Three charges on a plane in a magnetic field: Special trajectories

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Abstract

As a generalization and extension of JMP **54** (2013) 022901, the classical dynamics of three non-relativistic Coulomb charges (e_1, m_1), (e_2, m_2) and (e_3, m_3) on the plane placed in a perpendicular constant magnetic field is considered. Special trajectories for which the distances between the charges remain unchanged are presented and their corresponding constants of motion are indicated. For these special trajectories the number of constants of motion is larger than the dimension of the configuration space and hence they can be called *particularly superintegrable*. Three physically relevant cases are analyzed in detail, namely that of three electrons, a neutral system and a Helium-like system. The n-body case is discussed as well.

I. INTRODUCTION

The three-body problem, the classical motion of three particles with pairwise gravitational potential, has a monumental history capturing the attention of many prominent mathematicians, including Euler, Lagrange, Poincaré and Sundman [1]-[3]. This problem is surprisingly difficult to solve, even in the so called restricted three-body problem, corresponding to the simple case of three masses moving in a common plane. The quantum three-body problem is not less important. There exists a large number of states and processes which demand to be treated as three-particle systems and they are standard material in advanced quantum mechanics textbooks [4].

Moreover, under electromagnetic interactions a new physics emerges where the study of two-dimensional three-electron systems in a magnetic field has led to the experimental and theoretical discovery of the anomalous quantum Hall effect [5].

In a previous work [6] of one of the authors, to be referred below simply as paper I, the classical motion of two non-relativistic Coulomb charges placed on a plane subject to a perpendicular constant magnetic field was analyzed in detail. In this paper we will discuss the case of three charges. As a by product, this case provides an insight for the planar N body problem in magnetic field which will be addressed in the last section of the present work.

It is well known that the 3D motion of a classical charged particle subject to a constant magnetic field is characterized by four integrals of motion: the Hamiltonian and the three components of the pseudomomentum. Thus, the system is superintegrable. Furthermore, restricted to the transverse plane to the magnetic field direction, the 2D problem, the system becomes maximally superintegrable [7] and then all trajectories are closed and periodic [8]. Needless to say that superintegrabilty plays a fundamental role in the description of exactly solvable models in both classical and quantum mechanics (see [9] for a review).

We can ask the natural question: can three charges describe periodic trajectories?. The answer is affirmative although those trajectories are rather distinctive. They appear if certain initial conditions are chosen only, and not for any system of charged particles. This implies that special trajectories indicate the appearance of *particular* constants of motion [10]: they are conserved on certain trajectories only. We call these trajectories special or superintegrable. For the two-body case, some examples of these trajectories for two electrons

(an integrable system) and two particles with opposite charges (a chaotic system) were found in [11] and [12], respectively.

It may be remarked that for a neutral two-body system in 3D the motion is known to be chaotic. In general, trajectories are not closed (see, e.g., [13]). For this reason special trajectories represent a certain order. This behavior remains true for a three-body neutral system on the plane.

The main purpose of this paper is to determine integrals and *particular* constants of motion of three Coulomb charges to classify the initial data associated with special trajectories. It will set up the basis towards the quantum case. Here, we follow the philosophy behind the two body problem where exact analytical solutions of the Schrödinger equation were found [14]. In [14] the key point was to take a *particular* constant of motion derived in paper I [6], promoting it to a quantum operator and then seek for its common eigenfunctions with the Hamiltonian.

The structure of the paper is organized as follows: in Section II we start by looking at the integrals of motion of the Hamiltonian. The formulae of this section can be easily translated to the quantum case. Afterwards, in order to study the issue on separation of variables both the Hamiltonian and the corresponding Newton equations of motion are written in suitable variables, and by doing so the relevant parameters (effective and *coupling* charges) of this problem are revealed. The physical systems to focus in will naturally appear as well.

In Section III we show periodic trajectories for which the relative distance between particles remain unchanged during time evolution. They are not generic, they occur only for specific initial conditions associated with the *particular* constants of motion we are looking for. The three electron, neutral and the Helium-like systems are examples worked out explicitly.

Then, Section IV describes the extension of such special trajectories to the general case of n Coulomb charges in a constant magnetic field. Finally, in Section V we present a summary and discussion of the results.

II. THREE CHARGES IN A MAGNETIC FIELD

The Hamiltonian which describes three non-relativistic particles (e_i, m_i), $i = 1, 2, 3$, placed on the plane subject to a constant and uniform magnetic field $\mathbf{B} = B\,\hat{\mathbf{z}}$ perpendicular to the

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plane has the form

$$\mathcal{H} = \frac{(\mathbf{p}_1 - e_1 \mathbf{A}_{\boldsymbol{\rho}_1})^2}{2\,m_1} + \frac{(\mathbf{p}_2 - e_2 \mathbf{A}_{\boldsymbol{\rho}_2})^2}{2\,m_2} + \frac{(\mathbf{p}_3 - e_3 \mathbf{A}_{\boldsymbol{\rho}_3})^2}{2\,m_3}$$
$$+ \frac{e_1\,e_2}{|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2|} + \frac{e_1\,e_3}{|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_3|} + \frac{e_2\,e_3}{|\boldsymbol{\rho}_2 - \boldsymbol{\rho}_3|} \ , \tag{1}$$

where $\boldsymbol{\rho}_i$ denotes the position vector of particle i and $\mathbf{p}_i$ its corresponding canonical momentum. We shall henceforth stick to the so called symmetric gauge where the magnetic vector potential is $\mathbf{A_r} = \frac{1}{2}\,(\mathbf{B} \times \mathbf{r})$. It is well known that the total Pseudomomentum

$$\mathbf{K} \equiv (K_x, K_y) = \boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3 = (\mathbf{p}_1 + e_1 \mathbf{A}_{\boldsymbol{\rho}_1}) + (\mathbf{p}_2 + e_2 \mathbf{A}_{\boldsymbol{\rho}_2}) + (\mathbf{p}_3 + e_3 \mathbf{A}_{\boldsymbol{\rho}_3}) \ , \tag{2}$$

is an integral of motion [15]-[16], the Poisson bracket $\{\mathbf{K}, \mathcal{H}\} = 0$ vanishes. In (2), $\boldsymbol{k}_j$ is the individual pseudomomentum of particle $j = 1, 2, 3$. The total *canonical* momentum $\mathbf{L}_z^{total}$

$$\mathbf{L}_z^{total} \equiv \boldsymbol{\ell}_{z_1} + \boldsymbol{\ell}_{z_2} + \boldsymbol{\ell}_{z_3} = (\boldsymbol{\rho}_1 \times \mathbf{p}_1) + (\boldsymbol{\rho}_2 \times \mathbf{p}_2) + (\boldsymbol{\rho}_3 \times \mathbf{p}_3) \ , \tag{3}$$

is also conserved, $\{\mathbf{L}_z^{total}, \mathcal{H}\} = 0$. Hence, the problem is characterized by three integrals of motion $K_{x,y}$, L_z^{total}, where some (x, y)-coordinate system is introduced on the plane. The dimension of the configuration space is six. Therefore, in general the problem (1) is not integrable, the number of integrals is less than the dimension of the configuration space. The quantities $K_{x,y}$, L_z^{total} are not in involution, they obey the algebra

$$\{K_x, K_y\} = -Q\,B \ ,$$
$$\{L_z^{total}, K_x\} = K_y \ , \tag{4}$$
$$\{L_z^{total}, K_y\} = -K_x \ ,$$

where

$$Q = e_1 + e_2 + e_3 \ ,$$

is the total (net) charge. The Casimir operator $\mathcal{C}$ is given by

$$\mathcal{C} = K_x^2 + K_y^2 - 2\,Q\,B\,L_z^{total} \ . \tag{5}$$

For the case of a single charge the quantity $\mathcal{C}$ is, in fact, the Hamiltonian.

Let us introduce Jacobi variables in a standard way

$$\mathbf{R} = \mu_1\,\boldsymbol{\rho}_1 + \mu_2\,\boldsymbol{\rho}_2 + \mu_3\,\boldsymbol{\rho}_3 \quad , \quad \boldsymbol{\tau}_1 = \boldsymbol{\rho}_2 - \boldsymbol{\rho}_1 \quad , \quad \boldsymbol{\tau}_2 = \boldsymbol{\rho}_3 - (\nu_1\boldsymbol{\rho}_2 + \nu_2\,\boldsymbol{\rho}_1) \ ,$$
$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 \quad , \quad \mathbf{p}_{\boldsymbol{\tau}_1} = \nu_1\,\mathbf{p}_2 - \nu_2\,\mathbf{p}_1 \quad , \quad \mathbf{p}_{\boldsymbol{\tau}_2} = \mathbf{p}_3(\mu_1 + \mu_2) - \mu_3\,(\mathbf{p}_1 + \mathbf{p}_2) \ , \tag{6}$$

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where $\mu_i = \frac{m_i}{M}$ and $\nu_i = \frac{m_i}{m_1 + m_2}$ are dimensionless parameters, $M = m_1 + m_2 + m_3$ is the total mass of the system. In these coordinates the total Pseudomomentum (2) becomes

$$\mathbf{K} = \mathbf{P} + Q\,\mathbf{A_R} - e_{c1}\,\mathbf{A_{\tau_1}} - e_{c2}\,\mathbf{A_{\tau_2}} , \tag{7}$$

where the two coefficients

$$e_{c1} = \frac{m_1\,m_2}{m_1 + m_2}\left(\frac{e_1}{m_1} - \frac{e_2}{m_2}\right) \quad ; \quad e_{c2} = \frac{(m_1 + m_2)\,m_3}{m_1 + m_2 + m_3}\left(\frac{e_1 + e_2}{m_1 + m_2} - \frac{e_3}{m_3}\right) , \tag{8}$$

can be called *coupling charges*. In general, the Hamiltonian (1) expressed in Jacobi variables (6) does not admit separation of the center of mass (CM) motion. However, for particles with equal Larmor frequency, i.e. $\frac{e_1}{m_1} = \frac{e_2}{m_2} = \frac{e_3}{m_3} = \alpha$, both coupling charges vanish $e_{c1} = e_{c2} = 0$ and the CM motion do separate from the relative ones. In this case the CM motion is described by the elementary Hamiltonian

$$\mathcal{H}_{CM} = \frac{(\mathbf{P} - Q\,\mathbf{A_R})^2}{2\,M} ,$$

which corresponds to that of a particle with charge Q and mass M placed in a constant magnetic field. All CM trajectories are circular and possess the same Larmor frequency of the individual particles, $\frac{Q}{M} = \alpha$.

Now, for convenience we introduce another canonical change of variables

$$\mathbf{P}' = \mathbf{P} - e_{c1}\,\mathbf{A_{\tau_1}} - e_{c2}\,\mathbf{A_{\tau_2}} , \qquad \mathbf{p}'_{\tau_1} = \mathbf{p}_{\tau_1} + e_{c1}\,\mathbf{A_R} , \qquad \mathbf{p}'_{\tau_2} = \mathbf{p}_{\tau_2} + e_{c2}\,\mathbf{A_R} ,$$
$$\mathbf{R}' = \mathbf{R} , \qquad \boldsymbol{\tau}_1{}' = \boldsymbol{\tau}_1 , \qquad \boldsymbol{\tau}_2{}' = \boldsymbol{\tau}_2 , \tag{9}$$

in which the Hamiltonian (1), hereafter omitting the primes, takes the form

$$\mathcal{H} = \frac{(\mathbf{P} - Q\,\mathbf{A_R} + 2\,e_{c1}\,\mathbf{A_{\tau_1}} + 2\,e_{c2}\,\mathbf{A_{\tau_2}})^2}{2\,M} + \frac{(\mathbf{p}_{\tau_1}^2 - e_{1eff}\,\mathbf{A_{\tau_1}})^2}{2\,\nu_1\,\nu_2\,(m_1 + m_2)} + \frac{(\mathbf{p}_{\tau_2}^2 - e_{2eff}\,\mathbf{A_{\tau_2}})^2}{2\,m_3\,(\mu_1 + \mu_2)}$$
$$+ \frac{e_{c1}^2\,\mathbf{A}_{\tau_1}^2\,\mu_3}{2\,(m_1 + m_2)} + \frac{e_{c2}^2\,\mathbf{A}_{\tau_2}^2\,\mu_3^2}{2\,\nu_1\,\nu_2\,(m_1 + m_2)} + \frac{e_{c1}}{m_1 + m_2}\mathbf{A_{\tau_1}}\,\mathbf{p_{\tau_2}} - \frac{e_{c1}\,m_3}{m_1\,m_2}\,\mathbf{A_{\tau_2}}\,\mathbf{p_{\tau_1}}$$
$$+ \frac{e_{c1}}{m_1\,m_2}\,\mathbf{A_{\tau_1}}\,\mathbf{A_{\tau_2}}[e_3\,\mu_1\,\mu_2\,(m_1 + m_2) + e_2\,\mu_1\,\mu_3\,(m_1 + m_3) + e_1\,\mu_2\,\mu_3\,(m_2 + m_3)]$$
$$+ \frac{e_1\,e_2}{|\boldsymbol{\tau}_1|} + \frac{e_1\,e_3}{|\boldsymbol{\tau}_2 + \nu_2\,\boldsymbol{\tau}_1|} + \frac{e_2\,e_3}{|\boldsymbol{\tau}_2 - \nu_1\,\boldsymbol{\tau}_1|} , \tag{10}$$

where

$$e_{1eff} = e_2\,\nu_1^2 + e_1\,\nu_2^2 \qquad , \qquad e_{2eff} = e_3\,(\mu_1 + \mu_2)^2 + (e_1 + e_2)\,\mu_3^2 , \tag{11}$$

play the role of two *effective charges*, cf. (8). Because of the change of variables (9), the total Pseudomomentum

$$\mathbf{K} \;=\; \mathbf{P} \;+\; Q\,\mathbf{A_R}\;,$$

coincides with the CM Pseudomomentum. Thus, for a neutral system ($Q=0$) the new CM momentum $\mathbf{P}$ is conserved, $\{\mathbf{P},\mathcal{H}\}=0$, as one would expect for a free particle [17]. However, separation of CM motion and momentum-like conserved quantities are evidently distinct notions.

The first term in (10) describes the kinetic energy of the CM motion and it can be shown that it is gauge invariant. The second and the third terms correspond to the kinetic energy of two effective charges e_{1eff} and e_{2eff} (11), respectively. The Hamiltonian (10) is suitable to study finite mass corrections. In the present work we will focus in the general case rather than limiting cases such that the Born-Oppenheimer approximation where $M \to \infty$. Also, we do not consider a perturbative approach since an exact treatment is preferable.

For a quantum system of three Coulomb charges, the Hamiltonian $\mathcal{H}$, the Pseudomomentum $\mathbf{K}$ and the angular momentum L_z^{total} can be obtained replacing in the previous formulae the momenta by the corresponding differential operators. In particular, the canonical transformation (9) can be achieved by means of a unitary transformation.

A. Equations of motion

In this section we switch from the phase space (Hamiltonian) to the configuration space (equations of motion). From (1) we obtain the Newton equations of motion

$$
\begin{aligned}
m_1\,\ddot{\boldsymbol{\rho}}_1 \;&=\; e_1\,\dot{\boldsymbol{\rho}}_1 \times \mathbf{B} \;+\; \left(\frac{e_1\,e_2}{|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2|^3}(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2) \;+\; \frac{e_1\,e_3}{|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_3|^3}(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_3) \right) , \\
m_2\,\ddot{\boldsymbol{\rho}}_2 \;&=\; e_2\,\dot{\boldsymbol{\rho}}_2 \times \mathbf{B} \;-\; \left(\frac{e_1\,e_2}{|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2|^3}(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2) \;-\; \frac{e_2\,e_3}{|\boldsymbol{\rho}_2 - \boldsymbol{\rho}_3|^3}(\boldsymbol{\rho}_2 - \boldsymbol{\rho}_3) \right) , \qquad (12) \\
m_3\,\ddot{\boldsymbol{\rho}}_3 \;&=\; e_3\,\dot{\boldsymbol{\rho}}_3 \times \mathbf{B} \;-\; \left(\frac{e_1\,e_3}{|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_3|^3}(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_3) \;+\; \frac{e_2\,e_3}{|\boldsymbol{\rho}_2 - \boldsymbol{\rho}_3|^3}(\boldsymbol{\rho}_2 - \boldsymbol{\rho}_3) \right) ,
\end{aligned}
$$

($\dot{\boldsymbol{\rho}} \equiv \frac{d}{dt}\boldsymbol{\rho}$). The equations (12) are invariant under the changes of parameters

$$(\, B \to -B,\; e_1 \to -e_1,\; e_2 \to -e_2,\; e_3 \to -e_3 \,) \;,$$

and reflections

$$(\, \boldsymbol{\rho}_1 \to -\boldsymbol{\rho}_1,\; \boldsymbol{\rho}_2 \to -\boldsymbol{\rho}_2,\; \boldsymbol{\rho}_3 \to -\boldsymbol{\rho}_3 \,) \;.$$

In Jacobi variables (6), the Newton equations (12) can be written as

$$M\ddot{\mathbf{R}} \;=\; Q\,\dot{\mathbf{R}} \times \mathbf{B} \;-\; \mathbf{E}_R(\dot{\boldsymbol{\tau}}_1,\,\dot{\boldsymbol{\tau}}_2)\,,$$

$$\tilde{m}_1\,\ddot{\boldsymbol{\tau}}_1 \;=\; e_{1eff}\,\dot{\boldsymbol{\tau}}_1 \times \mathbf{B} \;-\; \frac{e_{c1}^2\,B^2}{2M}\boldsymbol{\tau}_1 \;+\; \mathbf{E}_1(\mathbf{R},\,\boldsymbol{\tau}_2,\,\dot{\boldsymbol{\tau}}_2) \;+\; V_1(\boldsymbol{\tau}_1,\,\boldsymbol{\tau}_2)\,, \qquad (13)$$

$$\tilde{m}_2\,\ddot{\boldsymbol{\tau}}_2 \;=\; e_{2eff}\,\dot{\boldsymbol{\tau}}_2 \times \mathbf{B} \;-\; \frac{e_{c2}^2\,B^2}{2\,M}\boldsymbol{\tau}_2 \;+\; \mathbf{E}_2(\mathbf{R},\,\boldsymbol{\tau}_1,\,\dot{\boldsymbol{\tau}}_1) \;+\; V_2(\boldsymbol{\tau}_1,\,\boldsymbol{\tau}_2)\,,$$

where

$$\mathbf{E}_R(\dot{\boldsymbol{\tau}}_1,\,\dot{\boldsymbol{\tau}}_2) \;=\; e_{c1}\,\dot{\boldsymbol{\tau}}_1 \times \mathbf{B} \;+\; e_{c2}\,\dot{\boldsymbol{\tau}}_2 \times \mathbf{B}\,,$$

$$\mathbf{E}_1(\mathbf{R},\,\boldsymbol{\tau}_2,\,\dot{\boldsymbol{\tau}}_2) \;=\; \frac{e_{c1}\,Q\,B^2}{2\,M}\mathbf{R} \;-\; \frac{e_{c1}}{M}\mathbf{K} \times \mathbf{B} \;-\; \frac{e_{c1}\,e_{c2}\,B^2}{2\,M}\boldsymbol{\tau}_2 \;+\; \mu_3\,e_{c1}\,\dot{\boldsymbol{\tau}}_2 \times \mathbf{B}\,, \qquad (14)$$

$$\mathbf{E}_2(\mathbf{R},\,\boldsymbol{\tau}_1,\,\dot{\boldsymbol{\tau}}_1) \;=\; \frac{e_{c2}\,Q\,B^2}{2\,M}\mathbf{R} \;-\; \frac{e_{c2}}{M}\mathbf{K} \times \mathbf{B} \;-\; \frac{e_{c1}\,e_{c2}\,B^2}{2\,M}\boldsymbol{\tau}_1 \;+\; \mu_3\,e_{c1}\,\dot{\boldsymbol{\tau}}_1 \times \mathbf{B}\,,$$

make sense of electric fields, $\tilde{m}_1 = \frac{m_1\,m_2}{m_1+m_2}$, $\tilde{m}_2 = \frac{(m_1+m_2)\,m_3}{m_1+m_2+m_3}$ and

$$V_1(\boldsymbol{\tau}_1,\,\boldsymbol{\tau}_2) \;=\; \frac{e_1\,e_2}{|\boldsymbol{\tau}_1|^3}\,\boldsymbol{\tau}_1 \;-\; \frac{\nu_1\,e_2\,e_3}{|\boldsymbol{\tau}_2 - \nu_1\,\boldsymbol{\tau}_1|^3}(\boldsymbol{\tau}_2 - \nu_1\,\boldsymbol{\tau}_1) \;+\; \frac{\nu_2\,e_1\,e_3}{|\boldsymbol{\tau}_2 + \nu_2\,\boldsymbol{\tau}_1|^3}(\boldsymbol{\tau}_2 + \nu_2\,\boldsymbol{\tau}_1)\,,$$

$$V_2(\boldsymbol{\tau}_1,\,\boldsymbol{\tau}_2) \;=\; \frac{e_2\,e_3}{|\boldsymbol{\tau}_2 - \nu_1\boldsymbol{\tau}_1|^3}(\boldsymbol{\tau}_2 - \nu_1\boldsymbol{\tau}_1) \;+\; \frac{e_1\,e_3}{|\boldsymbol{\tau}_2 + \nu_2\,\boldsymbol{\tau}_1|^3}(\boldsymbol{\tau}_2 + \nu_2\,\boldsymbol{\tau}_1)\,, \qquad (15)$$

describe Coulomb interactions. In (14) the Pseudomomentum

$$\mathbf{K} \;=\; M\,\dot{\mathbf{R}} \;-\; Q\,\mathbf{R} \times \mathbf{B} \;+\; e_{c1}\,\boldsymbol{\tau}_1 \times \mathbf{B} \;+\; e_{c2}\,\boldsymbol{\tau}_2 \times \mathbf{B}\,,$$

is conserved, $\dot{\mathbf{K}} = 0$. The CM motion is coupled to the internal motion via non-trivial electric fields $\mathbf{E}_R(\dot{\boldsymbol{\tau}}_1,\,\dot{\boldsymbol{\tau}}_2)$, $\mathbf{E}_1(\mathbf{R},\,\boldsymbol{\tau}_2,\,\dot{\boldsymbol{\tau}}_2)$ and $\mathbf{E}_2(\mathbf{R},\,\boldsymbol{\tau}_1,\,\dot{\boldsymbol{\tau}}_1)$ which vanish at $e_{c1} = e_{c2} = 0$. Also, the variables τ_1 and τ_2 themselves are strongly coupled with each other via the Coulomb potential. We emphasize that even in the Born-Oppenheimer approximation, where $M \to \infty$, the CM and relative variables continue to be coupled.

In the case of a neutral system ($Q = 0$) at rest ($\mathbf{K} = 0$) with two identical particles ($e_{c1} = 0$)

$$e_1 \;=\; e_2 \equiv e\,, \qquad m_1 \;=\; m_2 \equiv m\,,$$

the electric fields $\mathbf{E}_1$, $\mathbf{E}_2$ disappear. The corresponding quantum neutral system was shown to possess exact factorizable solutions in the absence of the Coulomb interaction [18]. In the present work, the origin of such solvability becomes more transparent.

To the best of the knowledge of the present authors, the equations (13) and the associated classical and quantum Hamiltonian (10) have not been discussed in the literature in full generality.

The above analysis of the equations of motion indicates the existence of three important particular cases, namely

(i) $Q = 0$ (neutral system): the components of $\mathbf{K}$ are in involution, $\{K_x, K_y\} = 0$, and a pseudo-separation of the CM motion can be achieved [16].

(ii) $e_{c1} = e_{c2} = 0$ (particles with the same charge-to-mass ratio): In the equations (13) the center of mass variable can be separated out.

(iii) $Q = 0$, $e_2 = e_3$; $m_2 = m_3$ (neutral system at rest $\mathbf{K} = 0$ with two identical particles): In the equations (13), a certain number of linear terms (electric fields) can be removed.

III. SPECIAL TRAJECTORIES

In this section special trajectories where the distances between the particles remain unchanged are presented. We focus on the classification of the initial data associated with these periodic orbits and the corresponding integrals and *particular* constants of motion. A precise distinction between integrals and *particular* constants is in order.

Integrals and *particular* constants of motion

A function $\mathcal{I} = \mathcal{I}(\boldsymbol{\rho}, \mathbf{p})$, defined in the phase space, such that

$$\{H, \mathcal{I}\} \;=\; F(\boldsymbol{\rho}, \mathbf{p}) \;\neq\; 0 \;,$$

is called a *particular* constant of motion[10] if there exists, within the domain where the problem is defined, a trajectory Σ for which $F\,|_\Sigma \;=\; 0$. Evaluated along Σ, the Poisson bracket vanishes $\{H, \mathcal{I}\} = 0$ and, consequently, $\mathcal{I}\,|_\Sigma$ is conserved. Of course, this implies the choice of specific initial conditions.

In the case of an integral of motion, like the Pseudomomentum $\mathbf{K}$ (7), the Poisson bracket $\{H, \mathbf{K}\} = 0$ is identically zero and thus independent of the initial conditions.

For the three body problem we will show that along the special trajectories: (1) unlike

the generic case (4), the integrals K_x, K_y, L_z^{total} are Poisson commuting invariants, and (2) *particular* constants of motion occur.

A. CONFIGURATION I

The Configuration I corresponds to the case when two charges e_1 and e_2 rotate, with the same angular frequency ω and opposite velocities, around the third charge e_3. In its own the charge e_3 rotates with frequency ω_3 around a fixed point. This Configuration I is the superposition of two independent circular motions, it is presented in Fig. 1. The explicit form, as a function of time, of the corresponding special trajectories is given by

$$
\begin{aligned}
\boldsymbol{\rho}_1(t) &= \frac{\mathrm{v}_1}{\omega}\left(\cos\omega t,\ -\sin\omega t\right)\ +\ \boldsymbol{\rho}_3(t)\ , \\
\boldsymbol{\rho}_2(t) &= -\frac{\mathrm{v}_2}{\omega}\left(\cos\omega t,\ -\sin\omega t\right)\ +\ \boldsymbol{\rho}_3(t)\ , \\
\boldsymbol{\rho}_3(t) &= \frac{\mathrm{v}_3}{\omega_3}\left(\cos\omega_3 t,\ -\sin\omega_3 t\right)\ ,
\end{aligned}
\tag{16}
$$

where $\mathrm{v}_1 > 0, \mathrm{v}_2 > 0, \mathrm{v}_3 \geq 0$, ω and ω_3 are real parameters to be determined by the Newton equations (12). For these trajectories (16) the relative distances between the particles remain unchanged during time evolution

$$
|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2| = \frac{\mathrm{v}_1 + \mathrm{v}_2}{\omega}\ , \qquad |\boldsymbol{\rho}_1 - \boldsymbol{\rho}_3| = \frac{\mathrm{v}_1}{\omega}\ , \qquad |\boldsymbol{\rho}_2 - \boldsymbol{\rho}_3| = \frac{\mathrm{v}_2}{\omega}\ , \tag{17}
$$

hereafter without loss of generality we assume $\omega > 0$. Substituting (16) in (12) we obtain the following set of algebraic equations

$$
\mathrm{v}_3\left(B\,e_1 - m_1\,\omega_3\right) = 0\ , \quad \mathrm{v}_3\left(B\,e_2 - m_2\,\omega_3\right) = 0\ , \quad \mathrm{v}_3\left(B\,e_3 - m_3\,\omega_3\right) = 0\ , \tag{18}
$$

$$
e_3\left(\frac{e_1}{\mathrm{v}_1^2} - \frac{e_2}{\mathrm{v}_2^2}\right)\omega^2 = 0\ , \tag{19}
$$

$$
B\,e_1\,\mathrm{v}_1 - \omega\left[m_1\mathrm{v}_1 + \frac{e_1(e_2\mathrm{v}_1^2 + e_3(\mathrm{v}_1 + \mathrm{v}_2)^2)\,\omega}{\mathrm{v}_1^2\,(\mathrm{v}_1 + \mathrm{v}_2)^2}\right] = 0\ , \tag{20}
$$

$$
B\,e_2\,\mathrm{v}_2 - \omega\left[m_2\mathrm{v}_2 + \frac{e_2(e_1\mathrm{v}_2^2 + e_3(\mathrm{v}_1 + \mathrm{v}_2)^2)\,\omega}{\mathrm{v}_2^2\,(\mathrm{v}_1 + \mathrm{v}_2)^2}\right] = 0\ . \tag{21}
$$

We consider fixed charges and masses, the goal is to find v_i ($i = 1, 2, 3$), the magnetic field B and the two angular frequencies ω and ω_3 such that the equations (18)-(21) are satisfied.
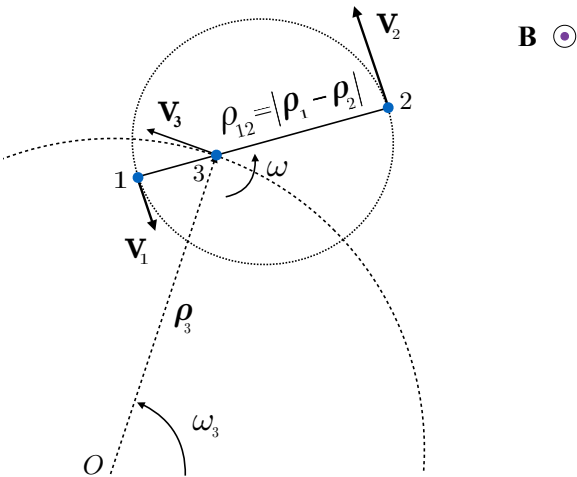
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FIG. 1. Configuration I. Two charges e_1 and e_2 rotate, with angular frequency ω and opposite velocities, around e_3 which by itself rotates with frequency ω_3. At $v_3 = 0$ (the inner charge at rest), these special trajectories are realized by the three electron, neutral and Helium-like systems. For $v_3 \neq 0$, this configuration appears only for charges of the same sign.

The solutions of Eqs. (18)-(21) determine completely the initial conditions for which the Configuration I occurs.

Now, notice that in (18) all the equations are proportional to v_3 while Eqs. (19)-(21) do not depend on v_3 at all. Thus, we distinguish two cases $v_3 = 0$ and $v_3 \neq 0$.

1. Case $v_3 = 0$

This corresponds to the situation when the third charge (e_3) is at rest, $v_3 = 0$ and thus $\boldsymbol{\rho}_3(t) = 0$, while the two charges (e_1, e_2) move around it in clockwise (or counterclockwise) direction with frequency ω and antiparallel velocities. For $v_3 = 0$, the equations (18) are satisfied trivially. The Eq. (19) implies $e_1 e_2 > 0$ (charges of the same sign) and it leads to the following expression for v_2

$$v_2 = v_1 \sqrt{\frac{e_2}{e_1}} \ . \tag{22}$$

For arbitrary magnetic field B, to determine a closed analytical expression for v_1 solution

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of the remaining equations (20)-(21) is not only a difficult task but unnecessary. In fact, we can indicate the value of magnetic field $B \equiv B_I$ for which these special trajectories occur, it is derived from the compatibility condition of the Eqs. (20)-(21)

$$B_I = \frac{(e_2\,m_1 - e_1\,m_2)\,r\,(1+r)^2\,(m_1 - m_2\,r)\,\mathrm{v}_1^3}{e_1\,(e_1 - r\,e_2)^2\,(e_2 + e_3\,(1+r)^2)} \neq 0\,, \tag{23}$$

where we used (22) and $r = \sqrt{\frac{e_2}{e_1}}$. For the magnetic field B_I, the frequency ω is

$$\omega = \frac{(e_2\,m_1 - e_1\,m_2)\,r\,(1+r)^2\,\mathrm{v}_1^3}{e_1\,(e_1 - r\,e_2)\,(e_2 + e_3\,(1+r)^2)} > 0\,.$$

Notice that both B_I and ω vanish at $(e_2\,m_1 - e_1\,m_2) = 0$, i.e. at $e_{c1} = 0$. Therefore the case $e_{c1} = 0$ should be considered separately.

Two particles with equal Larmor frequency ($e_{c1} = 0$)

Direct analysis of (19)-(21) shows that for $e_{c1} = 0$ there exist special trajectories for two identical particles only. More precisely, for two identical particles $e_1 = e_2 = e$, $m_1 = m_2 = m$, and $\mathrm{v}_1 = \mathrm{v}_2 = \mathrm{v}$, the Eqs. (20)-(21) do coincide. Then, the Eqs. (20)-(21) reduce to a single equation

$$4\,B\,e\,\mathrm{v}^3 - \omega\,[4\,m\,\mathrm{v}^3 + e\,(e + 4\,e_3)\,\omega] = 0\,. \tag{24}$$

From Eqs. (24) and (17) we immediately find

$$\mathrm{v} = \frac{e\,B\,\rho_{12}}{4\,m}\left(1 \pm \sqrt{1 - \frac{8\,m\,(e + 4\,e_3)}{e\,B^2\,\rho_{12}^3}}\right)\,. \tag{25}$$

Thus, for given ρ_{12} there exist two different initial velocities v leading to the same circular trajectory presented in Fig. 1. However, for given v there exists a single circular trajectory with a certain ρ_{12}. It corresponds to rotation with frequency equal to $\frac{2\,\mathrm{v}}{\rho_{12}}$. It is interesting that for a given magnetic field there exists a minimal circular trajectory with $\rho_{12} = \rho_{min} = \left(\frac{8\,m}{e\,B^2}(e + 4\,e_3)\right)^{\frac{1}{3}}$ (when the square root in (25) vanishes).

Conserved quantities ($\mathrm{v}_3 = 0$)

At $\mathrm{v}_3 = 0$, the system is *particularly superintegrable*. Evaluated along the special trajectories (16), the six quantities $(\mathcal{H},\,\mathbf{K}^2,\,\mathbf{L}_z^{total},\,\ell_{z3},\,T_1,\,T_2)$ are in involution where $T_i = \frac{(\mathbf{p}_i - e_i\,\mathbf{A}_{\rho_i})^2}{2\,m_i}$,

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$i = 1, 2$. Moreover, the function $\mathcal{I} = \boldsymbol{\tau}_1 \cdot \mathbf{p}_{\boldsymbol{\tau}_1}$ is an extra *particular* constant of motion. We emphasize that the three quantities $(\mathcal{H}, \mathbf{K}^2, \mathbf{L}_z^{total})$ are integrals, they are conserved for any trajectory (arbitrary initial conditions) while $(\ell_{z_3}, T_1, T_2, \mathcal{I})$ are constants only for special ones (constraint on initial conditions (18)-(21)).

Explicitly, they take the values

$$\mathcal{H} = \frac{\mathrm{v}_1^2}{2}\left[m_1 + m_2\, r^2 + \frac{2\, e_1\, r^2\, (1+r)|m_2 - m_1\, r^2|}{|e_1\,(r^3-1)\,(e_1\, r^2 + e_3(1+r))|}(e_1\, r + e_3(1+r)) + \frac{2\, e_1\, e_3\,(1+r)^2|m_2\, r - m_1\, r^3|}{|e_1\,(r^3-1)\,(e_1\, r^2 + e_3(1+r))|}\right] ,$$

$$\mathbf{K}^2 = 0 ,$$

$$L_z^{total} = \frac{e_1(r^3-1)(e_1 r^2 + e_3(1+r)^2)[B_I e_1^2(r^3 - 1 - r^4 + r^7)(e_1 r^2 + e_3(1+r)^2) + 2r(1+r)^2 \mathrm{v}_1^3(m_1 r^2 - m_2)(m_2 r^2 + m_1)]}{2r^2(1+r)^4 \mathrm{v}_1^4(m_2 - m_1 r^2)^2} ,$$

$$T_1 = \frac{1}{2}m_1\, \mathrm{v}_1^2 , \qquad T_2 = \frac{1}{2}m_1\, r^2\, \mathrm{v}_1^2 , \qquad \ell_{z_3} = 0 , \qquad \mathcal{I} = 0 ,$$

$$\tag{26}$$

where $\mathrm{v}_1 > 0$, $e_{c1} \neq 0$, $B_I = \frac{r\,(1+r)^2\,(m_1 - m_2\, r)(m_1\, r^2 - m_2)\,\mathrm{v}_1^3}{e_1^2\,(r^3-1)^2\,(e_1\, r^2 + e_3\,(1+r)^2)}$ and $r^2 = \frac{e_2}{e_1}$.

2. *Case* $\mathrm{v}_3 \neq 0$

For $\mathrm{v}_3 \neq 0$ the Eq. (18) implies that both coupling charges $e_{c1} = e_{c2} = 0$ vanish, namely, the three charges must possess the same charge-to-mass ratio

$$\frac{e_1}{m_1} = \frac{e_2}{m_2} = \frac{e_3}{m_3} \equiv \alpha . \tag{27}$$

Substituting (16) into the Newton equations of motion (12) we find the algebraic equations

$$\omega_3 = B\,\alpha , \tag{28}$$

$$\frac{e_1}{\mathrm{v}_1^2} = \frac{e_2}{\mathrm{v}_2^2} , \tag{29}$$

$$B\,\alpha - \omega\left(1 + \frac{e_1(e_2\mathrm{v}_1^2 + e_3(\mathrm{v}_1 + \mathrm{v}_2)^2)\,\omega}{m_1\,\mathrm{v}_1^3(\mathrm{v}_1 + \mathrm{v}_2)^2}\right) = 0 , \tag{30}$$

$$B\,\alpha - \omega\left(1 + \frac{e_2(e_1\mathrm{v}_2^2 + e_3(\mathrm{v}_1 + \mathrm{v}_2)^2)\,\omega}{m_2\,\mathrm{v}_2^3(\mathrm{v}_1 + \mathrm{v}_2)^2}\right) = 0 . \tag{31}$$

In particular, the condition (28) implies that the motion of the charge e_3 corresponds to that of a free particle in a constant magnetic field B, thus it rotates on a circular path with frequency $\omega_3 = B\,\alpha$ and arbitrary $\mathrm{v}_3 > 0$. Moreover, the Eqs. (30)-(31) are satisfied for two

identical particles $e_1 = e_2 = e$, $m_1 = m_2 = m$, and $v_1 = v_2 = v$ only. The corresponding motion was already described in detail in the previous section, see (25). By virtue of (27), all three charges must be of the same sign and $(e_3\, m - e\, m_3) = 0$. An example is presented in Fig. 2.
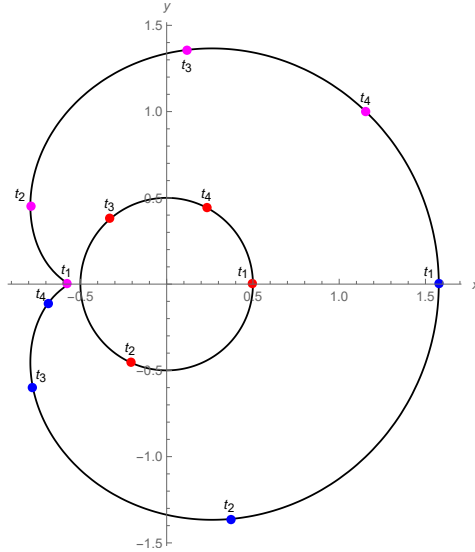

FIG. 2. Configuration I ($v_3 \neq 0$): special trajectories for a three electron system, $e_1 = e_2 = e_3 = -1$ and $m_1 = m_2 = m_3 = 1$. The numerical solutions of the Newton equations (12) are displayed for the values $B = -2$, $v_1 = v_2 = \sqrt[3]{\frac{5}{4}}$, $v_3 = 1$ and $\omega = 1$, $\omega_3 = 2$ obtained by solving the algebraic equations (28)-(31). The successive positions of the third charge, which moves in a circular trajectory, are indicated by red points at different times $t_1 < t_2 < t_3 < t_4$. Similarly, the locations of the first charge (blue points) and the second one (magenta points) are presented. The relative distances between particles remain constant during time evolution.

Conserved quantities $v_3 \neq 0$

For the case $v_3 \neq 0$, two particles are identical $e_1 = e_2 = e$, $m_1 = m_2 = m$ and $v_1 = v_2 = v$. The system is again *particularly superintegrable*. Along any special trajectory the six quantities $(\mathcal{H}, \mathbf{K}^2, \mathbf{L}_z^{total}, \ell_{z_3}, T_3, k_3)$ where $k_3 = (\mathbf{p}_3 + e_3\, \mathbf{A}_{\boldsymbol{\rho}_3}) \cdot \hat{\mathbf{x}}$ are in involution. Again, the function $\mathcal{I} = \boldsymbol{\tau}_1 \cdot \mathbf{p}_{\boldsymbol{\tau}_1}$ is an extra *particular* constant of motion.

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Explicitly, they take the values

$$\mathcal{H} = \frac{m_3\, v_3^2}{2} + \frac{e^2\, B^2\, \rho_{12}^2}{16\, m}\left[1 \pm \sqrt{1 - \frac{8\, m\,(e + 4\, e_3)}{e\, B^2\, \rho_{12}^3}}\,\right]^2 + \frac{e\,(e + 4\, e_3)}{\rho_{12}}\ ,$$

$$\mathbf{K}^2 = 0\ ,$$

$$L_z^{total} = \frac{e\, B\, \rho_{12}^2}{4} - \frac{(2\, m + m_3)\, m_3\, v_3^2}{2\, e_3\, B} - \frac{e\, B\, \rho_{12}^2}{4}\left(1 \pm \sqrt{1 - \frac{8\, m\,(e + 4\, e_3)}{e\, B^2\, \rho_{12}^3}}\,\right)\ , \qquad (32)$$

$$\ell_{z3} = -\frac{m_3^2\, v_3^2}{2\, e_3\, B}\ ,$$

$$T_3 = \frac{m_3\, v_3^2}{2}\ ,$$

$$k_3 = 0\ , \qquad \mathcal{I} = 0\ ,$$

where ρ_{12} is taken from (25). The allowed values of the magnetic field are given by $B^2 \geq \frac{8\, m\,(e + 4\, e_3)}{e\, \rho_{12}^3}$.

B. CONFIGURATION II

The Configuration II corresponds to three particles rotating clockwise (or counterclockwise) in phase, around a fixed common center, all with the same frequency ω. This Configuration II is presented in Fig. 3.
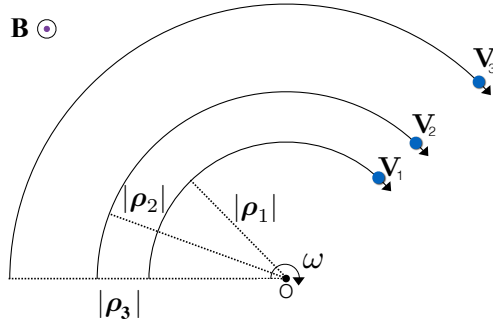

FIG. 3. Configuration II. Three particles rotate clockwise (or counterclockwise) in phase, around a fixed common center, with the same frequency ω. This Configuration is not admitted for three-electron systems.

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As a function of time, the trajectories are given by

$$
\begin{aligned}
\boldsymbol{\rho}_1(t) &= \frac{v_1}{\omega}\left(\cos\omega t,\ -\sin\omega t\right), \\
\boldsymbol{\rho}_2(t) &= \frac{v_2}{\omega}\left(\cos\omega t,\ -\sin\omega t\right), \\
\boldsymbol{\rho}_3(t) &= \frac{v_3}{\omega}\left(\cos\omega t,\ -\sin\omega t\right),
\end{aligned}
\tag{33}
$$

where without losing generality we have assumed

$$
v_3 > v_2 > v_1 > 0 ; \qquad \omega > 0 .
$$

The concrete value of these quantities, as in the previous case, is determined by the Eqs. (12). The constant relative distances between the particles read

$$
\left|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2\right| = \frac{v_2 - v_1}{\omega}, \quad \left|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_3\right| = \frac{v_3 - v_1}{\omega}, \quad \left|\boldsymbol{\rho}_2 - \boldsymbol{\rho}_3\right| = \frac{v_3 - v_2}{\omega},
\tag{34}
$$

they remain unchanged during time evolution.

Putting (33) into the Newton equations (12) we arrive to three algebraic equations

$$
B\,e_1 v_1 - m_1\,v_1\,\omega + e_1\,\omega^2\left(\frac{e_2}{(v_1 - v_2)^2} + \frac{e_3}{(v_1 - v_3)^2}\right) = 0 ,
\tag{35}
$$

$$
B\,e_2 v_2 - m_2\,v_2\,\omega + e_2\,\omega^2\left(\frac{e_3}{(v_2 - v_3)^2} - \frac{e_1}{(v_1 - v_2)^2}\right) = 0 ,
\tag{36}
$$

$$
B\,e_3 v_3 - m_3\,v_3\,\omega - e_3\,\omega^2\left(\frac{e_1}{(v_1 - v_3)^2} + \frac{e_2}{(v_2 - v_3)^2}\right) = 0 .
\tag{37}
$$

From (35)-(37) it follows that for particles with equal charge to mass ratio

$$
\frac{e_1}{m_1} = \frac{e_2}{m_2} = \frac{e_3}{m_3} = \alpha ,
$$

($\alpha \neq 0$ is a real constant) the Configuration II does not occur. Therefore, three electrons cannot perform this Configuration II.

Now, the sum of the equations (35)-(37) gives the following relation between the magnetic field B and the frequency ω

$$
\omega = \left(\frac{e_1\,v_1 + e_2\,v_2 + e_3\,v_3}{m_1\,v_1 + m_2\,v_2 + m_3\,v_3}\right) B .
\tag{38}
$$

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The above relation (38) is a necessary (not sufficient) condition for the existence of special trajectories in the Configuration II. Substituting (38) into the equation (36) and solving for $B \equiv B_{II}$ we obtain

$$B_{II} = \frac{(v_1 - v_2)^2 v_2 (v_2 - v_3)^2 (m_1 v_1 + m_2 v_2 + m_3 v_3) (e_2 (m_1 v_1 + m_3 v_3) - m_2 (e_1 v_1 + e_3 v_3))}{e_2 (e_1 (v_2 - v_3)^2 - e_3 (v_1 - v_2)^2) (e_1 v_1 + e_2 v_2 + e_3 v_3)^2} .$$

$$(39)$$

From a physical point of view we have to impose $B_{II} \neq 0$ and finite.

In general, substituting (39) into (37) we get a multivariate polynomial equation of sixth order in the variables v_1, v_2, v_3

$$\mathbb{P}_6(v_1, v_2, v_3) \equiv \sum_{i+j+k=6} a_{ijk} v_1^i v_2^j v_3^k = 0 , \qquad (40)$$

where the coefficients $a_{ijk} = a_{ijk}(e_1, e_2, e_3, m_1, m_2, m_3)$ are completely determined by the system we deal with, (see Appendix). For fixed charges and masses in (40), it is not possible to write its general solution analytically, meaning a relation of the form $v_1 = (e_1, e_2, e_3, m_1, m_2, m_3; v_2, v_3)$ such that (40) is satisfied. However, we can do so in the following physically relevant systems.

Particles with special charge to mass ratio

When the charge to mass ratio of the particle e_1 is equal to the ratio of a composite particle (formed by the other charges), namely

$$\frac{e_1}{m_1} - \frac{(e_2 v_2 + e_3 v_3)}{(m_2 v_2 + m_3 v_3)} = 0 , \qquad (41)$$

$(\frac{e_1}{m_1} \neq \frac{e_3}{m_3})$ then the polynomial (40) simplifies. In this case we obtain effectively a quadratic polynomial in v_1 solutions of which are given by

$$v_1 = \frac{e_1 v_3 (m_2 v_2 + m_3 v_3) + e_3 m_1 (v_2^2 - v_3^2) \pm (v_2 - v_3) \sqrt{e_3 m_1 v_2 (e_3 m_1 v_3 - e_1 (m_2 v_2 + m_3 v_3))}}{e_3 m_1 (v_2 - v_3) + e_1 (m_2 v_2 + m_3 v_3)} ,$$

$$(42)$$

and they give rise to the Configuration II. There exist special values of velocities and masses for which the neutral system $Q = 0$ occurs in this Configuration II. The corresponding magnetic field B is given by (39) with v_1 taken from (42).

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Helium-like system

Here we consider in detail the physically important case of a neutral system

$$Q \; = \; e_1 \; + \; e_2 \; + \; e_3 \; = \; 0 \; ,$$

with two identical particles (Helium-like system)

$$e_2 \; = \; e_3 \; \equiv \; e \; , \qquad m_2 \; = \; m_3 \; \equiv \; m \; , \qquad \frac{e_1}{m_1} \; \neq \; \frac{e}{m} \; .$$

For any initial v_2 and v_3 such that

$$v_3 \; \geq \; \lambda \; > \; v_2 \; > \; 0 \; , \tag{43}$$

where λ obeys the cubic equation

$$\lambda^3 \; - \; 117 \, v_2 \, \lambda^2 \; - \; 81 \, v_2^2 \, \lambda \; - \; 27 \, v_2^3 \; = \; 0 \; , \tag{44}$$

thus $\lambda = \lambda(v_2)$, there exists an initial velocity v_1 for which the Configuration II occurs. This $v_1 = v_1(v_2, v_3)$ obeys the following quartic equation

$$v_1^4 \, (v_3 + v_2) \; - \; 2 \, v_1^3 \, (v_3^2 + 2v_3v_2 + v_2^2) \; + \; v_1^2 \, (3v_3^3 - v_3^2v_2 + 11v_3v_2^2 - v_2^3)$$
$$+ \, 2 \, v_1 \, (3v_3^3v_2 - 2v_3^2v_2^2 - 5v_3v_2^3 + 2v_2^4 - 2v_3^4) + (2v_3^5 - 4v_3^4v_2 + 3v_3^3v_2^2 - v_3^2v_2^3 + 4v_3v_2^4 - 2v_2^5) \; = \; 0 \; . \tag{45}$$

The corresponding magnetic field B is given by

$$B_{II}^{Helium-system} \; = \; \frac{(2m + m_1) \, v_1 v_2 \, (v_1 - v_2)^2 \, (v_2 - v_3)^2 \, (m_1 v_1 + m \, (v_2 + v_3))}{e^3 \, (2v_1 - v_2 - v_3)^2 \, (v_1^2 - 2v_2v_1 + 3v_2^2 + 2v_3^2 - 4v_2v_3)} \; ,$$

and the frequency takes the form

$$\omega \; = \; \frac{(2m + m_1) \, v_1 \, (v_1 - v_2)^2 v_2 \, (v_2 - v_3)^2}{e^2 \, (2v_1 - v_2 - v_3) \, (v_1^2 - 2v_2v_1 + 3v_2^2 + 2v_3^2 - 4v_2v_3)} \; .$$

Therefore, a two-parametric family of special trajectories occur. They are functions of the parameters v_2 and v_3, see (43).

Conserved quantities

For the Configuration II shown in Fig. (3), the system is again *particularly superintegrable*. Along any special trajectory (33) the six quantities $(\mathcal{H}, \mathbf{K}^2, \mathbf{L}_z^{total}, \ell_{z_2}, T_1, T_2)$ are in involution. The function $\mathcal{I} = \boldsymbol{\tau}_1 \cdot \mathbf{p}_{\boldsymbol{\tau}_1}$ is an extra *particular* constant of motion as well.

Explicitly, the integrals and *particular* constants take the values

$$\mathcal{H} = \frac{1}{2} \left[\frac{2\, B_{II} \left(\frac{e_2 e_3}{v_2 - v_3} + \frac{e_1 e_2}{v_1 - v_2} + \frac{e_1 e_3}{v_1 - v_3} \right) (e_1\, v_1 + e_2\, v_2 + e_3\, v_3)}{m_1\, v_1 + m_2\, v_2 + m_3\, v_3} + m_1\, v_1^2 + m_2\, v_2^2 + m_3\, v_3^2 \right] \,,$$

$$\mathbf{K}^2 = 0 \,,$$

$$L_z^{total} = \frac{(m_1 v_1 + m_2 v_2 + m_3 v_3)^2}{2\, B_{II}\, (e_1 v_1 + e_2 v_2 + e_3 v_3)^2} \left[e_1 v_1^2 + e_2 v_2^2 + e_3 v_3^2 - \frac{2\, (e_1 v_1 + e_2 v_2 + e_3 v_3)\, (m_1 v_1^2 + m_2 v_2^2 + m_3 v_3^2)}{m_1 v_1 + m_2 v_2 + m_3 v_3} \right] \,,$$

$$\ell_{z_2} = \frac{v_2^2\, (m_1 v_1 + m_2 v_2 + m_3 v_3) \left[\, e_2\, (m_1 v_1 - m_2 v_2 + m_3 v_3) - 2 m_2\, (e_1 v_1 + e_3 v_3)\, \right]}{2\, B_{II}\, (e_1 v_1 + e_2 v_2 + e_3 v_3)^2} \,,$$

$$T_i = \frac{m_i\, v_i^2}{2} \,, \qquad (i = 1, 2) \,,$$

$$\mathcal{I} = 0 \,,$$

$$(46)$$

where B_{II} is given in (39) and v_1, v_2, v_3 are solutions of the polynomial equation (40).
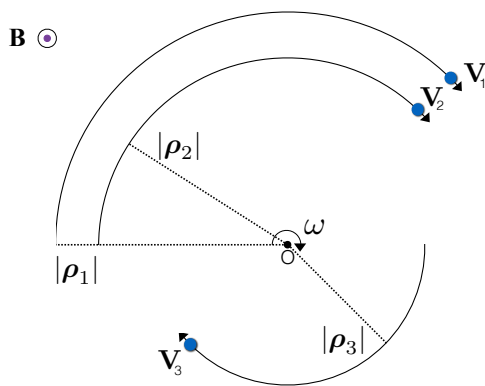
C.   CONFIGURATION III

FIG. 4. Configuration III. Two charges move in phase (counter)clockwise on two concentric circles with a relative phase π with respect to the other charge.

The Configuration III corresponds to three particles rotating clockwise (or counterclockwise) with the same frequency ω, in which two of them move in phase and the third one is shifted with a phase π. In this trajectory the not-in-phase particle can be located: a) along the inner radius, b) at the outer radius or c) in the central radius. The Configuration III for the case a) is presented in Fig. 4.

For the case a), the explicit form of the trajectories is given by

$$
\begin{aligned}
\boldsymbol{\rho}_1(t) &= \frac{\mathrm{v}_1}{\omega}\left(\cos\omega t, -\sin\omega t\right), \\
\boldsymbol{\rho}_2(t) &= \frac{\mathrm{v}_2}{\omega}\left(\cos\omega t, -\sin\omega t\right), \\
\boldsymbol{\rho}_3(t) &= -\frac{\mathrm{v}_3}{\omega}\left(\cos\omega t, -\sin\omega t\right).
\end{aligned}
\tag{47}
$$

For convenience we assume the following conditions

$$
\mathrm{v}_1 > \mathrm{v}_2 > \mathrm{v}_3 > 0 ; \qquad \omega > 0 .
$$

The concrete value of these parameters, as in the previous cases, is determined by the equations of motion (12). The relative distances between the particles take the form

$$
|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2| = \frac{\mathrm{v}_1 - \mathrm{v}_2}{\omega}, \qquad |\boldsymbol{\rho}_1 - \boldsymbol{\rho}_3| = \frac{\mathrm{v}_1 + \mathrm{v}_3}{\omega}, \qquad |\boldsymbol{\rho}_2 - \boldsymbol{\rho}_3| = \frac{\mathrm{v}_2 + \mathrm{v}_3}{\omega} .
\tag{48}
$$

For the circular trajectories (47), the Newton equations (12) lead to the following system of algebraic equations

$$
B\, e_1\, \mathrm{v}_1 - m_1\, \mathrm{v}_1\, \omega - e_1\, \omega^2 \left(\frac{e_2}{(\mathrm{v}_1 - \mathrm{v}_2)^2} + \frac{e_3}{(\mathrm{v}_1 + \mathrm{v}_3)^2} \right) = 0 ,
\tag{49}
$$

$$
B\, e_2\, \mathrm{v}_2 - m_2\, \mathrm{v}_2\, \omega - e_2\, \omega^2 \left(\frac{e_3}{(\mathrm{v}_2 + \mathrm{v}_3)^2} - \frac{e_1}{(\mathrm{v}_1 - \mathrm{v}_2)^2} \right) = 0 ,
\tag{50}
$$

$$
-B\, e_3\, \mathrm{v}_3 + m_3\, \mathrm{v}_3\, \omega + e_3\, \omega^2 \left(\frac{e_1}{(\mathrm{v}_1 + \mathrm{v}_3)^2} + \frac{e_2}{(\mathrm{v}_2 + \mathrm{v}_3)^2} \right) = 0 .
\tag{51}
$$

By adding the three equations (49)-(51) we arrive to the relation between the magnetic field B and the frequency ω

$$
\omega = \left(\frac{e_1\, \mathrm{v}_1 + e_2\, \mathrm{v}_2 - e_3\, \mathrm{v}_3}{m_1\, \mathrm{v}_1 + m_2\, \mathrm{v}_2 - m_3\, \mathrm{v}_3} \right) B .
\tag{52}
$$

The relation (52) is a necessary (not sufficient) compatibility condition of the Eqs. (49)-(51). Substituting (52) into the equation (50) and solving for B we obtain

$$B_{III} = -\frac{(v_1 - v_2)^2 v_2 (v_2 + v_3)^2 (m_1 v_1 + m_2 v_2 - m_3 v_3)(e_2 (m_1 v_1 - m_3 v_3) + m_2 (e_3 v_3 - e_1 v_1))}{e_2 (e_1 (v_2 + v_3)^2 - e_3 (v_1 - v_2)^2)(e_1 v_1 + e_2 v_2 - e_3 v_3)^2} .$$

(53)

Notice that by replacing $v_3 \to -v_3$ in B_{III} we obtain $-B_{II}$ (see (39)).

Finally, putting Eqs. (52)-(53) into the Eq. (51) we also get a sixth order polynomial equation in the variables v_1, v_2, v_3 which coincides with (40) when $v_3 \to -v_3$,

$$\sum_{i+j+k=6} a_{ijk} v_1^i v_2^j (-v_3)^k = 0 .$$

(54)

Therefore, its solutions can be obtained from those of (40). The system is *particularly superintegrable*, and the corresponding integrals and *particular* constants of motion are given by (46) with the substitution $B_{II} \to B_{III}$ and $v_3 \to -v_3$.

It is worth to note that for particles with the special charge to mass ratio

$$\frac{e_1}{m_1} - \frac{(e_2 v_2 - e_3 v_3)}{(m_2 v_2 - m_3 v_3)} = 0 , \qquad \left(\frac{e_1}{m_1} \neq \frac{e_3}{m_3} \right) ,$$

(55)

the sixth order polynomial (67) becomes a second order polynomial in the v_1 variable. The two independent solutions are given by the Eqs. (42) with the substitution $v_3 \to -v_3$. There exist special values of velocities and masses for which the neutral system $Q = 0$ occurs in this Configuration III. The corresponding magnetic field B is given by (53).

For the cases in which the not-in-phase particle (see Fig. 4) is located either at the outer radius or in the central radius the corresponding special trajectories can be obtained straightforwardly.

IV. N-BODY CASE: SPECIAL TRAJECTORIES

Now, we proceed to study the case of $n \geq 2$ Coulomb charges on the plane in a constant perpendicular magnetic field. The Hamiltonian is of the form

$$\mathcal{H} = \sum_{i=1}^{n} \frac{(\mathbf{p}_i - e_i \mathbf{A}_{\boldsymbol{\rho}_i})^2}{2 m_i} + \sum_{i,j=1,2,\ldots,n \, ; \, j>i} \frac{e_i e_j}{|\boldsymbol{\rho}_i - \boldsymbol{\rho}_j|} ,$$

(56)

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where $\boldsymbol{\rho}_i$ is the position vector of particle i, $\mathbf{p}_i$ is the associated canonical momentum and $\mathbf{A_r} = \frac{1}{2}(\mathbf{B} \times \mathbf{r})$. The total Pseudomomentum

$$\mathbf{K} \equiv (K_x, K_y) = \sum_{i=1}^{n} \boldsymbol{k}_i = \sum_{i=1}^{n} (\mathbf{p}_i + e_i \mathbf{A}_{\boldsymbol{\rho}_i}) , \tag{57}$$

is a constant of motion [16], the Poisson bracket $\{\mathbf{K}, \mathcal{H}\} = 0$ vanishes. The total *canonical* momentum $\mathbf{L}_z^{total}$

$$\mathbf{L}_z^{total} \equiv \sum_{i=1}^{n} \boldsymbol{\ell}_{z_i} = \sum_{i=1}^{n} \boldsymbol{\rho}_i \times \mathbf{p}_i , \tag{58}$$

is also conserved, $\{\mathbf{L}_z^{total}, \mathcal{H}\} = 0$. Hence, the problem is characterized at least by three conserved quantities (integrals) $K_{x,y}$, L_z^{total}. The dimension of the configuration space is $2n$. The problem (56) is not integrable, the number of integrals (including the Hamiltonian) is much less than the dimension of the configuration space. The integrals $K_{x,y}$, L_z^{total} are not in involution, they obey the commutation relations (4) with $Q = e_1 + e_2 + \cdots + e_n$. From (56) we obtain the Newton equations

$$m_i \ddot{\boldsymbol{\rho}}_i = e_i \dot{\boldsymbol{\rho}}_i \times \mathbf{B} + \sum_{j=1,2,\ldots,n \,; \, j \neq i} \frac{e_i e_j}{|\boldsymbol{\rho}_i - \boldsymbol{\rho}_j|^2} (\boldsymbol{\rho}_i - \boldsymbol{\rho}_j) , \qquad i = 1, 2, \ldots, n . \tag{59}$$

From (59) it follows that

$$\sum_{i=1}^{n} \left[\frac{m_i}{e_i} \ddot{\boldsymbol{\rho}}_i - \dot{\boldsymbol{\rho}}_i \times \mathbf{B} \right] = 0 . \tag{60}$$

Therefore, from (60) we arrive to the following interesting result: for the Hamiltonian $\mathcal{H}$ (56) with particles of the same charge-to-mass ratio (equal Larmor frequencies)

$$\frac{e_i}{m_i} = \alpha , \qquad i = 1, 2, \ldots, n , \tag{61}$$

exact separation of the CM occurs. Its motion is described by the equation $M\ddot{\mathbf{R}} = Q\dot{\mathbf{R}} \times \mathbf{B}$ and possesses the same Larmor frequency equal to αB. For the physically important n-electron system, the condition (61) is realized. This is the analog to the well known separation of variables for the free field case $B = 0$.

A. Special trajectories

The generalization of the Configuration II to the case of n particles corresponds to the situation in which all the charges rotate clockwise (or counterclockwise) in phase with the same angular frequency ω, (see Fig.(5)). These special circular trajectories read

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$$\boldsymbol{\rho}_i(t) \ = \ \frac{\mathrm{v}_i}{\omega} \left(\cos \omega t, \ - \sin \omega t \right) \quad , \qquad i = 1, 2, \ldots, n \ , \tag{62}$$

where without losing generality, we assume

$$\mathrm{v_i} > \mathrm{v_j} > 0 \ , \quad (\mathrm{i} > \mathrm{j}) \quad ; \qquad \omega > 0 \ .$$

The concrete value of these quantities are determined by the equations of motion (59). The constant relative distances between the particles are given by

$$|\boldsymbol{\rho}_i - \boldsymbol{\rho}_j| = \frac{\mathrm{v_i} - \mathrm{v_j}}{\omega} \ , \quad i, j = 1, 2, \ldots, n \ ; \quad i > j \ , \tag{63}$$
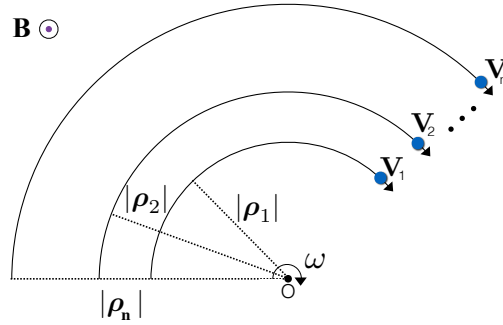
all remain unchanged during time evolution.

FIG. 5. Configuration II: special trajectories where the n particles rotate clockwise (or counter-clockwise) in phase with the angular frequency ω.

Putting (62) into the Newton equations (59) we arrive to the system of n coupled algebraic equations

$$B \, e_i \, \mathrm{v}_i \ - \ m_i \, \mathrm{v}_i \, \omega \ + \ e_i \, \omega^2 \sum_{j=1,2,\ldots,n \, ; \, i \neq j} \frac{e_j \, s}{(\mathrm{v}_i - \mathrm{v}_j)^2} \ = \ 0 \quad , \quad i = 1, 2, \ldots, n \ , \tag{64}$$

where $s = 1$ for $j > i$ and $s = -1$ for $j < i$.

From (64) it follows that for particles with equal charge to mass ratio

$$\frac{e_i}{m_i} \ \equiv \ \alpha \quad , \qquad i = 1, 2, \ldots n \ ,$$

with $\alpha \neq 0$ a real constant, the Configuration II does not occur. Therefore, n electrons cannot perform this type of configuration.

Now, the sum of the n equations (64) gives the following relation between the magnetic field B and the frequency ω

$$\omega = \left(\frac{e_1\, \mathrm{v}_1 + e_2\, \mathrm{v}_2 + \ldots + e_n\, \mathrm{v}_n}{m_1\, \mathrm{v}_1 + m_2\, \mathrm{v}_2 + \ldots + m_n\, \mathrm{v}_n} \right) B \; . \tag{65}$$

The above relation (65) is a necessary (not sufficient) condition for the existence of special trajectories in the Configuration II. Substituting (65) into the equation of motion in (64) with $i = 2$ and then solving for $B \equiv B_{II}$ we obtain

$$B_{II} = \frac{\mathrm{v}_2\, (m_2\, \kappa - e_2)}{e_2\, \kappa^2} \left[\sum_{j=3}^{n} \frac{e_j}{(\mathrm{v}_2 - \mathrm{v}_j)^2} - \frac{e_1}{(\mathrm{v}_2 - \mathrm{v}_1)^2} \right]^{-1} , \tag{66}$$

$\kappa = \frac{e_1\, \mathrm{v}_1 + e_2\, \mathrm{v}_2 + \ldots + e_n\, \mathrm{v}_n}{m_1\, \mathrm{v}_1 + m_2\, \mathrm{v}_2 + \ldots + m_n\, \mathrm{v}_n}$.

In general, substituting (66) into (64) we get a system of $(n-2)$ multivariate polynomial coupled equations of order $(4n-6)$ for the parameters $\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_n$

$$\sum_{g_1 + g_2 + \ldots + g_n = 4n-6} a^{(l)}_{g_1 g_2 \ldots g_n}\, \mathrm{v}_1^{g_1}\, \mathrm{v}_2^{g_2}\, \ldots\, \mathrm{v}_n^{g_n} = 0 \; , \qquad l = 1, 2, \ldots, n-2 \; , \tag{67}$$

where the coefficients $a^l_{ijk} = a^l_{ijk}(e_1, e_2, \ldots, e_n, m_1, m_2, \ldots, m_n)$. Therefore, formally there exist a two-parametric family of initial conditions for which special trajectories appear.

As for the integrals and *particular* constants of motion, in addition to the three integrals $K_{x,y}$, L_z^{total} (57)-(58) there exist $2\,(n-1)$ *particular* constants of motion, namely $(n-1)$ individual angular momenta ℓ_{z_i} and $(n-1)$ kinetic terms $\frac{1}{2} m_i\, \mathrm{v}_i^2$. The system is *particularly superintegrable*.

V. CONCLUSIONS

A classification of systems with three charged particles on the plane placed in a perpendicular constant magnetic field B, which admit special trajectories was presented. In general, these trajectories describe concentric circles of finite radii. Their main characteristic is that relative distances between particles remain unchanged under the time evolution. Similar to the two body case, it corresponds to the existence of *particular* constants of motion.

These special periodic trajectories are characterized by seven conserved quantities. The three integrals K_x, K_y, L_z^{total} which are conserved for any trajectory (arbitrary initial conditions) and four *particular* constants that emerge only for certain values of initial data. Hence, these trajectories are *particularly superintegrable*. The complete classification of such initial data was presented in detail.

There are three important physical systems admitting special trajectories:

- $Q = 0$ (neutral system), the special trajectories of all Configurations I, II and III appear.

- $\frac{e_1}{m_1} = \frac{e_2}{m_2} = \frac{e_3}{m_3} = \alpha$ (particles with equal Larmor radius): the special trajectories of Configuration I occur only.

- $e_1 = -2\,e\,;e_2 = e_3 = e$ (Helium-like system), all Configurations I, II and III appear.

Along these lines, results for the n-body problem in a constant magnetic field were presented as well. The separation of the center of mass for particles with the same charge to mass ratio (an n-electron system) and, in general, the existence of a non-trivial two-parametric family of special periodic trajectories were indicated.

The issue about the stability of the special trajectories was not addressed in the present work since it is not relevant for the quantum case which is the main goal we are interested in. After the standard quantization of the Hamiltonian $\mathcal{H}$ (1), the Pseudomomentum (2), the angular momentum (3), the *particular* constants of motion, i.e. upon replacing the momenta by the corresponding differential operators, one can ask whether there exist eigenstates which are common for $\mathcal{H}$ and one of the *particular* constants. Such common eigenfunctions may exist, as in the two-body problem [14], for certain discrete values of the magnetic field B and even for systems that classically are known to be chaotic like the neutral system. In this paper we have identified all the physical systems, the integrals and *particular* constants of motion that may lead to exact solutions of the quantum three body Coulomb problem in a magnetic field. This construction would imply a certain particular integrability: the commutator (the Lie bracket) of the Hamiltonian and an operator vanishes on a subspace of the Hilbert space [10].

VI. ACKNOWLEDGMENTS

VII. APPENDIX

The multivariate polynomial equation of sixth order in the variables v_1, v_2, v_3 solutions of which give rise to the Configuration II (Fig. 3) is given by

$$
\begin{aligned}
\mathbb{P}_6 &= e_2 e_3 \left(e_2 m_1 - e_1 m_2\right) v_2 v_1^5 + e_2 e_3 \left(e_3 m_1 - e_1 m_3\right) v_3 v_1^5 + 2 e_2 e_3 \left(e_1 m_2 - e_2 m_1\right) v_2^2 v_1^4 + 2 e_2 e_3 \left(e_1 m_3 - e_3 m_1\right) v_3^2 v_1^4 \\
&+ 2 e_2 e_3 \left(e_1 \left(m_2 + m_3\right) - e_2 m_1 - e_3 m_1\right) v_2 v_3 v_1^4 - \left(e_1 + e_2\right) e_3 \left(e_1 m_2 - e_2 m_1\right) v_2^3 v_1^3 + e_2 \left(e_1 - e_3\right) \left(e_1 m_3 - e_3 m_1\right) v_3^3 v_1^3 \\
&+ \left(e_2 e_3 \left(3 m_1 - m_2 - 4 m_3\right) e_1 - \left(e_3 m_2 + 2 e_2 m_3\right) e_1^2 + e_2 e_3 \left(e_2 + 4 e_3\right) m_1\right) v_2 v_3^2 v_1^3 \\
&+ \left(\left(2 e_3 m_2 + e_2 m_3\right) e_1^2 - e_2 e_3 \left(3 m_1 + 4 m_2 + m_3\right) e_1 + e_2 e_3 \left(4 e_2 + e_3\right) m_1\right) v_2^2 v_3 v_1^3 \\
&+ 2 e_1 e_3 \left(e_1 m_2 - e_2 m_1\right) v_2^4 v_1^2 + 2 e_1 e_2 \left(e_3 m_1 - e_1 m_3\right) v_3^4 v_1^2 \\
&+ \left[4 e_2 m_3 e_1^2 + \left(m_3 e_2^2 - e_3 \left(4 m_1 + m_2 - 3 m_3\right) e_2 - e_3^2 m_2\right) e_1 - 2 e_2 e_3^2 m_1\right] v_2 v_3^3 v_1^2 \\
&- 2 \left[\left(e_3 m_1 + e_1 m_3\right) e_2^2 + \left(m_3 e_1^2 - 2 e_3 m_2 e_1 + e_3^2 m_1\right) e_2 - e_1 e_3 \left(e_1 + e_3\right) m_2\right] v_2^2 v_3^2 v_1^2 \\
&+ \left[\left(m_3 e_2^2 + e_3 \left(4 m_1 + m_2 + m_3\right) e_2 - e_3^2 m_2\right) e_1 - 4 e_3 m_2 e_1^2 - 2 e_2^2 e_3 m_1\right] v_2^3 v_3 v_1^2 \\
&+ e_1 e_3 \left(e_2 m_1 - e_1 m_2\right) v_2^5 v_1 + e_1 e_2 \left(e_1 m_3 - e_3 m_1\right) v_3^5 v_1 + 2 e_1 e_2 \left(e_3 \left(m_1 + m_2\right) - \left(e_1 + e_2\right) m_3\right) v_2 v_3^4 v_1 \\
&+ \left[e_2 m_3 e_1^2 + \left(4 m_3 e_2^2 - e_3 \left(m_1 + 4 m_2 + 3 m_3\right) e_2 + 2 e_3^2 m_2\right) e_1 + e_2 e_3^2 m_1\right] v_2^2 v_3^3 v_1 \\
&+ \left[\left(e_3 \left(m_1 + m_2 + 4 m_3\right) e_2 - 2 m_3 e_2^2 - 4 e_3^2 m_2\right) e_1 - e_3 m_2 e_1^2 + e_2^2 e_3 m_1\right] v_2^3 v_3^3 v_1 \\
&+ 2 e_1 e_3 \left(\left(e_1 + e_3\right) m_2 - e_2 \left(m_1 + m_3\right)\right) v_2^4 v_3 v_1 + e_1 e_2 \left(e_2 m_3 - e_3 m_2\right) v_2 v_3^5 + 2 e_1 e_2 \left(e_3 m_2 - e_2 m_3\right) v_2^2 v_3^4 \\
&+ e_1 \left(e_2 + e_3\right) \left(e_2 m_3 - e_3 m_2\right) v_2^3 v_3^3 + 2 e_1 e_3 \left(e_3 m_2 - e_2 m_3\right) v_2^4 v_3^2 + e_1 e_3 \left(e_2 m_3 - e_3 m_2\right) v_2^5 v_3 = 0 \ .
\end{aligned}
\tag{68}
$$

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