# Three-coloring and list three-coloring of graphs without induced paths on seven vertices

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**Abstract**

In this paper we present a polynomial time algorithm that determines if an input graph containing no induced seven-vertex path is 3-colorable. This affirmatively answers a question posed by Randerath, Schiermeyer and Tewes in 2002. Our algorithm also solves the list-coloring version of the 3-coloring problem, where every vertex is assigned a list of colors that is a subset of $\{1, 2, 3\}$, and gives an explicit coloring if one exists.

Moreover, we present an independent algorithm that works in the special case when triangles are forbidden in addition to induced seven-vertex paths. Its running time is significantly faster compared to the general case.

**Keywords:** 3-coloring, list 3-coloring, $P_7$-free graph, polynomial time algorithm.

## 1 Introduction

A *k-coloring* of a graph $G = (V, E)$ is a function $f : V \to \{1, \ldots, k\}$ such that $f(v) \neq f(w)$ whenever $vw \in E$. The *vertex coloring problem*, whose input is a graph $G$ and a natural number $k$, consists of deciding whether $G$ is $k$-colorable or not. This well-known problem is one of Karp's 21 NP-complete problems [16] (unless $k = 2$; then the problem is solvable in linear time). Stockmeyer [24] proved that the problem remains NP-complete even if $k \geq 3$ is fixed, and Maffray and Preissmann proved that it remains NP-complete for triangle-free graphs [19].

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List variations of the vertex coloring problem can be found in the literature. For a survey on that kind of related problems, see [25]. In the *list-coloring problem*, every vertex $v$ comes equipped with a list of permitted colors $L(v)$, and we require the coloring to respect these lists, i.e., $f(v) \in L(v)$ for every $v$ in $V$. For a positive integer $k$, the *$k$-list-coloring problem* is a particular case in which $|L(v)| \leq k$ for each $v$ in $V$, but the union of the lists can be an arbitrary set. If the size of the list assigned to each vertex is at most two (i.e., 2-list-coloring), the instance can be solved in $O(|V| + |E|)$ time [6, 7, 26], by reducing the problem to a 2-SAT instance, which Aspvall, Plass and Tarjan [1] showed can be solved in linear time (in the number of variables and clauses). The *list $k$-coloring problem* is a particular case of $k$-list-coloring, in which the lists associated to each vertex are a subset of $\{1, \ldots, k\}$. Since list $k$-coloring generalizes $k$-coloring, it is NP-complete as well.

Because of the notorious hardness of $k$-coloring, efforts were made to understand the problem on restricted graph classes. Some of the most prominent such classes are the classes of $H$-free graphs, i.e., graphs containing no induced subgraph isomorphic to $H$, for some fixed graph $H$. Kamiński and Lozin [15] and independently Král, Kratochvíl, Tuza, and Woeginger [17] proved that for any fixed $k, g \geq 3$, the $k$-coloring problem is NP-complete for the class of graphs containing no cycle of length less than $g$. As a consequence, if the graph $H$ contains a cycle, then $k$-coloring is NP-complete for $k \geq 3$ for the class of $H$-free graphs.

The *claw* is the complete bipartite graph $K_{1,3}$. A theorem of Holyer [12] together with an extension due to Leven and Galil [18] imply that if a graph $H$ contains a claw, then for every fixed $k \geq 3$, the $k$-coloring problem is NP-complete for the class of $H$-free graphs.

Combined, these two results only leave open the complexity of the $k$-coloring problem for the class of $H$-free graphs where $H$ is a fixed acyclic claw-free graph, i.e., a disjoint union of paths. There is a nice recent survey by Hell and Huang on the complexity of coloring graphs without paths and cycles of certain lengths [10] and another nice survey by Golovach et al. [8]. We denote a path and a cycle on $t$ vertices by $P_t$ and $C_t$, respectively.

The strongest known results related to our work are due to Huang [13], who proved that 4-coloring is NP-complete for $P_7$-free graphs, and that 5-coloring is NP-complete for $P_6$-free graphs. On the positive side, Hoàng, Kamiński, Lozin, Sawada, and Shu [11] have shown that $k$-coloring can be solved in polynomial time on $P_5$-free graphs for any fixed $k$. Huang [13] conjectures that 4-coloring is polynomial-time solvable for $P_6$-free graphs. This conjecture, if true, thus settles the last remaining open case of the complexity of $k$-coloring $P_t$-free graphs for any fixed $k \geq 4$. On the other hand, for $k = 3$ it is not known whether there exists a $t$ such that 3-coloring is NP-complete for $P_t$-free graphs. Randerath and Schiermeyer [21] gave a polynomial time algorithm for 3-coloring $P_6$-free graphs. Later, Golovach et al. [9] showed that the list 3-coloring problem can be solved efficiently for $P_6$-free graphs. Some of these results are summarized in Table 1.

We show that the 3-coloring problem for $P_7$-free graphs is polynomial, answering positively a question first posed in 2002 by Randerath et al. [21, 22]. Our algorithm even works for the list 3-coloring problem. This is not trivial: there are cases where $k$-coloring and list $k$-coloring have different complexities. For instance, in the class of $\{P_6, C_5\}$-free graphs, 4-coloring can be solved in polynomial time [3] while list 4-coloring is NP-complete [14]. Our main theorem reads as follows.

**Theorem 1.** *One can decide whether a given $P_7$-free graph $G$ has a list 3-coloring, and find such a coloring (if it exists) in polynomial time. The running time of the proposed algorithm is $O(|V(G)|^{21}(|V(G)| + |E(G)|))$.*

| k\t | 4 | 5 | 6 | 7 | 8 | ... |
|---|---|---|---|---|---|---|
| 3 | $O(m)$ [5] | $O(n^\alpha)$ [20] | $O(mn^\alpha)$ [21] | **P** | ? | ... |
| 4 | $O(m)$ [5] | P [11] | ? | NPC [13] | NPC | ... |
| 5 | $O(m)$ [5] | P [11] | NPC [13] | NPC | NPC | ... |
| 6 | $O(m)$ [5] | P [11] | NPC | NPC | NPC | ... |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋱ |

Table 1: Table of known complexities of the $k$-coloring problem in $P_t$-free graphs. Here, $n$ is the number of vertices in the input graph, $m$ the number of edges, and $\alpha$ is the matrix multiplication exponent known to satisfy $2 \leq a < 2.376$ [4]. The boldfaced complexity is the topic of this paper, while '?' stands for open problems.

The algorithm given by Theorem 1 is based on the following ideas. First we compute a small 2-dominating set (i.e., a set such that every vertex has distance at most two to some vertex of the set) and apply some preprocessing techniques. Then we use a controlled enumeration based on a structural analysis of the considered graphs, in order to reduce the problem to a polynomial number of instances of list 3-coloring in which the size of the list of each vertex is at most two. These instances, in turn, can be solved via 2-SAT.

As one would expect, the problem of finding a list 3-coloring is easier if the attention is restricted to triangle-free graphs. We can prove the following result in this special case.

**Theorem 2.** *One can decide whether a given $\{P_7$, triangle$\}$-free graph $G$ has a list 3-coloring, and find such a coloring (if it exists) in $O(|V(G)|^5(|V(G)| + |E(G)|))$ time. If $G$ is bipartite, the time complexity drops to $O(|V(G)|^2(|V(G)| + |E(G)|))$.*

The algorithm proposed by Theorem 2 is independent of the algorithm in Theorem 1, although it shares some of its underlying ideas. It is primarily based on a structural discussion followed by a reduction to polynomially many 2-SAT instances.

## 2 Proof of the main result

We start by establishing some notation and preliminary results. A *stable set* in a graph $G$ is a subset of pairwise non-adjacent vertices of $G$. Let $X$ and $Y$ be two sets of vertices of $G$. We say that $X$ is *complete to* $Y$ if every vertex in $X$ is adjacent to every vertex in $Y$, and that $X$ is *anticomplete to* $Y$ if no vertex of $X$ is adjacent to a vertex of $Y$.

If in a graph coloring context each of the vertices $v$ in $G$ is assigned a list $L(v) \subseteq \{1, 2, 3\}$ of possible colors, we call $L := \{L(v) : v \in V(G)\}$ a *palette* of $G$. A palette $L'$ is a *subpalette* of $L$ if $L'(v) \subseteq L(v)$ for each $v \in V(G)$. Given a graph $G$ and a palette $L$, we say that a 3-coloring $c$ of $G$ is a *coloring of* $(G, L)$ if $c(v) \in L(v)$ for all $v \in V(G)$. We also say that $c$ is a coloring of $G$ *for the palette* $L$. We say that $(G, L)$ is *colorable* if there exists a coloring of $(G, L)$. We denote by $(G, \mathcal{L})$ a graph $G$ and a collection $\mathcal{L}$ of palettes of $G$. We say $(G, \mathcal{L})$ is *colorable* if $(G, L)$ is colorable for some $L \in \mathcal{L}$. Further, $c$ is a *coloring* of $(G, \mathcal{L})$ if $c$ is a coloring of $(G, L)$ for some $L \in \mathcal{L}$.

An *update* of the list of a vertex $v$ *from* $w$ means we delete an entry from the list of $v$ that appears as the unique entry of the list of a neighbor $w$ of $v$. Clearly, such an update does not change the colorability of the graph. If a palette $L'$ is obtained from a palette $L$ by updating repeatedly until for every vertex $v$, if $v$ has a neighbor $u$ with $L'(u) = \{i\}$, then $i \notin L'(v)$, we say we obtained $L'$ from $L$ by *updating*. For a fixed $w \in V(G)$ if a palette $L'$ is obtained from a palette $L$ by repeatedly updating vertices $v$ from vertices $w'$ that are connected to $w$ by a path all whose vertices have current lists of size one, and continuing to do so until no candidates for updating are left, then we say we obtained palette $L'$ from palette $L$ by *updating from* $w$. Finally, if in either of these two procedures we update all vertices $v$ except those from a fixed set $T$, we say we obtained $L'$ by *updating except on* $T$.

Let us illustrate these notions with a quick example. Consider $C_6$ with lists $\{1\}$, $\{2,3\}$, $\{2\}$, $\{1,2\}$, $\{2,3\}$, $\{1,2\}$ (in this order). Then updating from $v_1$ gives lists $\{1\}$, $\{2,3\}$, $\{2\}$, $\{1,2\}$, $\{3\}$, $\{2\}$, while updating from $v_1$ except on $\{v_6\}$ leaves us with the initial lists. Note that updating can be carried out in $O(|V(G)| + |E(G)|)$ time.

By reducing to an instance of 2-SAT, which can be solved in linear time in the number of variables and clauses [1], several authors [6, 7, 26] independently proved the following.

**Lemma 3.** *If a palette $L$ of a graph $G$ is such that $|L(v)| \leq 2$ for all $v \in V(G)$, then a coloring of $(G, L)$, or a determination that none exists, can be obtained in $O(|V(G)| + |E(G)|)$ time.*

Let $G$ be a graph. A subset $S$ of $V(G)$ is called *monochromatic* with respect to a given coloring $c$ of $G$ if $c(u) = c(v)$ for all $u, v \in S$. Let $L$ be a palette of $G$, and $X$ a set of subsets of $V(G)$. We say that $(G, L, X)$ is *colorable* if there is a coloring $c$ of $(G, L)$ such that $S$ is monochromatic with respect to $c$ for all $S \in X$.

A triple $(G', L', X')$ is a *restriction* of $(G, L, X)$ if $G'$ is an induced subgraph of $G$, palette $L'$ is a subpalette of $L|_{V(G')}$, and $X'$ is a set of subsets of $V(G')$ such that if $S \in X$ then $S \cap V(G') \subseteq S' \in X'$. Let $\mathcal{P}$ be a set of restrictions of $(G, L, X)$. We say that $\mathcal{P}$ is *colorable* if at least one element of $\mathcal{P}$ is colorable. If $\mathcal{L}$ is a set of palettes of $G$, we write $(G, \mathcal{L}, X)$ to mean the set of restrictions $(G, L', X)$ where $L' \in \mathcal{L}$.

Note that if two sets $S$ and $S'$ are monochromatic and have a non-empty intersection, then $S \cup S'$ is monochromatic, too. Thus, for each triple $(G, L, X)$ there is an equivalent triple $(G, L, Y)$ such that $Y$ contains only mutually disjoint sets. During our algorithm, we compute the set family $X$ along the way and such that the sets are mutually disjoint. Under this assumption, the proof of Lemma 3 can be easily modified to obtain the following generalization [23].

**Lemma 4.** *If a palette $L$ of a graph $G$ is such that $|L(v)| \leq 2$ for all $v \in V(G)$, and $X$ is a set of subsets of $V(G)$, then a coloring of $(G, L, X)$, or a determination that none exists, can be obtained in $O(|V(G)| + |E(G)|)$ time.*

We write $N(S)$ for the set of vertices of $V(G) \setminus S$ with a neighbor in $S$. For disjoint sets of vertices $S, T$ of $V(G)$, let $N_T(S) = N(S) \cap T$. If $S = \{s\}$ we just write $N_T(s)$. For a vertex set $S$, let $\overline{S} := S \cup N(S)$. If $\overline{S} = V(G)$, we say that $S$ is *dominating* $G$, or is a *dominating set*. Moreover, if $S$ is dominating and the subgraph induced by $S$ is connected, then we call $S$ a *connected dominating set*. If $\overline{S}$ dominates $G$, we call $S$ *2-dominating*. For a graph $G$ with a palette $L$, call a (nonempty) 2-dominating set $S \subseteq V(G)$ which induces a connected subgraph a *seed* of $(G, L)$, if $|L(v)| = 1$ for each $v \in S$ and $|L(v)| = 2$ for each $v \in N(S)$. Note that we do not require the palette $L$ to be updated.

Observe that for any seed $S$, and for any two non-adjacent vertices $v, w \in N(S)$ the following holds.

There is an induced $v$–$w$ path of at least 3 vertices whose inner vertices all lie in $S$. (1)

The next result is essential to our proof.

**Theorem 5** (Camby and Schaudt [2]). *For all $t \geq 3$, any connected $P_t$-free graph has a connected dominating set whose induced subgraph is either $P_{t-2}$-free, or isomorphic to $P_{t-2}$.*

We use the following easy corollary of Theorem 5.

**Corollary 6.** *Every connected $P_7$-free graph $G$ has either a connected 2-dominating set of size at most 3 or a complete subgraph of 4 vertices. The set or the subgraph can be found in $O(|V(G)|^3|E(G)|)$ time.*

*Proof.* Apply Theorem 5 to the graph in question. If the outcome is a $P_5$, the assertion follows, so assume otherwise. Then we can apply Theorem 5 again to obtain either a $P_3$ or a $P_3$-free 2-dominating connected subgraph. Observing that a $P_3$-free connected graph is a complete graph, we are done.

To detect such a set, we run through all triples $T$ of vertices, and check if there is common neighbor $v$ of $T$ such that $T \cup \{v\}$ induces a complete subgraph. If not, we check whether $T$ induces a connected subgraph and all vertices of the graph are within distance 2 from $T$. We can test the second property by using two steps of a breadth-first-search. □

This corollary will help us to reduce in the next section the original instance to a polynomial number of simpler instances. In each of these, the vertices having lists of size 1 or 2 satisfy some structural properties and the vertices having lists of size 3 form a stable set. We will in turn solve these special instances in Section 2.2 by reducing them to a polynomial number of instances to which we can apply Lemma 4.

## 2.1 Proof of Theorem 1

Let $G$ be a connected $P_7$-free graph with a palette $L^*$. The following claim will be useful.

**Claim 7.** *Let $v \in V(G)$. We may assume that if $|L^*(v)| = 3$, then the neighborhood of $v$ is disconnected.*

*Proof.* Indeed, suppose this fails for some vertex $v$. We first check if the neighborhood $N(v)$ of $v$ induces a bipartite graph (if not we abort since in that case $G$ is not 3-colorable). Let $U, W$ be a bipartition of $G[N(v)]$. Then, we can solve the problem for the graph $G'$ we obtain from $G$ by deleting $v$ and replacing the neighborhood of $v$ with an edge $uw$, where $N_{G'}(u) \cap V(G) = N_G(U) \cap V(G')$, and $N_{G'}(w) \cap V(G) = N_G(W) \cap V(G')$. In the case that $W$ is empty, say, we know $U = \{u\}$, and we just define $G' = G - \{v\}$. The list of $u$ is the intersection of all lists of vertices from $U$, and similar for $w$. Clearly, $G$ admits a coloring for $L^*$ if and only $G'$ admits a coloring for the new palette.

Now it is sufficient to show that $G'$ is still $P_7$-free. Suppose $Q$ is a copy of $P_7$ in $G'$. Since $G$ is $P_7$-free, it follows that $V(Q) \cap \{u, w\}$ is non-empty. Note that if $Q$ contains both $u$ and $w$, then $u, w$ are consecutive on $Q$. So (in any case) we can write $Q$ as $Q_1 - Q_2 - Q_3$, where $V(Q_2) \subseteq \{u, w\}$ and $Q_1, Q_3$ avoid $\{u, w\}$. We can assume that $Q_1, Q_3$ are not empty, as otherwise it is easy to substitute $Q_2$ with one or two vertices in $U \cup W$, and thus find a $P_7$ in $G$, a contradiction.

Observe that $Q_1, Q_3$ each have exactly one vertex $q_1, q_3$ in $N(V(Q_2))$. If $|V(Q_2)| = 1$, we may assume both these vertices lie in $N(U)$, and we can substitute $Q_2 = u$ with either a common neighbor of $q_1, q_3$, or with a path $u_1 - v - u_2$ with $u_1 \in U \cap N(q_1)$ and $u_2 \in U \cap N(q_3)$. This gives a $P_7$ in $G$, a contradiction.

So assume $|V(Q_2)| = 2$. Since we assume that $U \cup W$ induces a connected graph, also $Z := U \cup W \cup N(U \cup W)$ is connected. We can substitute $q_1 - Q_2 - q_3 = q_1 - u - w - q_3$ with any induced path connecting $q_1, q_3$ in $Z$ to obtain a $P_7$ in $G$, yielding the final contradiction. $\square$

In order to simplify the algorithm, in the first phase we do not take the input lists into account. Instead, we consider the list $L$ with $L(v) = \{1, 2, 3\}$ for each $v \in V(G)$. At the final steps and before applying any reduction like the one of Claim 7, we intersect the current lists with the original input lists.

We now describe the algorithm. We first apply Corollary 6 to $G$. Notice that if $G$ contains a complete subgraph of 4 vertices then it is not 3-colorable. So we may assume we obtain a 2-dominating connected set $S_1$ of size at most 3. As we may add a neighbor, if necessary, we can assume that $|S_1| \geq 2$. We will go through all possible 3-colorings of $S_1$, and check for each whether it extends to a coloring of $G$ which respects the palette $L^*$. This is clearly enough for deciding whether $(G, L^*)$ is colorable.

So from now on, assume the coloring on $S_1$ is fixed. We update the lists of all remaining vertices. Note that updating can be done in $O(|V(G)| + |E(G)|)$ time, because each edge $vw$ needs to be checked at most once (either updating $v$ from $w$ or updating $w$ from $v$). After updating to palette $L^2$, consider the largest connected set $S_2$ of vertices with lists of size 1 that contains $S_1$. We claim that $S_2$ is a seed for $(G, L^2)$. Indeed, since $\overline{S_1}$ dominates $G$, so does $\overline{S_2}$. Also, all vertices in $N(S_2)$ must have lists of size 2, since they are adjacent, but do not belong to $S_2$. So $S_2$ is a seed.

In the case that two adjacent vertices of $S_2$ have the same entry on their list, we abort the process immediately.

**Claim 8.** *For every vertex $v$ in $N(S_2)$ there is an induced path on at least 3 vertices contained in $S_2 \cup \{v\}$ having $v$ as an endpoint.*

*Proof.* This holds since $S_2$ is connected, $|S_2| \geq |S_1| \geq 2$, and $v$ is not adjacent to two vertices of $S_2$ that have different entries on their lists (because $|L^2(v)| = 2$). $\square$

Now, in two steps $j = 3, 4$, we will refine the set of subpalettes of $L$ we are looking at, starting with $\mathcal{L}^2 := \{L^2\}$. At each step we replace the set $\mathcal{L}^{j-1}$ of palettes from the previous step with a set $\mathcal{L}^j$. More precisely, each element $L$ of $\mathcal{L}^j$ is a subpalette of some element $Pred(L)$ of $\mathcal{L}^{j-1}$. We will argue below why it is sufficient to check colorability for the new set of palettes.

For each of the palettes $L$ in $\mathcal{L}^j$, we will define a seed $S_L$ and a set $T_L \subseteq N(S_L)$. We start with $S_{L^2} := S_2$ and $T_{L^2}$ being the set of vertices $x \in N(S_{L^2})$ for which there does not exist an induced path $x - y - z$ with $|L^2(y)| = 3$ and $z \notin \overline{S_{L^2}}$. We will ensure for each palette $L$ that $S_L \supseteq S_{Pred(L)}$ and $T_L \supseteq T_{Pred(L)}$. Furthermore, the seeds $S_L$ and the sets $T_L$ will have the following properties:

(A) for all $x \in N(S_L) \setminus T_L$, there is an induced path $x - y - z$ with $|L(y)| = 3$ and $z \notin \overline{S_L}$, and for no $x \in T_L$ is there such a path; and

(B) for each vertex $v \in V(G) \setminus \overline{S_L}$ either $|L(v)| = 1$ or $|L(v)| = 3$.

Let us now get into the details of the procedure. Successively, for $j = 3, 4$, we consider for each $L \in \mathcal{L}^{j-1}$ a set of subpalettes of $L$ obtained by partitioning the possible colorings of induced paths $x - y - z$ with $x \in N(S_L) \setminus T_L$, $|L(y)| = 3$ and $z \notin \overline{S_L}$ into a polynomial number of cases. The set $\mathcal{L}^j$ will be the union of all the sets of subpalettes corresponding to lists $L$ in $L^{j-1}$. The idea behind is to make the seed grow, and after these two steps, obtain a set of palettes we can deal with, and such that the graph admits a coloring for the original palette if and only if it admits a coloring for one of the palettes in the set.

For each $i \in \{1, 2, 3\}$, let $\mathcal{P}_i$ be the set of paths $x - y - z$ with $x \in N(S_L) \setminus T_L$, $|L(y)| = 3$ and $z \notin \overline{S_L}$, and such that $i \notin L(x)$. We will order the paths of $\mathcal{P}_i$ non-increasingly by $|N(x) \setminus (N(y) \cup N(z) \cup \overline{S_L})|$, i.e., the number of vertices $w$ (if any) such that $w - x - y - z$ is an induced path.

We can compute and sort the paths of $\mathcal{P}_i$ in $O(|V(G)|^4)$ time. Moreover, this order of the paths induces an order on the set $Y_i$ of vertices $y$ that are midpoints of paths $x - y - z$ in $\mathcal{P}_i$. The vertices in $Y_i$ are ordered by their first appearance as midpoints of the ordered paths in $\mathcal{P}_i$. Let $n_i = |Y_i|$, and $Y_i = \{y_{i,1}, \ldots, y_{i,n_i}\}$.

For each $i \in \{1, 2, 3\}$, simultaneously and independently, we will consider the following cases.

(a) All vertices in $Y_i$ are colored $i$.

(b) There is a $k$, $1 \leq k \leq n_i$, such that the first $k - 1$ vertices of $Y_i$ are colored $i$, and the first path $x - y_{i,k} - z$ in $\mathcal{P}_i$ is colored such that the color of $y_{i,k}$ is different from $i$, the color of every vertex in $W = N(x) \setminus (N(y_{i,k}) \cup N(z) \cup \overline{S_L})$ is $i$, and the color of $z$ is $i$ if $W$ is empty.

(c) There is a $k$, $1 \leq k \leq n_i$, such that the first $k - 1$ vertices of $Y_i$ are colored $i$, and the first path $x - y_{i,k} - z$ in $\mathcal{P}_i$ is colored such that the color of $y_{i,k}$ is different from $i$, the color of $z$ is different from $i$ if $W = N(x) \setminus (N(y_{i,k}) \cup N(z) \cup \overline{S_L})$ is empty, and if $W$ is nonempty, there is a vertex $w$ of $W$ that gets a color different from $i$.

In order to do that, we consider all choices of functions $f : \{1, 2, 3\} \to \{a, b, c\}$. For each of these choices, we generate a set $\mathcal{L}_f$ of subpalettes of $L$, and $\mathcal{L}^j$ will be the union of all sets $\mathcal{L}_f$. For fixed $f$ the first step to obtain $\mathcal{L}_f$ consists of doing the following simultaneously for $i = 1, 2, 3$.

If $\mathcal{P}_i$ is empty, then set $\mathcal{L}_{i,f} := \{L\}$. Otherwise, the set is as follows.

If $f(i) = a$, set $\hat{L}(y) := \{i\}$ for every $y \in Y_i$ and $\hat{L}(v) := L(v)$ for every $v \in V(G) \setminus Y_i$. Set $\mathcal{L}_{i,f} := \{\hat{L}\}$.

If $f(i) \neq a$, for each $k \in \{1, \ldots, n_i\}$, let $x$ and $z$ such that $x - y_{i,k} - z$ is the first path in $\mathcal{P}_i$ having $y_{i,k}$ as midpoint, and let $W = N(x) \setminus (N(y_{i,k}) \cup N(z) \cup \overline{S_L})$.

If $f(i) = b$, consider all subpalettes $\hat{L}$ of $L$ which only differ from $L$ on $W \cup \{y_{i,1}, \ldots, y_{i,k}, z\}$, and satisfy $\hat{L}(y_{i,k}) = \{i'\}$ for some $i' \neq i$, $\hat{L}(v) = \{i\}$ for all $v \in W \cup \{y_{i,1}, \ldots, y_{i,k-1}\}$, $|\hat{L}(z)| = 1$, and $\hat{L}(z) = \{i\}$ if $W$ is empty. Update these palettes $\hat{L}$ from $y_{i,k}$ except on $T_L$ and let $\mathcal{L}_{i,f}$ be the set of all palettes found in this way, for every choice of $k$. Note that, in each palette, the updated list of $x$ has size 1, and that the number of palettes generated this way is $O(|V(G)|)$.

If $f(i) = c$, if $W$ is nonempty, for each $w \in W$ consider all subpalettes $\hat{L}$ of $L$ which only differ from $L$ on $\{y_{i,1}, \ldots, y_{i,k}, z, w\}$, and satisfy $\hat{L}(v) = \{i\}$ for all $v \in \{y_{i,1}, \ldots, y_{i,k-1}\}$,

$|\hat{L}(y_{i,k})| = |\hat{L}(z)| = |\hat{L}(w)| = 1$, $\hat{L}(y_{i,k}) \neq \{i\}$, and $\hat{L}(w) \neq \{i\}$. If $W$ is empty, consider all subpalettes $\hat{L}$ of $L$ which only differ from $L$ on $\{y_{i,1}, \ldots, y_{i,k}, z\}$, and satisfy $\hat{L}(v) = \{i\}$ for $v \in \{y_{i,1}, \ldots, y_{i,k-1}\}$, $|\hat{L}(y_{i,k})| = |\hat{L}(z)| = 1$, $\hat{L}(y_{i,k}) \neq \{i\}$, and $\hat{L}(z) \neq \{i\}$. Update these palettes $\hat{L}$ from $y_{i,k}$ except on $T_L$ and let $\mathcal{L}_{i,f}$ be the set of all palettes found in this way, for every choice of $k$ and of $w$ (if such a $w$ exists). Note that again, in each palette, the updated list of $x$ has size 1, and that the number of palettes generated this way is $O(|V(G)|^2)$.

Finally, for each triple $(L_1, L_2, L_3) \in \mathcal{L}_{1,f} \times \mathcal{L}_{2,f} \times \mathcal{L}_{3,f}$ consider the palette $\tilde{L}$ obtained from intersecting the lists of $L_1, L_2, L_3$, taking intersections at each vertex. Update the palette $\tilde{L}$ from any vertex in $S_L$, except on $T_L$. Let $\mathcal{L}_f$ be the set of all palettes $\tilde{L}$ thus generated.

Observe that $|\mathcal{L}_f| = O(|V(G)|^6)$, since $|\mathcal{L}_{i,f}| = O(|V(G)|^2)$ for $i = 1, 2, 3$.

For each $L' \in \mathcal{L}_f$, let $S_{L'}$ be a maximal connected set of vertices with list size 1 that contains $S_L$. Then $S_{L'}$ is a seed.

Note that for each $L' \in \mathcal{L}_f$, all vertices $v$ in $T_L$ satisfy $|L'(v)| = 2$, since they were never updated. Let $T_{L'}$ be the union of $T_L$ with all vertices $x \in N(S_{L'})$ which are not the starting point of an induced path $x - y - z$ with $|L'(y)| = 3$ and $z \notin \overline{S_L}$.

Clearly, $T_{L'} \subseteq N(S_{L'})$. Property (A) holds because of the way we defined $T_{L'}$, and because there are no new paths of the type described in (A) that start at vertices in $T_L$, as seeds grow and lists shrink. Property (B) holds because $S_L$ was a seed satisfying Properties (A) and (B), and when defining palettes in $\mathcal{L}_f$ by the cases $(a)$, $(b)$, and $(c)$, we have reduced the size of some vertex lists from 3 to 1, never to 2; then we only updated *from vertices* in $S_L$ except on $T_L$, thus every vertex that got a list of size 1 by updating is connected to $S_L$ by a path all whose vertices have current lists of size one, and is now in $S_{L'}$ and, consequently, every vertex that got a list of size 2 by updating is in $N(S_{L'})$.

**Claim 9.** *There is a coloring of $G$ for the palette $L^2$ if and only if $G$ has a coloring for at least one of the palettes in $\mathcal{L}^4$.*

*Proof.* Indeed, observe that when obtaining $\mathcal{L}^j$ from $\mathcal{L}^{j-1}$, we consider for each $L \in \mathcal{L}^{j-1}$ and for each $i \in \{1, 2, 3\}$ the possibility that all induced 3-vertex-paths that start in $N(S_L)$ and then leave $\overline{S}$ have their second vertex colored $i$ (when $f(i) = a$). We also consider the possibility that there is such a path whose second vertex is colored with a different color (when $f(i) = b$ or $f(i) = c$). In that case, we consider separately the possible colorings of a fourth vertex $w$, if such a $w$ exists. □

Note that $|\mathcal{L}^{j+1}| = O(|\mathcal{L}^j| \cdot |V(G)|^6)$ for each $j = 2, 3$. Since $|\mathcal{L}^2| = 1$, the number of palettes in $\mathcal{L}^4$ is $O(|V(G)|^{12})$.

Next we prove that during the above described process, the union of our seed with the set $T_L$ actually grows.

**Claim 10.** *For each $L \in \mathcal{L}^j$, we have $N(S_{Pred(L)}) \subset S_L \cup T_L$.*

*Proof.* In order to see Claim 10, suppose there is a vertex $x' \in N(S_{Pred(L)}) \setminus (S_L \cup T_L)$. As $x' \notin S_L$ and $S_L \supseteq S_{Pred(L)}$, we know that $x' \in N(S_L)$. Furthermore, since $x' \notin T_L$, there is an induced path $x' - y' - z'$ with $|L(y')| = 3$ and $z' \notin \overline{S_L}$. So $f(i) \neq a$, where $i$ is such that $i \notin L(x')$. Thus $f(i) \in \{b, c\}$, and there is an induced path $x - y - z$ with $x \in N(S_{Pred(L)})$ such that $|L(x)| = |L(y)| = |L(z)| = 1$, and so $x, y, z \in S_L$. Since $y', z' \notin \overline{S_L}$, it follows that there are no edges between $\{y', z'\}$ and $\{x, y, z\}$. Also, since $x' \in N(S_L)$, there are no edges from $x'$ to vertices $v \in \{x, y, z\}$ with $L(v) \subsetneq L(x')$. In other words, the only possible edge between $\{x, y, z\}$ and $\{x', y', z'\}$ is $x'z$, and if this edge is present, we have that $L(z) = \{i\}$. On the other hand, by (1), there is a path $Q$ of at least 3 vertices connecting $x$ and $x'$ whose interior lies in $S_{Pred(L)}$. So, since $G$ is $P_7$-free, the edge $x'z$ has to be present and thus we have $L(z) = \{i\}$.

Now, assume there is an extension of $x - y - z$ to an induced path $w - x - y - z$. Then, as the sequence $w - x - y - z - x' - y' - z'$ is not an induced $P_7$, there is a edge from $w$ to one of $x', y', z'$. Observe if $|L(w)| = 1$, then $w \in S_L$ and neither $wy'$ nor $wz'$ is an edge. Hence either $|L(w)| \geq 2$, or $L(w) = \{i\}$, and in the latter case the only edge from $w$ to $\{x', y', z'\}$ is $wx'$. As this happens for all possible choices of $w$, we see that $f(i) \neq c$, and thus $f(i) = b$. This means that for all possible $w$, $w$ is adjacent to $x'$. But now, observe that

$$N(x) \setminus (N(y) \cup N(z) \cup \overline{S_L}) \subsetneq N(x') \setminus (N(y') \cup N(z') \cup \overline{S_L}),$$

a contradiction to the choice of the path $x - y - z$.

We conclude that there is no extension of $x - y - z$ to an induced path $w - x - y - z$. But then, the fact that $L(z) = \{i\}$ implies that again, $f(i) \neq c$, and thus, $f(i) = b$. The existence of the edge $x'z$ gives a contradiction to the choice of the path $x - y - z$. This proves Claim 10. □

Next, we prove that two steps of performing the above procedure suffice to take care of all paths on three vertices that start in the boundary of the current seed, and then leave the seed.

**Claim 11.** *For each $L \in \mathcal{L}^4$, we have $N(S_L) \subset T_L$.*

*Proof.* Suppose there are $L \in \mathcal{L}^4$ and $x \in N(S_L)$ such that $x \notin T_L$. Then by (A) there is a path $x - y - z$ with $|L(y)| = 3$ and $z \notin \overline{S_L}$. Clearly $y$ and $z$ are anticomplete to $S_L$. Let $L' := Pred(L)$ and $L'' := Pred(L')$. Choose a path $P$ from $x$ to $N(S_{L''})$ with all interior vertices in $S_L$, say it ends in $x'' \in N(S_{L''})$. Note that $P$ contains a vertex $x' \in N(S_{L'})$. As $x' \in S_L$ (by the choice of $P$), we know that $x' \notin T_L$. Thus Claim 10 implies that $x \neq x' \neq x''$. So $P$ together with the path $x - y - z$ and the path provided by Claim 8 for $x''$ gives a path on at least 7 vertices, a contradiction. □

By Claim 9, $(G, L^2)$ is colorable if and only if $(G, L)$ is colorable for some $L \in \mathcal{L}^4$. For each $L \in \mathcal{L}^4$, we check whether there is a coloring of $(G, L)$, with the help of Lemma 12 below. So from now on, let $L \in \mathcal{L}^4$ be fixed. Let $X$ be the set of all vertices in $V(G) \setminus \overline{S_L}$ with lists of size 1, and set $Y := V(G) \setminus (\overline{S_L} \cup X)$. By construction, $|L(y)| = 3$ for each $y \in Y$.

By Claim 11, no vertex of $N(S_L)$ is the starting point of an induced path $x - y - z$ with $y \in Y$ and $z \in X \cup Y$. In other words, for each $y \in Y$, all edges between $N_{\overline{S_L}}(y)$ and $N_{V(G)\setminus\overline{S_L}}$ are present.

Now we intersect $L$ with the given input palette $L^*$, we apply the reduction suggested by Claim 7, and then update. Let $L'$ be the resulting palette. We may assume that $|L'(v)| \geq 1$ for all $v \in V(G)$, otherwise we detect that the palette $L$ does not lead to a feasible solution to $L^*$. Let $Y'$ be the set of vertices $y$ of $Y$ such that $|L'(y)| = 3$. We noticed that for each $y \in Y$, all edges between $N_{\overline{S_L}}(y)$ and $N_{V(G)\setminus\overline{S_L}}$ are present. Since we have applied the reduction of Claim 7, for $y \in Y'$ one of these sets is empty, and since $\overline{S_L} \supseteq \overline{S_2}$ is dominating, we conclude that $N_{V(G)\setminus\overline{S_L}}(y) = \emptyset$, and thus

$$N(y) \subseteq \overline{S_L} \text{ for each } y \in Y'. \qquad (2)$$

Consider the set $S'$ of all vertices that are connected to $S_L$ by a (possibly trivial) path containing only vertices with lists $L'$ of size 1. Note that $S'$ is a seed. In particular, $S_L \subseteq S'$ and by (2), we have $N(y) \subseteq \overline{S}'$ for every $y \in Y'$. That is, $Y'$ is a stable set anticomplete to $V(G) \setminus (\overline{S}' \cup Y')$.

We are now in a situation where the following lemma applies, solving the remaining problem.

**Lemma 12.** *Let $G$ be a connected $P_7$-free graph with a palette $L$. Let $S$ be a seed of $G$ such that if $v \in S$ and $w \in N(S)$ are adjacent, then they do not share list entries. Assume that the set $X$ of vertices having lists of size 3 is stable and anticomplete to $V(G) \setminus (\overline{S} \cup X)$. Then we can decide whether $G$ has a coloring for $L$ in $O(|V(G)|^9(|V(G)| + |E(G)|))$ time.*

The next subsection is devoted to the proof of Lemma 12. Since we have $|\mathcal{L}^4| = O(|V(G)|^{12})$ many lists to consider, the total running time amounts to $O(|V(G)|^{21}(|V(G)| + |E(G)|))$.

## 2.2 Proof of Lemma 12

Let $G$, $L$, $S$ and $X$ be as in the statement of the lemma. We assume also by Claim 7 that no vertex in $X$ has a connected neighborhood.

Our aim is to define a set $\mathcal{P}$ of restrictions of $(G, L)$ with the property that in any element of $\mathcal{P}$ there are no vertices with list of size 3, and $(G, L)$ is colorable if and only if $(G, \mathcal{P})$ is colorable. Moreover, $\mathcal{P}$ has polynomial size and is computable in polynomial time.

If $X = \emptyset$, we simply put $\mathcal{P} = \{(G, L)\}$. Otherwise, for all $i = 1, 2, 3$, let $D_i$ be the set of vertices $v \in N(S)$ with $L(v) = \{1, 2, 3\} \setminus \{i\}$, and for $x \in X$, let $N_i(x) = N(x) \cap D_i$, for $i = 1, 2, 3$. Observe that for every $d \in D_i$ and for every $s \in S \cap N(d)$, we have $L(s) = \{i\}$.

If $N(x) \subseteq D_i$ for some $x \in X$ and some $i \in \{1, 2, 3\}$, then setting $L(x) = \{i\}$ does not change the colorability of $(G, L)$, so we may assume that for every $x \in X$ at least two of the sets $N_1(x), N_2(x), N_3(x)$ are non-empty. Let $X_1$ be the set of vertices $x \in X$ for which $N_2(x)$ is not complete to $N_3(x)$; for every $x \in X_1$ fix two vertices $n_2(x) \in N_2(x)$ and $n_3(x) \in N_3(x)$ such that $n_2(x)$ is non-adjacent to $n_3(x)$. Define similarly $X_2$ and $n_1(x), n_3(x)$ for every $x \in X_2$, and $X_3$ and $n_1(x), n_2(x)$ for every $x \in X_3$. Since no vertex of $X$ has a connected neighborhood and $X$ is a stable set, it follows that $X = X_1 \cup X_2 \cup X_3$.

Before we state the coloring algorithm, we need some auxiliary statements. For a path $P$ with ends $u, v$ let $P^* = V(P) \setminus \{u, v\}$ denote the interior of $P$.

**Claim 13.** *Let $i, j \in \{1, 2, 3\}$, $i \neq j$, and let $u_i, v_i \in D_i$ and $u_j, v_j \in D_j$, such that $\{u_i, v_i, u_j, v_j\}$ is a stable set. Then there exists a path $P$ with ends $a, b \in \{u_i, v_i, u_j, v_j\}$ such that*
*(a) $\{a, b\} \neq \{u_i, u_j\}$ and $\{a, b\} \neq \{v_i, v_j\}$,*
*(b) $|L(v)| = 1$ for every $v \in P^*$, and*
*(c) $P^*$ is anticomplete to $\{u_i, v_i, u_j, v_j\} \setminus \{a, b\}$.*

*Proof.* Note that each of $u_i, u_j, v_i, v_j$ has a neighbor in $S$, and $G[S]$ is connected. Let $P$ be an induced path with $P^* \subseteq S$ that connects $u_i$ with $v_i$. If $P$ is not as desired, at least one of $u_j, v_j$ has a neighbor on $P$. Let $p$ be the neighbor of $u_j$ or $v_j$ on $P$ that is closest to $v_i$; by symmetry we may assume $p$ is a neighbor of $u_j$. Note that $p$ is not adjacent to $u_i, v_i \in D_i$, because $p$ is already adjacent to $u_j \in D_j$. Hence, if $u_j - p - P - v_i$ is not as desired, then $v_j$ must have a neighbor on $p - P - v_i$. Among all such neighbors, let $p'$ be the one that is closest to $p$ (possibly $p' = p$). As before, $p'$ is not adjacent to any of $u_i, v_i \in D_i$, and thus, $u_j - p - P - p' - v_j$ is the desired path. □

**Claim 14.** *Let $\{i,j,k\} = \{1,2,3\}$. Let $x, y \in X_i$, let $n_j \in N_j(x)$ and $n_k \in N_k(x)$ such that $n_j$ is non-adjacent to $n_k$. Then there is an edge between $\{x, n_j, n_k\}$ and $\{y, n_j(y), n_k(y)\}$.*

*Proof.* Assume there is no such edge. Then in particular, vertices $n_j, n_j(y), n_k, n_k(y)$ are distinct, and we can apply Claim 13 to obtain a path $P$ with $P^* \subseteq S$ that connects two vertices from $\{n_j, n_j(y), n_k, n_k(y)\}$ in way that $P^*$, together with $n_j - x - n_k$ and $n_j(y) - y - n_k(y)$, forms an induced path of length at least 7, a contradiction. □

Next we distinguish between several types of colorings of $G$, and show how to reduce the list sizes assuming that a coloring of a certain type exists. For this, let $\{i,j,k\} = \{1,2,3\}$. We call a coloring $c$ of a restriction $(G, L'', Z'')$ of $(G, L, Z)$ a *type A coloring with respect to $i$* if there exists an induced path $n_j - x - n_k - z - m_j$ with $x, z \in X_i$, $n_j \in N_j(x)$, $m_j \in N_j(z)$, and $n_k \in N_k(x) \cap N_k(z)$ such that $c(n_j) = i$, $c(x) = j$ and $c(z) = k$ (this implies $c(n_k) = c(m_j) = i$), or the same with the roles of $j$ and $k$ reversed.

**Claim 15.** *Let $(G', L', Z')$ be a restriction of $(G, L, Z)$. There exists a set $\mathcal{L}_i$ of $O(|V(G)|^3)$ subpalettes of $L'$ such that*
*(a) $|L''(v)| \leq 2$ for every $L'' \in \mathcal{L}_i$ and $v \in X_i \cap V(G')$, and*
*(b) $(G', L', Z')$ admits a type A coloring with respect to $i$ if and only if $(G', \mathcal{L}_i, Z')$ is colorable. Moreover, $\mathcal{L}_i$ can be constructed in $O(|V(G)|^4)$ time.*

*Proof.* For every $x, z \in X_i \cap V(G')$ and $n_j \in N_j(x)$ for which there are $n_k \in N_k(x) \cap N_k(y)$ and $m_j \in N_j(z)$ such that $n_j - x - n_k - z - m_j$ is an induced path, we construct a palette $L'' = L_{x,z,n_j}$ depending on $x, z, n_j$; for the same case with triples $x, z \in X_i \cap V(G')$, $n_k \in N_k(x)$, and the roles of $j$ and $k$ reversed, we construct in an analogous way a palette $L'' = L'_{x,z,n_k}$ depending on $x, z, n_k$. The set $\mathcal{L}_i$ will be the set of all palettes $L''$ obtained in this way. So the number of palettes in $\mathcal{L}_i$ is $O(|V(G)|^3)$.

For $x, z, n_j$ as above (we will assume the first case in the definition, the other case is analogous), we define $L''$ by setting $L''(x) = \{j\}$, $L''(z) = \{k\}$, $L''(n_j) = \{i\}$, and leaving $L''(v) = L'(v)$ for all $v \in V(G') \setminus \{x, z, n_j\}$. Update $N_j(z)$ from $z$, and $N_k(x)$ from $x$. Let $n_k$ and $m_j$ such that $n_k \in N_k(x) \cap N_k(y)$, $m_j \in N_j(z)$, and $n_j - x - n_k - z - m_j$ is an induced path. Note that after updating, $L''(n_k) = L''(m_j) = \{i\}$. Now, for each vertex $v \in D_j \cup D_k$ that has a neighbor $v' \in \{x, z, n_j, n_k, m_j\}$, update $v$ from each such neighbor $v'$. Next, for every vertex $y \in X_i \cap V(G')$, if $n_j(y)$ or $n_k(y)$ now has list size 1, then update $y$ from both $n_j(y)$ and $n_k(y)$, and also update $y$ from $m_j, n_j$ and $n_k$ in the case that $y$ is adjacent to any of them. Call the obtained palette $L''$ (slightly abusing notation). By the way we updated, it only takes $O(|V(G)|)$ time to compute this palette. The total time for constructing all palettes for $\mathcal{L}_i$ thus amounts to $O(|V(G)|^4)$.

In order to see Claim 15 (a), we need to show that $|L''(y)| \leq 2$ for all $y \in X_i \cap V(G')$. For contradiction, suppose $|L''(y)| = 3$ for some $y \in X_i \cap V(G')$. By Claim 14, there are edges between $\{x, n_j, n_k\}$ and $\{y, n_j(y), n_k(y)\}$, and also between $\{z, m_j, n_k\}$ and $\{y, n_j(y), n_k(y)\}$. By the way we updated $L''$, the only possibly edges between these sets are those connecting $n_j(y)$ with $x$, and $n_k(y)$ with $z$. But now $m_j - z - n_k(y) - y - n_j(y) - x - n_j$ is a $P_7$, a contradiction.

For Claim 15 (b), first note that by construction, if $(G', \mathcal{L}_i, Z')$ is colorable then $(G', L', Z')$ has a type A coloring with respect to $i$. On the other hand, if $c$ is a type A coloring of

$(G', L', Z')$ with respect to $i$, then there is an induced path $n_j - x - n_k - z - m_j$ with $x, z \in X_i$, $n_j, m_j \in N_j(x)$, and $n_k \in N_k(x)$ such that $c(n_j) = c(m_j) = c(n_k) = i$, $c(x) = j$, and $c(z) = k$ (or the same with the roles of $j$ and $k$ reversed). Since updating does not change the set of possible colorings for a list, $c$ is a coloring for the list $L'' = L_{x,z,n_j}$ (respectively, $L'' = L_{x,z,n_k}$). So $\mathcal{L}_i$ is as required for Claim 15 (b). □

Let $\{i, j, k\} = \{1, 2, 3\}$. A coloring $c$ of a restriction $(G', L', Z')$ of $(G, L, Z)$ is a *type B coloring with respect to* $i$ if it is not a type A coloring with respect to $i$, and there exists an induced path $x - n_k - z - m_j$ with $x, z \in X_i \cap V(G')$, $m_j \in N_j(z)$, $n_k \in N_k(x) \cap N_k(z)$ such that $c(x) = j$ and $c(z) = k$ (this implies $c(n_k) = c(m_j) = i$), or the same with the roles of $j$ and $k$ reversed.

**Claim 16.** *Let $(G', L', Z')$ be a restriction of $(G, L, Z)$ that does not admit a type A coloring. There exists a set $\mathcal{L}_i$ of $O(|V(G)|^2)$ subpalettes of $L'$ such that*
*(a) $|L''(v)| \le 2$ for every $L'' \in \mathcal{L}_i$ and $v \in X_i \cap V(G')$, and*
*(b) $(G', L', Z')$ admits a type B coloring with respect to $i$ if and only if $(G', \mathcal{L}_i, Z')$ is colorable. Moreover, $\mathcal{L}_i$ can be constructed in $O(|V(G)|^3)$ time.*

*Proof.* For every $x, z \in X_i \cap V(G')$ for which there exist $n_k \in N_k(x) \cap N_k(z)$ and $m_j \in N_j(z)$ such that $x - n_k - z - m_j$ is an induced path, we construct a palette $L'' = L_{x,z}$, depending on $x$ and $z$. For the case with the roles of $j$ and $k$ reversed, we construct analogously a palette $L'' = L'_{x,z}$. The set $\mathcal{L}_i$ will be the set of all palettes $L''$ obtained in this way. So the number of palettes in $\mathcal{L}_i$ is $O(|V(G)|^2)$.

Given a pair of vertices $x$, $z$ in $X_i \cap V(G')$ satisfying the hypothesis, let $n_k$ and $m_j$ such that $n_k \in N_k(x) \cap N_k(z)$, $m_j \in N_j(z)$, and $x - n_k - z - m_j$ is an induced path. Let $M$ be the set of all $n \in N_j(x)$ for which $n - x - n_k - z - m_j$ is an induced path.

Define $L''$ by setting $L''(x) = \{j\}$, $L''(z) = \{k\}$, $L''(n_k) = L''(m_j) = \{i\}$, and $L''(n) = \{k\}$ for all $n \in M$, and leaving $L''(v) = L'(v)$ for all $v \in V(G') \setminus (\{x, z, n_k, m_j\} \cup M)$. Now, for each vertex $v \in D_j \cup D_k$ that has a neighbor $v'$ in $\{x, z, m_j, n_k\}$, update $v$ from each such neighbor $v'$. Next, for every vertex $y \in X_i \cap V(G')$, if $n_j(y)$ or $n_k(y)$ now has list size 1, then update $y$ from both $n_j(y)$ and $n_k(y)$, and also update $y$ from $m_j$ and $n_k$ in the case that $y$ is adjacent to either of them. Call the obtained palette $L''$. Note that by the way we updated, it takes $O(|V(G)|)$ time to compute this palette. The total time for constructing all palettes for $\mathcal{L}_i$ thus amounts to $O(|V(G)|^3)$.

In order to see Claim 16 (a), we need to show that $|L''(y)| \le 2$ for all $y \in X_i \cap V(G')$. For contradiction, suppose $|L''(y)| = 3$ for some $y \in X_i \cap V(G')$. Then $n_j(y) \notin M \cup \{m_j\}$ and $n_k(y) \ne n_k$. By Claim 14, it follows that $n_k(y)$ is adjacent to $z$, and by the way we updated $L''$, the only other possible edge between $\{x, n_k, z, m_j\}$ and $\{y, n_j(y), n_k(y)\}$ would be $xn_j(y)$. However, since $n_j(y) \notin M$, we deduce that $n_j(y)$ is non-adjacent to $x$. Let $s$ be a neighbor of $n_j(y)$ in $S$ with $L(s) = \{j\}$. Then $s$ is anticomplete to $\{n_k, x, y, z, n_k(y)\}$. So $x - n_k - z - n_k(y) - y - n_j(y) - s$ is a $P_7$, a contradiction.

For Claim 16 (b), note that by construction, if $(G', \mathcal{L}_i, Z')$ is colorable then $(G', L', Z')$ has a type B coloring with respect to $i$. On the other hand, if $c$ is a type B coloring of $(G', L', Z')$ with respect to $i$, then there is an induced path $x - n_k - z - m_j$ with $x, z \in X_i$, $m_j \in N_j(x)$, and $n_k \in N_k(x) \cap N_k(z)$ such that $c(m_j) = c(n_k) = i$, $c(x) = j$, and $c(z) = k$ (or the same with the roles of $j$ and $k$ reversed). Since $c$ is not a type A coloring, it follows that $c(v) = k$ for all $v$ in $M$. Since updating does not change the set of possible colorings for a list, $c$ is a coloring for $L'' = L_{x,z}$. So $\mathcal{L}_i$ is as required for Claim 16 (b). □

Let $\{i, j, k\} = \{1, 2, 3\}$. We call a coloring $c$ of a restriction $(G', L', Z')$ of $(G, L, Z)$ a *type C coloring with respect to* $i$ if it is not a type A or type B coloring, and there exist $z \in X_i \cap V(G')$, $m_j \in N_j(z)$ and $n_k \in N_k(z)$ such that $c(m_j) = c(n_k) = i$.

**Claim 17.** *Let $(G', L', Z')$ be a restriction of $(G, L, Z)$ that does not admit a type A or type B coloring. There exists a set $\mathcal{L}_i$ of $O(|V(G)|^2)$ subpalettes of $L'$ such that*
*(a) $|L''(v)| \leq 2$ for every $L'' \in \mathcal{L}_i$ and $v \in X_i \cap V(G')$, and*
*(b) $(G', L', Z')$ admits a type C coloring with respect to $i$ if and only if $(G', \mathcal{L}_i, Z')$ is colorable.*
*Moreover, $\mathcal{L}_i$ can be constructed in $O(|V(G)|^4)$ time.*

*Proof.* For every $z \in X_i \cap V(G')$ having non-adjacent neighbors $m_j \in N_j(z)$ and $n_k \in N_k(z)$, we construct two families of palettes, one for each of the possible colors $j, k$ of $z$ in a type C coloring, $z$, $m_j$, $n_k$ are as in the definition of a type C coloring. We only describe how to obtain the family of palettes $L''$ with $L''(z) = \{k\}$; the definition of the family of palettes $L''$ with $L''(z) = \{j\}$ is analogous, with the roles of $j$ and $k$ reversed.

Let $N_z$ be the set of vertices $n_k$ in $N_k(z)$ having a non-neighbor in $N_j(z)$. For each such vertex $n_k$, let $W = W_{z,n_k}$ be the set of all $w \in X_i \cap V(G')$ such that there exists an induced path $w - n_k - z - m_j$ with $m_j \in N_j(z)$. We will order the vertices of $N_z$ non-increasingly by $|W|$. We can compute and sort the vertices of $N_z$ in $O(|V(G)|^3)$ time.

For each $n_k \in N_z$, define $L'' = L_{z,n_k}$ by setting $L''(z) = L''(w) = \{k\}$ for all $w \in W$, $L''(n_k) = \{i\}$, $L''(n'_k) = \{j\}$ for every $n'_k \in N_z$ having an index lower than the index of $n_k$ in $N_z$, and leaving $L''(v) = L'(v)$ for all the remaining vertices. Update each vertex of $N_j(z)$ from $z$. Now, for each vertex $v$ that has a neighbor in $\{z\} \cup N_k(z) \cup N_j(z) \cup W$, update $v$ from each such neighbor $v'$. Next, for every vertex $y \in X_i \cap V(G')$, if $n_j(y)$ or $n_k(y)$ now has list size 1, then update $y$ from both $n_j(y)$ and $n_k(y)$. Call the obtained palette $L''$. Note that by the way we updated, it takes $O(|V(G)|^2)$ time to compute this palette. The number of palettes $L_{z,n_k}$ is $O(|V(G)|^2)$, and the same for the case with the roles of $j$ and $k$ reversed. Then $\mathcal{L}_i$, the set of all palettes obtained in this way, has cardinality $O(|V(G)|^2)$, and can be constructed in $O(|V(G)|^4)$ time. We may assume that $|L''(v)| \geq 1$ for all $v \in V(G')$, otherwise we detect that the palette $L''$ does not lead to a feasible solution to $L'$.

In order to see Claim 17 (a), we need to show that $|L''(y)| \leq 2$ for all $y \in X_i \cap V(G')$. For contradiction, suppose $|L''(y)| = 3$ for some $y \in X_i \cap V(G')$. Let $m_j$ be a non-neighbor of $n_k$ in $N_j(z)$. Note that by the way we updated, $L''(m_j) = \{i\}$. Claim 14 guarantees an edge between $\{z, m_j, n_k\}$ and $\{y, n_j(y), n_k(y)\}$. By the way we updated $L''$, $n_j(y) \neq m_j$, $n_k(y) \neq n_k$, $z$ is not adjacent to $n_j(y)$, and there is no edge between $\{m_j, n_k\}$ and $\{y, n_j(y), n_k(y)\}$. So $z$ is adjacent to $n_k(y)$. Since $n_k(y)$ is not adjacent to $m_j$, $n_k(y)$ belongs to $N_z$, and as it has two colors in its list $L''$, its index is greater than the index of $n_k$ in $N_z$. As $y$ is adjacent to $n_k(y)$ and not to $\{m_j, n_k\}$, $y \in W_{z,n_k(y)} \setminus W_{z,n_k}$. Since $|W_{z,n_k}| \geq |W_{z,n_k(y)}|$, there is a vertex $x \in W_{z,n_k} \setminus W_{z,n_k(y)}$. By definition, $L''(x) = \{k\}$, thus $x$ is not adjacent to $\{y, n_j(y)\}$. Let $s$ be a neighbor of $n_j(y)$ in $S$ with $L(s) = \{j\}$. Then $s$ is anticomplete to $\{n_k, x, y, z, n_k(y)\}$. So $x - n_k - z - n_k(y) - y - n_j(y) - s$ is a $P_7$, a contradiction.

For Claim 17 (b), note that by construction, if $(G', \mathcal{L}_i, Z')$ is colorable then $(G', L', Z')$ has a type C coloring with respect to $i$. On the other hand, if $c$ is a type C coloring of $(G', L', Z')$ with respect to $i$, then there is a path $n_k - z - m_j$ with $z \in X_i \cap V(G')$, $m_j \in N_j(z)$, $n_k \in N_k(z)$, and $c(m_j) = c(n_k) = i$. Assume $c(z) = k$ (the case $c(z) = j$ is analogous), and consider the path $n_k - z - m_j$ that minimizes the index of $n_k$ in $N_z$. Since $c(m'_j) = i$ for every $m'_j$ in $N_j(z)$, it follows that $c(n'_k) = j$ for every $n'_k \in N_z$ having a lower index than the index of $n_k$ in $N_z$.

Since $c$ is not a type A or B coloring, for every vertex $w \in W_{x,m_j,n_k}$ (respectively $w \in W'_{x,m_j,n_k}$) we have $c(w) = c(x)$. Since updating does not change the set of possible colorings for a list, $c$ satisfies the palette $L'' = L_{x,n_j,n_k}$ (respectively $L'' = L'_{x,n_j,n_k}$). So $\mathcal{L}_i$ is as required for Claim 17 (b). □

**Claim 18.** *Let $(G', L', Z')$ be a restriction of $(G, L, Z)$. Assume that $(G', L', Z')$ does not admit a type A, type B, or type C coloring with respect to $i$ (i.e., no coloring with a vertex $x$ of $X_i \cap V(G')$ having neighbors colored $i$ both in $N_j(x)$ and $N_k(x)$). Let $Y_i$ be the set of vertices $x \in X_i \cap V(G')$ such that $N_i(x) = \emptyset$, and let $Z_i = \bigcup_{y \in Y_i}\{N_j(y), N_k(y)\}$. Then $(G', L', Z')$ is colorable if and only if $(G' - Y_i, L', Z' \cup Z_i)$ is colorable, and any 3-coloring of $(G' \setminus Y_i, L', Z' \cup Z_i)$ can be extended to a 3-coloring of $(G', L', Z')$ in $O(|V(G)|)$ time.*

*Proof.* It is enough to prove that for every coloring $c$ of $(G', L', Z')$ and every $x \in X_i \cap V(G')$ such that $N_i(x) = \emptyset$, the sets $N_j(x)$ and $N_k(x)$ are monochromatic with respect to $c$. Supposing this is false, we may assume that for some coloring $c$ there are vertices $u, v \in N_j(x)$ with $c(u) = i$ and $c(v) = k$. Since there are no type A or type B colorings and $c$ is not of type C, it follows that $c(w) = j$ for every $w \in N_k(x)$. But then $x$ has neighbors of all three colors, contrary to the fact that $c$ is a coloring. □

Recall that our aim was to define a set $\mathcal{P}$ of restrictions of $(G, L)$ with the property that in any element of $\mathcal{P}$ there are no vertices with list of size 3. We now construct $\mathcal{P}$ as follows. Let $Z = \emptyset$. Apply Claims 15, 16, 17 and 18 with $i = 1$ to $(G, L, Z)$ to create sets $\mathcal{P}_2, \ldots, \mathcal{P}_5$, each consisting of $O(|V(G)|^3)$ restrictions of $(G, L, Z)$. For every $x \in X_1$ and every $(G', L', Z') \in \mathcal{P}_2 \cup \mathcal{P}_3 \cup \mathcal{P}_4$, we have that $|L'(x)| \leq 2$. For $(G', L', Z') \in \mathcal{P}_5$, if $x \in X_1$ and $|L(x)| = 3$, then $N_j(x) \neq \emptyset$ for every $j \in \{1, 2, 3\}$. Repeat this with $i = 2$ for every restriction in $\mathcal{P}_2 \cup \mathcal{P}_3 \cup \mathcal{P}_4 \cup \mathcal{P}_5$, and then again with $i = 3$ for every restriction obtained with $i = 2$. This creates a set $\mathcal{P}'$ of $O(|V(G)|^9)$ restrictions. Finally, we construct $\mathcal{P}$ from $\mathcal{P}'$ by removing all restrictions that still contain lists which have size three for some vertex. Following Claims 15, 16, 17 and 18, the whole computation can be done in $O(|V(G)|^9 \cdot |V(G)|) = O(|V(G)|^{10})$ time.

Let us say that $x \in X$ is *wide* if $N_1(x) \neq \emptyset, N_2(x) \neq \emptyset$ and $N_3(x) \neq \emptyset$. Due to the construction of $\mathcal{P}'$ it holds that if $|L'(x)| = 3$ for some $(G', L', Z') \in \mathcal{P}'$, then $x$ is wide.

It remains to show that $(G, L, Z)$ is colorable if and only if $\mathcal{P}$ is colorable. By Claims 15, 16, 17 and 18, we know that if $\mathcal{P}$ is colorable then $(G, L, Z)$ is colorable. Now assume that $(G, L, Z)$ is colorable, and let $c$ be a coloring of $(G, L, Z)$. Consider any wide vertex $x$. Since the neighborhood of $x$ can only have two distinct colors in total, there are two vertices $n_j \in N_j(x)$ and $n_k \in N_k(x)$ such that $c(n_j) = c(n_k) = i$, for some distinct $j, k \in \{1, 2, 3\}$. Then $c$ is a type A, type B, or type C coloring with respect to $i$. Consequently, by Claims 15, 16, 17, there is a restriction $(G', L', Z') \in \mathcal{P}'$ that is colorable, and where for every wide vertex $y \in X \cap V(G')$ it holds that $|L'(y)| \leq 2$. Therefore $(G', L', Z') \in \mathcal{P}$.

We now come to the running time analysis. Using Lemma 4, we can check in $O(|V(G)| + |E(G)|)$ time whether a given restriction $(G', L', Z')$ of $(G, L, Z)$ is colorable. Since we have $O(|V(G)|^9)$ many restrictions to consider, and these can be computed in $O(|V(G)|^{10})$ time, the total running time amounts to $O(|V(G)|^9(|V(G)| + |E(G)|))$. This completes the proof.

# 3 The triangle-free case

We now focus on triangle-free input graphs. After showing how the 3-colorability problem can be efficiently solved, in the subsequent sections, we sketch in Section 3.4 how the list 3-coloring problem can be solved.

## 3.1 The structure of the graph $G$

Let a $\{P_7, \text{triangle}\}$-free graph $G$ be given, say with $n$ vertices and $m$ edges. We may assume that $G$ is connected. A subgraph $H$ of $G$ will be called *trivial* if $|V(H)| = 1$.

### 3.1.1 The core structure

If $G$ is bipartite, which can be decided in $O(m)$ time, then $G$ is 2-colorable and we are done. So assume $G$ is not bipartite. Since $G$ is $\{P_7$,triangle$\}$-free, the shortest odd cycle of $G$ has either length 5 or length 7, and it is an induced cycle. We first discuss the case that $G$ contains no $C_5$. Two vertices $u$ and $v$ of a graph $G$ are *false twins* if and only if $N(u) = N(v)$ (in particular, they are non-adjacent).

**Claim 19.** *If $G$ is $C_5$-free, then after identifying false twins in $G$, the remaining graph is $C_7$.*

*Proof.* As argued above, since $G$ is not bipartite and contains no $P_7$, triangle, or $C_5$, we know $G$ contains an induced cycle $C = v_1 - \cdots - v_7 - v_1$ of length 7. Suppose some vertex $v \in V(G - C)$ has neighbors in $C$. If $v$ has only one neighbor, say $v_i$, then $G$ contains an induced $P_7$, namely $v - v_i - v_{i+1} - \cdots - v_{i-2}$. (As usual, index operations are done modulo 7). By the absence of triangles, $v$ has at most three neighbors in $C$, and they are pairwise non-consecutive. If $v$ has two neighbors $v_i$, $v_{i+3}$ at distance three in $C$, then $v_i$, $v_{i+1}$, $v_{i+2}$, $v_{i+3}$, and $v$ together induce $C_5$, a contradiction. So $v$ has only two neighbors and they are at distance two in $C$.

For $i = 1, \ldots, 7$, let $V_i$ be the set formed by $v_i$ and the vertices not in $C$ whose neighbors in $C$ are $v_{i-1}$ and $v_{i+1}$. As $G$ is triangle-free, $V_i$ is a stable set. Since $G$ is $P_7$-free, every vertex in $V_i$ is adjacent to every vertex in $V_{i+1}$. Moreover, since $G$ is connected and $P_7$-free, there are no vertices outside $\bigcup_{i=1}^{7} V_i$. As $G$ is $\{\text{triangle}, C_5\}$-free, there are no edges between $V_i$ and $V_j$, for $j \notin \{i+1, i-1\}$. So, for each $i$, the vertices of $V_i$ are false twins and, after identifying them, we obtain $C_7$. □

If $G$ is a 'blown-up' $C_7$, then it is clearly 3-colorable (as false twins can use the same color). Thus, Claim 19 enables us to assume $G$ has an induced cycle $C$ of length 5, say its vertices are $c_1, c_2, c_3, c_4, c_5$, in this order. From now on, all index operations will be done modulo 5. Because $G$ has no triangles, the neighborhood $N_C$ of $V(C)$ in $G \setminus V(C)$ is comprised of 10 sets (some of these possibly empty):

- sets $T_i$, whose neighborhood on $C$ is equal to $\{c_{i-1}, c_{i+1}\}$;
- sets $D_i$, whose only neighbor on $C$ is $c_i$;

where the indices $i$ go from 1 to 5. Note that, because of $G$ being triangle-free, the sets $T_i$ and $D_i$ are each stable. We set $S := V(C) \cup \bigcup_{i=1}^{5} T_i \cup \bigcup_{i=1}^{5} D_i$. Note that $V(C)$ has no neighbors in $G - S$.

### 3.1.2 The non-trivial components of $G - S$

The following list of claims narrows down the structure of the non-trivial components of $G-S$.

**Claim 20.** *If $xy$ is an edge in $G - S$, then $x$ and $y$ have no neighbors in any of the sets $D_i$.*

*Proof.* Suppose that $x$ has a neighbor $u$ in $D_1$. Then as $G$ is triangle-free, $y$ is not adjacent to $u$. So $y - x - u - c_1 - c_2 - c_3 - c_4$ is an induced $P_7$, a contradiction. The other cases are symmetric. □

**Claim 21.** *If $x - y - z$ is an induced $P_3$ in $G - S$, then the neighborhoods of $x$ and $z$ inside each $T_i$ are identical.*

*Proof.* Suppose that $x$ has a neighbor $u$ in $T_1$ that is not a neighbor of $z$. Then, as $G$ is triangle-free, $y$ is not adjacent to $u$, either. Also, $x$ and $z$ are not adjacent. So $z-y-x-u-c_2-c_3-c_4$ is an induced $P_7$, a contradiction. We argue similarly for all other $T_i$'s. □

**Claim 22.** *$G - S$ is bipartite.*

*Proof.* Assume $G-S$ has an odd cycle $C'$. Take a shortest path $P = c' - p_1 - \cdots - p_k - s$ from $V(C')$ to $S$. Since $G$ is triangle-free, and $C'$ is odd, there is a induced path $c'_1 - c'_2 - c'_3 - p_1$ with $c'_i \in V(C')$ for $i = 1, 2, 3$. Furthermore, as $P$ was chosen to be a shortest path, there are no edges of the form $c'_i p_j$ except for $c'_3 p_1$, and no edges of the form $p_j s$ except for $p_k s$. So we can complete $c'_1 - c'_2 - c'_3 - p_1 - \cdots - p_k - s$ with three vertices from $C$ to obtain an induced path of length at least 7, a contradiction. □

**Claim 23.** *Let $M$ be a non-trivial component of $G - S$. Then there is a partition of $V(M)$ into stable sets $U_1, U_2$ such that all vertices in $U_i$ have the same set $N_i$ of neighbors in $S$, at least one of $N_1$, $N_2$ is non-empty, and $N_1 \cap N_2 = \emptyset$.*

*Proof.* This follows directly from the two previous claims, and the fact that $G$ is connected and triangle-free. □

We need one more claim about independent edges outside $S$. For this, let $2K_2$ denote the graph that is the disjoint union of two edges. Moreover, let $N_{T_i}(v)$ denote the set $N(v) \cap T_i$ for each $v \in V(G)$.

**Claim 24.** *Suppose $vw, xy \in E(G)$ induce $2K_2$ in $G - S$. Then, for every $i = 1, \ldots, 5$, $N_{T_i}(x) \cup N_{T_i}(y) \subseteq N_{T_i}(v) \cup N_{T_i}(w)$, or $N_{T_i}(v) \cup N_{T_i}(w) \subseteq N_{T_i}(x) \cup N_{T_i}(y)$.*

*Proof.* If none of these inclusions holds, then there are vertices $u, z \in T_i$ such that $uv, yz \in E(G)$ and $u \notin N_{T_i}(x) \cup N_{T_i}(y)$, $z \notin N_{T_i}(v) \cup N_{T_i}(w)$ (after possibly swapping some names). Since $G$ is triangle-free, also $u \notin N_{T_i}(w)$ and $z \notin N_{T_i}(x)$. So, after possibly swapping some names, $x - y - z - c_{i+1} - u - v - w$ is an induced $P_7$, a contradiction. □

Observe that we can not extend the last claim to the neighborhood in all of $S$, because then $u$ and $z$ might be adjacent.

### 3.1.3 The trivial components of $G - S$

Let $W$ be the set of isolated vertices in $G - S$. We will first prove some properties of the vertices in $W$ and their neighbors in $S$.

**Claim 25.** *There is no vertex in $W$ having neighbors in both $D_i$ and $D_{i+1}$, $i = 1, \dots, 5$.*

*Proof.* Suppose $w$ has neighbors $d_1$ in $D_1$ and $d_2$ in $D_2$. Then $d_1 - w - d_2 - c_2 - c_3 - c_4 - c_5$ is an induced $P_7$ in $G$, a contradiction. The other cases are symmetric. □

As both $W$ and $D_i$ are stable, $G[W \cup D_i]$ is bipartite, for every $i = 1, \dots, 5$. We now show some properties similar to the ones we showed for the non-trivial components of $G - S$ in Section 3.1.2.

**Claim 26.** *If $x - y - z$ is a $P_3$ in $G[W \cup D_i]$, then the neighborhoods of $x$ and $z$ inside $T_i$ are identical, for $i = 1, \dots, 5$.*

*Proof.* Suppose that $x$ has a neighbor $u$ in $T_i$ that is not a neighbor of $z$. Then as $G$ is triangle-free, $y$ is not adjacent to $u$, either. So $z - y - x - u - c_{i+1} - c_{i+2} - c_{i+3}$ is an induced $P_7$, a contradiction. □

**Claim 27.** *Let $M$ be a non-trivial component of $G[W \cup D_i]$, $i \in \{1, \dots, 5\}$. Then all vertices in $M \cap W$ have the same set of neighbors in $T_i$, and all vertices in $M \cap D_i$ have the same set of neighbors in $T_i$.*

*Proof.* Directly from the previous claim, and the fact that $G[W \cup D_i]$ is bipartite. □

Finally, we extend Claim 24 to connected components of $G[W \cup D_i]$ (for $i = 1, \dots, 5$).

**Claim 28.** *Let $i \in \{1, \dots, 5\}$. Suppose $vw, xy \in E(G)$ induce $2K_2$ in $G[(G - S) \cup D_i]$. Then, $N_{T_i}(x) \cup N_{T_i}(y) \subseteq N_{T_i}(v) \cup N_{T_i}(w)$, or $N_{T_i}(v) \cup N_{T_i}(w) \subseteq N_{T_i}(x) \cup N_{T_i}(y)$.*

*Proof.* If none of these inclusions holds, then there are vertices $u, z \in T_i$ such that $uv, yz \in E(G)$ and $u \notin N_{T_i}(x) \cup N_{T_i}(y)$, $z \notin N_{T_i}(v) \cup N_{T_i}(w)$ (after possibly swapping some names). Since $G$ is triangle-free, also $u \notin N_{T_i}(w)$ and $z \notin N_{T_i}(x)$. So, after possibly swapping names, $x - y - z - c_{i+1} - u - v - w$ is an induced $P_7$, a contradiction. □

## 3.2 The algorithm

When the input graph contains an induced $C_5$, we will use the structural properties described in Section 3.1 in order to reduce the 3-coloring problem to a polynomial number of instances of list-coloring where the lists have length at most two. The latter problem is then solved using Lemma 4. To be precise, we will reduce our problem to a polynomial number of instances of list-coloring where every vertex $v$ either has a list of size at most 2, or there is a color $j \in \{1, 2, 3\}$ missing in the list of each of its neighbors, and so $v$ can safely use color $j$.

First of all, we fix a coloring of the 5-cycle $C$ (there are 5 essentially different colorings). For each $D \in \{T_1, D_1, \dots, T_5, D_5\}$, the coloring of $C$ either determines the color of $D$, or the vertices of $D$ lose a color. Notice that, once we have fixed the coloring of $C$, three of the $T_i$ have their color determined, while two of them (consecutive sets, actually) have two possible colors left. For instance, if we color $c_1, c_2, c_3, c_4, c_5$ with $1, 2, 1, 2, 3$, respectively, then vertices in $T_1$ will be forced to have color 1, vertices in $T_4$ color 2, vertices in $T_5$ color 3, vertices in

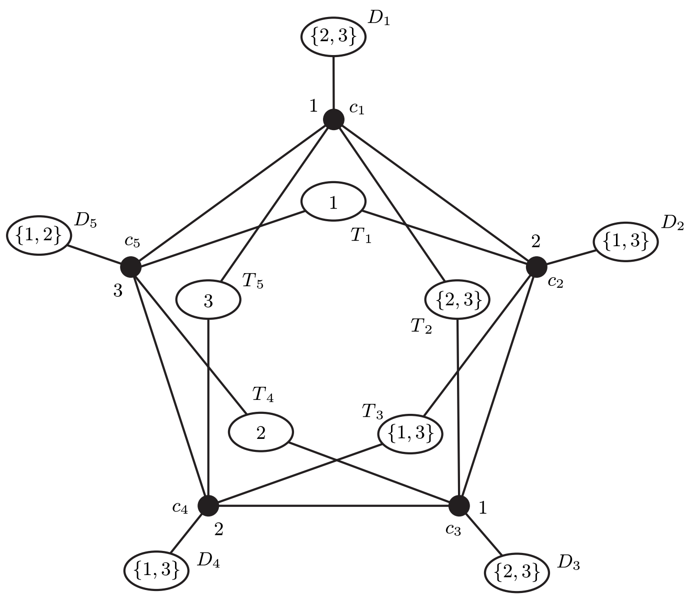


Figure 1: Scheme of the colored cycle and its neighbors with their possible colors.

$T_2$ have the options $\{2, 3\}$ and vertices in $T_3$ have the options $\{1, 3\}$. The vertices in $D_i$, $i = 1, \ldots, 5$, have lost one color each.

We will work with this coloring of $V(C)$, since the other 4 are totally symmetric. In addition to what we observed above about possible colorings of the sets $T_i$ and $D_i$, we know that each neighbor of $T_1 \cup T_4 \cup T_5$ has already lost a color. We have to deal now with the vertices having neighbors only in $T_2$, $T_3$, and $D_i$, for $i = 1, \ldots, 5$, and vertices having no neighbors in $S$.

Note that, by Claim 20, no connected component of $G[(G - S) \cup D_i]$ may contain at the same time edges from $G - S$ and vertices of $D_i$. So, from Claims 23, 27 and 28 we know that for each $i = 2, 3$, we can order the non-trivial components of $G - S$ and of $G[W \cup D_i]$ that have at least one neighbor in $T_i$ according to their neighborhood in $T_i$ by inclusion.

Let $N_1^i \subsetneq N_2^i \subsetneq \cdots \subsetneq N_{r_i}^i$, for $i = 2, 3$, be the non-empty neighborhoods of the non-trivial components of $G - S$ and $G[W \cup D_i]$ in $T_i$, respectively. For technical reasons, choose an arbitrary vertex $v_0^i \in N_1^i$, set $N_0^i := \{v_0^i\}$, and set $N_{r_i+1}^i := T_i$, for $i = 2, 3$.

Our algorithm now enumerates the following partial colorings. In each partial coloring, some vertices of $T_i$ receive a color, for both $i = 2, 3$ simultaneously and independently.

(a) There is a $k \in \{0, \ldots, r_i\}$ such that all vertices in $N_k^i$ get color 2 (resp. 1) and there is a vertex $w$ in $N_{k+1}^i \setminus N_k^i$ getting color 3.

(b) There is a $k \in \{0, \ldots, r_i\}$ such that all vertices in $N_k^i$ get color 3 and there is a vertex $w$ in $N_{k+1}^i \setminus N_k^i$ getting color 2 (resp. 1).

(c) All vertices in $T_i$ get color 2 (resp. 1).

(d) All vertices in $T_i$ get color 3.

For example, one such partial coloring could be that every vertex of $N_6^2$ gets color 2 and a vertex $w$ in $N_7^2 \setminus N_6^2$ gets color 3, while every vertex in $T_3$ gets color 3, assuming that these are valid situations. Note that every 3-coloring that agrees with the coloring of the already colored induced $C_5$ is an extension of one of the above mentioned partial colorings.

In each of the partial colorings there might be vertices which have only one color left on their list. Successively we color all such vertices. We discard the cases when there are adjacent vertices receiving the same color, or when some vertex has no color left.

**Claim 29.** *For any partial coloring as above and for any vertex $x$ of some non-trivial component $M$ of $G - S$, we have the following. Either $x$ has at most two colors left on its list, or there is a color $j \in \{1, 2, 3\}$ missing in the list of each of the neighbors of $x$ (and so $x$ can safely use color $j$).*

*Proof.* By Claim 20, for each partial coloring as above, either there are two vertices of different color in $N(M) \cap S$, or $N(M) \cap S$ is completely colored. In the first case, either $x$ is adjacent to a colored vertex in $S$ and so it loses a color; or, by Claim 23, its neighbors in $M$ have two neighbors in $S$ of different color, and thus their color is fixed. In this case again, $x$ loses a color.

In the second case, if $x$ has neighbors in $S$, then it loses a color. If not, then again by Claim 23, all the neighbors of $x$ are in $M$ and have lost a common color $j$ from their colored neighbors in $S$, so we are done. □

**Claim 30.** *For any partial coloring as above and for any vertex $x$ of any non-trivial component of $G[W \cup D_i]$, we have the following. If $x$ has a neighbor in $T_i$, then $x$ has at most two colors left on its list.*

*Proof.* The proof is analogous to the proof of the previous claim, replacing Claim 23 with Claim 27, and $S$ with $T_i$. □

**Claim 31.** *In any partial coloring as above, every vertex $w \in W$ having neighbors both in $T_2$ and in $T_3$ loses a color.*

*Proof.* Notice that either $T_2$ is monochromatic with color 2, or $T_3$ is monochromatic with color 1 (in either case, $w$ loses a color), or there are two non-adjacent vertices $x_2 \in T_2$ and $x_3 \in T_3$ having color 3. Let $w$ in $W$ have neighbors $y_2$ in $T_2$ and $y_3$ in $T_3$. Observe that $y_2$ and $y_3$ are not adjacent because there are no triangles. Then either $w$ is adjacent to $x_2$ or to $x_3$ (and hence loses a color), or $y_2$ is adjacent to $x_3$, or $y_3$ is adjacent to $x_2$, since otherwise $x_3 - c_4 - y_3 - w - y_2 - c_1 - x_2$ is an induced $P_7$ in $G$, a contradiction. If $y_2$ is adjacent to $x_3$ then it has to be colored 2, and if $y_3$ is adjacent to $x_2$ then it has to be colored 1. In either case, $w$ loses a color. □

In total, we enumerate

$$O((|N_1^2|+|N_2^2 \setminus N_1^2|+\cdots+|N_{r_2+1}^2 \setminus N_{r_2}^2|)\cdot(|N_1^3|+|N_2^3 \setminus N_1^3|+\cdots+|N_{r_3+1}^3 \setminus N_{r_3}^3|)) = O(|T_2|\cdot|T_3|)$$

many partial colorings, that is, $O(n^2)$. For each of these, we will solve a set of instances of list-coloring with lists of size at most two, after dealing with the trivial components of $G - S$ as detailed in the next paragraphs.

Since $G$ is connected, each $w \in W$ has a neighbor in $S$. If $w$ has a neighbor in a set with fixed color, then $w$ has at most 2 colors on its list. On the other hand, if there is a color $j \in \{1, 2, 3\}$ missing in the list of each of its neighbors, then $w$ can safely use color $j$. So we have to deal only with vertices in $W$ having no neighbors in $T_1$, $T_4$ and $T_5$, and having neighbors in at least two sets with different color options in $\{T_2, T_3, D_1, \ldots, D_5\}$. Vertices of $W$ having neighbors in $T_2$ and $D_2$, or in $T_3$ and $D_3$, have already lost a color by Claim 30. Vertices of $W$ having neighbors in $T_2$ and $T_3$, have already lost a color by Claim 31.

So, the types of vertices in $W$ we need to consider are the following (a scheme of the situation can be seen in Figure 1).

- *Type 1:* vertices in $W$ having neighbors in $T_2$ and in $D_5$; symmetrically, vertices in $W$ having neighbors in $T_3$ and $D_5$; vertices in $W$ having neighbors in $T_2$ and $D_4$; vertices in $W$ having neighbors in $T_3$ and $D_1$.
- *Type 2:* vertices in $W$ having neighbors in $D_2$ and in $D_5$; symmetrically, vertices in $W$ having neighbors in $D_3$ and $D_5$; and vertices in $W$ having neighbors in $D_1$ and $D_4$.

Pick an arbitrary vertex $v_i \in D_i$, for $i = 1, 4, 5$. We now extend each partial coloring from above by further enumerating the coloring of some vertices in $D_i$, for all $i = 1, 4, 5$ simultaneously and independently. For this, say that $\{a_i, b_i\}$ are the possible colors in $D_i$.

(e) Vertex $v_i$ gets color $a_i$ and there is a vertex $v'$ in $D_i - \{v_i\}$ getting color $b_i$.

(f) Vertex $v_i$ gets color $b_i$ and there is a vertex $v'$ in $D_i - \{v_i\}$ getting color $a_i$.

(g) All vertices in $D_i$ get color $a_i$.

(h) All vertices in $D_i$ get color $b_i$.

Again, every 3-coloring of $G$ that agrees with the coloring of the induced $C_5$ is an extension of one of the enumerated partial colorings.

Notice that for each $D_i$ these are $O(|D_i|)$ cases, so the total number of combinations of cases is $O(n^3)$. Again, we discard a partial coloring if there are adjacent vertices receiving the same color, or when some vertex has no color left.

It remains to show that for each partial coloring, all the remaining vertices in $W$ (Type 1 and Type 2) lose a color.

First we discuss vertices of Type 1. Suppose some $w$ in $W$ has neighbors in $T_2$ and in $D_5$. For each case of (e)–(h), either $D_5$ is monochromatic with color 1 (so $w$ loses a color), or there is a vertex $x$ in $D_5$ with color 2. Then either $w$ is adjacent to $x$ (and loses a color), or every neighbor $z$ of it in $T_2$ is adjacent to $x$, otherwise, by triangle freeness, $x - c_5 - v - w - z - c_3 - c_2$, where $v$ is some neighbor of $w$ in $D_5$, is a $P_7$, a contradiction. Then $z$ has to be colored 3, and $w$ loses a color. The other cases are symmetric.

Now we discuss vertices of Type 2. Suppose some $w$ in $W$ has neighbors in $D_2$ and in $D_5$. Such a vertex can be dealt with in exactly the same way as the vertices of Type 1. The only difference is the path $P_7$, which here is given by $x - c_5 - v - w - z - c_2 - c_3$, where $x, v \in D_5$ and $z \in D_2$ are neighbors of $w$. The other cases are symmetric.

## 3.3 The overall complexity of the algorithm

### 3.3.1 Finding the $C_5$ and partitioning the graph

We check if $G$ is bipartite in $O(m)$ time. If it is not bipartite, we find either an induced $C_5$ or an induced $C_7$. If we find an induced $C_7$, let us call it $X$, we check the neighborhood of $N(X)$ in $X$: either we find a $C_5$ or we partition the vertices into sets that are candidates to be false twins of the vertices of $X$ (see Claim 19). The adjacencies must hold because of the $P_7$-freeness, and the sets are stable and there are no edges between vertices in sets corresponding to vertices at distance 2 in $X$ because of the triangle-freeness. While checking for edges between vertices in sets corresponding to vertices at distance 3 in $X$, we will either find a $C_5$ or conclude that the graph is $C_5$-free and after identifying false twins we obtain $C_7$. For this, we just need to check for each edge, if their endpoints are labeled with numbers corresponding to vertices at distance 3 in $X$. The whole step can be done in $O(m)$ time.

Assuming we found an induced $C_5$, partitioning the neighbors $S$ of this $C_5$ into the sets $T_1, D_1, \ldots, T_5, D_5$ can be also done in linear time. Moreover, finding the connected components of $S$ and their neighborhoods on each of the $T_i$'s can be done in $O(m)$ time.

#### 3.3.2 Exploring the cases to get the list-coloring instances

The number of distinct colorings of the $C_5$ to consider is 5. For each of them, using the notation of the coloring of the last section, there are $O(n^2)$ many combinations to be tested on $T_2$ and $T_3$ in order to deal with the non-trivial components of $G - S$, $G[W \cup D_2]$ and $G[W \cup D_3]$, and the vertices in $W$ having neighbors in both $T_2$ and $T_3$ (cases (a)–(d)). For each of them, we have to test all the combinations for $D_1$, $D_4$ and $D_5$ (cases (e)–(h)), that are $O(n^3)$ many, in order to deal with the remaining vertices in $W$. Finally, for each of these possibilities, we have to solve a list-coloring instance with lists of size at most 2, which can be done in $O(n + m)$ time by Lemma 3. Thus, the overall complexity of the algorithm is $O(n^5(n + m))$.

### 3.4 List 3-coloring

When there is no induced $C_5$, the graph is either bipartite or, according to Claim 19, a blown-up $C_7$. The discussion of the bipartite case is given below. In the case of a blown-up $C_7$, we may simply identify twins that have identical lists. Since we have at most 7 possible distinct non-empty lists, the remaining graph has at most 49 vertices and we are done.

When the input graph contains an induced $C_5$, we follow the steps described in Section 3.2 thus reducing the 3-coloring problem to a polynomial number of instances of list-coloring where every vertex $v$ either has a list of size at most 2, or there is a color $j \in \{1, 2, 3\}$ missing in the list of each of its neighbors, and so $v$ can safely use color $j$. In order to do that, we pre-color some vertices, in particular those of the $C_5$. To take into account the list restrictions, we have to consider all possible (feasible) colorings of the $C_5$ (at most 30 instead of 5), since the colors are not playing symmetric roles now. Also, we may simply close a branch in the enumeration whenever a vertex is assigned a color not contained in its list. Finally, for the cases in which our argument was that for some vertex $v$ there is a color $j \in \{1, 2, 3\}$ missing in the list of each of its neighbors so we can assign color $j$ to $v$, it may happen now that $j$ is not in the list of $v$. But then $v$ has a list of size at most 2, which is equally useful for our purposes.

The same analysis from Section 3.2 gives that, in this case, the overall complexity is $O(n^5(n + m))$.

In order to complete the proof of Theorem 2, it suffices to show the following.

**Theorem 32.** *Given a $P_7$-free bipartite graph $G$, the list 3-coloring problem can be decided, and a coloring can be found, in $O(|V(G)|^2(|V(G) + E(G)|))$ time.*

We will first preprocess the graph and then either find two vertices such that every vertex in the graph having a list of size 3 is adjacent to one of them, or find an induced $C_6$ with certain properties, and proceed in the same spirit of the algorithm of Section 3.2, but with different arguments.

We will use the following simple statement below.

**Claim 33.** *Let $H$ be a $C_6$-free graph and let $A, B \subseteq V(H)$ be disjoint stable sets of $H$. If every two vertices of $A$ have a common neighbor in $B$, then some vertex of $B$ is complete to $A$.*

*Proof.* We prove this by induction on $|A|$. Inductively we may assume that, for $i = 1, 2$, there exist $a_i \in A$ and $b_i \in B$ such that $a_i$ is non-adjacent to $b_i$, while $b_i$ is complete to $A \setminus a_i$. Let $b_3 \in B$ be a common neighbor of $a_1$ and $a_2$. We may assume that $b_3$ has a non-neighbor in $A$, $a_3$ say. But now there is an induced $C_6$ on $a_1 - b_3 - a_2 - b_1 - a_3 - b_2 - a_1$, a contradiction. □

From now on we assume that $G = (V, E)$ is a connected $P_7$-free bipartite graph with bipartition $(A, B)$, on $n$ vertices and $m$ edges. Moreover, each vertex $v \in V$ is equipped with a list $L(v) \subseteq \{1, 2, 3\}$.

We call a vertex $v \in V$ *dominated* if there is some $u \in V$ such that $N(v) \subseteq N(u)$. In particular, $uv \notin E$. We also say $u$ *dominates* $v$. We remark that this notion is not to be confused with the notion of a dominating set given in the first part of the paper.

We may assume that if $u$ dominates $v$, then $L(u) \not\subseteq L(v)$. In particular, we may assume that if $v$ is dominated, then $|L(v)| \leq 2$ (otherwise it is enough to test whether $G - v$ is colorable).

**Claim 34.** *Let $A'$ be the vertices in $A$ with lists of size at most 2, and let $A'' = A \setminus A'$. Define $B'$ and $B''$ similarly. Then either*

*(a) some vertex $w$ of $A$ is complete to $B''$, or*

*(b) there is an induced 6-cycle $C$ in $G$ with $V(C) \cap B \subseteq B''$.*

*The same holds with the roles of $A, B$ reversed. We can find $w$ or $C$, respectively, in $O(n^2)$ time.*

*Proof.* Suppose that both (a) and (b) fail to hold. We may thus apply Claim 33 to the graph $G[A \cup B'']$. Since no vertex of $A$ is complete to $B''$, there exist $b_1, b_2 \in B''$ with no common neighbor in $A$. Clearly, a shortest path from $b_1$ to $b_2$ in $G$, $P$ say, is of the form $b_1 - a_1 - b_3 - a_2 - b_2$. Since $b_1, b_2 \in B''$, $b_3$ does not dominate either of them.

For $i = 1, 2$ let $x_i$ be adjacent to $b_i$ and not to $b_3$. Since $b_1$ and $b_2$ have no common neighbor in $A$, $x_1 \neq x_2$ and, since $G$ is bipartite, $x_1 - b_1 - a_1 - b_3 - a_2 - b_2 - x_2$ is an induced $P_7$, a contradiction.

It is easy to check if there is a vertex of $A$ that is complete to $B''$ in $O(n^2)$ time. If there is not, a subset $B'''$ of $B''$ that is minimal with respect to the property of having no vertex of $A$ that is complete to it can be found in $O(n^2)$ time. Notice that $|B''| \geq 3$, since we previously observed that every pair of vertices of $B''$ have a common neighbor in $A$. So we can proceed as in the proof of Claim 33 in order to find the induced $C_6$. □

**Claim 35.** *If $G$ has no induced $C_6$ in which all the vertices of some parity have lists of size 3, then we can test in $O(n + m)$ time whether $G$ is colorable.*

*Proof.* With the notation of Claim 34, we may guess the colors of common neighbors of $A''$ and of $B''$, respectively. This reduces the size of all lists, and list-coloring where the lists have length at most two can be solved in $O(n + m)$ time by Lemma 3. □

So, from now on, we will assume that $G$ contains an induced $C_6$ such that all the vertices of some parity have lists of size 3. Let $C = c_1 - \cdots - c_6 - c_1$ be an induced $C_6$ in $G$. Let $L_i$ be the vertices of $G$ whose unique neighbor in $C$ is $c_i$, and let $T_i$ be the vertices adjacent to exactly $c_{i-1}$ and $c_{i+1}$ in $C$. Let $S_1$ be the vertices adjacent to all of $c_1, c_3, c_5$, and $S_2$ the vertices adjacent to all of $c_2, c_4, c_6$.

In order to further discuss the structure of $G$ and its colorings, we need the following notions Let $L = \bigcup_{i=1}^{6} L_i$, $T = \bigcup_{i=1}^{6} T_i$ and $S = S_1 \cup S_2$. Moreover, let $W = V \setminus (V(C) \cup L \cup T \cup S)$ and let $D$ be the vertices in $W$ with a neighbor in $T$. Let $X_1$ be the vertices in $W \setminus D$ with a neighbor in $S$, $X_2$ vertices in $W \setminus (D \cup X_1)$ with a neighbor in $X_1$, and let $X_3 = W \setminus (D \cup X_1 \cup X_2)$.

**Claim 36.** *The following assertions hold.*

*(a) $L$ is anticomplete to $W$.*

*(b) $D$ is stable, anticomplete to $W \setminus D$, and every vertex in $D$ is dominated.*

*(c) $X_3$ is stable, and every vertex of $X_3$ is dominated.*

*(d) Let $x, y, z \in X_1 \cup X_2 \cup X_3$ and $s_1 \in S_1$ be such that $s_1 - x - y - z$ is an induced path. Then either $s_1$ dominates $c_2$, or $y \in X_1$ and there exists $s_2 \in S_2$ non-adjacent to $s_1$ and adjacent to $y$.*

*(e) Assertion* (d) *holds with the roles of $S_1$ and $S_2$ reversed.*

*Proof.* Assertion (a) follows from the fact that $G$ is $P_7$-free.

Now we prove (b). Let $d \in D$, and suppose that some $w \in W \setminus \{d\}$ is adjacent to $d$. We may assume that there is some $c \in T_1 \cap N(d)$. But then $w - d - c - c_2 - c_3 - c_4 - c_5$ is an induced $P_7$, which proves that $D$ is stable and anticomplete to $W \setminus D$.

Again pick $d \in D$ and assume that there is some $c \in T_1 \cap N(d)$. We may assume that $d$ is not dominated by $c_2$, so $d$ has a neighbor $v$ that is non-adjacent to $c_2$. By (a) and since $D$ is stable and anticomplete to $W \setminus D$, it follows that $v$ is in $T \cup S$, and by parity $v$ is in $T_1 \cup T_3 \cup T_5 \cup S_2$. Since $v$ is non-adjacent to $c_2$, we deduce $v \in T_5$. But now $c_1 - c_2 - c - d - v - c_4 - c_5$ is an induced $P_7$, a contradiction. This proves (b).

To see (c), let $x, y \in X_3$ be adjacent, let $x_2 \in X_2$, $x_1 \in X_1$ and $s \in S$ be such that $x - x_2 - x_1 - s$ is an induced path. We may assume that $s \in S_1$. But then $y - x - x_2 - x_1 - s - c_1 - c_2$ is an induced $P_7$, a contradiction. This proves that $X_3$ is stable.

Let $x \in X_3$, and note that $N(x) \subseteq X_2$. Let $n \in N(x)$ and let $x_1 \in X_1$ be a neighbor of $n$. We may assume that $x$ is not dominated by $x_1$, and so there is some $n' \in N(x) \setminus N(x_1)$. Moreover, we may assume that $x_1$ has a neighbor $s_1$ in $S_1$. But now $n' - x - n - x_1 - s_1 - c_1 - c_2$ is an induced $P_7$, a contradiction. This proves (c).

Finally, we prove (d), and (e) follows by symmetry. Let $s_1, x, y, z$ be as in (d). Assuming $s_1$ does not dominate $c_2$, there exists $v$ adjacent to $c_2$ and not to $s_1$. Since $v - c_2 - c_1 - s_1 - x - y - z$ is not a $P_7$, since $G$ is bipartite, it follows that $v$ is adjacent to $y$. By (b), $y \in X_1$ and thus $v \in S_2$. This proves (d), and completes the proof of Claim 36. □

Notice that, by Claim 36 and our assumptions about dominated vertices, after fixing a coloring of the cycle $C$ and updating the lists, the only vertices that may still have lists of size 3 are in $X_1 \cup X_2$.

We say a coloring of an induced 6-cycle is of *type 1* if its sequence of colors is $1, 2, 3, 1, 2, 3$ (possibly permuting the colors); and of *type 2* if its sequence is $1, 2, 3, 1, 3, 2$ (possibly permuting the colors, and shifting the starting vertex). Otherwise, its color sequence reads $1, 2, 1, 2, 1, *$, where $*$ is either 2 or 3 (and possibly permuting the colors, and shifting the starting vertex). If the cycle furthermore contains a vertex with a list of size 3 that is in a color class of size at most two, we call the coloring *type 3*; and else we call it *type 4*.

We can sketch our algorithm as follows. We will show first that we can test if a type 1 coloring of a cycle can be extended to the whole graph in $O(n + m)$ time. Next, we will deal with the case in which all vertices with lists of size 3 have the same parity, showing that if this is the case, we can test if a type 2 or type 3 coloring of a cycle can be extended to the whole graph in $O(n + m)$ time. Further, we will show that the list 3-coloring problem in which all vertices with lists of size 3 have the same parity can be reduced to testing $O(n)$ times if a type 1, type 2 or type 3 coloring of a cycle extends to the whole graph, thus giving a time complexity of $O(n(n + m))$ for that case. Then we will go for the general case, showing that testing if a type 2 or type 3 coloring of a cycle can be extended to the whole graph reduces to the list 3-coloring problem in which all vertices with lists of size 3 have the same parity, thus it is solvable in $O(n(n + m))$ time. Finally, we will show that the list 3-coloring problem (the general case) can be reduced to testing $O(n)$ times if a type 1, type 2 or type 3 coloring of a cycle extends to the whole graph, thus giving a time complexity of $O(n^2(n + m))$ in total.

**Claim 37.** *Given an induced 6-cycle $C$ of $G$ with a coloring of type 1, we can test in $O(n+m)$ time if the coloring extends to $G$.*

*Proof.* If there is vertex with three neighbors on $C$, the coloring does not extend. Otherwise, in the language from above, the set $S$ is empty and thus $W = D$. So Claim 36 implies that after updating, that can be performed in time $O(m)$, all lists have size at most two. We can check whether $G$ is colorable with the updated lists in the required time, by Lemma 3. □

**Claim 38.** *Let $C = c_1 - \cdots - c_6 - c_1$ be an induced $C_6$ with $|L(c_2)| = 3$, where $c_2$ has parity $A$. Then no vertex of $X_2$ of parity $A$ has a list of size 3.*

*Proof.* Let $x_2 \in X_2$ with $|L(x_2)| = 3$ such that $x_2$ has parity $A$. Let $x_1 \in X_1$ and $s_1 \in S$ be such that $s_1 - x_1 - x_2$ is a path. By parity, $s_1 \in S_1$. Since $s_1$ does not dominate $x_2$, there is a neighbor $z$ of $x_2$ that is non-adjacent to $s_1$. Since $|L(c_2)| = 3$, $s_1$ does not dominate $c_2$, and so by Claim 36(d), there is $s_2$ in $S_2$ adjacent to $x_2$, contrary to the fact that $x_2 \in X_2$. □

**Claim 39.** *Assume that some induced 6-cycle of $G$ is given with a pre-coloring of type 2 or 3, that contains a vertex $c$ from a color class of size 1 or 2 which has parity $A$ and a list of size 3. Then, after updating all lists, no vertex of parity $A$ has a list of size 3.*

*Proof.* Suppose some vertex $v$ of parity $A$ has a list of size 3. Then $v \notin X_1$, since every vertex of $S$ that is adjacent to $c$ has only one color left in its list, as $c$ is in a color class of size 1 or 2 in the pre-coloring. Now by Claim 36(b), Claim 36(d), and since no dominated vertex has a list of size 3, we have that $v \in X_2$, contrary to Claim 38. □

We will deal now with the subcase in which all the vertices having lists of size 3 have the same parity on $G$. We will use then the algorithm for this subcase as a subroutine in the general case.

**Claim 40.** *Suppose all vertices of $G$ with lists of size 3 have parity $A$. Further, let $C$ be an induced 6-cycle of $G$ with a pre-coloring of type 2 or 3, that contains a vertex $c$ from a color class of size 1 or 2 which has parity $A$ and a list of size 3. Then we can test in $O(n+m)$ time whether the coloring of $C$ extends to $G$.*

*Proof.* After updating all the lists, no vertex of parity $A$ has a list of size 3, because of Claim 39. Therefore, no vertex has a list of size 3, and so we can check if the coloring extends to the whole graph in the required time. □

**Proposition 41.** *If all vertices with lists of size 3 have parity $A$, we can test in $O(n(n+m))$ time if $G$ admits a coloring.*

*Proof.* Preprocessing the graph with respect to dominated vertices can be done in time $O(n^2)$. Also, by Claim 34, we can find in time $O(n^2)$ either a vertex $v$ that is adjacent to all of $A''$, or an induced 6-cycle $C$ with $V(C) \cap A \subseteq A''$. In the former case, we can check for all feasible pre-colorings of $v$ whether they extend to $G$ in $O(n+m)$ time (since after updating again, all lists have size at most 2 and we have Lemma 3).

So assume we found $C = c_1 - c_2 - \cdots - c_6 - c_1$ as above. By Claims 37 and 40, we can check for every feasible pre-coloring of $C$ of type 1,2, or 3 whether it extends to $G$, in $O(n+m)$ time. Now we go through all type 4 colorings of $C$.

Consider any such coloring, and for simplicity assume $c_2, c_4$ and $c_6$ have color 1 and parity $A$. After updating the lists, all vertices of parity $A$ with lists of size 3 are in $X_1$, by Claims 36 and 38. Let $X_1'$ be the set of all such vertices. If there is a vertex that is adjacent to all of $X_1'$, then we can check for all feasible pre-colorings of $v$ whether they extend to $G$ in $O(n+m)$ time (since after updating again, all lists have size at most 2 and we have Lemma 3). So assume there is no such vertex.

Then, since by parity, every vertex of $X_1'$ has a neighbor in $S_2$, there exist $x_1, x_1' \in X_1'$ and $s_2, s_2' \in S_2$ such that $x_1 s_2$ and $x_1' s_2'$ are edges, but $x_1 s_2'$ and $x_1' s_2$ are not edges. Since $c_2$ does not dominate $x_1$ or $x_1'$, there exist $y, y'$ such that $y$ is adjacent to $x_1$ and not to $c_2$, and $y'$ is adjacent to $x_1'$ and not to $c_2$. Since $G$ is bipartite, and since $y - x_1 - s_2 - c_2 - s_2' - x_1' - y'$ is not an induced $P_7$, it follows that $y = y'$.

Consider now the induced 6-cycle $C' = c_2 - s_2 - x_1 - y - x_1' - s_2' - c_2$. By Claims 37 and 40, we can check for every feasible pre-coloring of $C'$ of type 1,2, or 3 whether it extends to $G$, in $O(n+m)$ time. If none of them extends, we can deduce that in every coloring of $G$ (that extends the current pre-coloring of $C$), the cycle $C'$ will have a type 4 coloring and thus the color of both $x_1$ and $x_1'$ must be 1. We update the lists (and thus the set $X_1'$) in time $O(n+m)$, and repeat the procedure from above. That is, we first try to find common neighbor of $X_1'$, and if we do not find such a vertex, we find an induced 6-cycle. As above, this either gives a coloring of $G$, or we fix the colors of two more vertices from $X_1'$. So the procedure repeats at most $n$ times. This gives the total complexity of $O(n(n+m))$. □

We now deal with the general setting, where both $A$ and $B$ may contain vertices with lists of size 3.

**Claim 42.** *Let $C$ be an induced 6-cycle of $G$ with a pre-coloring of type 2 or 3, that contains a vertex $c$ from a color class of size 1 or 2 which has parity $A$ and a list of size 3. Then we can test in $O(n(n+m))$ time whether the coloring extends to $G$.*

*Proof.* Start by updating all lists. By Claim 39, no vertex of parity $A$ has a list of size 3. Now by Proposition 41 (exchanging the roles of $A$ and $B$) we can test in $O(n(n+m))$ time if the graph with the new lists has a coloring that extends the coloring of $C$. □

We are now ready to prove Theorem 32.

Preprocessing the graph with respect to dominated vertices can be done in $O(n^2)$. Also, by Claims 34 and 35, in $O(n^2)$ time we can either find an induced $C_6$ with all vertices of some parity having lists of size 3, or check if $G$ is colorable.

So assume we found $C = c_1 - c_2 - \cdots - c_6 - c_1$ as above, and say the vertices of parity $A$ have lists of size 3. By Claims 37 and 42, we can check for every feasible pre-coloring of $C$ of type 1,2, or 3 whether it extends to $G$, in $O(n(n+m))$. Now we go through all type 4 colorings of $C$.

Consider any such coloring, and for simplicity assume $c_2, c_4$ and $c_6$ have color 1 and parity $A$. After updating the lists, all vertices of parity $A$ with lists of size 3 are in $X_1$, by Claims 36 and 38. Let $X_1'$ be the set of all such vertices. If there is a vertex that is adjacent to all of $X_1'$, then we can check for all feasible pre-colorings of $v$ whether they extend to $G$ in $O(n(n+m))$ time (since after updating again, all vertices of parity $A$ have lists of size at most 2, so we can apply Proposition 41 after exchanging the roles of $A$ and $B$). So assume there is no such vertex.

Then, since by parity, every vertex of $X_1'$ has a neighbor in $S_2$, there exist $x_1, x_1' \in X_1'$ and $s_2, s_2' \in S_2$ such that $x_1 s_2$ and $x_1' s_2'$ are edges, but $x_1 s_2'$ and $x_1' s_2$ are not edges. Since $c_2$ does not dominate $x_1$ or $x_1'$, there exist $y, y'$ such that $y$ is adjacent to $x_1$ and not to $c_2$, and $y'$ is adjacent to $x_1'$ and not to $c_2$. Since $G$ is bipartite, and since $y - x_1 - s_2 - c_2 - s_2' - x_1' - y'$ is not an induced $P_7$, it follows that $y = y'$.

Consider now the induced 6-cycle $C' = c_2 - s_2 - x_1 - y - x_1' - s_2' - c_2$. By Claims 37 and 42, we can check for every feasible pre-coloring of $C'$ of type 1,2, or 3 whether it extends to $G$, in $O(n(n+m))$ time. If none of them extends, we can deduce that in every coloring of $G$ (that extends the current pre-coloring of $C$), the cycle $C'$ will have a type 4 coloring and thus the color of both $x_1$ and $x_1'$ must be 1. We update the lists (and thus the set $X_1'$), and repeat the procedure from above. That is, we first try to find common neighbor of $X_1'$, and if we do not find such a vertex, we find an induced 6-cycle. As above, this either gives a coloring of $G$, or we fix the colors of two more vertices from $X_1'$. So the procedure repeats at most $n$ times. This gives the total complexity of $O(n^2(n+m))$. This completes the proof of both Theorem 32 and Theorem 2.

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