fix-and-relax to solve maritime inventory routing problems. *TOP* 21(2):355–377.
- Viswanathan S, Mathur K (1997) Integrating routing and inventory decisions in one-warehouse multiretailer multiproduct distribution systems. *Management Sci.* 43(3):294–312.
- Zhao Q-H, Chen S, Zang C-X (2008) Model and algorithm for inventory/routing decisions in a three-echelon logistics system. *Eur. J. Oper. Res.* 19(3):623–635.
- Zhao Q-H., Wang S-Y, Lai KK (2007) A partition approach to the inventory/routing problem. *Eur. J. Oper. Res.* 177(2):786–802.
- Zhong H, Hall RW, Dessouky M (2007) Territory planning and vehicle dispatching with driver learning. *Transportation Sci.* 41(1):74–89.