The Journal of Educational Research

ISSN: 0022-0671 (Print) 1940-0675 (Online) Journal homepage: http://www.tandfonline.com/loi/vjer20

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To cite this article: Mitchell J. Nathan, Matthew Wolfgram, Rachaya Srisurichan, Candace Walkington & Martha W. Alibali (2017) Threading mathematics through symbols, sketches, software, silicon, and wood: Teachers produce and maintain cohesion to support STEM integration, The Journal of Educational Research, 110:3, 272-293

To link to this article: http://dx.doi.org/10.1080/00220671.2017.1287046

Published online: 21 Mar 2017.

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THE JOURNAL OF EDUCATIONAL RESEARCH
2017, VOL. 110, NO. 3, 272–293
http://dx.doi.org/10.1080/00220671.2017.1287046

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Threading mathematics through symbols, sketches, software, silicon, and wood: Teachers produce and maintain cohesion to support STEM integration

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ABSTRACT

This classroom-based investigation sought to document how, in real time, STEM teachers and students attempt to locate the invariant mathematical relations that are threaded through the range of activities and representations in these classes, and how highlighting this common thread influences student participation and learning. The authors conducted multimodal discourse analyses of teacher–student interactions during multiday observations in 3 urban high school STEM classes. The focal lessons were in electrical engineering and mechanical engineering (within Project Lead the Way), and precollege geometry. Across 3 cases, teachers and students actively built and maintained cohesion of invariant mathematical relations across activities and representations. Pre- and postlesson interviews revealed that teachers intentionally managed cohesion to provide the continuity across the curricular activities that teachers believed would promote student understanding. The findings contribute to ways of fostering STEM integration and ways of grounding abstractions to promote meaning making and transfer.

ARTICLE HISTORY
Received 30 May 2016
Revised 20 December 2016
Accepted 20 January 2016

KEYWORDS
Career and technical education; coordination, digital electronics; high school geometry; instruction; integrated STEM; precollege engineering; projection

In response to the clarion call for effective science, technology, engineering, and mathematics (STEM) education, policy makers in and outside of education seek to foster approaches to integrated STEM instruction that more accurately reflect societal needs than do the traditional silos of individual STEM fields (Honey, Pearson, & Schweingruber, 2014; National Research Council, 2011). As noted in the National Research Council/National Academy of Engineering report, "STEM Integration in K–12 Education: Status, Prospects, and an Agenda for Research" (Honey et al., 2014), students' abilities to perceive, produce, and manipulate concepts across different contexts and disciplines are fundamental to their acquisition and application of integrated STEM knowledge. However, students seldom spontaneously make such deep connections (Honey et al., 2014; Katehi, Person, & Feder, 2009). Consequently, STEM integration often depends on instruction that promotes students' making connections among concepts and practices in science, technology, engineering, and mathematics, while also contributing to knowledge and skill in the individual STEM disciplines (Nathan et al., 2013).

In service of integrated STEM education, students encounter a wide range of ideas and practices in project-based classrooms (Barron, 1998; Blumfeld et al., 1991). These include specialized vocabulary and representational systems, such as symbolic equations and diagrams; digital media, such as software simulations and electronic circuits; raw materials such as silicon, metal, plastic, and wood; and designed objects, tools, and measurement instruments. Furthermore, these varied encounters involve a host of activity settings, such as those that commonly occur in classroom lectures and demonstrations, computer lab work, small group work, wood and metal shops, and so forth. Learners operating in project-based STEM settings must recognize the relatedness of ideas across a broad range of material and representational forms and settings, and they must realize how concepts (e.g., a quadratic relation) encountered in one form (e.g., an equation) relate to those same concepts encountered elsewhere (e.g., the behavior of a ballistic device).

In this research we seek to (a) describe the challenges K–12 students face in producing cohesion of concepts across the many representations, material forms, and activity settings encountered during STEM lessons; and (b) document ways that cohesion of concepts is dynamically established during real-time interactions and instruction in STEM classrooms.

In this context, *cohesion* refers to the ways elements of the externalized, observable learning environment are connected to one another. Our use of the term is similar to its use by scholars of reading comprehension to refer to connections among ideas and symbols in texts (e.g., Graesser et al., 2004; McNamara, Louwerse, & Graesser, 2005). By extending cohesion to the analysis of teaching, learning, and the management of effective learning environments, we recognize cohesion as a mediator of STEM integration (Nathan et al., 2013) and we highlight ways that meaning-making occurs in authentic vocational and precollege STEM education settings.

This study makes several contributions to education research. Our central contribution is an analytic framework for the study of teaching as it takes place in real time within collaborative, project-based, resource-intensive learning environments that are

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organized around the monitoring, creation, and maintenance of cohesion. We present three cases of classroom learning and instruction—multiday lessons centered on mechanical engineering, electrical engineering, and geometric proof—to illustrate how this framework offers a valuable lens for analyzing the rich sociocognitive interactions that take place among students, teachers, and the varieties of technological resources and representations that are used in STEM settings. The cases share some common properties: geometry and digital electronics share a reliance on logic and proof, mechanical engineering and geometry share spatial reasoning and geometric construction, and digital electronics and mechanical engineering share technical education and engineering design. These cases provide the empirical basis of a comparative analysis that reveals both commonalities and variability in how cohesion is managed in STEM classrooms in real time. The comparison across cases also highlights common processes for fostering cohesion that go beyond specific STEM content and settings.

Establishing cohesion in the classroom: The *where* of mathematics

Many essential skills arise out of engagements with tools, materials, and other people, as well as with algorithms and inscriptions (Johri & Olds, 2011). Lave (1988) posited, "'Cognition' observed in everyday practice is distributed—stretched over, not divided among—mind, body, activity and culturally organized settings (which include other actors)" (p. 1). There are many challenges in perceiving and maintaining cohesion within one's complex, technical environment. Hutchins (1995), for example, provided an account of how representational states used during the computation of ship navigation are propagated by naval personnel across a range of physical and semiotic media. Stevens and Hall (1998) underscored the disciplining of a learner's visual perception when teaching the Cartesian coordinate system to "learn to see with and through their inscriptions" (p. 108). In these analyses of collaborative navigation and perception training with many forms of media, the key role of mathematical ideas in fostering a connected experience is noteworthy. Others (e.g., Fairweather, 2008; Prevost, Nathan, Stein, Tran, & Phelps, 2009) have shown that instruction and peer collaboration can at times foster the needed integration of mathematical ideas in STEM projects, providing cohesion when curricular materials do not.

We have identified three ways that transitions across activities, representations, and settings are managed by teachers and students to establish and maintain cohesion in project-based STEM classrooms. One is that participants are frequently asked to make an ecological shift, a reorientation of the activity context that can include different spaces, tools, instructional media, and participation structures. By changing the physical environment, spatial layout, and array of available resources and representations, ecological shifts alter social norms and practices. This can have profound consequences for the ways in which STEM concepts are enacted, tracked, and understood by students, often presenting challenges to teachers and students to establish cohesion and recognize the continuity of STEM concepts across contexts. Ecological shifts, though common, can make it challenging for participants to preserve cohesion in the learning environment.

A second way to manage cohesion is projection, the use of multimodal language to connect events in the present to past or future events. Projections to the past can link across an ecological shift that has already occurred, while projections to the future can anticipate the need to bridge a coming shift. Projections can take many forms (cf. Engle, 2006). Some are brief utterances, as when a teacher foreshadows future activities, or simple pointing gestures, as when a teacher points to an empty white board to reinvoke the mathematical derivations from a prior lesson. Teachers and students use projection to reflect upon the history of a concept as it unfolds in their classrooms, and to plan for, motivate, and bridge to future manifestations of the concept.

A third way to manage cohesion is coordination, the juxtaposing and linking of material and representational forms. Some scholars (e.g., diSessa, 2004; Hutchins, 1995; Stevens & Hall, 1998) have described coordination across agents, physical objects, and representations. For example, Nathan, Wolfgram, Srisurichan, and Alibali (2011) showed that high school digital electronics students initially exhibited difficulty mapping the electronic "chips" needed to wire a digital circuit to the corresponding symbolic expressions (i.e., Boolean algebra expressions of logical operations in which the values of the variables are true [1] and false [0]). When students used color-coded wires to visually coordinate chip locations for the inputs and outputs to the operations indicated by the symbolic expressions, this led to changes in the students' design and interpretation of the circuits, and subsequent improvement in their troubleshooting behaviors.

Coordination and projection often co-occur, as when a teacher makes a connection between a device in the immediate context and a previously encountered equation (Nathan et al., 2013). Projection and coordination serve to integrate STEM concepts over time and across ecological shifts, including shifts that occur across activity contexts, and across material and representational forms.

Identifying locally invariant relations: The what *of mathematics*

In addition to identifying where STEM concepts are located through projection and coordination, and describing transitions across ecological shifts, it is important to be able to say what teachers and learners must attend to across shifting social configurations, physical settings, and material and symbolic forms. Mathematics often serves as the underlying language of STEM. We focus here on invariant mathematical relations as the *what* of the mathematics that threads through STEM learning, because, despite changes in surface forms and notations, the underlying mathematical relationships are consistent and unchanging.

In framing the cohesion of STEM learning environments around the continuity of invariant mathematical relations, we recognize that measurements sufficiently localized in space and time provide mathematical experiences that only approximate invariant relations—much like assuming a flat Earth when constructing a house. Thus, even though many of these STEM

activities only approximate the associated mathematical relations, we posit that they instantiate relations that are locally invariant. We define locally invariant relations as the key, unchanging concepts shared among diverse notations and material forms. Learning to construct and identify locally invariant mathematical relations across representations and contexts is challenging for students, in part because of the abstract nature of mathematical content, but also because of the visual and semiotic complexity of the systems of representations employed.

STEM experts can reliably identify and produce locally invariant relations across representations and contexts (Gainsburg, 2006). Students can also learn to notice and produce locally invariant relations (Stevens & Hall, 1998). One way to do so is by using relation- and inference-preserving cognitive mechanisms, such as analogical mapping and conceptual metaphor (Gentner & Markman, 1997; Lakoff & Johnson, 1980). For example, in a case we describe in detail subsequently, in a unit on projectile motion in a high school engineering class, there was a need to characterize theta, the angle of ascent of a projectile, across a range of activities and forms (Figure 1). In the classroom, the teacher and the students worked to represent the invariant mathematical relation labeled theta in several ways: as a Greek symbol in and equation, as a numeric measure yielded by a sextant, as a tangent line meeting a plane in a geometric diagram, as the teacher's raised arm in his lecture, and as the relation between the trajectory of an object and the ground in a drawing. Starting with relations formalized in a trigonometric equation, students and teacher worked to constitute similar relations in devices built from wood and other materials, which—done correctly—materialized those relations in a working catapult, trebuchet, slingshot, or other ballistic

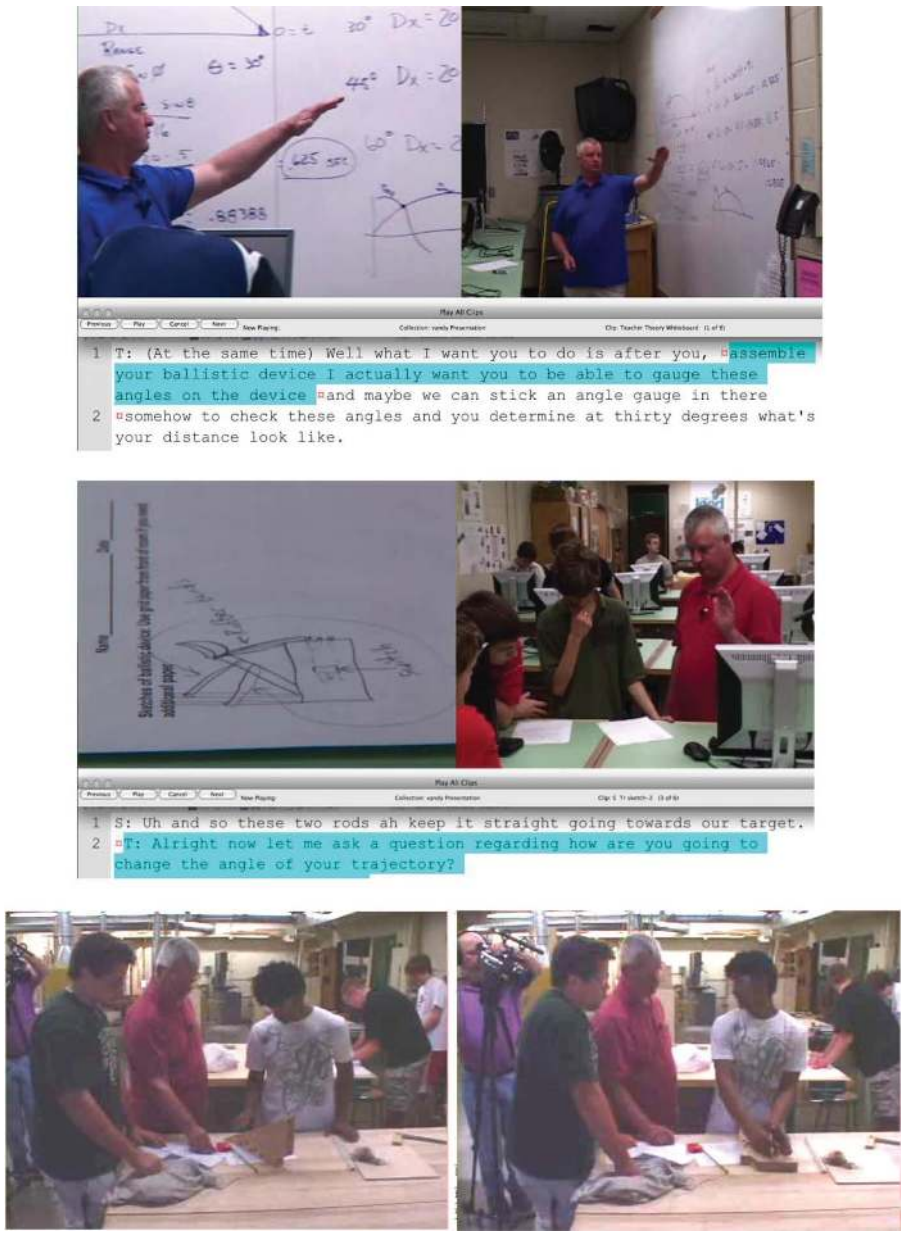


Figure 1. (a; top two panels) The mathematics and physics of kinematics that model ballistic motion must also be connected to (b; middle two panels) the two-dimensional design sketch, and (c; bottom two panels) the construction, testing, and redesign of the ballistic device. Note that the teacher attempts to connect the design sketch to the wood in the construction phase (bottom left), but the student focuses on the wood, to the exclusion of any cross-modal connections (bottom right).

device. In this way, class participants actively built and maintained cohesion of locally invariant relations across representations and shifting contexts, thereby enabling STEM integration within and across classroom settings.

Focus of research

Students' identification and tracking of locally invariant relations cannot be assumed, even when it is suggested by the classroom curriculum. Novices need to be socialized into perceiving the same invariants that are salient to experts (Stevens & Hall, 1998). Thus, we set out to show that cohesion of locally invariant relations across contexts is produced and maintained by the participants. We provide evidence that teachers' actions are at times intentionally directed at producing and maintaining cohesion. We posit that many features of STEM curriculum and instruction exist to highlight relations, with the goal of advancing students' perceptions of locally invariant properties so that they serve as a cohesive thread through activities, contexts, and representations.

Our analyses focus on ways teachers and students establish cohesion of locally invariant relations (the *what* of mathematics) as these relations are projected and coordinated across various modalities and ecological contexts (the *where* of mathematics). Thus, we ask: How can we describe the challenges STEM students face in recognizing cohesion in the classroom, and the instructional processes through which cohesion of ideas (locally invariant relations) is produced and maintained in STEM classrooms? Second, we explore how teachers intentionally design or regulate their instruction to foster greater cohesion. We ask: How are teachers' instructional moves shaped by the need to establish and maintain a cohesive STEM learning environment?

We explore these questions in three classroom cases. In the first two cases, mathematics concepts arise in project-based engineering within career and technical education (CTE) classes. In the third case, we investigate a technology-enhanced, precollege geometry classroom, and we discuss how cohesion is a considerable challenge even in non–project-based settings where mathematical abstraction (e.g., generating a theorem, rather than a designed product) is the ultimate goal. The utility of the concept of cohesion production across these cases provides compelling evidence for the applicability of our framework to describe the persistent challenges of integration in STEM learning environments. In the final section, we consider how this work informs two core issues: managing learning environments that foster STEM integration and grounding abstractions to promote comprehension and learning.

Theoretical framework: Cohesion production and maintenance

In these STEM units, invariant mathematical relations provide a connective thread through the activities and representations. We ask first how mathematical relations are realized in STEM classrooms. To address this question, we focus on both the *where* and the *what* of mathematics as it is stretched over the varieties of ecological shifts, notations, and objects common to collaborative, project-based activities. In addressing the *where* of mathematics, we examine a broad range of student and teacher behaviors as we consider how locally invariant mathematical relations are realized in forms such as project artifacts and symbol systems as they are encountered across time and learning spaces (Hall & Nemirovsky, 2012).

We also address the *what* of mathematics by exploring how mathematical relations can be preserved when they are manifest in markedly different ways. Our analysis foregrounds the mathematical relations that are deemed by STEM experts to be locally invariant, regardless of their changing outward forms. This investigation raises important questions about how mathematical notation both facilitates and obfuscates the integration of concepts for learners across scientific fields and across phases of project-based learning activities.

To foreshadow, we find that these connections are often only tacitly manifest in the lessons and not readily apparent to students. Consequently, the teacher and the students must continually manage and negotiate cohesion. To do so, they rely heavily on language, gesture, and other forms of verbal and nonverbal scaffolding. Speech provides cohesion using resources such as labels and explanations. However, simply referring to mathematical ideas using consistent labels across different contexts is not sufficient for most students to establish the cohesion necessary to understand how the mathematics permeates the various activities and representations. Along with speech, teachers also frequently use gestures to establish and maintain cohesion. Gestures provide cohesion by connecting related ideas or visual representations (Alibali & Nathan, 2012; Alibali et al., 2014; McNeill & Duncan, 2000; Nathan, 2008; Williams, 2012). Teachers also use other forms of visual signaling (Ozogul, Reisslein, & Johnson, 2011), including written inscriptions such as equations, diagrams and words that reify concepts, and relationships and plans in a manner that is (relatively) enduring and that highlights connections.

We also examine how teachers' instructional moves are shaped by their perceived need to establish cohesion in the learning environment in service of STEM integration. We explore teachers' intentional uses of speech, gesture, and body-based and environmental resources, both for establishing cohesion and for identifying breaks in cohesion that can be disruptive for students. We draw on pre- and postlesson interviews to understand teachers' efforts to focus attention on invariant relations and to provide cohesion that may enable students to thread relations through their classroom experiences, yielding a deep, integrated understanding of fundamental ideas.

Method

Participants and research sites

To document the constituent processes used to establish connections between and among STEM content and representations, we conducted multiday observations and collected dual-camera videos of teacher-student interactions in three urban high school classes with lessons focused on electrical engineering (Elm High School [EHS]), mechanical engineering (Maple High School [MHS]), and technology-rich geometry (Lake High School [LHS]). The three schools are located in a mid-sized city in the American Midwest. The schools and classrooms at EHS

and LHS exhibit a typical urban and semiurban demographic profile, with high racial diversity (with a Caucasian student majority and African-American, Hispanic, and Hmong student minorities), and a high rate of students qualifying for free and reduced-price lunch. MHS has less racial and socioeconomic diversity. The school site demographics are shown in Table 1. The geometry classroom observed in LHS was half girls, while the engineering-technology class at MHS had one female student, and the EHS classroom had two female students in the digital electronics course.

Data collection

Our observations are drawn from one classroom from each school site. We observed precollege engineering classes from the Project Lead the Way (PLTW) course sequence in EHS and MHS. At MHS, we observed a four-day unit on mechanical engineering (ballistics devices) during the PLTW class named Principles of Engineering, which ran over four 60-min class sessions. At EHS, we observed a four-day unit on electrical engineering (logic circuit design) during the PLTW class named Digital Electronics, which ran over four 60-min class sessions. Our observations at LHS were of a three-day precollege honors-level geometry unit on proof and properties of angles, which ran over three 90-min class sessions.

We coordinated our video and audio recording with our research questions, following recommendations by Roschelle (2000). Our research focus was on the role of teacher-student communication during STEM learning and instruction. We coordinated our placement and framing of the two video cameras and microphones accordingly. One camcorder-and-microphone system continuously tracked the teacher's movements around the classroom, such as during group help and classroom lectures and demonstrations. The other camcorder was focused with a wide frame of the classroom as a whole. To further contextualize our classroom observations, we conducted teacher interviews before and after each classroom observation, to clarify the a priori purposes of each lesson, and to debrief about the classroom observations with the classroom instructor.

Data coding and analysis

Lesson videos were transcribed in Transana (Woods & Dempster, 2011), a platform that allows for integrated viewing of multiple audio and video feeds (i.e., multiple camera angles) and transcripts. Transcripts and their associated video were segmented into event-focused episodes following the method described by van Dijk (1982). We formed a video clip from each episode as our primary unit of analysis. Such episodes in the classroom discourse encompass a portion of the transcript that conveys one consistent idea unit (Chi, 1997; Ericsson & Simon, 1980) with a single activity-centered goal (e.g., Lave & Wenger, 1991). Episodes often included a social-spatial coordination of participants and co-present tools, written inscriptions, and other material forms. Elsewhere (Nathan et al., 2013; Walkington, Nathan, Wolfgram, Alibali, & Srisurichan, 2014), these kinds of episodes of social-spatial co-orientation toward the tools, inscriptions, and material forms of an activity have been referred to as modal engagements.

A multidisciplinary research team, including team members who directly observed the classroom events, met regularly to code the video transcripts in Transana. Our coding of the video transcript data involved creating categories designed summarize the events and ideas, and help identify patterns of interconnections in the data (Saldaña, 2015).

Following the analytical tradition of multimodal discourse analysis (Erickson, 2004; O'Halloran, 2011), through multiple iterations with the video data, we created increasingly refined and detailed transcription annotations of features of the discourse, visual-spatial activities, and gesture and bodily communication. We interpreted the episodes in relation to the social context of the participants (Gumperz, 1982) and other features of the emergent context of the interaction (Kendon, 1990). Through this procedure of iterations of description and analysis with selected segments of video data, we identified teachers' use of projection and coordination to manage mathematical relations and ecological shifts, and we discerned the importance of locally invariant mathematical relations (Table 2).

Each video clip was coded with the material and representational forms that arose during the interaction (e.g., a sketch, a gesture, an interactive shape in Geometer's Sketchpad [GSP; McGraw-Hill Education, Columbus, OH, USA]), as well as the locally invariant relations being discussed (e.g., properties of inscribed angles). Clips were also coded with the ecological context (physical and social setting) in which they occurred. We then applied these and other codes to the larger corpus of video data, which allowed us to establish whether the features

Table 1. Demographics and percentages for each school sites (EHS, MHS, and LHS).

	EHS Electrical	MHS Mechanical	LHS Geometry
Enrollment	1,600	1,900	1,700
White (Non-Hispanic Caucasian)	39%	51%	41%
African-American	24%	17%	23%
Hispanic	18%	13%	21%
Asian	9%	13%	5%
Native American	0%	0%	1%
Two or more races	9%	6%	10%
Free/reduced-price lunch	57%	25%	58%

Note. Due to rounding, these numbers may not sum precisely to 100%. EHS = Elm High School; LHS = Lake High School; MHS = Maple High School.

Table 2. Coding criteria for the production and maintenance of cohesion.

Transition	Coding criteria
Ecological Shift	Evidence of a major reorientation of classroom activity such that understanding of invariant relations to involves different physical settings, social participation structures, and material and representational forms, tools, or actions.
Projection	Evidence that participants refer to an absent (past, planned, or imagined) material or representational form.
Coordination	Evidence that participants link two or more copresent material or representational forms.
Projection + coordination	Evidence that participants make a projection to an absent material or representational form while also linking it to a currently present material or representational form.

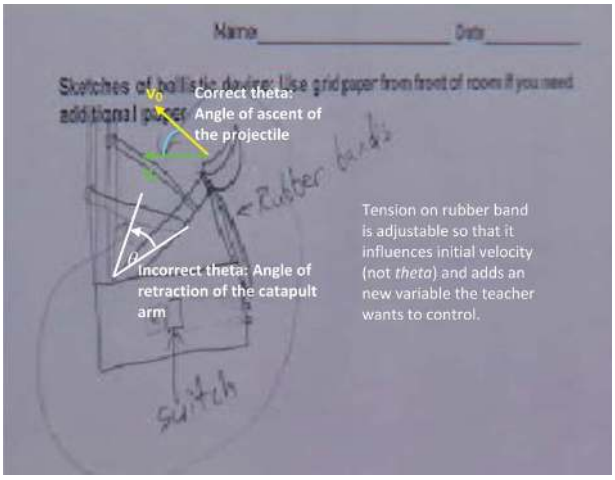


Figure 2. A group design sketch (with verbal and mathematical elaborations added).

of the discourse were recurrent, whether they were manifest across different days and STEM contexts, and whether we had captured adequate variation in the behavior (Saldaña, 2015; the results of this coding of the larger corpus are reported in Nathan et al., 2013). Clips showing projection and coordination were further transcribed to include detailed descriptions of teacher and student gesture use. The series of clips for each multiday unit was sequenced and visually mapped (Figures 3, 5, and 8) to illustrate how invariant relations become threaded through different ecological contexts via the cohesion mechanisms of projection and coordination.

Transcripts from the videos of teacher interviews were analyzed to examine instances in which the teacher discussed the use of projection and coordination, or instances in which the teacher reflected on a classroom episode that involved projection or coordination. These instances were then examined in the context of the lesson being described to provide additional insight into the classroom interactions.

Findings from three case studies

The three classrooms that we describe each illustrate how teachers and students thread concepts through rich ecological contexts to produce and maintain cohesion in complex, project-based STEM lessons. The cases show students designing ballistics devices, debugging digital circuits, and exploring general mathematical properties of circles. We present descriptions of the classrooms along with transcript excerpts, and we discuss connections to our theoretical framework.

Case 1. Theta in symbols, paper, and wood: Engineering a ballistic device

The ballistics project challenge

> "[W]hatever distance that is I'm gonna decide that at the time, we're gonna set the basket so many feet away and you have to try to hit it. So by doing some calculations on, what your, um, ballistic device fires you can kinda set your angle hopefully to get, to get that distance." (Principles of engineering teacher, Day 1).

On Day 1 of this lesson, students in a second-year, precollege engineering course were presented with the mathematics and physics of calculating projectile motion using the laws of kinematics and trigonometric relations. The teacher highlighted the angle of ascent of the projectile—which he labeled theta—as the key variable that students needed to represent in their design sketches (Figure 2), to parameterize, and ultimately to build—using wood, metal, plastic, and other materials—into catapults, trebuchets, guns, or other ballistic devices. When a device properly parameterizes theta, and permits adjusting the angle of release while holding other influential variables (e.g., initial velocity) constant, students using the laws of kinematics are able to predictably modify the distance that the projectile will travel to hit its target.

This case illustrates how easily students—even those who are deeply engaged—can lose sight of the central mathematical concept, and how this results in a breakdown that leads to weak STEM integration and poor engineering design. The events depicted in the transcript below took place on Day 2, after a group of students presented their sketch of a catapult to the teacher. The discussion is sandwiched between the teacher's formal lecture on kinematics (including algebra, trigonometry, and the idealized behavior of an object in free fall with a constant horizontal velocity) and the group-led material construction activity. This discussion includes projections toward future contexts where the students will use their sketches to guide construction, as well as projections back to the earlier lecture on kinematics.

Over the course of the discussion, a breakdown of cohesion of the concept of theta is apparent: students have confused the angle of ascent of the projectile with the angle of retraction of the arm of the catapult (Figure 2). The students' design sketch misidentifies the relevant angle. The teacher points out this confusion in Lines 1 and 3 (see Case 1 Transcript, Figure 3), saying, "I'm wondering if the further you, pull your rubber band down, is gonna affect your, velocity, more than your angle." In Lines 4–8, the students try to salvage their sketch, focusing on the placement of the moveable arm. Based on their constrained use of gestures and eye gaze, and restricted references in their speech, it appears the students are focused almost exclusively on the sketch itself, to the near exclusion of the invariant mathematical relations used to model the object's ballistic behavior.

In Line 9, the teacher explains that students need to control the angle of ascent of the projectile with respect to the (horizontal) ground, stating, "you might be able to adjust your angle by, by having some type, by controlling where this [arm] stops." In Lines 12–15, a student defends the group's choice to focus instead on the placement of the fulcrum, stating, "cuz the two sides stay put but then the top part can, tilt… right there. So the fulcrum can change positions basically." With this utterance, the student further confirms that the group is not considering the distance traveled as a function of angle of ascent, but rather is focusing on the placement of the fulcrum.

In lines 16–23, the teacher coordinates and projects the students' design sketch backward in time to the mathematical relations presented the day before on the whiteboard (which are still present in the front of the room), emphasizing that students

1 T: Well I'm wondering if the further you, pull your rubber band down-

2 S Mm-hm.

3 T: is gonna affect your, velocity, more than your angle.

((Teacher points to diagram))

4 S: Yeah it's. Well no this is the velocity but what we're sayin' is that this is how hard it pulls

((Student points fingers at different parts of diagram))

5 S: but then right here, where it where it…

((Student makes flat-hand gesture on top of diagram))

6 S: where the fulcrum is like this actually you can tilt it.

((Student makes arm into lever))

7 S: The rubber bands control the tension but the placement is what really controls...

8 S: Like. See what we're saying?

9 T: So it's it okay so, if I could, suggest, I think that, you might be able to adjust your angle by, by having some type, by controlling where this stops.

((Teacher positions flat hand over diagram, and moves fingertips up and down while keeping base of hand stationary))

10 S: Yeah.

11 T: But that's probably also gonna affect your, maybe affect your velocity. What I'm saying is. Either that or else you have to tip the whole thing.

((Teacher places flat hand over diagram, this time moving both ends of his arm back and forth))

12 S: No we don't. That's why cuz the two sides stay put but then the top part can, tilt…

((Student places to flat hands parallel to each other, and then places flat hand perpendicular to other hand, tilting palm upwards and downards))

Figure 3. Case #1, Transcript 1. (NB. Speech transcript is complete but only gestures relevant to this analysis are shown. Square boxes denote the start and end of the gesture, red arrows indicate direction of movement, and green arrows indicate the location of pointing.)

13 S: right there.

((Student points to diagram))

14 T: Okay.

15 S: So the fulcrum can change positions basically.

((Student traces back and forth on table with finger))

16 T: Alright. So I think maybe what you need to do is is, take into consideration what I just said about-

17 S: Yeah.

18 T: being able to control the ang- …

((Teacher moves flat palm back and forth in the air))

19 T: that's why we did everything we did here

((Teacher points to board))

20 S: Mm-hm.

21 T: -with the math. Because we wanna-

22 S: the math yeah.

23 T: -be able to adjust the angle of the trajectory. I would try to keep, the velocity, the same, consistent, throughout the whole every test that you do that that's consistent and so all you're gonna change once you once you decide what that velocity has to be all you're gonna change is your angle.

24 S: Yeah.

25 T: Okay?

26 S: Mm-hm.

27 T: I don't really want you to use the tension on the rubber bands, as, the only control. I want you to have an angle adjustment.

Figure 3. (*Continued*)

need to be able to control the angle, and stating (using backward projection), "that's why we did everything we did here (pointing to board) with the math." The teacher also projects forward to the future behavior of the yet-to-be-realized device: "Because we wanna be able to adjust the angle of the trajectory" He coordinates and projects with coexpressive speech and gesture. The first gesture on Line 18 is a flat-palmed hand lined up horizontally with the diagram, iconically representing theta as an angle relating the initial trajectory of the projectile to the ground, though translated into the plane of the paper. This hand shape reinvokes a similar gesture that he used during the lecture on the mathematics of projectile motion, thus making a gestural catchment in

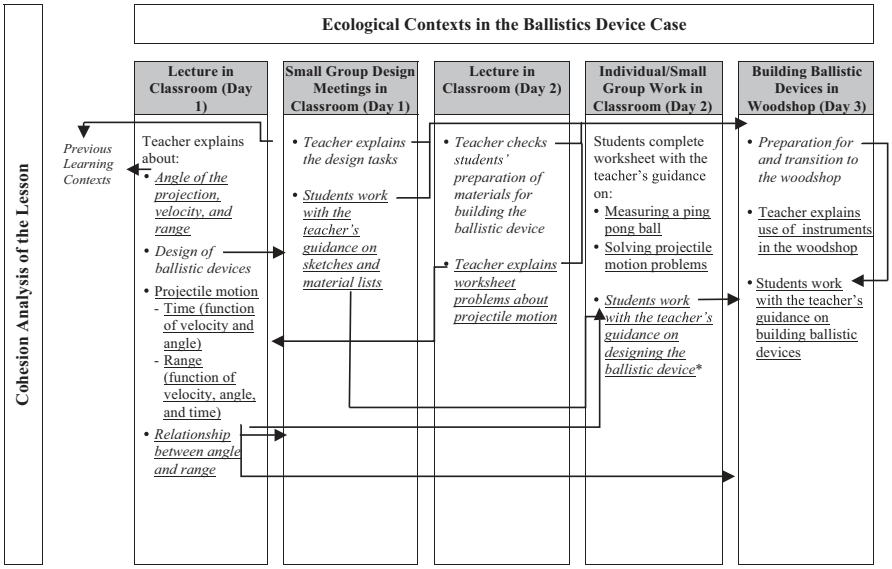


Figure 4. Case 1 cohesion analysis of the ballistic device lesson. Boxes show the major ecological contexts in which activity was embedded throughout the case. Bullets show the main modalities in the lesson occurring sequentially in the case. Italics indicates projection, underline indicates coordination, and italics and underline indicates projection + coordination. Arrows show the main backward or forward projections with wedge-shaped ends pointing to projected past or future modal form(s). The asterisk indicates the modalities discussed in the transcript of the ballistic device case. Modalities are nested within ecological contexts. Columns show the various ecological contexts in which the activities were embedded throughout the project. Arrows in the figure illustrate the roles of projection (italicized text) and coordination (underlined text) in the management of ecological shifts. The figure shows how the teacher often used backward (left arrow) and forward (right arrow) projections together to bridge present modal forms (bulleted entries within each context), such as working with design sketches, to those in the past (e.g., the physics and mathematics of projectile motion) and those that will be used in the future. The figure also indicates more frequent use of forward projection as the teacher prepared students to build and test the ballistic device. The enactment of coordination and project are insufficient, however, if these are not attended to or taken up by learners.

which a speaker repeats gestural features, such as hand shape or motion, to establish cohesion in discourse (McNeill & Duncan, 2000). Through the second gesture on Line 19 the teacher coordinates the calculations on the whiteboard with the students' diagram in an effort to locate theta in the design sketch and reinstate its original meaning. The students are fixated on their own work and give little attention to either the iconic angle gesture or the overt point to the whiteboard. Then, for the first time during the discussion, one of the students (line 22) acknowledges the relevance of the mathematical relations for their design, saying "the math yeah." Even so, little was taken up by the students, and their design remained largely unchanged.

We reflect on this case in the language of our theoretical framework. We identify the locally invariant relation (the *what* of mathematics) as the angle of ascent of a projectile with respect to the ground, represented initially by the symbol *theta*, and the role it plays in predicting the distance traveled. We describe the *where* of mathematics in terms of ecological shifts and transitions between modal forms, which are the various material and representational forms in which a concept or idea is manifested. To foster cohesion, the teacher threads the mathematics through the various symbols, drawings, and objects using speech and gesture to coordinate the angle theta and its meaning for projectile motion with elements of the design sketch. The teacher uses temporal projection to signal the historical role of the design sketch. He makes backward projections to the mathematical formalisms that model projectile motion, which he presented in the previous class.

Figure 4 illustrates the cohesion analysis, showing how the sequence of events in the ballistics project was coded for ecological shifts, modalities, and transitions, revealing the temporal and hierarchical structure of the lessons. Following the lecture on Day 1, the teacher often used projections to bridge present modal forms, such as the design sketches, to those in the past (e.g., the physics and mathematics of projectile motion), and those that will be used in the future (e.g., the target competition between teams). However, students' fixations with their own work during crucial moments thwarted the teacher's efforts to produce cohesion. The teacher employs coordination, often with projection, to bolster cohesion by identifying the invariant relation in students' own sketch during the Day 1 small group session, and relating the design sketch to the forthcoming session in the wood shop.

In summary, this first case illustrates challenges to fostering cohesion in the engineering education environment. The post-lesson interview from Day 2 provides evidence that this teacher was sensitive to the challenges students faced in grasping cohesion across modal forms.

> [I]n some cases [the designs] were too simple and, and not very complete in that sense that they didn't really indicate to me what the, how they would do that. How are they gonna change their angle. How are they gonna sh- show what the angles are. So I was looking for something and I related it to the fact that, you know, we talked about it yesterday that we were going to have to propel this ball to a certain distance… [I]n reference to the work we did yesterday, they could– we could see that the angle of the trajectory is going to affect the distance that they are able to shoot it.

The teacher was quite explicit about the needed STEM integration—the connections students needed to make between the mathematics and physics presented the previous day and the design sketch (Lines 16–19). In critiquing students' designs, the teacher wanted them to recognize that a seemingly unimportant aspect of their sketch was in fact the key parameter of the design. In this way, the sketches did not merely resemble the

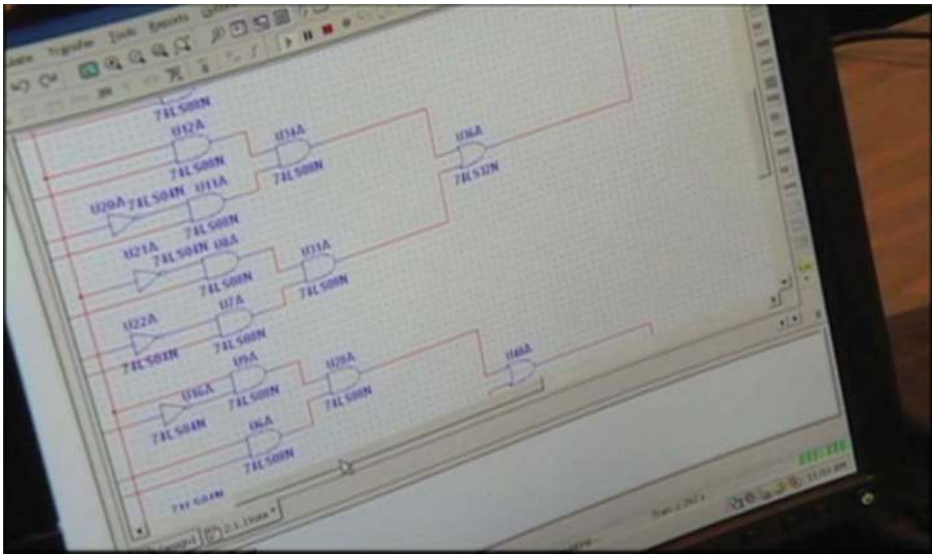


Figure 5. A screenshot of a sample MultiSim diagram. Chip model numbers are used to label the various logic gates in the digital schematic.

devices—they also modeled central kinematic relations of a working ballistics device.

Case 2. Logic enacted through Boolean algebra, simulation, and silicon: Designing a secure digital voting booth

The main activity of this digital electronics lesson was to design a voting booth privacy monitoring system. An effective monitoring circuit is indicated by two outputs: a green light-emitting diode (LED) that lights up whenever a particular voting booth is available for use and a red LED that lights up whenever privacy is at risk and entry is denied.

The circuit design involved implementing the basic set of logical constraints and conditions into a working electronic circuit that outputs a green light when all of the conditions are met or a red light (alarm) when any condition is violated. The activity also involved using Boolean algebraic expressions of logical operations in which the values of the variables are true (1) and false (0), representing green and red in the circuit, respectively. The process unfolds across the following activities: verbally introducing the problem, along with an equipment list and a block diagram representing the monitoring system; discussing a completed truth table with entries composed of ones and zeros accounting for all possible states of the circuit (voting booth occupancy and LED output), and a related, spatial Karnaugh map (K-map, which is a diagram that uses arrays of squares to represent the Boolean algebraic expressions,); generating and manipulating a set of Boolean algebraic expressions consistent with the K-map; drawing an AOI circuit (which is composed only of AND, OR, and Inverter/NOT gates or logical operations); modeling the circuit in the MultiSims software (National Instruments Corporation, Austin, TX, USA) to create computer generated circuit diagrams showing the flow of information through the circuit (Figure 5); and building and debugging a working electronic circuit made of a breadboard (i.e., base), integrated circuits, resistors and capacitors, wires, a power source, and colored LEDs. The mathematics in this lesson, Boolean logic, is manifest in symbolic and physical forms with instructional contexts, representations, and a range of materials traversed by ecological shifts. The teacher sought to establish cohesion between the relations modeled by the Boolean algebra and different forms of materials and representations by using coordination with both past and future projections.

To use the integrated circuits as a source for specific Boolean logic operations (e.g., AND, NOT, OR), the teacher and students consulted a data sheet, which was a set of documents affixed on a poster board. This printed material illustrated the formal specifications of different logic gates and their layouts in each integrated circuit, which varied by manufacturer. The data sheets established the connections between idealized logic symbols for Boolean operations, and the actual locations of the inputs and outputs of specific integrated circuit components. Furthermore, the computer based MultiSim diagrams (Figure 5) are tied to and constrained by the layout of the specific chips as determined by the manufacturer. The MultiSim diagram sets spatial locations for each of the logical operations and shows the paths of information flow, with input-to-output relations generally moving from left to right.

The following transcript is an excerpt from Day 4 of the lesson in which the teacher initiates an ecological shift (line 1, Figure 6) by calling students to gather at the lab station of the student group that had made the most progress on their voting booth monitor. The class then witnessed the conversation between the teacher and a student about checking the circuit for accuracy and discussing how the circuit could be improved. At the end of Line 1 the teacher forecasts that he has opinions about the organization of the wires in the board, saying, “I want to make some comments about the board.” Anticipating this (line 2), the student acknowledges, “it’s messy. I get it.” Regardless, in lines 3–11 the teacher addresses this practical matter that is not evident in the symbolic or simulation-based representations—the need for an orderly and “clean” wiring job. He makes this point with statements such as “to try to solve problems and you got stuff running all over it’s much harder to do” (Line 7), and “it’s hard just look at this as a whole picture, it’s just the spaghetti mess” (line 15).

In line 16 the teacher models how to coordinate the physical arrangement of wires, integrated circuits ("chips"), and electronic components, using speech and gesture, with the simulated circuit shown in the information flow (MultiSim) diagram. In explaining this coordination, he states, "If I know that you want to do something I can look [*pointing to the SIM diagram*]. Look at the number and say oop you're in Hole 2 when it should be Hole 3. You just put it in the wrong hole and that's your troubleshooting. That's a checklist by putting it on here."

In Line 17, the student takes up the teacher's troubleshooting practice. However, the student works from the truth table ("we'll just do it with this thing") rather than the MultiSim diagram. Practically speaking, this allows the student to turn each input switch on and off to model the occupancy state of the voting booth (ON = occupied, OFF = vacant). By mapping entries in the truth table directly to the circuit, the student bypasses the conceptual connection of the circuit to the Boolean expressions that are central to the SIM diagram and to the original problem context.

The dialogue from Lines 18–24 shows the teacher's troubleshooting method. The teacher establishes the meaning of the symbolic table entries by coordinating between the table and the circuit. He models how each entry in the truth table maps directly to a physical state of the circuit, running his finger from one row of the table to the next. Initially (Line 22), the

1 T: Guys everybody stop come over here 'cause we're gonna stop here and then we're gonna do an exercise, up in the front. So but I need you everybody stop what you're doing leave it come over here. (Name) come on. Okay. It says ninety percent working but I want to make some comments about the board.

2 Steven: Yeah it's messy. I get it.

3 T: I don't have to make comments about the board you just did it.

4 Steven: Yeah.

5 T: Right? What's uh the term I'm always giving you is spaghetti.

6 Steven: Spaghetti.

7 T: To try to solve problems and you got stuff running all over it's much harder to do but I'm glad for the most part you've got it working. So just demonstrate to me that what you've got working but you need to put your wires-

8 Steven: I just need-

9 T: -so they-

10 Steven: -to put the switch.

11 T: -they're not at angles try to get them all square so you can follow a path, laying right next to each other. Nothing goes over switches, nothing goes over the integrated circuits, get 'em straight, and if you got a long wire and you've got to make a bubble out of it shorten the wire. And I'm always saying if you have like these here are going at angles those could have been shortened straightened out. Kay.

12 Steven: Oh yeah.

Figure 6. Case #2 Transcript 2. (*NB. Speech transcript is complete but only gestures relevant to this analysis are shown. Square boxes denote the start and end of the gesture, red arrows indicate direction of movement, and green arrows indicate the location of pointing.*)

13 T: And on your paperwork when you're doing the check, you have numbers and letters here. What hole is that in? There's a l- number and a letter. Use 'em

((Teacher points to two ends of breadboard))

14 T: And to check things off. Write 'em right on here. I did this one, this one's hooked up, go to the next one, look, put the number on here.

((Teacher points to SIM diagram))

15 T: You know 1A. You know is it 10B. What are the things plugged into? Well that's your checklist. Otherwise it's hard just look at this as a whole picture, it's just the spaghetti mess.

((Teacher points to position on breadboard, then indicates SIM diagram))

16 T: But uh now I can follow this. If I know that you want to do something I can look. Look at the number and say oop you're in Hole 2 when it should be Hole 3. You just put it in the wrong hole and that's your troubleshooting. That's a checklist by putting it on here. Alright go through and show me what does work.

((Teacher points to SIM diagram and then points to breadboard))

17 Steven: Oh okay, we'll uh we'll just go with this thing.

((Steven points to truth table))

18 T: Alright.

19 Steven: Okay uh.

20 T: So we...

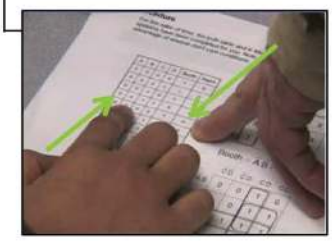
21 Steven: …Booth alarm all of this is…

Figure 6. (*Continued*)

((Steven points to one end of breadboard))!

22 Steven: (at the same time) off right now?

T: (at the same time) Kay so you're doing zero zero zero.



((Steven flips switches on breadboard and points to truth table with two different hands, while teacher points to truth table only. These gestures continue through the end of line 28))!

23 Steven: So zero zero zero booth is on alarm is off.

24 T: Okay.

25 Steven: Zero zero one booth is on alarm is off. Uh zero zero one (at the same time)...

T: (at the same time) Zero one.

26 Steven: zero booth alarm same thing. This too so the green one should come on here and it does and the red one doesn't matter.

27 T: Yeah.!

28 Steven: Uhhh (pause) yeah green one's okay, so so far it works. Oh see that one's the one that oh so the green one doesn't work but the red one works for that one over there. I'll keep that a mental note okay.

29 Steven: Uh so these two doesn't work uh the third one eh this one works. The thir- that one works. Uhhh (pause) okay this one should be off but yeah. Uh this one works. That one works. That one works. This uh that one works. That one works. And that one works.

((Teacher removes finger from truth table and Steven continues flipping switches on breadboard and pointing to truth table throughout Line 29))

30 Steven: So there's three doesn't work.

Figure 6. (*Continued*)

teacher calls out the circuit inputs, while the student sets the switches appropriately ("zero, zero, zero" means all of the switches are in the OFF position because none of the booths are occupied); thus, the coordination occurs across participants in the interaction. The student echoes the teacher in Line 23, reporting on the state of the input switches ("So zero zero zero booth is on"), and then describes the output ("alarm is off," indicating that the red alarm LED is off and entry is permitted). By Line 25 the student has taken up the reporting of the inputs and output himself, though the teacher is still tracking the entries and guiding the process by moving his finger to each successive row in the truth table. This is fortunate, because the student appears to repeat the previous entry at the end of Line 25. The teacher corrects the entry, and the student, seeing the problem, immediately initiates a repair (Line 26), saying, "This too so the green one should come on here and it does and the red one doesn't matter." In Line 28 the student notes the circuit gives the incorrect output, revealing a bug in implementing the

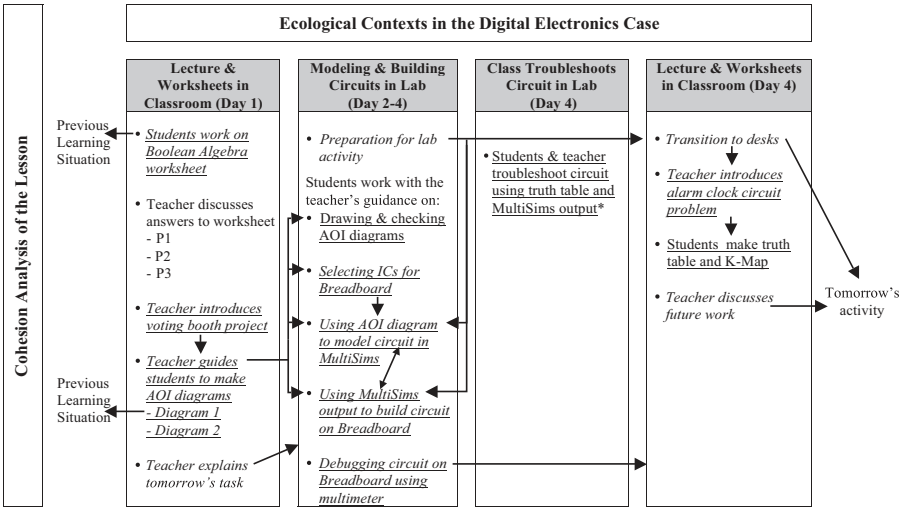


Figure 7. Case 2 cohesion analysis of the digital electronics lesson. Columns show the major ecological contexts. Bullets show the main modalities in the lesson. Italics indicates projection, underline indicates coordination, and italics and underline indicates projection + coordination. Arrows show the main backward or forward projections. The asterisk indicates the modalities discussed in the transcript.

logic electronically, saying "oh so the green one doesn't work but the red one works for that one over there." In Line 29 the student notes another error ("Uh so these two doesn't work"). The teacher withdraws his finger and the student autonomously coordinates the entries of the truth table with the state of the circuit. The student then rapidly completes the coordination of the table and the circuit, glancing repeatedly between the two material forms, noting more successes and one more error. Finally, the student sums up the accuracy of the circuit for implementing the logic of the monitoring system, saying "So there's three doesn't work" (Line 30).

In the language of our theoretical framework, the *what* of mathematics in this case is the propositional logic that instantiates the privacy conditions of the voting booth, which is reified in the truth table and in a simplified Boolean expression. The mathematics can be traced across various representational forms of the truth table (the students' preferred representation), the Boolean algebra equations, the MultiSim diagram (the teacher's preferred representation), and the physical circuit, which yields a series of outputs in the form of lighted LEDs for a given set of inputs. The teacher seeks to establish cohesion in several ways. The discussion of proper versus messy circuit wiring highlights the practical consideration of constructing a well-organized circuit board to provide a clear mapping between the physical circuit and the symbolic representation of the design—a mapping that is more easily traced and debugged. Cohesion is also established by showing that the trajectory of building and troubleshooting a correct circuit is not linear; rather, verification of the circuit involves cycling back to a representational form that was encountered earlier (in this case, truth table entries). The teacher modeled attending to the immediately present representational and material forms.

Figure 7 provides an analytic view of the cohesion in the digital electronics case, showing the sequence of modalities as participants used Boolean algebraic expressions, an AOI diagram, a simulation of the circuit in MultiSims, and a working electronic circuit, as they engaged in the troubleshooting process. In the days leading up to the debugging activity in the transcript, the teacher regularly used forward projection along with coordination to connect the various representations and objects to those that would be used at future stages of the project, striving to establish cohesion by communicating the interrelated nature of the various modal forms to the overall Boolean relations conveyed in the original design statement.

During the postlesson interview on Day 4, the teacher was explicit about his intentions to promote cohesion.

> Teacher: Now again the big one we were talking about is going from that to the breadboard. Going from the, um, schematic to the actual… and trying to understand how it works. When they get that, then I know they've learned.

Yet in Figure 8 (Case #2 Transcript 3), he recognizes that students' typical practices can impede this understanding:

In describing his actions (Figure 8), the teacher demonstrates that he is aware of the need to explicitly coordinate these modal forms for students, using his gestures and staccato speech to reenact the indexical role of his pointing actions. In addition, he recognizes that his acts of coordination are part of a complex set of communicative interactions that are not complete unless students also fulfill their roles. By stating that he must "make sure their eyes are following," the teacher reveals his sensitivity to the shared attention that is needed to establish a cohesive learning environment. In this way, the teacher ties the success of his actions to promote cohesion across symbolic logic, simulation software, and circuit wiring to the behaviors of the students.

Case 3. Circles enacted in diagrams, software, and gestures: Proof in an advanced high school geometry class

The final case is drawn from a high school geometry lesson about properties of circles and their associated theorems. The lesson alternated between three ecological contexts: individual seatwork, where a worksheet was used to practice a theorem on which the current activity builds; the computer lab, where students used an interactive dynamic geometry software environment; and a teacher-led whole-class discussion in the regular classroom. In the computer lab, students worked in dyads while each student had

1 T: Kids want a shortcut all the time, and that's what's getting them in trouble when they can't get from one to the other.

((Teacher moves his left hand, and then his right hand))

2 T: If they take the time to break it down they get the job done in half the time.

But they want a shortcut that takes longer. Uh, and sometimes it's just a matter of just standing over them and point! Point! Point! Point!…

((*Teacher makes four dramatic pointing gestures on either side of his body, alternating between pointing with left index finger and right index finger*))

3 T: …and make sure their eyes are following.

Figure 8. Case #2 Transcript 3. Teacher interview.

access to the program GSP. The software allows students to construct, measure, and control relations between geometric objects.

The first excerpt takes place in the computer lab (Figure 9). During this interaction, the teacher assisted several students as they generated explanations about why opposite angles in a quadrilateral inscribed in a circle are always supplementary (i.e., together total 180°). To solve the problem, students need an essential mathematical relation: the measure of any angle inscribed in a circle will always equal one-half the length of its intercepted arc. This idea had been encountered in an earlier lab task and in a worksheet activity earlier that day.

This first excerpt starts with the teacher responding to a difficulty exhibited by a student named Jordan. Jordan is stuck on the notion that any inscribed angle must intercept exactly half of the circle (as would be the case only for a right angle). In her explanation, the teacher coordinates between the display in GSP and her gestures of the angles and arcs, and also projects back to concepts addressed in the earlier classroom activity relating inscribed angles to the length of their intercepted arcs.

The teacher starts (Figure 10, Line 1) with a hint directing students' attention to the relevant parts of the angles and circles using speech ("Okay so here's my hint. Look at angle A.") and an explicit pointing gesture to the computer screen. Pointing to an angle on the student's worksheet, she asks, "What arc does that intercept?" In Lines 6–9, Jordan's response indicates a misconception that an angle and its opposing angle must each necessarily intercept a semicircle (180°): "Oh, it intercepts- it intercepts the other half." To address Jordan's misconception, the teacher used the dynamic nature of GSP to clarify what it means for the angle to intercept "half" a circle (Lines 9 and 10). She creates a chord on the circle and then drags one point so that the chord passes through the circle's center, forming a diameter.

Turning to another group in Lines 11–18, the teacher shows how to apply the central mathematical idea relating opposite inscribed angles of a quadrilateral to the current diagram. At the end of Line 13 the teacher leaves her statement incomplete ("the angle across from that, angle C, intercepts…"), as a prompt for the student to name the arc intercepted by angle C. When the student makes a new misstatement, where the student identifies angle BCA, the teacher is more concrete in Line 16, using a pointing gesture (with pen in hand) to coordinate the term *angle A* with a specific location on the GSP image. In Line 18, the teacher traces with her hand along the intercepted arc from vertex B past C to D as she says "Intercepts from B to D."

The teacher asks (Line 20) if this arc BD is the same as the one she traced in Line 16, saying, "Is it the same BD?" thereby providing a projection back in time. Some students respond "no" (Line 21), so the teacher scaffolds them further to consider putting the parts together, saying, "Together those two angles inter-

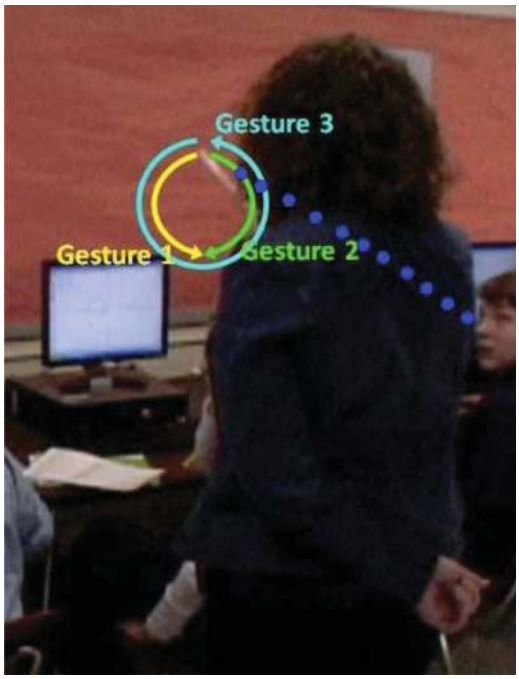
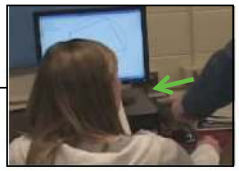


Figure 9. Zoomed out view of image from Line 24 showing Jordan (at right) attending to the teacher's circular gestures.

1 T: Okay so here's my hint. Look at angle A. What arc does- well even on your picture there, well you didn't label- okay well

look at (Name)'s angle A over here on her picture.



((*Teacher points with index finger to student's worksheet*))

What arc does that intercept? Angle A should intercept- oh you've got an E there huh?

2 Jordan: Yeah

3 T: For you it's BE and for these guys it's BD. And now look at the angle across from A having the opposite angle, right?

4 S: Mm-hm.

5 T: In yours that's angle C.

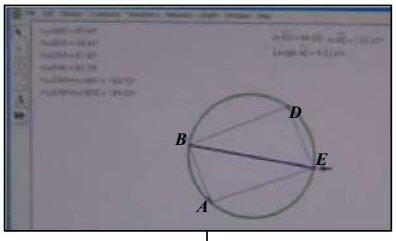
6 Jordan: Oh, it intercepts- it intercepts the other half.

7 T: What do you mean the other half?

8 Jordan: Okay well.

9 T: These are half?

10 Group: (* loud background chatter, inaudible *)



(At the same time) (* Teacher works individually with Jordan on his screen

((*Teacher creates $\overline{BE}$ on Jordan's sketch and drags Point E so that $\overline{BE}$ is a diameter*))

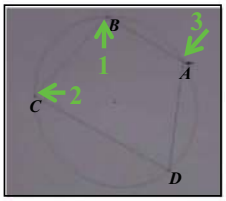
before coming back to work with the whole group *)

11 T: ((*To a different student with rectangle ABCD)*) So, wait, angle A intercepts the arc from B all the way around to D, right?

12 S: Yeah.

13 T: And the angle across from that, angle C, intercepts …

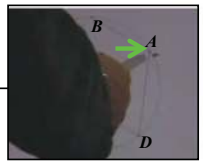
14 S: B (pause) CA



((*Student moves the pointer on her sketch to Points B, C, and A*))

15 T: What?

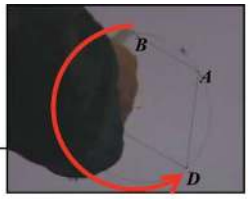
16 T: Angle A, right?



((*Teacher points with a pen to Point A*

Figure 10. Case#3 Transcript 4a. (*NB. Speech transcript is complete but only relevant gestures are shown.*)

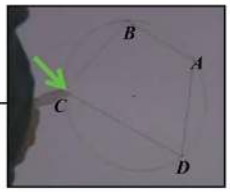
while Student moves her pointer to the same position))



17 Jordan: Intercepts B-

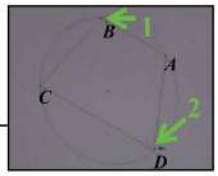
18 T: Intercepts from B to D.

((*Teacher sweeps pen from Point B to Point D toward Point C while Student tracks her pen with the pointer*))



Angle C intercepts …

((*Teacher points with pen to Point C*))



19 S: BD

((*Student moves the pointer to Point B and then Point D*))

20 T: Is it the same BD?

21 Jordan and S: No.

22 T: No. Together those two angles intercept … what …

23 S: BD.

24 T: But it's BD on this part

((*Teacher draws a left arc in the air with pen as though tracing a portion of an imaginary circle*))

and BD on this part, which is?

((*Teacher draws an arc with pen on the right side of the imaginary circle in the air*))

The entire circle right?

((*Teacher forms in the air a complete circle with the pen starting approximately where Point B is located near the top of the imaginary circle*))

25 S: Yeah.

26 T: Cool. And so then what does that mean about those angles? What kind of angles are they?

27 S: Supplementary.

Figure 10. (*Continued*)

28 T: Why?

29 S: 'Cause.

30 Jordan: They are.

31 T: What kind of angles are they?

32 S: (* talking inaudibly *)

33 T: Okay they are opposite, but in the circle?

((*Teacher moves her hand in circles*))

On that first page you talked about-

((*Teacher points with finger to student's worksheet*))

34 S: Inscribed.

35 T: So think about what you know about inscribed angles. Alright I think you're almost there.

Figure 10. (*Continued*)

cept… what…?" (Line 22), again prompting students to name the intersected arcs on their respective screens. She employs a gestural catchment in the air by reenacting the trace she performed earlier on the computer screen (Line 16) with a counterclockwise motion in the air that starts at point B and traces an arc along an imaginary circle that ends at point D, all the while speaking about arc BD. Then, she reenacts tracing a clockwise arc along the imaginary circle, again starting at point B and ending at D. Finally, in one continuous gesture in the air (Line 24) she provides two complete counterclockwise traversals of the circumference of the circle, starting and ending approximately at the location of vertex B, thereby demonstrating that the two arcs necessarily compose the whole circle. She says "But it's BD on this part and BD on this part, which is the entire circle, right?" Jordan attends carefully to these gestures (Figure 9).

Thus, the teacher establishes this thread of cohesion between the earlier theorem work, the part-whole relations of the circles depicted by her gestures, the GSP displays, and the worksheet (which she references in Line 33, saying "On that first page" and pointing to the student's worksheet). With these cohesion-producing acts, the teacher helped students to recognize that the two opposing angles of any inscribed rectangle are each inscribed. By repeatedly reinvoking resources such as the worksheet and gestures tracing the arcs, the teacher produced a cohesive account of how the many activities manifest the same invariant relation between angle and arc measures.

Back in the classroom, away from the computers, the teacher prompted students to make statements about the relations they uncovered between the inscribed angles and the arcs they intercept. In framing this session, the teacher projected backward in time to the quadrilaterals that students had inscribed in the circles, both by referring to the lab activity and by drawing a diagram resembling what students had constructed (Figure 11). She then coordinated the drawing with a gestural catchment—a repetition of the circular gestures she had made in the lab—reinvoking the GSP diagrams.

In Line 36 (Figure 13), Jordan seems to repeat his earlier misconception when he says, "we know that the inverted angle's one half the…," but he quickly makes his own repair. He also stumbles over the terminology of "inscribed angle," saying "inverted angle" instead. In acknowledging this error, he elaborates his line of explanation (Line 40). His language seems at first to reflect the initial misconception ("That would be three hundred and sixty divided by two"); however, he then clarifies that it is not necessarily half, but "you have to figure it out," suggesting that he is aware that it is some unknown, supplementary value. Thus, with support from the teacher's coordination and projections of different modal forms, Jordan catches his own error and corrects his mathematical reasoning. The

Figure 11. The teacher projected and coordinated the lab activity with the diagram on the whiteboard and highlighted arcs BAD and BCD in blue (similar to her gestures in the computer lab) to help students recognize that the two arcs together form the entire circle.

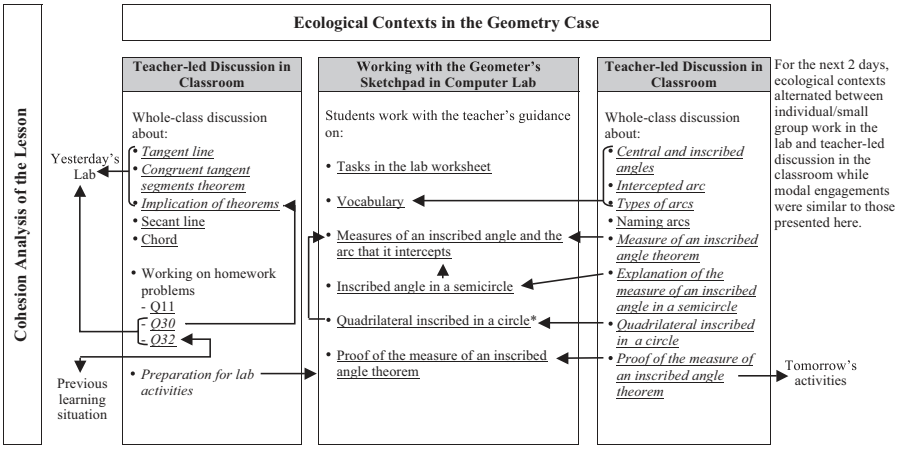


Figure 12. Case 3 cohesion analysis of the precollege geometry lesson. Columns show the major ecological contexts. Bullets show the main modalities in the lesson. Italics indicates projection, underline indicates coordination, and italics and underline indicates projection + coordination. Arrows show the main backward or forward projections to past or future modal form(s). The asterisk indicates the modalities discussed in the transcript of the geometry case.

result is that Jordan's understanding of the relationship of inscribed angles to their intercepted arcs can apply to any angle.

During a postlesson interview, the teacher described her use of projection to connect students' experience in the computer lab with their mathematical explanations in the classroom.

> I think they'll understand it better 'cause you know we can refer to 'Well, remember in the lab when we did this and what did you notice?' and, you know, I think they're … making those connections better than 'Oh look at your notes yesterday. What was that theorem?' (Day 2 postlesson interview).

This teacher views cohesion production as a pedagogical tool to help foster the learning of mathematical concepts across modal forms and ecological shifts between the lab, student notes, and the classroom.

The summative cohesion analysis of the geometry case shown in Figure 12 reveals the cycle of activities that occur as geometry concepts are threaded through the ecological contexts of the classroom and the lab. Unlike the ballistic device and digital electronics cases where forward projections were often employed, the geometry teacher regularly used coordination with backward projection, invoking the geometric relations discovered during lab activities to support the more formal discussion of concepts in classroom lectures. Such backward projections can be used to foster reflection and cohesion. The figure also reveals the heavy use of coordination during lab activities, in which participants mapped the diagrams on the computer screen to the explanations on their worksheets and to iconic gestures, recognizing the core invariant relation of inscribed angles as instantiated across multiple representational forms and ecological contexts.

The analysis of each case yields insights about how invariant mathematical relations are threaded through the transitions across mathematical and technical notational systems, computer

((Students and the teacher are now back in the classroom, with a teacher led discussion using the whiteboard.))

36 Jordan: Okay so since we know a circle's three hundred-sixty degrees and if those two angles take up an entire circle and we know that the inverted angle's one half the-

37 T: Inverted angle? Inscribed angle?

38 Jordan: Yeah. Yeah.

39 T: Okay.

40 Jordan: You all know what I'm talkin' about. That would be three hundred and sixty divided by two divided by two. Because you have- well divided by two and then you have to figure it out.

Figure 13. Case #3 Transcript 4b. The following conversation between the teacher and Jordan shows that the teacher's efforts paid off. Here Jordan, while still struggling to select the proper mathematical terminology, exhibits understanding of the central mathematical relationship of the earlier lesson.

simulations, physical objects and changing classroom spaces. These three cases illustrate ways that cohesion is supported in real time by teachers and students through their speech, gestures, and actions, which are used to project both backward and forward in time, and to coordinate across representational forms. In the context of so may changing representations and ecological contexts, highlighting the invariance is crucial to preserve cohesion and to offer learners a sense of continuity and integration.

Discussion and conclusions

In this article, we have argued that one of the primary goals—and one of the central challenges—of integrated STEM education is threading mathematical concepts and ideas through the various ecological contexts, representations, and modal forms that are commonplace in mathematics, science and engineering classrooms. A focus on the *what* and *where* of mathematics underscores the importance of identifying the common invariant relations, as well as "locating the mathematics" to produce and maintain cohesion. We have highlighted three processes that teachers (and students) use to facilitate STEM learning by producing and maintaining cohesion: guiding attention and behavior around ecological shifts, coordinating ideas across different spaces, and projecting ideas across time. As demonstrated in postlesson interviews, teachers intentionally use these approaches to support student understanding.

In this final section we consider two of the broad issues that this work touches on, specifically, fostering STEM integration and grounding abstractions to promote meaning making and transfer.

Instructional implications for STEM integration

STEM integration is a form of understanding that supports awareness of the relations of ideas and representations from different disciplines, while also deepening one's discipline-specific skills and knowledge. By explicitly addressing cohesion in instruction, we address what Latour (1999) would call the "sequence of mediators" for promoting STEM integration. We highlight three implications of this research for instruction to promote STEM integration: (a) cohesion is particularly effective when it is produced by learners themselves, (b) projection to abstract principles is especially important for fostering cohesion, and (c) teachers naturally appreciate the importance of cohesion.

Cohesion is particularly effective when it is produced by learners

Effective instruction does not simply inform students of the cohesion of the invariant relations, it allows students to experience it by directly participating in moves such as projection and coordination. In the case of the ballistics devices, the teacher clearly understands how theta must be realized across the different stages of the project, but a breakdown occurs because students struggle to take this idea up on their own terms. Even as the teacher enacts projection and coordination in an effort to repair the situation through discussion with the students, little common ground is achieved. We posit that it is especially important for students themselves to take up coordination and projection in meaningful ways. In the digital electronics case, the student enacts the coordination between different representations as the teacher fades his support. In the geometry case, Jordan produces his own coordination by formulating the desired mathematical argument in his own words. These examples contrast with other cases in which students are not involved in producing cohesion, and display few gains in understanding. For example, Walkington et al. (2014) reported that for high school engineering students participating in a bridge-building project, direct instruction by the teacher on the invariant relations seemed to have little effect.

Projection to abstract principles is especially important to foster cohesion

In project-based learning environments in which a product design is the stated goal, abstract principles of mathematics may be left behind once students start to focus on the materiality of their designs. As in the ballistics case, this can result in key principles being forgotten as the excitement of later project stages ensues (for another example, see Walkington et al., 2014). When students focus on the physicality of the projects they may do so at the expense of the deep STEM concepts that run throughout the project (Nathan et al., 2013; Walkington et al., 2014). In each of the first two cases in which the mathematics was part of an integrated STEM design project, we saw relatively few instances of backward projection, as the life cycle of the project seemed to move from abstract laws to concrete devices. The geometry classroom had roughly the opposite sequence, and we observed many more backward projections in this context. The explicit coordination of abstract principles with concrete instantiations is an important element of cohesion production that may be challenging to accomplish in practice, especially in project-based learning settings. However, using projection and coordination to connect abstract and concrete principles, and focusing particularly on backward projection during design activities, may be key to successfully producing cohesion. Our findings about teachers' use of projection are reminiscent of work on expansive framing of learning contexts, which has been shown to promote transfer (Engle, Nguyen, & Mendelson, 2011). Teachers' projections to past and future instantiations of a concept help establish the "temporal horizon" (Engle et al., 2011, p. 610) of a lesson.

Teachers naturally appreciate the importance of cohesion

In postlesson interviews, all three teachers acknowledged the importance of cohesion informally and in their own words. They recognized that cohesion was a key element of what they were trying to accomplish and a key barrier to student understanding. The fact that experienced STEM teachers already have ideas about cohesion production suggests that professional development could productively focus on formalizing these ideas, and providing recommendations and critical discussion about how to enact these ideas. An important starting point, especially in integrated STEM settings, is making sure the teacher has a clear understanding of which invariant relations are most critical to teach because there are often multiple possible connections to mathematics or science that could be made.

Connecting abstract and concrete ideas

The notion of grounding is often invoked in project-based instruction and reform approaches to education, on the

argument that context, materials, and activity structures—tangible aspects of the learning environment—help to establish the meaning and appropriate uses of abstract ideas in concrete ways, and help to make schooling more relevant (Blumenfeld et al., 1991; Hmelo, Holton, & Kolodner, 2000; Jurow, 2005). Yet we observed in these cases that grounding contexts and activities cannot be assumed to enhance understanding and learning. Indeed, tracking the underlying invariant relations across their many manifestations is a central challenge for many students operating in rich, project-based learning environments. This is because grounding contexts also introduce extra demands for establishing cohesion across the familiar and new contexts and modal forms (Kozma, 2003). To realize the potential of grounding as a means to facilitate meaning making and the application of abstract concepts, learners must grasp the relation of the abstract ideas to the grounding objects and activities themselves. Teachers and curriculum developers can improve students' prospects when they explicitly attend to these links (Honey et al., 2014).

One of the central issues to emerge from the analysis of cohesion production is a greater appreciation of the challenges of STEM integration from the learner's perspective (Lobato, 2012). There is a tendency to see hands-on activities and authentic contexts as powerful ways to ground new ideas and abstract representations. However, the cohesion analyses presented here underscore the demands placed on learners working in multimodal learning environments. Why should this be so? The philosopher Martin Heidegger argued that the vast majority of our everyday behaviors are forms of skillful coping, rather than theoretically driven action and critical reflection (Dreyfus, 1991). When we eat, or build something, we do not continually experience the world mediated through the representations or tools we use, but instead as actions applied directly upon the world through our state of being (or *Dasein*), employing background practices in the manners in which we were socialized to act. This pertains to schooling as well as home and work life. The implements we use to function (forks, hammers) are invisible to us as tools most of the time. Little in the way of learning can be expected when people are engaged in skillful coping. However, when skillful coping is insufficient, or when our mediational tools break, we then engage in critical reflection and intentional thinking mediated by representations of the world. One implication is that to achieve the lofty objectives of integrated STEM education, teachers may want to create intentional disruptions of everyday school behaviors to facilitate moments of critical reflection on the mediating representations through which learning is likely to take place. In this way, abstractions can become meaningful ways of describing general relations that model the real world while also fostering transfer of learning.

Conclusion

One of the central challenges of STEM education is students' recognition of cohesion in the rich STEM learning environment. This work highlights the ways in which mathematical concepts and ideas can be threaded through the various modal forms that are commonplace in STEM disciplines. We believe that this focus on the *what* and *where* of mathematics provides a valuable new lens with which to consider classroom discourse and activities, as well as teachers' intentions and goals. A focus on cohesion—including how it can be described and enacted in the classroom—provides a new framework for conceptualizing the challenges that teachers and students face in STEM integration every day.

Acknowledgments

The authors gratefully acknowledge the technical assistance of Dan Klopp in conducting this research and preparing this manuscript. They are indebted to the teachers and students at our research sites. They benefitted greatly from our discussions with members of the Tangibility for the Teaching, Learning, and Communicating of Mathematics project at San Diego State University, University of Wisconsin–Madison, and Vanderbilt University's Peabody College.

Funding

This work was made possible through the financial support of the National Science Foundation REESE Program (DRL-0816406) and the wonderful resources and staff of the Wisconsin Center for Education Research. Support for this research was also provided by the U.S. Department of Education, Institute of Education Sciences Research Training Program under grant number R305B130007 as part of the Postdoctoral Training Program in Mathematical Thinking, Learning, and Instruction at the Wisconsin Center on Education Research at the University of Wisconsin–Madison.

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