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Thom Isotopy Theorem for Nonproper Maps and Computation of Sets of Stratified Generalized Critical Values

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Abstract
Let $X \subset \mathbb{C}^n$ be an affine variety and $f: X \to \mathbb{C}^m$ be the restriction to X of a polynomial map $\mathbb{C}^n \to \mathbb{C}^m$. We construct an affine Whitney stratification of X. The set $K(f)$ of stratified generalized critical values of f can also be computed. We show that $K(f)$ is a nowhere dense subset of $\mathbb{C}^m$ which contains the set $B(f)$ of bifurcation values of f by proving a version of the Thom isotopy lemma for nonproper polynomial maps on singular varieties.

Keywords Isotopy lemma · Affine varieties · Nonproper polynomial mapping · Local trivial fibration

Mathematics Subject Classification Primary 32B20; Secondary 14PXX

1 Introduction

Ehresmann's fibration theorem [3] states that a proper smooth surjective submersion $f: X \to N$ between smooth manifolds is a locally trivial fibration. With some extra assumptions, this result has been considered in different contexts.

Firstly, if we remove the assumption of properness or submersiveness, in general, Ehresmann's fibration theorem does not hold, since f might have "local singularities" or "singularities at infinity." The set of points in N where f fails to be trivial, denoted by

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$B(f)$, is called the *bifurcation set* of f, which is the union of the set $K_0(f)$ of *critical values* and the set $B_\infty(f)$ of *bifurcation values at infinity* of f. To date, characterizing $B_\infty(f)$ remains an open problem. In general, a larger (but easier to describe) set is used, viz. the *set of asymptotic critical values of* f (see Definition 3.1), denoted by $K_\infty(f)$, to control $B_\infty(f)$. The set $K_\infty(f)$ is always a nowhere dense subset of $\mathbb{C}^m$ and is a good approximation of the set $B_\infty(f)$. For a dominant map $f\colon X \to \mathbb{C}^m$ on a smooth complex affine variety X, the computation of $K_\infty(f)$, and hence of the *set of generalized critical values*, $K(f) := K_0(f) \cup K_\infty(f)$, is given in [8–10].

Now, if X is singular, one must partition it into disjoint smooth manifolds and then apply Ehresmann's fibration theorem on each part. However, if we do not require any extra assumptions, then the trivialization on the parts may not match. This obstacle can be overcome by introducing the Whitney conditions [21,22]. Indeed, if f is proper and X admits a Whitney stratification, then f is locally trivial if it is a submersion on strata [13,18,20]. Moreover, if f is nonproper and nonsubmersive, we can also define the bifurcation set of f such that f is locally trivial outside $B(f)$. However, to date, to the best of the authors' knowledge, no connection between $B(f)$ and the *set of stratified generalized critical values of* f, defined by $K(f,\mathcal{S}) := \bigcup_{X_\alpha \in \mathcal{S}} K(f, X_\alpha)$, for a Whitney stratification $\mathcal{S}$ of X, has been established. Here $K(f, X_\alpha) = K_0(f, X_\alpha) \cup K_\infty(f, X_\alpha)$, where $K_0(f, X_\alpha)$ is the closure of the set of critical values of $f_{|X_\alpha}$ and $K_\infty(f, X_\alpha) = \{y \in \mathbb{C}^m$: there is a sequence $x^k \to \infty$, $x^k \in X_\alpha$ such that $\|x^k\|\,\nu(\mathrm{d}_{x^k}(f|_{X_\alpha})) \to 0$ and $f(x^k) \to y\}$ (ν denotes the Rabier function, see Sect. 5).

Let $X \subset \mathbb{C}^n$ be a singular algebraic set of dimension $n-r$ with $I(X) = (g_1, \ldots, g_\omega)$, and let $f := (f_1, \ldots, f_m)\colon X \to \mathbb{C}^m$ be a polynomial dominant map, i.e., $\overline{f(X)} = \mathbb{C}^m$. Now, restricting ourselves to the cases of dominant polynomial maps on singular affine varieties, the main goals of this paper are the following:

- Construct an affine Whitney stratification $\mathcal{S}$ of X.
- Establish some version of the Thom isotopy lemma for f which yields the inclusion $B(f) \subset K(f,\mathcal{S})$.
- Calculate the set $K(f,\mathcal{S})$ of stratified generalized critical values of f.

The remainder of this manuscript is organized as follows: In Sect. 2, we recall the definitions of Whitney regularity and Whitney stratification, then construct an affine stratification from a filtration of X by means of some hypersurfaces, and refine it to obtain an affine Whitney stratification. Some versions of the Thom isotopy lemma for nonproper polynomial maps (Theorem 3.4 and Corollary 3.11) are given in Sect. 3. We then compute the set of stratified generalized critical values of f, which contains the bifurcation values of f, where $f := (f_1, \ldots, f_m)\colon X \to \mathbb{C}^m$ is a polynomial dominant map, in the final Sects. 4 and 5.

For the remainder of the paper, the differential of f at a point x is identified with its (row) matrix, so we write $\mathrm{d}_x f = \big(\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_n}(x)\big)$. Let

$$\nabla f(x) := \begin{bmatrix} \overline{\frac{\partial f}{\partial x_1}(x)} \\ \vdots \\ \overline{\frac{\partial f}{\partial x_n}(x)} \end{bmatrix},$$

the Hermitian transpose of $\mathrm{d}_x f$. For $v, w \in \mathbb{C}^n$, denote by $\langle v, w\rangle = \sum_{i=1}^n \overline{v}_i w_i$ the Hermitian product, and let $v \cdot w = \sum_{i=1}^n v_i w_i$. For a set $A \subset \mathbb{C}^n$, let $\overline{A}$ and $\overline{A}^{\mathcal{Z}}$ denote respectively the topological closure and the Zariski closure of A. For an algebraic variety X, the singular part and the regular part of X are denoted respectively by $\mathrm{sing}(X)$ and $\mathrm{reg}(X)$.

2 Affine Whitney Stratifications

2.1 Preliminaries

For any two different points $x, y \in \mathbb{C}^n$, define the secant $\overline{xy}$ to be the line passing through the origin which is parallel to the line through x and y.

A *stratification* $\mathcal{S}$ of X is a decomposition of X into a locally finite disjoint union $X = \bigsqcup_{\alpha\in I} X_\alpha$ of nonempty, nonsingular, connected, locally closed subvarieties, called strata, such that the boundary $\partial X_\alpha := \overline{X}_\alpha \backslash X_\alpha$ of any stratum X_α is a union of strata. If, in addition, for each α, the closure $\overline{X}_\alpha$ and the boundary ∂X_α are affine varieties, then we call $\mathcal{S}$ an *affine stratification*. It is obvious that any affine stratification is finite.

For linear subspaces $F, G \subseteq \mathbb{C}^n$, let

$$\delta(F, G) := \sup_{\substack{x\in F\\ \|x\|=1}} \mathrm{dist}(x, G),$$

where $\mathrm{dist}(x, G)$ is the Hermitian distance.

Let (X_α, X_β) be a pair of strata of $\mathcal{S}$ such that $X_\beta \subset \overline{X}_\alpha$ and let $x \in X_\beta$. We recall some regularity conditions:

(a) The pair (X_α, X_β) is said to be *Whitney* (a) *regular at* $x \in X_\beta$ if it satisfies the following Whitney condition (a) at x: if $x^k \in X_\alpha$ is any sequence such that $x^k \to x$ and $T_{x^k}X_\alpha \to T$, then $T \supset T_x X_\beta$.

(w) The pair (X_α, X_β) is said to be (w) *regular* at $x \in X_\beta$ (or *strictly Whitney* (a) *regular at* x *with exponent* 1) if it satisfies the following condition (w) at x: there exist a neighborhood U of x in $\mathbb{C}^n$ and a constant $c > 0$ such that, for any $y \in X_\alpha \cap U$ and $x' \in X_\beta \cap U$, we have $\delta(T_{x'}X_\beta, T_y X_\alpha) \leqslant c\|y - x'\|$.

(b) The pair (X_α, X_β) is said to be *Whitney regular at* $x \in X_\beta$ if it satisfies the following Whitney condition (b) at x: for any sequences $x^k \in X_\alpha$ and $y^k \in X_\beta$, $y^k \neq x^k$, such that $x^k \to x$, $y^k \to x$, $T_{x^k}X_\alpha \to T$, and $\overline{x^k y^k}$ converges to a line ℓ in the projective space $\mathbb{P}^{n-1}$, we have $\ell \subset T$.

We say that the pair (X_α, X_β) is *Whitney* (a) *regular* (resp. *Whitney regular*) if it is Whitney (a) regular (resp. Whitney regular) at every point of X_β. We say that $\mathcal{S}$ is a *Whitney* (a) *stratification* (resp. a *Whitney stratification*) if any pair of strata (X_α, X_β) of $\mathcal{S}$ with $X_\beta \subset \overline{X}_\alpha$ is Whitney (a) regular (resp. Whitney regular). It is well known that Whitney regularity implies Whitney (a) regularity [21,22]. Moreover, in light of [17, V.1.2], the Whitney condition (b) is equivalent to the condition (w) for the category of complex analytic sets, so to check Whitney regularity, we can verify either condition (w) or condition (b).

For the purpose of this paper, we also need the following notion of Whitney (resp. Whitney (a)) regularity along a stratum: Let X_β be a stratum of $\mathcal{S}$ and let $x \in X_\beta$. We say that X_β is *Whitney regular* (resp. *Whitney* (a) *regular*) at x if, for any stratum X_α such that $X_\beta \subset \overline{X}_\alpha$, the pair (X_α, X_β) is Whitney (resp. Whitney (a)) regular at x. The stratum X_β is *Whitney regular* (resp. *Whitney* (a) *regular*) if it is Whitney (resp. Whitney (a)) regular at every point of X_β. It is clear that $\mathcal{S}$ is a Whitney (resp. a Whitney (a)) stratification if and only if each stratum of $\mathcal{S}$ is Whitney (resp. Whitney (a)) regular.

2.2 Construction of Affine Stratifications

Let us, first of all, fix an affine stratification of X whose construction is based on the following proposition:

Proposition 2.1 *Let $X \subset \mathbb{C}^n$ be an affine subvariety of pure codimension r. Assume that $I(X) = (g_1, \ldots, g_\omega)$, where* $\deg g_i \le D$. *Let W be an affine subvariety of positive codimension in X with $I(W) = (g_1, \ldots, g_\omega, u_1, \ldots, u_\tau)$, where $u_i \notin I(X)$ and* $\deg u_i \leqslant D'$. *Then there exists a polynomial $p_{X,W}$ on $\mathbb{C}^n$ of degree less than or equal to $r(D-1)+D'$ such that $W \subseteq V(p_{X,W}) := \{x \in \mathbb{C}^n : \ p_{X,W}(x) = 0\}$ and $X \backslash V(p_{X,W})$ is a smooth, dense subset of X. Moreover, the polynomial $p_{X,W}$ can be constructed effectively.*

Proof Let $X = \bigcup_{i=1}^m Y_i$, where Y_i are irreducible (hence r-codimensional) components of X. Take sufficiently generic (random) rational numbers $\alpha_{ij} \in \mathbb{Q}, i = 1, \ldots, r$, $j = 1, \ldots, \omega$, and set

$$G_i = \sum_{j=1}^{\omega} \alpha_{ij} g_j, \quad i = 1, \ldots, r.$$

Here and in the following, to obtain a generic number, it is sufficient to take a random rational number and verify the genericity condition; the procedure is repeated until the genericity condition is satisfied. Note that the set $Z := V(G_1, \ldots, G_r)$ has pure codimension r and $X \subset Z$. Let $\gamma_1, \ldots, \gamma_\tau$ be some (random) generic rational numbers and set

$$H := \begin{cases} 1 & \text{if } \ W = \emptyset, \\ \sum_{i=1}^{\tau} \gamma_i u_i & \text{otherwise.} \end{cases}$$

Clearly $\dim\big(X \cap V(H)\big) < \dim X$. Moreover, for a sufficiently general linear r-dimensional subspace $L^r \subset \mathbb{C}^n$, the intersection $L^r \cap Z$ has only isolated smooth points and $L^r \cap Y_i \neq \emptyset$ for every $i = 1, \ldots, m$. We can assume that L^r is determined by the linear forms $l_i = \sum_{j=1}^n \beta_{ij} x_j$, $i = 1, \ldots, n-r$, where β_{ij} are sufficiently generic (random) rational numbers. Now take

$$p_{X,W} = |\mathrm{Jac}(G_1, \ldots, G_r, l_1, \ldots, l_{n-r})| \cdot H,$$

where Jac(·) denotes the Jacobian matrix. Then $p_{X,W}$ is a polynomial with the required properties. □

Remark 2.2 Theoretically, a random rational number is a generic rational number, but practically by random numbers we mean rational numbers produced by special random algorithms.

The polynomial $p_{X,W}$ can be found by using a probabilistic algorithm. First recall the following:

Definition 2.3 Let I be an ideal in $\mathbb{C}[x_1, \ldots, x_n]$. We define the homogenization of I to be the ideal I^{h} generated by $\{f^{\mathrm{h}} : f \in I\} \subset \mathbb{C}[x_0, \ldots, x_n]$, where f^{h} is the homogenization of f.

Theorem 2.4 ([2, Thm. 4, §4, Chap. 8, p. 388]) *Let I be an ideal in $k[x_1, \ldots, x_n]$ and let $G = \{g_1, \ldots, g_s\}$ be a Gröbner basis for I with respect to a graded lexicographic order in $k[x_1, \ldots, x_n]$ (i.e., the lexicographic order that first compares the total degree: $x^\alpha > x^\beta$ whenever $|\alpha| > |\beta|$). Then $G^{\mathrm{h}} = \{g_1^{\mathrm{h}}, \ldots, g_s^{\mathrm{h}}\}$ is a basis for $I^{\mathrm{h}} \subset k[x_0, x_1, \ldots, x_n]$.*

This theorem allows us to compute the set of points at infinity of an affine variety given by the ideal I; to this aim, it is enough to compute the Gröbner basis $\{g_1, \ldots, g_s\}$ of the ideal I and then to consider the ideal $I_\infty = (x_0, g_1^{\mathrm{h}}, \ldots, g_s^{\mathrm{h}})$. In particular, we can check effectively whether $L^r \cap \overline{X} \cap \{x_0 = 0\} = \emptyset$, which implies that $L^r \cap Y_i \neq \emptyset$ for $i = 1, \ldots, m$ (see the proof of Proposition 2.1). This is crucial for our computations.

Now we sketch the algorithm to compute the polynomial $p_{X,W}$. Note that, for a given ideal I, we can compute $\dim V(I)$ by [19].

INPUT: The ideal $I = I(X) = (g_1, \ldots, g_\omega)$ and the ideal $J = I(W) = (g_1, \ldots, g_\omega, u_1, \ldots, u_\tau)$
1) *repeat*
choose random rational numbers $\alpha_{i1}, \ldots, \alpha_{i\omega}$, $i = 1, \ldots, r$;
put $G_i := \sum_{k=1}^{\omega} \alpha_{ik} g_k$, $i = 1, \ldots, r$;
put $I = (G_1, \ldots, G_r)$;
until $\dim V(I) = n - r$.
2) *repeat*
choose random rational numbers $\beta_{i1}, \ldots, \beta_{in}$, $i = 1, \ldots, n - r$;
put $l_i := \sum_{k=1}^{n} \beta_{ik} x_k$, $i = 1, \ldots, n - r$;
put $I = (G_1, \ldots, G_r, l_1, \ldots, l_{n-r})$;
compute the ideal $I_\infty = (H_1, \ldots, H_m) \subset k[x_0, \ldots, x_n]$;
if $\dim V(I_\infty) = 0$ *then*
begin
compute $V(G_1, \ldots, G_r, l_1, \ldots, l_r) := \{a_1, \ldots, a_p\}$
end

until $\dim V(I_\infty) = 0$ and $|\mathrm{Jac}(G_1,\dots,G_r,l_1,\dots,l_{n-r})(a_i)| \neq 0$ for $i = 1,\dots p$.
3) *repeat*
choose random rational numbers $\gamma_1,\dots,\gamma_\tau$;
put $H := \sum_{k=1}^{\tau} \gamma_i u_k$;
put $J = (G_1,\dots,G_r,H)$;
until $\dim V(J) < n-r$.
OUTPUT: $p_{X,W} = |\mathrm{Jac}(G_1,\dots,G_r,l_1,\dots,l_{n-r})| \cdot H$

Remark 2.5 Let us assume that $I(X)$ and $I(W)$ are generated by polynomials from the ring $\mathbb{F}[x_1,\dots,x_n]$, where $\mathbb{F}$ is a subfield of $\mathbb{C}$. Then we can choose a polynomial $p_{X,W}$ such that $p_{X,W} \in \mathbb{F}[x_1,\dots,x_n]$.

From the proof of Proposition 2.1, with no loss of generality, we can assume that $\operatorname{rank}\mathrm{Jac}(g_1,\dots,g_r) = r$ on some nonempty regular open subset X^0 of X and that $X = \overline{X^0}$. It is clear that $V(p_{X,W})$ contains $\mathrm{sing}(X) \cup W$ and the singular points of the projection $(l_1,\dots,l_{n-r})\colon X \to \mathbb{C}^{n-r}$. Now, to construct an affine stratification of X, it is enough to construct an affine filtration $X = X_0 \supset X_1 \supset \dots \supset X_{n-r} \supset X_{n-r+1} = \emptyset$ by induction with $X_{i+1} := X_i \cap V(p_{X_i,\emptyset})$, $i = 0,\dots,n-r$. The degree of each X_i can be calculated and depends only on D.

2.3 Construction of Affine Whitney Stratifications

In this section, we construct an affine Whitney stratification of a given affine variety X, with $I(X) = (g_1,\dots,g_r)$ and $\deg g_i \le D$, by refining the affine stratification given in Sect. 2.2 so that the resulting stratification is still affine and moreover satisfies the Whitney condition.

First of all, inspired by the construction in [5,17], let us describe the Whitney condition (b) algebraically. Assume that $Y \subset X$ is an affine subvariety of X with $\dim Y < \dim X$ defined by

$$Y := X \cap \{\widetilde{g}_{r+1} = \dots = \widetilde{g}_p = 0\}.$$

Set

$$\Gamma_1 := \left\{ \begin{array}{l} (x,y,w,v,\gamma,\lambda) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^r : \\ g_1(x) = \dots = g_r(x) = 0 \\ g_1(y) = \dots = g_r(y) = \widetilde{g}_{r+1}(y) = \dots = \widetilde{g}_p(y) = 0 \\ w = \gamma(x-y) \\ v = \sum_{i=1}^{r} \lambda_i \mathrm{d}_x g_i \end{array} \right\},$$

and let

$$\pi_1\colon \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^r \to \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n$$

be the projection on the first $4n$ coordinates. Let $C(X,Y) = \overline{\pi_1(\Gamma_1)}^{\mathcal{Z}} \subset (X \times Y \times \mathbb{C}^n \times \mathbb{C}^n)$, where the closure is taken in the Zariski topology (which coincides with the topological closure; see, e.g., [16, Prop. 7]). Of course, $C(X,Y)$ is an affine variety. We have the following:

Lemma 2.6 *For each* $(x,x,w,v)\in C(X,Y)$*, there are sequences* $x^k\in X^0$*,* $y^k\in Y$*,* $\gamma^k\in\mathbb{C}$*, and* $\lambda^k\in\mathbb{C}^r$ *such that:*

- $x^k\to x$,
- $y^k\to x$,
- $w^k:=\gamma^k(x^k-y^k)\to w$,
- $v^k:=\sum_{i=1}^r\lambda_i^k\mathrm{d}_{x^k}g_i\to v$.

Proof By construction, there are sequences $\bar{x}^k\in X$, $y^k\in Y$, $\bar{\gamma}^k\in\mathbb{C}$, and $\lambda^k\in\mathbb{C}^r$ such that $\bar{x}^k, y^k\to x$, $\bar{w}^k:=\bar{\gamma}^k(\bar{x}^k-y^k)\to w$, and $\sum_{i=1}^r\lambda_i^k\mathrm{d}_{\bar{x}^k}g_i\to v$. It is clear that, by taking subsequences if necessary, we may suppose that:

- either $\bar{x}^k=y^k$ for every k or $\bar{x}^k\neq y^k$ for every k,
- for each i, either $\lambda_i^k\neq 0$ for every k or $\lambda_i^k=0$ for every k.

Set

$$\gamma^k=\begin{cases}0 & \text{if } \bar{x}^k=y^k \text{ for every } k,\\ \bar{\gamma}^k & \text{if } \bar{x}^k\neq y^k \text{ for every } k.\end{cases}$$

Suppose that $\lambda_i^k\neq 0$ for $i=1,\dots,r'\leqslant r,\ k\in\mathbb{N}$ and $\lambda_i^k=0$ for $i=r'+1,\dots,r,\ k\in\mathbb{N}$. Since $\bar{x}^k\in\overline{X^0}$, there exists a sequence $x^k\in X^0$ such that

$$\|x^k-\bar{x}^k\|\leqslant\begin{cases}\frac{1}{k} & \text{if } \bar{x}^k=y^k \text{ for every } k,\\ \frac{\|\bar{x}^k-y^k\|}{k} & \text{if } \bar{x}^k\neq y^k \text{ for every } k,\end{cases}$$

so $x^k\to x$. By continuity, we can also choose x^k so that $\|\mathrm{d}_{x^k}g_i-\mathrm{d}_{\bar{x}^k}g_i\|<\frac{1}{k\lambda_i^k}$ if $\lambda_i^k\neq 0$. Set $v^k:=\sum_{i=1}^r\lambda_i^k\mathrm{d}_{x^k}g_i$. Then

$$\begin{aligned}\left\|v^k-\sum_{i=1}^r\lambda_i^k\mathrm{d}_{\bar{x}^k}g_i\right\| &= \left\|\sum_{i=1}^{r'}\lambda_i^k\big(\mathrm{d}_{x^k}g_i-\mathrm{d}_{\bar{x}^k}g_i\big)\right\|\\ &\leqslant\sum_{i=1}^{r'}|\lambda_i^k|\left\|\mathrm{d}_{x^k}g_i-\mathrm{d}_{\bar{x}^k}g_i\right\|<\frac{r'}{k}\to 0,\end{aligned}$$

i.e., $v^k\to v$. Set $w^k:=\gamma^k(x^k-y^k)$. Now, if $\bar{x}^k=y^k$ for every k, then $\gamma^k=0$ and $w=\bar{w}^k=0$, so we have $w^k=0=w$. If $\bar{x}^k\neq y^k$ for every k, then

$$\|w^k-\bar{w}^k\|=|\gamma^k|\cdot\|(x^k-\bar{x}^k)\|\leqslant|\gamma^k|\cdot\frac{\|\bar{x}^k-y^k\|}{k}=\frac{\|\bar{w}^k\|}{k}\to 0.$$

Hence $w^k\to w$. □

The following algebraic criterion permits us to check Whitney regularity on $Y^0 = Y \backslash V(p_{Y,W})$, where the notation $V(p_{Y,W})$ is from Proposition 2.1 and the affine set W will be determined later.

Lemma 2.7 *Let $x \in Y^0$. Then the pair (X^0, Y^0) satisfies the Whitney condition* (b) *at x if and only if, for any $(x, x, w, v) \in C(X, Y)$, we have $v \cdot w = 0$.*

Proof Suppose that (X^0, Y^0) is Whitney regular at x and assume for contradiction that there is $(x, x, w, v) \in C(X, Y)$ such that $v \cdot w \neq 0$. In view of Lemma 2.6, there are sequences $x^k \in X^0,\ y^k \in Y,\ \gamma^k \in \mathbb{C}$, and $\lambda^k \in \mathbb{C}^r$ such that

- $x^k \to x, y^k \to x$,
- $w^k := \gamma^k(x^k - y^k) \to w$,
- $v^k := \sum_{i=1}^r \lambda_i^k \mathrm{d}_{x^k} g_i \to v$.

Note that $w \neq 0$, so w determines the limit of the sequence of secants $\overline{x^k y^k}$ and it follows that $x^k \neq y^k$ for k large enough. By taking a subsequence if necessary, we may assume that $T_{x^k} X^0 \to T$. By assumption, $w \in T$. For each k, let $\{b_1^k, \dots, b_r^k\}$ be an orthonormal basis of $N_{x^k} X^0$; recall that $N_{x^k} X^0 := \operatorname{span}\{\overline{\mathrm{d}_{x^k} g_1}, \dots, \overline{\mathrm{d}_{x^k} g_r}\}$ is the normal space of X^0 at x^k. Obviously $\langle \overline{b_1^k}, \dots, \overline{b_r^k} \rangle^\perp = T_{x^k} X^0$. By compactness, each sequence b_i^k has an accumulation point b_i. Without loss of generality, suppose that $b_i^k \to b_i$. It is clear that the system $\{b_1, \dots, b_r\}$ is also orthonormal and $\langle \overline{b}_1, \dots, \overline{b}_r \rangle^\perp = T$. Let $\widetilde{\lambda}^k = (\widetilde{\lambda}_1^k, \dots, \widetilde{\lambda}_r^k)$ be such that $v^k := \sum_{i=1}^r \widetilde{\lambda}_i^k b_i^k$. Then $\widetilde{\lambda}^k$ is convergent to a limit $\widetilde{\lambda}$ and it is clear that $v = \sum_{i=1}^r \widetilde{\lambda}_i b_i$. Finally, we have $w \in T = \langle \overline{b_1}, \dots, \overline{b_r} \rangle^\perp \subset \langle \overline{v} \rangle^\perp$, i.e., $v \cdot w = 0$, which is a contradiction.

Now suppose that $v \cdot w = 0$ for any $(x, x, w, v) \in C(X, Y)$ and assume that (X^0, Y^0) is not Whitney regular at x. So, there are sequences $x^k \in X^0$ and $y^k \in Y^0$ with the following properties:

- $x^k \neq y^k, x^k \to x, y^k \to y$;
- $T_{x^k} X^0 \to T$;
- the sequence of secants $\overline{x^k y^k}$ tends to a line $\ell \not\subset T$.

For each k, let $\{b_1^k, \dots, b_r^k\}$ be an orthonormal basis of $N_{x^k} X^0$ so $\langle \overline{b_1^k}, \dots, \overline{b_r^k} \rangle^\perp = T_{x^k} X^0$. As above, we may assume that $b_i^k \to b_i$. Then the system $\{b_1, \dots, b_r\}$ is also orthonormal and $\langle \overline{b_1}, \dots, \overline{b_r} \rangle^\perp = T$. Let $w^k := \frac{x^k - y^k}{\|x^k - y^k\|}$; we can assume that the limit $w := \lim w^k$ exists, and clearly w is a direction vector of ℓ. By assumption, $w \notin T = \langle \overline{b}_1, \dots, \overline{b}_r \rangle^\perp$; i.e., there exists an index j such that $b_j \cdot w \neq 0$. To obtain a contradiction, it is enough to show that there is a sequence $v^k := \sum_{i=1}^r \lambda_i^k \mathrm{d}_{x^k} g_i$ such that $v^k \to b_j$, but this is clear since $b_j^k \in \operatorname{span}\{\mathrm{d}_{x^k} g_1, \dots, \mathrm{d}_{x^k} g_r\}$, so such a sequence always exists. □

Now, according to [11, Algorithm 3.3], [7, Algorithm 4.5.3] (see also [4,6]), it is possible to calculate a basis for the ideal $I(\Gamma_1)$ by calculating the radical of the following ideal in $\mathbb{C}[x, y, w, v, \gamma, \lambda]$:

$$\begin{pmatrix} g_1(x), \ldots, g_r(x), \\ g_1(y), \ldots, g_r(y), \widetilde{g}_{r+1}(y), \ldots, \widetilde{g}_p(y), \\ w - \gamma(x-y), \\ v - \sum_{i=1}^{r} \lambda_i \mathrm{d}_x g_i \end{pmatrix}.$$

Then, by Buchberger's algorithm, we can calculate a Gröbner basis of $I(\Gamma_1)$. So, in view of [10, Thm. 5.1], [14], we can compute a Gröbner basis of the ideal $I(C(X, Y))$. Now we give another criterion for Whitney regularity.

Lemma 2.8 *Let* $\{h_1(x, y, w, v), \ldots, h_q(x, y, w, v)\}$ *be a Gröbner basis of* $I(C(X, Y))$ *and set*

$$\Gamma_2 := \left\{ \begin{array}{l} (x, x, w, v, \gamma, \lambda) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \times \mathbb{C} : \\ h_1(x, x, w, v) = \cdots = h_q(x, x, w, v) = 0 \\ \gamma \sum_{j=1}^{n} v_j w_j = 1 \\ \lambda p_{Y,\emptyset}(x) = 1 \end{array} \right\},$$

where $p_{Y,\emptyset}(x)$ *is the polynomial determined in Proposition 2.1. Let* $Y^0 := Y \setminus V(p_{Y,\emptyset})$. *Then the pair* (X^0, Y^0) *is not Whitney regular at* x *if and only if there exists* $(w, v, \gamma, \lambda) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}$ *such that* $(x, x, w, v, \gamma, \lambda) \in \Gamma_2$.

Proof Note that $x \in Y^0$ if and only if $p_{Y,\emptyset}(x) \neq 0$, i.e., there exists $\lambda \in \mathbb{C}$ such that $\lambda p_{Y,\emptyset}(x) = 1$. In view of Lemma 2.7, the pair (X^0, Y^0) is not Whitney regular at x if and only if there exist w, v with $v \cdot w \neq 0$ such that $(x, x, w, v) \in C(X, Y)$. The lemma follows easily. □

Now we determine an algebraic set $W = W(X, Y)$ in Y with $\dim W < \dim Y$ and $V(p_{Y,\emptyset}) \subset W$ such that the pair $(X^0, Y \setminus W)$ is Whitney regular. Let

$$\pi_2 \colon \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}^n$$

be the projection on the first n coordinates. By Lemma 2.8, $\pi_2(\Gamma_2)$ is the set of points where the Whitney condition (b) fails to be satisfied. Let $\overline{\pi_2(\Gamma_2)}^{\mathcal{Z}}$ be the Zariski closure of $\pi_2(\Gamma_2)$, then $\overline{\pi_2(\Gamma_2)}^{\mathcal{Z}}$ is affine. It follows from [21, Thm. 8.5], [22, Lem. 19.3] that $\dim \pi_2(\Gamma_2) < \dim Y$, so $\dim \overline{\pi_2(\Gamma_2)}^{\mathcal{Z}} < \dim Y$. Set

$$W = W(X, Y) := \overline{\pi_2(\Gamma_2)}^{\mathcal{Z}};$$

then obviously $\dim W < \dim Y$. Again, applying [11, Algorithm 3.3] or [7, Algorithm 4.5.3] (see also [4,6]) to find a system of generators of $I(W)$, then applying [10, Thm. 5.1], [14], we can compute a Gröbner basis of the ideal $I(W)$. Finally, let

- $X_0 := X$,
- $X_1 := X_0 \cap V(p_{X_0,\emptyset})$,
- $X_2 := X_1 \cap V(p_{X_1, W(X_0, X_1)})$,

- $X_3 := X_2 \cap V(p_{X_2, W(X_0,X_2) \cup W(X_1,X_2)}), \ldots,$
- $X_i := X_{i-1} \cap V\big(p_{X_{i-1}, \bigcup_{j=0}^{i-2} W(X_j, X_{i-1})}\big), \ldots$

By induction, we can construct a finite filtration of algebraic sets $X = X_0 \supset X_1 \supset \cdots \supset X_{n-r} \supset X_{n-r+1} = \emptyset$ with $\dim X_i > \dim X_{i+1}$. Let $B_i := X_i \backslash X_{i+1}$. Then $\mathcal{S} := \{B_i\}_{i=1,\ldots,q}$ is a Whitney stratification of X. Note that the degree of X_i can be determined explicitly and depends only on D.

3 Thom Isotopy Lemma for Nonproper Maps

We start this section with:

Definition 3.1 Let $f\colon X \to \mathbb{C}^m$ be a polynomial dominant map, where X is an algebraic set. Let $\mathcal{S} = \{X_\alpha\}_{\alpha \in I}$ be a stratification of X. By $K_\infty(f, X_\alpha)$ we mean the set $\{y \in \mathbb{C}^m :$ there is a sequence $x^k \to \infty$, $x^k \in X_\alpha$ such that $\|x^k\|\, \nu(\mathrm{d}_{x^k}(f|_{X_\alpha})) \to 0$ and $f(x^k) \to y\}$ (here ν denotes the Rabier function; for details, see [10, Sect. 5]). Now, let $C(f, X_\alpha)$ denote the set of points where $f|_{X_\alpha}$ is not a submersion. By $\mathrm{sing}(f, \mathcal{S})$ we denote the set of stratified singular values of f, i.e.,

$$\mathrm{sing}(f, \mathcal{S}) = \bigcup_{\alpha \in I} K_0(f, X_\alpha), \tag{1}$$

where $K_0(f, X_\alpha) = \overline{f(C(f, X_\alpha))}$.

By [10, Thm. 3.3], we have that, for every α, the set $K_\infty(f|_{X_\alpha})$ is closed and has measure 0 in $\mathbb{C}^m$. In particular the set $K(f)$ defined below is also closed and has measure 0.

Definition 3.2 Let $K(f, \mathcal{S})$ be the set of stratified generalized critical values of f given by

$$K(f, \mathcal{S}) := \bigcup_{\alpha \in I} \big(K_0(f, X_\alpha) \cup K_\infty(f, X_\alpha)\big). \tag{2}$$

Remark 3.3 The set $K(f, \mathcal{S})$ is closed. Indeed, it is enough to see that $K_\infty(f, X_\alpha)$ is closed for every α. Assume that $y^k \in K_\infty(f, X_\alpha)$ and $y^k \to y$. So, for every k, there are suitable sequences $x^{kj} \in X_\alpha$, $j = 1, 2, \ldots$ such that $\lim_{j\to\infty} f(x^{kj}) = y^k$. Hence, for any k, we can choose x^{kj_k} such that:

- $\|f(x^{kj_k}) - y^k\| < 1/k$,
- $\|x^{kj_k}\| > k$,
- $\|x^{kj_k}\|\, \nu(\mathrm{d}_{x^{kj_k}} f) < 1/k$.

Set $z^k := x^{kj_k}$. Thus $z^k \in X_\alpha$, $z^k \to \infty$, $\|z^k\|\, \nu(\mathrm{d}_{z^k}(f|_{X_\alpha})) \to 0$, and $f(z^k) \to y$. Consequently $y \in K_\infty(f, X_\alpha)$.

Assuming that $\mathcal{S}$ is an affine Whitney stratification of X, we prove that $K(f, \mathcal{S})$ contains the set of bifurcation values of f.

Theorem 3.4 (First isotopy lemma for nonproper maps) *Let* $X \subset \mathbb{C}^n$ *be an affine variety with an affine Whitney stratification* $\mathcal{S}$, *and let* $f = (f_1, \ldots, f_m) \colon X \to \mathbb{C}^m$ *be a polynomial dominant map. Let* $K(f, \mathcal{S})$ *be the set of stratified generalized critical values of* f *given by* (2). *Then* f *is locally trivial outside* $K(f, \mathcal{S})$.

Before proving Theorem 3.4, recall that the Whitney condition (b) is equivalent to the condition (w) (see [17, V.1.2]), so it is more convenient to use the condition (w), since we will need to construct rugose vector fields in the sense of [20]. In what follows, it is more convenient to work with the underlying real algebraic set of X in $\mathbb{R}^{2n}$, also denoted by X; the affine Whitney stratification $\mathcal{S}$ of X induces a semialgebraic Whitney stratification of the underlying set with the corresponding strata denoted by the same notation X_β. We also identify the polynomial map f with the real polynomial map $(\mathrm{Re}\, f_1, \ldots, \mathrm{Re}\, f_m, \mathrm{Im}\, f_1, \ldots, \mathrm{Im}\, f_m) \colon X \to \mathbb{R}^{2m}$. Let us recall the definitions pertaining to rugosity. Let $\psi \colon X \to \mathbb{R}$ be a real function. We say that ψ is a *rugose function* if the following conditions are fulfilled:

- The restriction $\psi|_{X_\beta}$ to any stratum X_β is a smooth function.
- For any stratum X_β and for any $x \in X_\beta$, there exist a neighborhood U of x in $\mathbb{R}^{2n}$ and a constant $c > 0$ such that, for any $y \in X \cap U$ and $x' \in X_\beta \cap U$, we have $|\psi(y) - \psi(x')| \leqslant c\|y - x'\|$.

A *rugose map* is a map whose components are rugose functions. A vector field v on X is called a *rugose vector field* if v is a rugose map and $v(x)$ is tangent to the stratum containing x for any $x \in X$.

Proof of Theorem 3.4 Let $z \in \mathbb{C}^m \setminus K(f, \mathcal{S})$, where we identify $\mathbb{C}^m$ with $\mathbb{R}^{2m}$, and let B be an open box centered at z such that $\overline{B} \cap K(f, \mathcal{S}) = \emptyset$. To prove the theorem, it is enough to prove that f is trivial on B. Without loss of generality, we may suppose that $z = 0$ and $B = (-1, 1)^{2m}$. Let $\partial_1, \ldots, \partial_{2m}$ be the restrictions of the coordinate vector fields (on $\mathbb{R}^{2m}$) to $\overline{B}$. Set $U := f^{-1}(B),\ U_\beta := U \cap X_\beta$, and

$$I' := \{\beta \in I : U_\beta \neq \emptyset\}.$$

Clearly $\overline{U} = f^{-1}(\overline{B})$ and $I' = \{\beta \in I : \overline{U} \cap X_\beta \neq \emptyset\}$. First of all, let us give a sufficient condition for trivializing a rugose vector field.

Lemma 3.5 *For* $i = 1, \ldots, 2m$, *let* v_i *be vector fields on* X *which are rugose in* U. *Assume that* $\mathrm{d} f(v_i) = \partial_i$ *and there is a positive constant* $c > 0$ *such that* $\|v_i(x)\| \leqslant \frac{\|x\|+1}{c}$ *for any* $x \in U$. *Then* f *is a topologically trivial fibration over* B.

Proof It is enough to prove that there is a homeomorphism $\phi \colon f^{-1}(B) \to f^{-1}(0) \times B$ such that the following diagram commutes:

$$\begin{array}{ccc} f^{-1}(B) & \xrightarrow{\ \ \phi\ \ } & f^{-1}(0) \times B \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & B & \end{array}$$

where π denotes the projection on the second factor. We note the following facts:

(i) The flow of v_i preserves the stratification. This is a consequence of the rugosity. For more detail, see [20, Prop. 4.8].
(ii) For each i and any $x \in U$, there is a unique integral curve of v_i passing through x. This follows from the uniqueness of integral curves of smooth vector fields and the fact that v_i preserves the stratification.

Set $Y^i_t := (y_1, \dots, y_{i-1}, t, y_{i+1}, \dots, y_n)$ and $Y^i = \{Y^i_t : t \in (-1,1)\}$. First of all, we will prove that the flow of v_i induces a homeomorphism $\phi_i \colon f^{-1}(Y^i) \to f^{-1}(Y^i_0) \times (-1,1)$ such that the following diagram commutes:

$$\begin{array}{ccc} f^{-1}(Y^i) & \xrightarrow{\;\phi_i\;} & f^{-1}(Y^i_0) \times (-1,1) \\ & {\scriptstyle p_i \circ f} \searrow \quad \swarrow {\scriptstyle \pi_i} & \\ & (-1,1) & \end{array}$$

where π_i denotes the projection on the second factor and p_i denotes the projection on the i^{th} coordinate. This follows from the following claim which states that there is no trajectory of v_i going to infinity:

Claim 3.6 *For each $x \in f^{-1}(Y^i_0)$, let γ be the integral curve of v_i such that $\gamma(0) = x$. Then γ reaches any level $f^{-1}(Y^i_t)$ at time t for $t \in (-1,1)$.*

Proof By assumption, $\|\dot\gamma(t)\| \leqslant \frac{\|\gamma(t)\|+1}{c}$. Without loss of generality, suppose that $t > 0$. In light of the Gronwall lemma, by repeating the calculation of [1, Thm. 3.5], we obtain

$$\begin{aligned} \|\gamma(t)\| &\leqslant \|\gamma(0)\| + \int_0^t \frac{\|\gamma(s)\|+1}{c}\,\mathrm{d}s \\ &= \|x\| + \frac{t}{c} + \int_0^t \frac{\|\gamma(s)\|}{c}\,\mathrm{d}s \\ &\leqslant \left(\|x\| + \frac{t}{c}\right)\exp\int_0^t \frac{\mathrm{d}s}{c} = \left(\|x\| + \frac{t}{c}\right)\mathrm{e}^{t/c} < +\infty, \end{aligned}$$

which implies that the trajectory γ does not go to infinity at time t. Now we have

$$\begin{aligned} f(\gamma(t)) - f(x) &= \int_0^t \mathrm{d}\big[f(\gamma(t))\big] \\ &= \int_0^t \mathrm{d}_{\gamma(t)} f(\dot\gamma(t))\,\mathrm{d}t = \int_0^t \partial_i\,\mathrm{d}t = (0, \dots 0, t, 0, \dots, 0). \end{aligned}$$

Since $f(x) \in Y^i_0$, clearly $f(\gamma(t)) \in Y^i_t$, and the claim follows. □

For any $x \in f^{-1}(Y^i_0)$, let $h_i(x,t) = x + \int_0^t \dot\gamma(s)\,\mathrm{d}s$. Then h_i defines a homeomorphism $f^{-1}(Y^i_0) \times (-1,1) \to f^{-1}(Y^i)$. Then $\phi_i = h_i^{-1}$ is the required homeomorphism. Now, for $x \in f^{-1}(0)$, let $h \colon f^{-1}(0) \times B \to f^{-1}(B)$ be defined by

$$h(x, t_1, \dots, t_{2m}) = h_{2m}(\dots(h_2(h_1(x,t_1),t_2),\dots,t_{2m}).$$

Then $\phi := h^{-1}$ is a homeomorphism, as required. The lemma is proved. □

Now let us prove the following:

Lemma 3.7 *There is a constant $c > 0$ such that, for any $\beta \in I'$ and any $x \in \overline{U} \cap X_\beta$, we have*

$$(\|x\| + 1)\, \nu(\mathrm{d}_x(f|_{X_\beta})) \geqslant c.$$

Proof Assume for contradiction that there exist an index $\beta \in I'$ and a sequence $x^k \in \overline{U} \cap X_\beta$ such that $(\|x^k\| + 1)\, \nu(\mathrm{d}_{x^k}(f|_{X_\beta})) \to 0$. Taking a subsequence if necessary, we can suppose that $x^k \to x$, $T_{x^k} X_\beta \to T$ and $f(x^k) \to y \in \overline{B}$ with $x \in \mathbb{C}^n$ or $x = \infty$. If $x = \infty$, then by definition, $y \in K_\infty(f|_{X_\beta}) \subset K(f, \mathcal{S})$. This is a contradiction since $\overline{B} \cap K(f, \mathcal{S}) = \emptyset$. Thus $x \in \mathbb{C}^n$, and we get $\nu(\mathrm{d}_{x^k}(f|_{X_\beta})) \to 0$. In the case $x \in X_\beta$, in view of [15, Lem. 2.2], we have $\nu(\mathrm{d}_x(f|_{X_\beta})) = 0$, i.e., $y \in K_0(f, X_\beta)$, which is also a contradiction. Therefore $x \in \overline{X}_\beta \backslash X_\beta$. Denote by X_α the stratum containing x. Let $F = (f_1, \dots, f_m)\colon \mathbb{C}^n \to \mathbb{C}^m$ be the polynomial extending f on $\mathbb{C}^n$. Obviously $\nu(\mathrm{d}_{x^k} F|_{T_{x^k} X_\beta}) = \nu(\mathrm{d}_{x^k}(F|_{X_\beta})) = \nu(\mathrm{d}_{x^k}(f|_{X_\beta})) \to 0$. Moreover, since $f|_{X_\alpha}$ is a submersion at x, so F is a submersion at x. Hence $\nu(\mathrm{d}_x(F|_{X_\alpha})) \neq 0$. Since the stratification is Whitney, it implies that $T \supset T_x X_\alpha$. Consequently $\nu(\mathrm{d}_x F|_T) \geqslant \nu(\mathrm{d}_x(F|_{X_\alpha})) \neq 0$. To get a contradiction, we will need the following claim:

Claim 3.8 *Let $A_k\colon \mathbb{R}^q \to \mathbb{R}^p$ be a sequence of linear maps such that $A_k \to A$ as $k \to +\infty$ (i.e., the terms of the matrix of A_k tend to the corresponding terms of the matrix of A). Let $H_k \subset \mathbb{R}^q$ be a sequence of linear subspaces of same dimension such that $H_k \to H$ (i.e., $\delta(H_k, H) \to 0$, where $\delta(H_k, H) := \sup_{y \in H_k, \|y\|=1} \mathrm{dist}(y, H)$ is the distance between H_k and H; $\mathrm{dist}(\cdot, \cdot)$ is the Euclidean distance). Then $\nu(A_k|_{H_k}) \to \nu(A|_H)$.*

Proof It is clear that

$$\|\nu(A_k|_{H_k}) - \nu(A|_H)\| \leqslant \|\nu(A_k|_{H_k}) - \nu(A|_{H_k})\| + \|\nu(A|_{H_k}) - \nu(A|_H)\|.$$

In light of [15, Lem. 2.1 (iv)], we have

$$\|\nu(A_k|_{H_k}) - \nu(A|_{H_k})\| \leqslant \|A_k|_{H_k} - A|_{H_k}\| = \|(A_k - A)|_{H_k}\| \leqslant \|A_k - A\| \to 0. \quad (3)$$

Note that $\nu(A)$ is the length of a minimal semiaxis of $A(B)$, where B is the unit ball. Since $H_k \to H$, we have $V_k := B \cap H_k \to B \cap H := V$ and $A(V_k) \to A(V)$ by continuity of A. Hence also $\nu(A|_{H_k}) \to \nu(A|_H)$. □

Applying Claim 3.8 with $A_k = \mathrm{d}_{x^k} F$ and $H_k = T_{x^k} X_\beta$, we get

$$0 = \lim_{k \to \infty} \nu\big(\mathrm{d}_{x^k} F|_{T_{x^k} X_\beta}\big) = \nu(\mathrm{d}_x F|_T),$$

which is a contradiction. The lemma follows. □

For each $\beta \in I'$, it is clear that $f|_{X_\beta}$ is a submersion on $(f|_{X_\beta})^{-1}(\overline{B})$, so for $x \in \overline{U} \cap X_\beta$, the differential $\mathrm{d}_x(f|_{X_\beta})\colon T_xX_\beta \to \mathbb{R}^{2m}$ is surjective, which induces an isomorphism of vector spaces

$$\widetilde{\mathrm{d}_x}(f|_{X_\beta})\colon T_xX_\beta/\ker \mathrm{d}_x(f|_{X_\beta}) \cong \mathbb{R}^{2m}.$$

Thus, for each $i = 1, \ldots, 2m$, the vector field ∂_i can be lifted uniquely and smoothly on each stratum X_β with $\beta \in I'$ to the vector field called the horizontal lift of ∂_i and denoted by v_i^β. Clearly, $v_i^\beta(x)$ is the unique vector in T_xX_β which lifts ∂_i and is orthogonal to $\ker \mathrm{d}_x(f|_{X_\beta})$. Each v_i^β has the following important properties:

Lemma 3.9 *Let $c > 0$ be the constant in Lemma 3.7. Then, for each $x \in \overline{U} \cap X_\beta$ with $\beta \in I'$, we have*

$$\|v_i^\beta(x)\| \leqslant \frac{\|x\| + 1}{c}.$$

Proof Let $\mathbb{B}_\beta$ be the closed unit ball centered at the origin in T_xX_β. Then $\mathrm{d}_x(f|_{X_\beta})(\mathbb{B}_\beta)$ is an ellipsoid in $\mathbb{R}^{2m}$ with $\nu(\mathrm{d}_x(f|_{X_\beta}))$ as the length of the shortest semiaxis. Let $\mathbb{B}^{2m}$ be the closed unit ball centered at the origin in $\mathbb{R}^{2m}$. Then $(\widetilde{\mathrm{d}}_x(f|_{X_\beta}))^{-1}\big(\nu(\mathrm{d}_x(f|_{X_\beta}))\mathbb{B}^{2m}\big)$ is an ellipsoid in T_xX_β with 1 as the length of the longest semiaxis. Therefore the longest semiaxis of the ellipsoid $(\widetilde{\mathrm{d}}_x(f|_{X_\beta}))^{-1}(\mathbb{B}^{2m})$ is $1/\nu(\mathrm{d}_x(f|_{X_\beta}))$. Consequently,

$$\|v_i^\beta(x)\| \leqslant \frac{1}{\nu(\mathrm{d}_x(f|_{X_\beta}))} \leqslant \frac{\|x\| + 1}{c},$$

which yields the lemma. □

Note that, for fixed i, the vector field on U which coincides with v_i^β on each U_β is not necessarily a rugose vector field. In what follows, we will try to deform these vector fields to produce a rugose vector field which satisfies the assumption of Lemma 3.5. The process is carried out by induction on dimension.

For $2m \leqslant d \leqslant 2\dim_{\mathbb{C}} X$, let $I'_d := \{\beta \in I' : 2m \leqslant \dim X_\beta \leqslant d\}$ and $U_d := \bigcup_{\beta \in I'_d} X_\beta \cap U$. By induction on d, we construct a rugose vector field on $U_{2\dim_{\mathbb{C}} X}$ with the property of Lemma 3.5. For $d = 2m$, let v_i^{2m} be the restriction to U_{2m} of the smooth vector field on $\bigcup_{\beta \in I'_{2m}} X_\beta$ which coincides with each v_i^β on X_β for $\beta \in I'_{2m}$. Then v_i^{2m} is clearly rugose, $\mathrm{d}f(v_i^{2m}) = \partial_i$ and by Lemma 3.9, $\|v_i^{2m}(x)\| \leqslant \frac{\|x\|+1}{c}$ for any $x \in U_{2m}$.

For each i, assume that we have constructed a rugose vector field, denoted by v_i^d, on U_d such that $\mathrm{d}_x f(v_i^d(x)) = \partial_i$ and $\|v_i^d(x)\| \leqslant \frac{\|x\|+1}{c_d}$ for every $x \in U_d$, where c_d is a positive constant. We need to extend each v_i^d to a rugose vector field v_i^{d+2} on U_{d+2} such that $\|v_i^{d+2}(x)\| \leqslant \frac{\|x\|+1}{c_{d+2}}$ for every $x \in U_{d+2}$, where c_{d+2} is also a positive constant (recall that the strata of $\mathcal{S}$ have even dimension). Note that, to construct v_i^{d+2}, it is enough to construct v_i^{d+2} separately on each stratum X_α with $\alpha \in I'_{d+2} \backslash I'_d$. Without loss of generality, suppose that $I'_{d+2} \backslash I'_d = \{\alpha\}$. By [20, Lem. 4.4], for each $i = 1, \ldots, 2m$, there is a rugose vector field on U_{d+2}, denoted by $\widetilde{w}_i^{d+2}$, which extends v_i^d, so the restriction $\widetilde{w}_i^{d+2}|_{U_{d+2} \cap X_\alpha}$ is a smooth vector field. We need to adjust $\widetilde{w}_i^{d+2}$ to get a new rugose vector field w_i^{d+2} on U_{d+2} such that, for any $y \in X_\alpha \cap U_{d+2}$, we have $\mathrm{d}_y f(w_i^{d+2}(y)) = \partial_i$.

Lemma 3.10 *For $y \in U_{d+2} \cap X_\alpha$, write*

$$\widetilde{w}_i^{d+2}(y) = \sum_{j=1}^{2m} a_j(y)\, v_j^\alpha(y) + P(y),$$

where $P(y) \in \ker \mathrm{d}_y f$. Define

$$w_i^{d+2}(x) := \begin{cases} v_i^\alpha(x) + P(x) & \text{if } x \in X_\alpha \cap U_{d+2}, \\ v_i^d(x) & \text{if } x \in U_d. \end{cases}$$

Then w_i^{d+2} is a rugose vector field on U_{d+2} and $\mathrm{d}_x f(w_i^{d+2}(x)) = \partial_i$ for $x \in U_{d+2}$.

Proof For $x' \in U_d$, let

$$v_i^d(x') = \sum_{j=1}^{2m} b_j(x', y)\, v_j^\alpha(y) + Q(x', y) + S(x', y),$$

where $Q(x', y) \in \ker \mathrm{d}_y f$ and $S(x', y) \in (T_y X_\alpha)^\perp$. Since $\widetilde{w}_i^{d+2}$ is rugose, for each $x \in U_d$, there is a neighborhood W_x of x such that, for any $y \in W_x \cap X_\alpha$ and any $x' \in W_x \cap X_\beta$, we have:

- $\|P(y) - Q(x', y)\| < C\|y - x'\|$,
- $\|S(x', y)\| < C\|y - x'\|$

for some $C > 0$, where X_β is the stratum containing x. Shrinking W_x and increasing C if necessary, we can suppose that $\mathrm{d}F\colon x \mapsto \mathrm{d}_x F$ is Lipschitz on W_x, where F is a polynomial extension of f to $\mathbb{C}^n$. Hence $\|\mathrm{d}_y F - \mathrm{d}_{x'} F\| < C\|y - x'\|$. In particular,

$$\left\|\mathrm{d}_y F(v_i^d(x')) - \mathrm{d}_{x'} F(v_i^d(x'))\right\| < C\|y - x'\| \cdot \|v_i^d(x')\|,$$

i.e.,

$$\left\|\sum_{j=1}^{2m} b_j(x', y)\partial_j - \partial_i\right\| - \|\mathrm{d}_y F(S(x', y))\| < C\|y - x'\| \cdot \|v_i^d(x')\|.$$

Shrink W_x, if necessary, so that $\overline{W}_x \cap X_\gamma \subset U_{d+2}$ for any $\gamma \in I'_{d+2}$. Let $M := \sup_{z \in U_d} \|v_i^d\|$ and $N := \sup_{z \in W_x} \|\mathrm{d}_z F\|$. Then we have

$$\sum_{j \neq i} |b_j(x', y)| + |b_i(x', y) - 1| < C(M+N)\|y - x'\|$$

and

$$\begin{aligned}\|w_i^{d+2}(y) - v_i^d(x')\| < &\Bigl(\sum_{j \neq i} |b_j(x', y)| + |b_i(x', y) - 1|\Bigr) D \\ &+ \|P(y) - Q(x', y)\| + \|S(x', y)\|,\end{aligned}$$

where $D := \sup_{z \in W_x \cap X_\alpha} \|v_i^\alpha\|$. Thus

$$\|w_i^{d+2}(y) - v_i^d(x')\| < (2C + CD(M+N))\|y - x'\|.$$

Hence w_i^{d+2} is rugose and of course $\mathrm{d}_x f(w_i^{d+2}(x)) = \partial_i$ for $x \in U_{d+2}$. $\square$

Since w_i^{d+2} is rugose (in view of Lemma 3.10), it is continuous. Then, by shrinking W_x, where W_x is determined in the proof of Lemma 3.10, if necessary, we may assume that $W_x \subset \{z \in \mathbb{C}^n : \|z - x\| \leqslant 1\}$ and

$$\|w_i^{d+2}(y)\| < 2\|v_i^d(x)\| \tag{4}$$

for any $y \in W_x \cap X_\alpha$. Let $W_d := \bigcup_{x \in U_d} W_x$, then W_d is an open neighborhood of radius no larger than 1 of U_d. Let $F = (f_1, \ldots, f_m) \colon \mathbb{C}^n \to \mathbb{C}^m$ be the polynomial extending f on $\mathbb{C}^n$. Let $K := F^{-1}(B)$, which is considered as a subset of $\mathbb{R}^{2n}$ under the identification of $\mathbb{C}^n$ with $\mathbb{R}^{2m}$. Then U_d is a closed set in K for the induced topology from the Euclidean topology. By a smooth version of Urysohn's lemma, there is a smooth function $\varphi \colon K \to [0, 1]$ such that $\varphi^{-1}(0) = K \setminus W_d$ and $\varphi^{-1}(1) = U_d$. (Note that there may not exist such a function φ defined on the whole of $\mathbb{R}^{2n}$ since U_d is not closed in $\mathbb{R}^{2n}$; namely, there may be no smooth extensions of φ on $\mathbb{R}^{2n}$.) For $x \in U_{d+2}$, set

$$v_i^{d+2}(x) := (1 - \varphi(x))\, v_i^\alpha(x) + \varphi(x)\, w_i^{d+2}(x).$$

Clearly, the restriction of v_i^{d+2} on each stratum is a smooth vector field. Moreover, we have

$$\begin{aligned}\mathrm{d}_x f\bigl(v_i^{d+2}(x)\bigr) &= \mathrm{d}_x f\bigl((1 - \varphi(x))\, v_i^\alpha(x) + \varphi(x)\, w_i^{d+2}(x)\bigr) \\ &= (1 - \varphi(x))\, \mathrm{d}_x f(v_i^\alpha(x)) + \varphi(x)\, \mathrm{d}_x f\bigl(w_i^{d+2}(x)\bigr) \\ &= (1 - \varphi(x))\, \partial_i + \varphi(x)\, \partial_i = \partial_i.\end{aligned}$$

Let us prove that v_i^{d+2} is a rugose vector field. For any $x \in U_d$, let X_β be the stratum containing x. For $x' \in W_x \cap X_\beta$ and $y \in W_x \cap X_\alpha$, we have

$$\begin{aligned}
&\big\|v_i^{d+2}(y) - v_i^{d+2}(x')\big\| \\
&\quad = \big\|(1-\varphi(y))\, v_i^{\alpha}(y) + \varphi(y)\, w_i^{d+2}(y) - v_i^{d}(x')\big\| \\
&\quad = \big\|(1-\varphi(y))\, v_i^{\alpha}(y) - (1-\varphi(y))\, w_i^{d+2}(y) + w_i^{d+2}(y) - v_i^{d}(x')\big\| \\
&\quad \leqslant (1-\varphi(y))\big\|v_i^{\alpha}(y) - w_i^{d+2}(y)\big\| + \big\|w_i^{d+2}(y) - v_i^{d}(x')\big\| \\
&\quad \leqslant (1-\varphi(y))\big(\big\|v_i^{\alpha}(y)\big\| + \big\|w_i^{d+2}(y)\big\|\big) + \big\|w_i^{d+2}(y) - w_i^{d+2}(x')\big\|.
\end{aligned}$$

We note the following facts:

- Since $1-\varphi(y)$ is a smooth function, it is locally Lipschitz; with no loss of generality, assume that $1-\varphi(y)$ is Lipschitz on W_x with constant c_1. Then

$$1-\varphi(y) = (1-\varphi(y)) - (1-\varphi(x')) \leqslant c_1\|y-x'\|.$$

- By Lemma 3.9 and by the continuity of w_i^{d+2}, there is a positive constant c_2 depending only on x such that $\|v_i^{\alpha}(y)\| + \|w_i^{d+2}(y)\| \leqslant c_2$ (we can take $c_2 := \sup_{z\in \overline{W}_x\cap X_\alpha} \frac{\|z\|+1}{c} + \sup_{z\in \overline{W}_x\cap X_\alpha} \|w_i^{d+2}(z)\|$; note that $\overline{W}_x \cap X_\alpha \subset U_{d+2}$).
- Since w_i^{d+2} is rugose, it follows that there is a positive constant c_3 depending only on x such that $\|w_i^{d+2}(y) - w_i^{d+2}(x')\| \leqslant c_3\|y-x'\|$.

Hence

$$\big\|v_i^{d+2}(y) - v_i^{d+2}(x')\big\| \leqslant (c_1c_2+c_3)\|y-x'\|,$$

i.e., v_i^{d+2} is rugose. Now it remains to show that there is a positive constant c_{d+2} such that $\|v_i^{d+2}(y)\| \leqslant \frac{\|y\|+1}{c_{d+2}}$ for every $y \in U_{d+2}$. Obviously, the statement holds for $y \in U_d$ by the induction assumption and for $y \in U_{d+2}\backslash W_d$ by Lemma 3.9, so we can suppose that $y \in W\backslash U_d$, which clearly implies that $y \in X_\alpha$. In light of Lemma 3.9, we get

$$\big\|v_i^{\alpha}(y)\big\| \leqslant \frac{\|y\|+1}{c}, \tag{5}$$

where c is the constant in the same lemma. In view of (4) and the induction assumption, we have

$$\begin{aligned}
\big\|w_i^{d+2}(y)\big\| &< 2\big\|v_i^{d}(x)\big\| \\
&\leqslant 2\,\frac{\|x\|+1}{c_d} \leqslant 2\,\frac{\|y\|+\|x-y\|+1}{c_d} \leqslant 2\,\frac{\|y\|+2}{c_d} \leqslant 4\,\frac{\|y\|+1}{c_d}.
\end{aligned} \tag{6}$$

Thus (5) and (6) yield

$$\begin{aligned}\left\|v_i^{d+2}(y)\right\| &= \left\|(1-\varphi(y))\,v_i^{\alpha}(y)+\varphi(y)\,w_i^{d+2}(y)\right\| \\ &\leqslant (1-\varphi(y))\|v_i^{\alpha}(y)\|+\varphi(y)\|w_i^{d+2}(y)\| \\ &\leqslant (1-\varphi(y))\,\frac{\|y\|+1}{c}+\varphi(y)\,4\,\frac{\|y\|+1}{c_d} \\ &< \left(\frac{1}{c}+\frac{4}{c_d}\right)(\|y\|+1).\end{aligned}$$

Set $c_{d+2} = \min\bigl\{\frac{1}{1/c+4/c_d}, c, c_d\bigr\}$, then $\|v_i^{d+2}(y)\| \leqslant \frac{\|y\|+1}{c_{d+2}}$ for every $y \in U_{d+2}$. By induction, there exists a rugose vector field on $U_{2\dim_{\mathbb{C}} X}$ with the property of Lemma 3.5. Then the theorem follows by applying Lemma 3.5. □

The following corollary follows immediately from Theorem 3.4:

Corollary 3.11 *Let $X \subset \mathbb{C}^n$ be an affine variety with an affine Whitney stratification $\mathcal{S}$, and let $f: X \to \mathbb{C}^m$ be a polynomial dominant map. Assume that, for any stratum $X_\beta \in \mathcal{S}$, the restriction $f|_{X_\beta}$ is a submersion and $K_\infty(f, X_\beta) = \emptyset$. Then f is a locally trivial fibration.*

4 Computation of the Sets of Stratified Generalized Critical Values

In this section, we compute the set $K(f, \mathcal{S})$ of stratified generalized critical values of f, for which we need to construct an affine Whitney stratification of X and then apply [10] for each stratum of this stratification. The process is slightly different from the construction in Sect. 2.3, since we only need to construct such an affine Whitney stratification "partially," by noting the following facts:

- As the construction of Whitney stratifications is by induction on dimension, we only need to proceed until the dimension shrinks below m, since the restriction of f to any stratum of dimension $< m$ is always singular.
- For any algebraic set $Z \subseteq X$, let

$$\begin{aligned} r_Z &:= \max_{x\in Z\setminus V(p_{Z,\emptyset})} \operatorname{rank} \operatorname{Jac}_x(f|_Z) \\ \text{and}\quad H(Z) &:= \overline{\{x \in Z\setminus V(p_{Z,\emptyset}) : \operatorname{rank}\operatorname{Jac}_x(f|_Z) < r_Z\}}^{\mathcal{Z}}. \end{aligned}$$

  Then, at any step of the induction process, the construction in Sect. 2.3 can be omitted if $r_Y < m$.

Let us now construct such a stratification. With the same notations as in Lemma 2.8, let

$$\Gamma_3 := \bigcup_{k=1}^{t} \left\{ \begin{array}{l} (x, x, w, v, \gamma, \lambda, \mu) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \times \mathbb{C} \times \mathbb{C}^t : \\ h_1(x, x, w, v) = \cdots = h_q(x, x, w, v) = 0 \\ \gamma \sum_{j=1}^{n} v_j w_j = 1 \\ \lambda p_{Y,\emptyset}(x) = 1 \\ \mu_k M_k^{(m,p)}(x) = 1 \end{array} \right\},$$

where each $M_k^{(m,p)}(x)$ is a minor of the matrix

$$A(x) := \begin{bmatrix} \mathrm{d}_x f_1 \\ \vdots \\ \mathrm{d}_x f_m \\ \mathrm{d}_x g_1 \\ \vdots \\ \mathrm{d}_x g_r \\ \mathrm{d}_x \widetilde{g}_{r+1} \\ \vdots \\ \mathrm{d}_x \widetilde{g}_p \end{bmatrix},$$

obtained by deleting $n-m-p$ columns. So Γ_3 differs from Γ_2 in the last t equations, since we are only interested in finding the points where the Whitney condition (b) is not satisfied, outside $P(Y, \emptyset)$. Let

$$\pi_3 \colon \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \times \mathbb{C} \times \mathbb{C}^t \to \mathbb{C}^n$$

be the projection on the first n coordinates. By Lemma 2.8, $\pi_3(\Gamma_3)$ is the set of points where the Whitney condition (b) fails. Obviously $\pi_3(\Gamma_3) \subset \mathrm{reg}(f|_{Y\setminus P(Y)})$ and $\dim \pi_3(\Gamma_3) < \dim Y$. Set $\widetilde{W} := \overline{\pi_3(\Gamma_3)}^{\mathcal{Z}}$. Then obviously $\dim \widetilde{W} < \dim Y$. Again, we can compute a Gröbner basis of the ideal $I(\widetilde{W})$. Finally, set

- $X_0 := X$,
- $X_1 := X_0 \cap V(p_{X_0,\emptyset}),\ S_1 = K_0(f, X_0 \setminus X_1), \ldots,$
- $X_i := X_{i-1} \cap V(p_{X_{i-1}, \bigcup_{j=0}^{i-2} \widetilde{W}(X_j, X_{i-1})}),\ S_i = K_0(f, X_{i-1} \setminus X_i), \ldots$

By induction, we can construct a finite filtration of algebraic sets $X = X_0 \supset X_1 \supset \cdots \supset X_q \supset X_{q+1} \supseteq \emptyset$ with $\dim X_i > \dim X_{i+1}$ and $r_{X_{q+1}} < m$. It is clear that this filtration does not induce an affine Whitney stratification $\mathcal{S}$ of X. However, it shows that there is an affine Whitney stratification $\mathcal{S}$ such that

$$K_0(f, \mathcal{S}) = \bigcup_{i=1}^{q} S_i \cup \overline{f(X_{q+1})}.$$

Let $Z_i := X_i \setminus X_{i+1}$. Then $\{Z_i\}_{i=0,\ldots,q}$ is an affine Whitney stratification of $X \setminus X_{q+1}$. Every variety Z_i can be realized as a closed affine variety $\widetilde{Z}_i$ in $\mathbb{C}^{n+1}$, by the embedding $Z_i \ni x \mapsto \big(x, 1/P_{X_i, \bigcup_{j=0}^{i-1} \widetilde{W}(X_j, X_i)}(x)\big) \in \mathbb{C}^{n+1}$ for $i > 0$ or the embedding $Z_0 \ni$

$x \mapsto (x, 1/P_{X_0,\emptyset}(x)) \in \mathbb{C}^{n+1}$. Let $K_\infty(f, Z_i)$ be the set of asymptotic critical values of $f|_{Z_i}$, which now can be computed analogously as in [8,10]—this will be done in the next section. Then, from the construction, it is clear that the set of stratified generalized critical values of f is given by

$$K(f, \mathcal{S}) := \bigcup_{i=1}^{q} (K_\infty(f, Z_i) \cup K_0(f, Z_i)) \cup \overline{f(X_{q+1})}$$

and $K(f, \mathcal{S})$ can be computed effectively. Note that Remark 2.5 and elementary properties of Gröbner bases imply:

Corollary 4.1 *Let $X \subset \mathbb{C}^n$ be an affine variety of pure dimension and let $f = (f_1, \ldots, f_m) \colon X \to \mathbb{C}^m$ be a polynomial mapping. Let $\mathbb{F} \subset \mathbb{C}$ be a subfield generated by coefficients of generators of $I(X)$ and all coefficients of polynomials f_i, $i = 1, \ldots, m$. Then there is a nowhere dense affine variety $K(f, \mathcal{S}) \subset \mathbb{C}^m$, which is described by polynomials from $\mathbb{F}[x_1, \ldots, x_m]$ such that all bifurcation values $B(f)$ of f are contained in $K(f, \mathcal{S})$. In particular, for $m = 1$, if X and f are described by polynomials from $\mathbb{Q}[x_1, \ldots, x_n]$, then all bifurcation values of f are algebraic numbers.*

5 Computation of $K_0(f, Z_i) \cup K_\infty(f, Z_i)$

Let $k = \mathbb{R}$ or $k = \mathbb{C}$. Let $X \cong k^n$, $Y \cong k^m$ be finite-dimensional vector spaces (over k). We consider those spaces equipped with the canonical scalar (hermitian) products. Let us denote by $\mathcal{L}(X, Y)$ the set of linear mappings from X to Y and by $\Sigma = \Sigma(X, Y) \subset \mathcal{L}(X, Y)$ the set of nonsurjective mappings. In this section, we give several different expressions for a distance of an $A \in \mathcal{L}(X, Y)$ to the space Σ of singular operators. Let us first recall the following [15]:

Definition 5.1 Let $A \in \mathcal{L}(X, Y)$. Set

$$\nu(A) = \inf_{\|\phi\|=1} \|A^*(\phi)\|,$$

where $A^* \in \mathcal{L}(Y^*, X^*)$ is the adjoint operator and $\phi \in Y^*$.

Let $\alpha, \beta \colon \mathcal{L}(X, Y) \to \mathbb{R}_+$ be two nonnegative functions. We shall say that α and β are *equivalent* (we write $\alpha \sim \beta$) if there are constants $c, d > 0$ such that

$$c\alpha(A) \le \beta(A) \le d\alpha(A)$$

for any $A \in \mathcal{L}(X, Y)$. We give below several functions equivalent to ν. Let $A = (A_1, \ldots, A_m) \in \mathcal{L}(X, Y)$, and let $\overline{A_i} = \operatorname{grad} A_i$. Denote by $\langle (\overline{A_j})_{j \neq i} \rangle$ the linear space generated by vectors $(\overline{A_j})$, $j \neq i$. Let

$$\kappa(A) = \min_{1\le i\le m} \operatorname{dist}(\overline{A_i}, \langle(\overline{A_j})_{j\ne i}\rangle)$$

be the *Kuo number of* A.

Proposition 5.2 ([12]) *The Kuo function* κ *is equivalent to the* ν *of Rabier. More precisely,*

$$\nu(A) \le \kappa(A) \le \sqrt{m}\,\nu(A).$$

Definition 5.3 Let $A \in \mathcal{L}(X, Y)$, and let $H \subset X$ be a linear subspace. We set

$$\nu(A, H) = \nu(A|_H), \quad \kappa(A, H) = \kappa(A|_H),$$

where $A|_H$ denotes the restriction of A to H.

From Proposition 5.2 we immediately get the following corollary:

Corollary 5.4 *We have* $\nu(A, H) \sim \kappa(A, H)$.

In fact we also have the following explicit expression for $\kappa(A, H)$ (see [9,10]):

Proposition 5.5 *Let* $A = (A_1, \dots, A_m) \in \mathcal{L}(X, Y)$*, and let* $H \subset X$ *be a linear subspace. Assume that* H *is given by a system of linear equations* $B_j = 0$*,* $j = 1, \dots, r$*. Then*

$$\kappa(A, H) = \min_{1\le i\le m} \operatorname{dist}\big(\overline{A_i}, \langle(\overline{A_j})_{j\ne i}; (\overline{B_j})_{j=1,\dots,r}\rangle\big),$$

where $\overline{A_i} = \operatorname{grad} A_i$ *and* $\overline{B_j} = \operatorname{grad} B_j$.

Finally, we introduce a function g' which will be useful in the explicit description of the set of generalized critical values.

Definition 5.6 Let $A \in \mathcal{L}(k^n, k^m)$, where $n \ge m + r$, and let $H \subset k^n$ be a linear subspace given by a system of independent linear equations $B_l = \sum b_{lk}x_k$, $l = 1, \dots, r$. By abuse of notation, we denote by A the matrix (in the canonical bases in k^n and k^m) of the mapping A. Let C be an $(m + r) \times n$ matrix given by rows $A_1, \dots, A_m; B_1, \dots, B_r$ (we identify $A_i = \sum a_{ik}x_k$ with the vector $(a_{j1}, \dots, a_{jn})$, similarly for B_l). Let M_I, where $I = (i_1, \dots, i_{m+r})$, denote an $(m+r)\times(m+r)$ minor of C given by columns indexed by I. Let $M_J(j)$ denote an $(m + r - 1) \times (m + r - 1)$ minor given by columns indexed by J and by deleting the j^{th} row, where $1 \le j \le m$. Note that we delete only A_j rows. We set

$$g'(A, H) = \max_I \left\{ \min_{\{J\subset I,\, 1\le j\le m\}} \frac{|M_I|}{|M_J(j)|} \right\}$$

(where we consider only numbers with $M_J(j) \ne 0$; if all numbers $M_J(j)$ are zero, we put $g'(A) = 0$).

In particular, we have the following (see [9,10]):

Proposition 5.7 *We have* $g'(A, H) \sim \nu(A, H)$.

Now we can prove the following theorem:

Theorem 5.8 *Let* Z_i *be a stratum of* X *as in Sect. 4. Then the set* $K(f, Z_i) = K_0(f, Z_i) \cup K_\infty(f, Z_i)$ *is a nowhere dense algebraic subset of* $\mathbb{C}^m$.

Proof It is a standard fact that $K_0(f, Z_i)$ is algebraic and nowhere dense (for details see the end of Sect. 5.1). Hence, it is enough to focus on $K_\infty(f, Z_i)$.

By construction, the set $X := Z_i \subset \mathbb{C}^n$ is a subset of complete intersection, $X \subset \{b_1 = 0, \ldots, b_s = 0\}$, and $\operatorname{rank}\{\nabla b_k : k = 1, \ldots, s\} = s$ (X has codimension s). Let us recall notation of Definition 5.6. For $x \in \mathbb{C}$, let $A = \mathrm{d}_x f$ and $B_l = \mathrm{d}_x b_l$, $l = 1, \ldots, s$. Let $A \in \mathcal{L}(k^n, k^m)$, where $n \geq m + s$, and let $T_x X = H \subset k^n$ be a linear subspace given by a system of independent linear equations $B_l = \sum b_{lk} x_k$, $l = 1, \ldots, s$. By abuse of notation, we denote by A the matrix (in the canonical bases in k^n and k^m) of the mapping A. Let C be an $(m + s) \times n$ matrix given by rows $A_1, \ldots, A_m; B_1, \ldots, B_s$ (we identify $A_i = \sum a_{ik} x_k$ with the vector $(a_{j1}, \ldots, a_{jn})$, similarly for B_l).

For an index $I = (i_1, \ldots, i_{m+r}) \subset \{1, \ldots, n\}$ let $M_I(x)$ denote the $(m+s)\times(m+s)$ minor of C given by columns indexed by I. For integers $j \in I$, $1 \leq k \leq m$ we denote by $M_{I(k,j)}(x)$ the $(m+s-1)\times(m+s-1)$ minor obtained by deleting the j^{th} column and the k^{th} row. Note that we delete only A_k, $1 \leq k \leq m$ rows.

Hence, M_I and $M_{I(k,j)}$ are regular (restriction of polynomials) functions on X. We now define a family of rational functions on X:

$$W_{I(k,j)}(x) = M_I(x)/M_{I(k,j)}(x),$$

where, for $M_{I(k,j)} \equiv 0$, we put $W_{I(k,j)} \equiv 0$. We write $b = (b_1, \ldots, b_s)$ and $(f, b)\colon \mathbb{C}^n \to \mathbb{C}^m \times \mathbb{C}^s$; here, we consider $f_1, \ldots, f_m$ and $b_1, \ldots, b_s$ as polynomials on $\mathbb{C}^n$.

Let $w = \binom{n}{m+s}$ and let $M_{I_1}, \ldots, M_{I_w}$ be all possible main minors of a matrix of $\mathrm{d}_x(f, b)$. For every index I_l, take a pair (k_l, j_l) which determines an $(m + s - 1) \times (m + s - 1)$ minor of M_{I_l} (we consider here only minors which are not identically zero). We denote a sequence $(k_1, j_1), \ldots, (k_w, j_w)$ by $(k, j) \in \mathbb{N}^w \times \mathbb{N}^w$, and we consider a rational function

$$\Phi_{(k,j)} = \Phi((k_1, j_1), \ldots, (k_w, j_w))\colon X \to \mathbb{C}^m \times \mathbb{C}^N,$$

where the first component of $\Phi_{(k,j)}$ is f and the next components are $W_{I_p(k_p,j_p)}$, $p = 1, \ldots, w$, and all products $x_l W_{I_p(k_p,j_p)}$, $p = 1, \ldots, w$; $l = 1, \ldots, n$.

We can assume that, for some choice of l, we have $W_{I_l(k_l,j_l)} \not\equiv 0$, and consequently $\dim \mathrm{cl}(\Phi_{(k,j)}(X)) = \dim X = n - s$. Here $\mathrm{cl}(Y)$ stands for the closure of Y in the strong (or which is the same, in the Zariski) topology. Let $\Gamma(k, j) = \mathrm{cl}(\Phi_{(k,j)}(X))$ (by $\Phi_{(k,j)}(X)$ we mean the set $\Phi_{(k,j)}(X \setminus P)$, where P is a set of poles of $\Phi_{(k,j)}$).

Now, for a given $r \in \{1, \ldots, n\}$, consider the set $X_r := X \setminus \{x_r = 0\}$. Finally, let $\Phi_{(k,j),r}(x) := (\Phi_{(k,j)}(x), 1/x_r)$ and $\Gamma((k,j),r) := \mathrm{cl}(\Phi_{(k,j),r}(X_r))$. Let us recall that $y \in K_\infty(f, Z_i)$ if there exists a sequence $x \to \infty;\ x \in Z_i$ such that

$$f(x) \to y \quad \text{and} \quad \|x\|\, g'(x) \to 0,$$

where $g'(x) = g'(\mathrm{d}_x f, T_x Z_i)$. We have

Lemma 5.9

$$K_\infty(f, X) = \mathbb{C}^m \cap \bigcup_{(k,j),r} \Gamma((k,j),r),$$

where we identify $\mathbb{C}^m$ *with* $\mathbb{C}^m \times (0, \ldots, 0)$.

Proof We identify X with $\widetilde{Z}_i \subset \mathbb{C}^{n+1}$, hence we can assume that X is closed in $\mathbb{C}^{n+1}$. Let $y \in K_\infty(f, X)$. Hence, there is a sequence $x^l \to \infty$ such that $x^l \in X$ and $f(x^l) \to y$ and $\|x^l\|\, g'(x_l) \to 0$. Moreover, if $x = (x_1, \ldots, x_n)$, then there is at least one r, $1 \le r \le n$, such that $x^l_r \to \infty$. If $\{x^l : l = 1, 2, \ldots\} \subset C(f, X)$ ($C(f, X)$ denotes the set of critical points of $f_{|X}$), then it is easy to see that $y \in \mathbb{C}^m \cap \Gamma((k,j),r)$ for every (k,j) (we can choose a close sequence x'^l such that $f(x'^l) \to y$ and $\|x'^l\|\, g'(x'^l) \to 0$ and functions $W_{I(k,j)}$ are defined). Consequently, we can assume that $\{x^l : l = 1, 2, \ldots\} \cap C(f, X) = \emptyset$. Thus, there is a sequence $x^l \to \infty$ such that, for every I_i, there are integers (k_i, j_i) such that $\|x^l\|\, M_{I_i}/M_{I_i(k_i,j_i)}(x_l) \to 0$ and $f(x^l) \to y$. This also gives $y \in \Gamma((k,j),r) \cap \mathbb{C}^m$ with $((k,j),r) = ((k_1,j_1), \ldots, (k_w,j_w), r)$.

Conversely, if $y \in \Gamma((k,j),r) \cap \mathbb{C}^m$, then we can choose a sequence $x^l \to \infty$, $x^l \in X_r$ such that $f(x^l) \to y$ and $\|x^l\|\, M_{I_i}/M_{I_i(k_i,j_i)}(x^l) \to 0$. It is easy to observe that this implies $\|x^l\|\, g'(x^l) \to 0$ and $f(x^l) \to y$, i.e. $y \in K_\infty(f, X)$. □

Now, in light of [10, Thm. 3.3], we have that $K_\infty(f, X) \neq \mathbb{C}^m$ hence $\mathbb{C}^m \cap \bigcup_{((k,j),r)} \Gamma((k,j),r) \neq \mathbb{C}^m$. By Lemma 5.9, $K_\infty(f, X)$ is an algebraic set. The theorem follows. □

5.1 A Sketch of an Algorithm

Let $X := Z_i \subset \mathbb{C}^n$ be a smooth affine variety of dimension $n - s$. Let $f = (f_1, \ldots, f_m)\colon X \to \mathbb{C}^m$ be a polynomial dominant mapping. Then the set $K_\infty(f, X)$ can be computed as follows:

By construction, X is a subset of complete intersection, hence we can choose polynomials $b_1, \ldots, b_s \in I(X)$ such that $\mathrm{rank}\{\mathrm{grad}\, b_1, \ldots, \mathrm{grad}\, b_s\} = s$ on X. Let us consider the rational mapping

$$\begin{aligned}
&\Phi((k_1,j_1), \ldots, (k_w,j_w), r)\colon X \ni x \\
&\quad\mapsto \big(f(x), W_{I_1(k_1,j_1)}(x), x_1 W_{I_1(k_1,j_1)}(x), \ldots, x_n W_{I_1(k_1,j_1)}(x), \\
&\qquad \ldots, W_{I_s(k_w,j_w)}(x), x_1 W_{I_w(k_w,j_w)}(x), \ldots, x_n W_{I_w(k_w,j_w)}(x), 1/x_r\big) \in \mathbb{C}^m \times \mathbb{C}^N,
\end{aligned}$$

which is constructed exactly as in the proof of Theorem 5.8. Recall that

$$\Gamma((k_1, j_1), \dots, (k_w, j_w), r) = \mathrm{cl}\big(\Phi((k_1, j_1), \dots, (k_w, j_w), r)(X)\big).$$

We know also that

$$K_\infty(f, X) = L \cap \left(\bigcup_{((k_1,j_1),\dots,(k_w,j_w)),r} \Gamma((k_1, j_1), \dots, (k_w, j_w), r) \right),$$

where $L = \mathbb{C}^m \times (0, \dots, 0)$. First we compute the ideal of the set $\Gamma((k_1, j_1), \dots, (k_w, j_w), r)$. Now we can consider a variety X as a closed affine variety in $\mathbb{C}^{n+1}$ (see the end of Sect. 4). Let $I(X) = (b_1, \dots, b_q)$. To this end, we restrict the mapping $\Phi((k, j), r)$ to an open dense subset $U \subset X$ on which this mapping is regular. In particular, we can choose the set $U = X \backslash \big(\bigcup_{l=1}^{w} \{M_{I_l(k_l, j_l)} = 0\} \cup \{x_r = 0\}\big)$. The set U can be identified with the set

$$\begin{aligned} &V((k_1, j_1), \dots, (k_w, j_w), r) \\ &\quad := \big\{(x, t, z_1, \dots, z_w) \in X \times \mathbb{C} \times \mathbb{C}^w : j = 1, \dots, w;\ x_r t = 1; \\ &\qquad\qquad M_{I_p(k_p, j_p)} z_p = 1;\ p = 1, \dots, w\big\}. \end{aligned}$$

Now we can consider a morphism

$$\Psi((k_1, j_1), \dots, (k_w, j_w))\colon V((k_1, j_1), \dots, (k_p, j_p), r) \to \mathbb{C}^m \times \mathbb{C}^N$$

defined by

$$\begin{aligned} (x, t, z) \to \big(&f(x), z_1 M_{I_1}(x), x_1 z_1 M_{I_1}(x), \dots, x_n z_1 M_{I_1}(x), \\ &\dots, z_p M_{I_w}(x), x_1 z_w M_{I_w}(x), \dots, x_n z_w M_{I_w}(x), t\big). \end{aligned}$$

Denote $\Psi((k_1, j_1), \dots, (k_w, j_w), r) := (\psi_1(x, z), \dots, \psi_{m+N}(x, z))$. It is easy to see that

$$\Gamma((k_1, j_1), \dots, (k_w, j_w), r)$$

is the closure of

$$\Psi((k_1, j_1), \dots, (k_w, j_w), q)(V((k_1, j_1), \dots, (k_w, j_w)), r).$$

Let $G((k_1, j_1), \dots, (k_w, j_w), r) = \mathrm{graph}(\Psi((k_1, j_1), \dots, (k_w, j_w), r))$. A basis of the ideal I of the set $G((k_1, j_1), \dots, (k_w, j_w), r)$ in the ring $\mathbb{C}[x_1, \dots, x_n, x_{n+1}, t, z_1, \dots, z_w; y_1, \dots, y_{m+N}]$ is given by the polynomials

$$\begin{aligned} \{b_j : j = 1, \dots, w\} \cup \{z_r M_{I_r(k_r, j_r)}(x) - 1 : r = 1, \dots, s\} \cup \{t x_r - 1\} \\ \cup\ \{y_i - \psi_i(x, z) : i = 1, \dots, m + N\}. \end{aligned}$$

To compute a basis $\mathcal{B}((k_1, j_1), \dots, (k_w, j_w), r)$ of the ideal of the set $\mathrm{cl}(\Gamma((k_1, j_1), \dots, (k_w, j_w), r)$, it is enough to compute a Gröbner basis $\mathcal{A}((k_1, j_1), \dots, (k_w, j_w), r)$ of the ideal I in $\mathbb{C}[x, t, z, y]$ with respect to the lexicographic order in which $y < x, t, z$ (see, e.g., [14]) and then to take

$$\mathcal{B}((k_1, j_1), \dots, (k_w, j_w), r) = \mathcal{A}((k_1, j_1), \dots, (k_w, j_w), r) \cap \mathbb{C}[y_1, \dots, y_{m+N}].$$

Consequently,

$$K_\infty(f, X) = \bigcup_{((k_1,j_1),\dots,(k_w,j_w)),r} \big\{y \in \mathbb{C}^m : h(y, 0, \dots, 0) = 0 \\ \text{for every } h \in \mathcal{B}((k_1, j_1), \dots, (k_w, j_w), r)\big\}.$$

The computation of the set $K_0(f, X)$ is standard. Let $I(X) = (b_1, \dots, b_q)$. Consider the set

$$U := \big\{x \in \mathbb{C}^{n+1} : b_j = 0,\ j = 1, \dots, w;\ M_{I_r} = 0;\ r = 1, \dots, s\big\}.$$

Now we can consider a morphism $f : U \to \mathbb{C}^m$. We have $K_0(f, X) = \overline{f(U)}$. Let Γ be a graph of $f|_U$ and $I = I(\Gamma)$. A basis of the ideal I is given by the polynomials

$$\{b_j : j = 1, \dots, w; \} \cup \{M_{I_r}(x) : r = 1, \dots, p\} \cup \{y_i - f_i : i = 1, \dots, m\}.$$

To compute a basis $\mathcal{B}$ of the ideal I it is enough to compute a Gröbner basis $\mathcal{A}$ of the ideal I in $\mathbb{C}[x_1, \dots, x_{n+1}; y_1, \dots, y_m]$ and then to take

$$\mathcal{B} = \mathcal{A} \cap \mathbb{C}[y_1, \dots, y_m].$$

Consequently, $K_0(f, X) = \bigcup\{y \in \mathbb{C}^m : h(y, 0, \dots, 0) = 0 \text{ for every } h \in \mathcal{B}\}$.

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