Three-coloring graphs with no induced seven-vertex path II : using a triangle

Maria Chudnovsky*, Peter Maceli†, and Mingxian Zhong‡

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Abstract

In this paper we give a polynomial time algorithm which determines if a given graph containing a triangle and no induced seven-vertex path is 3-colorable, and gives an explicit coloring if one exists. This is the second paper in a series of two. The first one, [3] is also submitted to this journal. In [2, 3], a polynomial time algorithm is given for three-coloring triangle-free graphs with no induced seven-vertex path. Combined, this shows that three-coloring a graph with no induced seven-vertex path can be done in polynomial time, thus answering a question of [13].

1 Introduction

We start with some definitions. All graphs in this paper are finite and simple. Let G be a graph and X be a subset of $V(G)$. We denote by $G[X]$ *the subgraph of* G *induced by* X, that is, the subgraph of G with vertex set X such that two vertices are adjacent in $G[X]$ if and only if they are adjacent in G. We denote by $G \setminus X$ the graph $G[V(G) \setminus X]$. If $X = \{v\}$ for some $v \in V(G)$, we write $G \setminus v$ instead of $G \setminus \{v\}$. Let H be a graph. If G has no induced subgraph isomorphic to H, then we say that G is H*-free.* For a family $\mathcal{F}$ of graphs, we say that G is $\mathcal{F}$*-free* if G is F-free for every $F \in \mathcal{F}$. If G is not H-free, then G *contains* H. If $G[X]$ is isomorphic to H, then we say that X *is an* H *in* G.

For $n \geq 0$, we denote by P_{n+1} the path with $n+1$ vertices, that is, the graph with distinct vertices $\{p_0, p_1, ..., p_n\}$ such that p_i is adjacent to p_j if and only if $|i-j| = 1$. We call the set $\{p_1, \dots, p_{n-1}\}$ the *interior* of P. For $n \geq 3$, we denote by C_n the cycle of length n, that is, the graph with distinct vertices $\{c_1, ..., c_n\}$ such that c_i is adjacent

*Princeton University, Princeton, NJ 08544, USA. E-mail: mchudnov@math.princeton.edu. Partially supported by NSF grants IIS-1117631, DMS-1001091 and DMS-1265803.

†Columbia University, New York, NY 10027, USA. E-mail: plm2109@columbia.edu.

‡Columbia University, New York, NY 10027, USA. E-mail: mz2325@columbia.edu.

to c_j if and only if $|i-j| = 1$ or $n-1$. When explicitly describing a path or a cycle, we always list the vertices in order. Let G be a graph. When $G[\{p_0, p_1, ..., p_n\}]$ is the path P_{n+1}, we say that $p_0 - p_1 - ... - p_n$ *is a* P_{n+1} *in* G or just a *path*, when there is no danger of confusion. Similarly, when $G[\{c_1, c_2, ..., c_n\}]$ is the cycle C_n, we say that $c_1 - c_2 - ... - c_n - c_1$ *is a* C_n *in* G. We also refer to a cycle of length three as a *triangle*. A *clique* in a graph is a set of pairwise adjacent vertices. A *stable set* is a set of vertices no two of which are adjacent.

A k-*coloring* of a graph G is a mapping $c : V(G) \to \{1, ..., k\}$ such that if $x, y \in V(G)$ are adjacent, then $c(x) \neq c(y)$. For $X \subseteq V(G)$, we define by $c(X) = \bigcup_{x \in X}\{c(x)\}$. If a k-coloring exists for a graph G, we say that G is k-*colorable*. The COLORING problem is determining the smallest integer k such that a given graph is k-colorable, and it was one of the initial problems R.M.Karp [9] showed to be NP-complete. For fixed $k \geq 1$, the k-COLORING problem is deciding whether a given graph is k-colorable. Since Stockmeyer [15] showed that for any $k \geq 3$ the k-COLORING problem is NP-complete, there has been much interest in deciding for which classes of graphs coloring problems can be solved in polynomial time. In this paper, the general approach that we consider is to fix a graph H and consider the k-COLORING problem restricted to the class of H-free graphs.

We call a graph *acyclic* if it is C_n-free for all $n \geq 3$. The *girth* of a graph is the length of its shortest cycle, or infinity if the graph is acyclic. Kamiński and Lozin [8] proved:

1.1. *For any fixed $k, g \geq 3$, the k-COLORING problem is NP-complete for the class of graphs with girth at least g.*

As a consequence of 1.1, it follows that if the graph H contains a cycle, then for any fixed $k \geq 3$, the k-COLORING problem is NP-complete for the class of H-free graphs. The *claw* is the graph with vertex set $\{a_0, a_1, a_2, a_3\}$ and edge set $\{a_0a_1, a_0a_2, a_0a_3\}$. A theorem of Holyer [6] together with an extension due to Leven and Galil [11] imply the following:

1.2. *If a graph H contains the claw, then for every fixed $k \geq 3$, the k-COLORING problem is NP-complete for the class of H-free graphs.*

Hence, the remaining problem of interest is deciding the k-COLORING problem for the class of H-free graphs where H is a fixed acyclic claw-free graph. It is easily observed that every connected component of an acyclic claw-free graph is a path. And so, we focus on the k-COLORING problem for the class of H-free graphs where H is a connected acyclic claw-free graph, that is, simply a path. Hoàng, Kamiński, Lozin, Sawada, and Shu [5] proved the following:

1.3. *For every k, the k-COLORING problem can be solved in polynomial time for the class of P_5-free graphs.*

Additionally, Randerath and Schiermeyer [12] showed that:

1.4. *The* 3*-COLORING problem can be solved in polynomial time for the class of* P_6*-free graphs.*

In [12] and [13] the question of the complexity of 3-coloring P_7-free graphs was posed.

On the other hand, Huang [7] recently showed that:

1.5. *The following problems are NP-complete:*

1. *The* 5*-COLORING problem is NP-complete for the class of* P_6*-free graphs.*
2. *The* 4*-COLORING problem is NP-complete for the class of* P_7*-free graphs.*

For our purposes, it is convenient to consider the following more general coloring problem. A *palette* L of a graph G is a mapping which assigns each vertex $v \in V(G)$ a finite subset of $\mathbb{N}$, denoted by $L(v)$. A *subpalette* of a palette L of G is a palette L' of G such that $L'(v) \subseteq L(v)$ for all $v \in V(G)$. We say a palette L of the graph G has *order* k if $L(v) \subseteq \{1, ..., k\}$ for all $v \in V(G)$. Notationally, we write (G, L) to represent a graph G and a palette L of G. We say that a k-coloring c of G is a *coloring of* (G, L) provided $c(v) \in L(v)$ for all $v \in V(G)$. We say (G, L) is *colorable*, if there exists a coloring of (G, L). We denote by $(G, \mathcal{L})$ a graph G and a collection $\mathcal{L}$ of palettes of G. We say $(G, \mathcal{L})$ is *colorable* if (G, L) is colorable for some $L \in \mathcal{L}$, and c is a *coloring* of $(G, \mathcal{L})$ if c is a coloring of (G, L) for some $L \in \mathcal{L}$.

Let G be a graph. We denote by $N_G(v)$ (or by $N(v)$ when there is no danger of confusion) the set of neighbors of v in G. Given (G, L), consider a subset $X, Y \subseteq V(G)$. We say that we *update the palettes of the vertices in* Y *with respect to* X (or simply *update* Y *with respect to* X), if for all $y \in Y$ we set

$$L(y) = L(y) \setminus \Big(\bigcup_{u \in N(y) \cap X \text{ with } |L(u)|=1} L(u) \Big).$$

When $Y = V(G)$ and X is the set of all vertices x of G with $|L(x)| = 1$, we simply say that we *update* L. Note that updating can be carried out in time $O(|V(G)|^2)$. By reducing to an instance of 2-SAT, which Aspvall, Plass and Tarjan [1] showed can be solved in linear time, Edwards [4] proved the following:

1.6. *There is an algorithm with the following specifications:*

Input: *A palette* L *of a graph* G *such that* $|L(v)| \leq 2$ *for all* $v \in V(G)$.

Output: *A coloring of* (G, L), *or a determination that none exists.*

Running time: $O(|V(G)|^2)$.

Let G be a graph. A subset S of $V(G)$ is called *monochromatic* with respect to a given coloring c of G if $c(u) = c(v)$ for all $u, v \in S$. Let L be palette of G, and X a set of subsets of $V(G)$. We say that (G, L, X) is *colorable* if there is a coloring c of (G, L) such that S is monochromatic with respect to c for all $S \in X$. A triple (G', L', X') is a *restriction* of (G, L, X) if G' is an induced subgraph of G, L' is a subpalette of $L|_{V(G')}$, and X' is a set of subsets of $V(G')$ such that if $S \in X$ then $S \cap V(G') \in X'$. Let $\mathcal{P}$ be a set of restrictions of (G, L, X). We say that $\mathcal{P}$ is *colorable* if at least one element of $\mathcal{P}$ is colorable. If $\mathcal{L}$ is a set of palettes of G, we write $(G, \mathcal{L}, X)$ to mean the set of restrictions (G, L', X) where $L' \in \mathcal{L}$. The proof of 1.6 is easily modified to obtain the following generalization [14]:

1.7. *There is an algorithm with the following specifications:*

Input: *A palette L of a graph G such that $|L(v)| \leq 2$ for all $v \in V(G)$, together with a set X of subsets of $V(G)$.*

Output: *A coloring of (G, L, X), or a determination that none exists.*

Running time: $O(|X||V(G)|^2)$.

A subset D of $V(G)$ is called a *dominating set* if every vertex in $V(G) \setminus D$ is adjacent to at least one vertex in D. Applying 1.6 yields the following general approach for 3-coloring a graph. Let G be a graph, and suppose $D \subseteq V(G)$ is a dominating set. Initialize the order 3 palette L of G by setting $L(v) = \{1, 2, 3\}$ for all $v \in V(G)$. Consider a fixed 3-coloring c of $G[D]$, and let L_c be the subpalette of L obtained by updating the palettes of the vertices in $V(G) \setminus D$ with respect to D. By construction, (G, L_c) is colorable if and only if the coloring c of $G[D]$ can be extended to a 3-coloring of G. Since $|L_c(v)| \leq 2$ for all $v \in V(G)$, 1.6 allows us to efficiently test if (G, L_c) is colorable. Let $\mathcal{L}$ to be the set of all such palettes L_c where c is a 3-coloring of $G[D]$. It follows that G is 3-colorable if and only if $(G, \mathcal{L})$ is colorable. Assuming we can efficiently produce a dominating set D of bounded size, since there are at most $3^{|D|}$ ways to 3-color $G[D]$, it follows that we can efficiently construct $\mathcal{L}$ and test if $(G, \mathcal{L})$ is colorable, and so we can decide if G is 3-colorable in polynomial time. This method figures prominently in the polynomial time algorithms for the 3-COLORING problem for the class of P_ℓ-free graphs where $\ell \leq 5$. However, this approach needs to be modified when considering the class of P_ℓ-free graphs when $\ell \geq 6$, since a dominating set of bounded size may not exist. Very roughly, the techniques used in this paper may be described as such a modification.

In [2, 3], the following was shown:

1.8. *There is an algorithm with the following specifications:*

Input: *A $\{P_7, C_3\}$-free graph G.*

Output: *A 3-coloring of G, or a determination that none exists.*

Running time: $O(|V(G)|^7)$.

In this paper, we consider the case when the input graph contains a triangle and prove the following:

1.9. *There is an algorithm with the following specifications:*

Input: *A P_7-free graph G which contains a triangle.*

Output: *A 3-coloring of G, or a determination that none exists.*

Running time: $O(|V(G)|^{24})$.

Together, 1.8 and 1.9 give:

1.10. *There is an algorithm with the following specifications:*

Input: *A P_7-free graph G.*

Output: *A 3-coloring of G, or a determination that none exists.*

Running time: $O(|V(G)|^{24})$.

Given a graph G and disjoint subsets A and B of $V(G)$, we say that A is *complete* to B if every vertex of A is adjacent to every vertex of B, and that A is *anticomplete* to B if every vertex of A is non-adjacent to every vertex of B. If $|A| = 1$, say $A = \{a\}$, we write "a is complete (or anticomplete) to B" instead of "$\{a\}$ is complete (or anticomplete) to B".

Here is a brief outline of our algorithm 1.9. We take advantage of the simple fact that all three-colorings of a triangle are the same (up to permuting colors), and, moreover, starting with the coloring of a triangle, the colors of certain other vertices are forced. In this spirit, we define a *tripod* in a graph G as a triple (A_1, A_2, A_3) of disjoint subsets of $V(G)$ such that

- $A_1 \cup A_2 \cup A_3 = \{a_1, ..., a_m\}$,
- $a_i \in A_i$ for $i = 1, 2, 3$,
- $a_1 - a_2 - a_3 - a_1$ is a triangle in G, and
- letting $\{i, j, k\} = \{1, 2, 3\}$, for every $s \in \{1, ..., m\}$, if $a_s \in A_i$, then a_s has a neighbor in $A_j \cap \{a_1, .., a_{s-1}\}$ and a neighbor in $A_k \cap \{a_1, .., a_{s-1}\}$.

Let G be a P_7-free graph which contains a triangle. The first step of the algorithm is to choose a maximal tripod (A_1, A_2, A_3) in G. It is easy to see that in every 3-coloring of G, each of the sets A_1, A_2, A_3 is monochromatic, thus if one of A_1, A_2, A_3 is not a stable set, the algorithm stops and outputs a determination that no 3-coloring exists. Let $A = A_1 \cup A_2 \cup A_3$. We analyze the structure of G relative to (A_1, A_2, A_3) and efficiently construct polynomially many subsets D of $V(G)$ such that for each of them $G[A \cup D]$ only has a bounded number of 3-colorings, and *almost* all vertices of $V(G) \setminus (A \cup D)$ have a neighbor in D. Ignoring the *almost* qualification, we are now done using 1.6 in polynomially many subproblems. In order to complete the proof, we guess a few more vertices that need to be added to D to create a dominating set in G, or show that certain subsets of $V(G)$ are monochromatic in all coloring of G, which allows us to delete some vertices of G without changing colorability. The last step is polynomially many applications of 1.7.

This paper is organized as follows. In section 2 we prove 2.4 and in section 3 we prove 3.1, both of which are pre-processing procedures. In section 4 we prove 4.1, which reduces the sizes of the lists of all the vertices in the graph except for a special stable set. In section 5 we prove a lemma, 5.1, that we will use to deal with the vertices of this special stable set. In Section 6 we verify that 5.1 can be applied in our situation. Finally, in Section 7 we put all the results together, and show that we have reduced the problem to polynomially many subproblems, each of which can be solved using 1.7.

2 Tripods

In this section, we introduce a way to partition a graph that contains a triangle so that we begin to gain understanding into monochromatic sets this triangle forces. Additionally, we show that further simplifications are possible in the case that the graph we are considering is P_7-free.

Let (A_1, A_2, A_3) be a tripod in a graph G. We say (A_1, A_2, A_3) is *maximal* if there does not exist a vertex in $V(G) \setminus (A_1 \cup A_2 \cup A_3)$ which has a neighbor in two of A_1, A_2, A_3.

2.1. *For any tripod (A_1, A_2, A_3) in a graph G, for $\ell = 1, 2, 3$ each A_ℓ is monochromatic with respect to any 3-coloring of G. Moreover, no color appears in two of A_1, A_2, A_3.*

Proof. Let $A_1 \cup A_2 \cup A_3 = \{a_1, ..., a_m\}$ and c be a 3-coloring of G. We proceed by induction. Since $a_1 - a_2 - a_3 - a_1$ is a triangle, it follows that $\{c(a_1), c(a_2), c(a_3)\} = \{1, 2, 3\}$. Suppose 2.1 holds for $\{a_1, ..., a_{s-1}\}$, where $s > 3$. Let $\{i, j, k\} = \{1, 2, 3\}$ so that $a_s \in A_i$. Since (A_1, A_2, A_3) is a tripod, it follows that a_s has a neighbor in $A_j \cap \{a_1, .., a_{s-1}\}$ and a neighbor in $A_k \cap \{a_1, .., a_{s-1}\}$. Inductively, it follows that every vertex in $A_j \cap \{a_1, .., a_{s-1}\}$ is assigned color $c(a_j)$ and that every vertex in $A_k \cap \{a_1, .., a_{s-1}\}$ is assigned color $c(a_k)$. Since c is a 3-coloring, it follows that $c(a_s) = c(a_i)$. This proves 2.1.

□

We say a tripod (A_1, A_2, A_3) is *stable* if A_i is stable for $i = 1, 2, 3$. By 2.1, it follows that if graph is 3-colorable, then every tripod is stable.

2.2. *If (A_1, A_2, A_3) is a stable tripod in G, then $G[A_j \cup A_k]$ is a connected bipartite graph for all distinct $j, k \in \{1, 2, 3\}$.*

Proof. Since A_j and A_k are stable, we only need to prove that $G[A_j \cup A_k]$ is connected. Suppose $A \cup B$ is a partition of $A_j \cup A_k$ such that both A and B are non-empty and A is anticomplete to B. Since $a_1 - a_2 - a_3 - a_1$ is a triangle, by symmetry, we may always assume $a_j, a_k \in A$. Choose $a_s \in B$ such that s is minimal. It follows that $s > 3$. By symmetry, we may assume $a_s \in A_j$. By definition, there exists $a_{s'} \in A_k \cap \{a_1, ..., a_{s-1}\}$ adjacent to a_s. However, by minimality, $a_{s'} \in A$, contrary to A being anticomplete to B. This proves 2.2.

□

We say a tripod (A_1, A_2, A_3) in a graph G is *reducible* if for $\{i, j, k\} = \{1, 2, 3\}$ we have that A_i is anticomplete to $V(G) \setminus (A_j \cup A_k)$. Suppose (A_1, A_2, A_3) is a maximal reducible stable tripod in a graph G. By symmetry, we may assume that A_1 is anticomplete to $V(G) \setminus (A_2 \cup A_3)$. Let G_R be the graph obtained by deleting A_1 and contracting $A_2 \cup A_3$ to an edge, that is, $V(G_R) = (V(G) \setminus (A_1 \cup A_2 \cup A_3)) \cup \{a'_2, a'_3\}$ and

- $a'_2 a'_3 \in E(G_R)$,
- $xy \in E(G_R)$ if and only if $xy \in E(G)$ for distinct $x, y \in V(G_R) \setminus \{a'_2, a'_3\}$,
- $a'_2 z \in E(G_R)$ if and only if $N_G(z) \cap A_2$ is non-empty where $z \in V(G_R) \setminus \{a'_2, a'_3\}$, and
- $a'_3 z \in E(G_R)$ if and only if $N_G(z) \cap A_3$ is non-empty where $z \in V(G_R) \setminus \{a'_2, a'_3\}$.

Note, G_R can be constructed in time $O(|V(G)|^2)$. The following establishes the usefulness of the above reduction.

2.3. *Let (A_1, A_2, A_3) be a maximal reducible stable tripod in a graph G and assume that A_1 is anticomplete to $V(G) \setminus (A_2 \cup A_3)$. Then the following hold:*

1. *If G is a P_7-free graph, then G_R is P_7-free.*
2. *If G is connected, then G_R is connected.*
3. *G is 3-colorable if and only if G_R is 3-colorable, and specifically from a coloring of G_R we can construct a coloring of G in time $O(|V(G)|)$.*

Proof. First, we prove 2.3.1. Suppose P is a copy of P_7 in G_R. Since G is P_7-free, it follows that $V(P) \cap \{a'_2, a'_3\}$ is non-empty. First, suppose $|V(P) \cap \{a'_2, a'_3\}| = 1$. By symmetry, we may assume $a'_2 \in V(P)$. Since G is P_7-free, it follows that a'_2 is an interior vertex of P, and so we can partition P as $P' - p' - a'_2 - p'' - P''$, where P', P'' are paths, possibly empty. By construction, both p', p'' have a neighbor in A_2, and $V(P)$ is anticomplete to A_1. Since by 2.2 $G[A_1 \cup A_2]$ is connected, there exists a path Q with ends p' and p'' and interior in $A_1 \cup A_2$. But now $P' - p' - Q - p'' - P''$ is a path in G of length at least 7, a contradiction.

Thus, it follows that both $a_2', a_3' \in V(P)$, and so we can partition P as $S' - a_2' - a_3' - T'$, where S', T' are paths, possibly empty. If $V(S) \neq \emptyset$, let s' be the neighbor of a_2' in S'; define t' similarly. Now s' has a neighbor in A_2, t' has a neighbor in A_3, and $V(P) \setminus \{a_2, a_3, s', t'\}$ is anticomplete to $A_2 \cup A_3$. Since by 2.2 $G[A_2 \cup A_3]$ is connected, it follows that there is a path Q from s' to t' and with interior in $A_2 \cup A_3$. But now $S' - s' - Q - t' - T'$ is a path in G of length at least 7, a contradiction. This proves 2.3.1.

Next we prove 2.3.2. Suppose G_R is not connected, and let $V(G_R) = X \cup Y$ such that X, Y are non-empty and anticomplete to each other. Since a_2' is adjaent to a_3', we may assume that $a_2', a_3' \in X$. Let $X' = (X \setminus \{a_2', a_3'\}) \cup (A_1 \cup A_2 \cup A_3)$. Then $V(G) = X' \cup Y$, and X', Y are anticomplete to each other, and so G is not connected. This proves 2.3.2.

Finally, we prove 2.3.3. Suppose c is a 3-coloring of G. And so, we define the coloring c' of G as follows: For every $v \in V(G_R)$ set

$$c'(v) = \begin{cases} c(a_2) & , \quad \text{if } v = a_2' \\ c(a_3) & , \quad \text{if } v = a_3' \\ c(v) & , \quad \text{otherwise} \end{cases} .$$

By construction, it clearly follows that c' is a 3-coloring of G_R.

Next, suppose $\hat{c}$ is a 3-coloring of G_R. Since a_2' is adjacent to a_3', it follows that $\hat{c}(a_2') \neq \hat{c}(a_3')$. Take $\tilde{c}_1$ so that $\{\tilde{c}_1, \hat{c}(a_2'), \hat{c}(a_3')\} = \{1, 2, 3\}$. Define the coloring $\tilde{c}$ of G as follows: For every $v \in V(G)$ set

$$\tilde{c}(v) = \begin{cases} \tilde{c}_1 & , \quad \text{if } v \in A_1 \\ \hat{c}(a_2') & , \quad \text{if } v \in A_2 \\ \hat{c}(a_3') & , \quad \text{if } v \in A_3 \\ \hat{c}(v) & , \quad \text{otherwise} \end{cases} .$$

By construction, it clearly follows that $\tilde{c}$ is a 3-coloring of G and the construction of $\tilde{c}$ takes $O(|V(G)|)$. This proves 2.3.3.

□

We say a tripod (A_1, A_2, A_3) is *normal* if it is stable, maximal and not reducible.

2.4. *There is an algorithm with the following specifications:*

Input: *A connected graph G.*

Output:

1. *a determination that G is not 3-colorable, or*
2. *a connected triangle-free graph G' with $|V(G')| \leq |V(G)|$ such that G' is 3-colorable if and only if G is 3-colorable, or*

3. *a connected graph G' with $|V(G')| \leq |V(G)|$ such that G' is 3-colorable if and only if G is 3-colorable, together with a normal tripod (A_1, A_2, A_3) in G'.*

Running time: $O(|V(G)|^3)$.

Additionally, any 3-coloring of G' can be extended to a 3-coloring of G in time $O(|V(G)^2|)$.

Proof. In time $O(|V(G)|^3)$, we can determine if G is triangle-free. If so return the triangle-free graph $G' = G$ and halt. Otherwise, we may assume there exist $a_1, a_2, a_3 \in V(G)$ such that $a_1 - a_2 - a_3 - a_1$ is a triangle. Next, we try and grow this triangle into a normal tripod. Initialize $A_i = \{a_i\}$ for $i = 1, 2, 3$. Assume $A_1 \cup A_2 \cup A_3 = \{a_1, ..., a_m\}$ and consider $v \in V(G) \setminus (A_1 \cup A_2 \cup A_3)$ such that v has a neighbor in A_i and A_j for $\{i, j, k\} = \{1, 2, 3\}$. If v is anticomplete to A_k, then set $a_{m+1} = v$ and $A_k = A_k \cup \{a_{m+1}\}$. If v has a neighbor in A_k, then, by 2.1, we may return that G is not 3-colorable and halt. Repeat this procedure again until either we determine that G is not 3-colorable or there does not exists any $v \in V(G) \setminus (A_1 \cup A_2 \cup A_3)$ such that v has a neighbor in A_i and A_j for $\{i, j, k\} = \{1, 2, 3\}$. By construction, this procedure either halts or yields a maximal, stable tripod (A_1, A_2, A_3) from the triangle $a_1 - a_2 - a_3 - a_1$. In time $O(|V(G)|^2)$, we can verify if A_i is anticomplete to $V(G) \setminus (A_j \cup A_k)$ for some $i \in \{1, 2, 3\}$, that is, if (A_1, A_2, A_3) is reducible. If not, then return the normal tripod (A_1, A_2, A_3) for $G' = G$ and halt. Otherwise, by symmetry, we may assume A_1 is anticomplete to $V(G) \setminus (A_2 \cup A_3)$. By 2.3.2, it follows that G_R is connected, G is 3-colorable if and only if G_R is 3-colorable, and a 3-coloring of G_R can be extended to a 3-coloring of G in time $|V(G)|$. Now, repeat the steps described above with G_R. This procedure can be carried out in time $O(|V(G)|^3)$. This proves 2.4.

□

3 Cleaning

In this section, we identify a configuration that, if present in G, allows us to efficiently find a graph G' with $|V(G')| < |V(G)|$ which is 3-colorable if and only G is 3-colorable.

Let G be a graph, and let (A_1, A_2, A_3) be a tripod in G. We say $v \in V(G)$ is a *connected vertex* if $G[N_G(v)]$ is connected. We say that a graph is (A_1, A_2, A_3)-*clean* if every connected vertex in $V(G) \setminus (A_1 \cup A_2 \cup A_3)$ has a neighbor in $A_1 \cup A_2 \cup A_3$, and (A_1, A_2, A_3) is a normal tripod in G.

3.1. *Let (A_1, A_2, A_3) be a normal tripod in G. There is an algorithm with the following specifications:*

Input: *A connected P_7-free graph G.*

Output: *A connected (A_1, A_2, A_3)-clean P_7-free graph G' with $|V(G')| \leq |V(G)|$ such that G' is 3-colorable if and only if G is 3-colorable, or a determination that G is not 3-colorable.*

Running time: $O(|V(G)|^4)$.

Additionally, any 3*-coloring of* G' *can be extended to a* 3*-coloring of* G *in time* $O(|V(G)|^2)$.

Proof. First, for every $v \in V(G)$, check if $G[N(v)]$ is 2-colorable. This can be done in time $O(|V(G)|^3)$, and if the answer is "no" for some v, we can stop and output that G is not 3-colorable.

Thus we may assume that $G[N(v)]$ is 2-colorable for every $v \in V(G)$. Let Y be the set of vertices of $V(G) \setminus (A_1 \cup A_2 \cup A_3)$ that are anticomplete to $A_1 \cup A_2 \cup A_3$. In time $O(|V(G)|^3)$, we can find a connected vertex in Y or determine that none exists. If no vertex in Y is connected, output $G' = G$.

Suppose $v \in Y$ is connected. Define G_v as follows. If $|N(v)| = 1$, let $G_v = G \setminus v$. Otherwise, let (A, B) be the unique bipartition of $G[N(v)]$. It follows that $\{v\} \cup A \cup B$ is a maximal reducible stable tripod in G. Let G_v be the graph obtained from G by deleting v and contracting $N_G(v)$ to an edge, that is, G_R with respect to $\{v\} \cup A \cup B$. Now, by 2.3, it follows that G_v is connected, and that G_v is colorable if and only if G is colorable. Moreover, since $v \in Y$, it follows that (A_1, A_2, A_3) is a normal tripod in G_v.

Now recursively applying the procedure to G_v, 3.1 follows. □

Given a graph G, we say that $X \subseteq V(G)$ is a *a homogeneous set in* G if $X \neq V(G)$, and every vertex of $V(G) \setminus X$ is either complete or anticomplete to X. We end the section with the following lemma.

3.2. *Let* X *be a homogeneous set in a connected graph* G *such that* $G[X]$ *is connected,* $X \neq V(G)$ *and* $|X| > 1$*. Then* X *contains a connected vertex.*

Proof. Consider $v \in X$ and define $X' = N(v) \cap X$ and $Y = N(v) \cap (V(G) \setminus X)$. Since G is connected, it follows that $V(G) \setminus X$ is not anticomplete to X, and so Y is non-empty. Since $G[X]$ is connected and $|X| > 1$, it follows that X' is non-empty. Since X is a homogeneous set, it follows that Y is complete to X', implying that $G[N(v)]$ is connected. This proves 3.2.

□

4 Reducing the Graph

The main result of this section is 4.1. It allows us (at the expense of branching into polynomially many subproblems) to reduce the lists of some of the vertices of the graph to size two, and get some control over the remaining vertices. More precisely, 4.1 reduces the problem to the case when the set of vertices whose list has size three is stable, and the neighbors of every such vertex satisfy certain technical conditions. These conditions are designed with the goal of using 5.1. In 6.1 we verify that the conclusion of 4.1 is in fact sufficient for applying 5.1.

For a fixed subset X of $V(G)$, we say that a vertex $v \in V(G) \setminus X$ *is mixed on an edge of* X, if there exist adjacent $x, y \in X$ such that v is adjacent to x and non-adjacent to y. Similarly, we say a vertex $v \in V(G) \setminus X$ *is mixed on a non-edge of* X, if there exist non-adjacent $x, y \in X$ such that v is adjacent to x and non-adjacent to y.

4.1. *Let $A = (A_1, A_2, A_3)$ be a normal tripod in a connected, (A_1, A_2, A_3)-clean P_7-free graph and partition $V(G) = A \cup X \cup Y \cup Z$, such that*

- $A = A_1 \cup A_2 \cup A_3$,
- *X is the set of vertices of $V(G) \setminus A$ with a neighbor in A,*
- *Y is the set of vertices of $V(G) \setminus (A \cup X)$ with a neighbor in X,*
- $Z = V(G) \setminus (A \cup X \cup Y)$.

For $i = 1, 2, 3$, let X_i be the set of vertices of $V(G) \setminus A$ with a neighbor in A_i.

There exists a set of $O(|V(G)|^{12})$ palettes $\mathcal{L}$ of G such that

(a) Each $L \in \mathcal{L}$ has order 3 and $|L(v)| \leq 2$ for every $v \in A \cup X$, and

(b) G has a 3-coloring if and only if $(G, \mathcal{L})$ is colorable.

Moreover, $\mathcal{L}$ can be computed in time $O(|V(G)|^{15})$.

For each $L \in \mathcal{L}$, let P_L be the set of vertices $y \in Y \cup Z$ with $|L(y)| = 3$. Then the following hold:

(c) P_L is stable.

(d) There exist subsets $X' \subset X$, $Y_0 \subset Y$, and vertices $s_\ell \in X_\ell \cap X'$ for $\ell = 1, 2, 3$, such that

- *$|L(x)| = 1$ for all $x \in X' \cup Y_0$, and*
- *Y_0 is complete to $\{s_1, s_2, s_3\}$, and*
- *letting Y' be the set of vertices in $Y \cup Z$ with a neighbor in $X' \cup Y_0$, we have that P_L is anticomplete to $(Y \cup Z) \setminus Y'$.*

Additionally, for every $(i, j, k) = (1, 2, 3)$ and $L \in \mathcal{L}$ the following hold:

(e) If $v \in Y'$ with $L(v) = \{i, j\}$, then there exists $u \in N(v) \cap (X' \cup Y_0)$ with $L(u) = \{k\}$,

(f) If $v \in X' \cap X_j$ with $L(v) = \{i\}$, then either there exists $u \in N(v)$ such that $L(u) = \{k\}$, or every $y \in Y$ with a neighbor in X_j has $L(y) = \{j\}$.

(g) If $v \in Y_0$ with $L(v) = \{i\}$, then there exist $u, w \in N(v) \cap \{s_1, s_2, s_3\}$ such that $L(u) = \{k\}$ and $L(w) = \{j\}$.

Proof. Since a normal tripod is maximal and not reducible, it follows that

- X_ℓ is non-empty for $\ell = 1, 2, 3$.
- $X_i \cap X_j = \emptyset$ for all distinct $i, j \in \{1, 2, 3\}$.
- $X_1 \cup X_2 \cup X_3 = X$.

Let $\ell \in \{1, 2, 3\}$. Let $\mathcal{S}_\ell$ be the set of all quadruples $S = (P, Q_1, Q_2, Q_3)$ such that

- $P = \{p\}$ and $p \in X_\ell$.
- For $i, j \in \{1, 2, 3\}$, if $Q_i \neq \emptyset$ and $j < i$, then $Q_j \neq \emptyset$.
- either $Q_1 = \emptyset$, or $Q_1 = \{q_1\}$, $q_1 \in Y$ and p is adjacent to q_1.
- either $Q_2 = \emptyset$, or $Q_2 = \{q_2\}$, $q_2 \in Y \cup Z$ and q_2 is adjacent to q_1 and not to p.
- either $Q_3 = \emptyset$, or $Q_3 = \{q_3\}$, $q_3 \in Y$, and q_3 is adjacent to p and anticomplete to $\{q_1, q_2\}$.

Let $E(S) = P \cup Q_1 \cup Q_2 \cup Q_3$. We write $P(S) = P$, and $Q_i(S) = Q_i$ for $i = 1, 2, 3$. Let $\mathcal{S} = \{(S_1, S_2, S_3)$ such that $S_\ell \in \mathcal{S}_\ell\}$. Then $|\mathcal{S}| = O(|V(G)|^{12})$.

Let us say that $y \in Y$ is an *i-cap* if there exist $x \in X_i$ and $y' \in (Y \cup Z) \setminus \{y\}$ such that $x - y - y'$ is a path. Initialize the palette L:

$$L(v) = \begin{cases} \{1\} & , \quad \text{if } v \in A_1 \\ \{2\} & , \quad \text{if } v \in A_2 \\ \{3\} & , \quad \text{if } v \in A_3 \\ \{2,3\} & , \quad \text{if } v \in X_1 \\ \{1,3\} & , \quad \text{if } v \in X_2 \\ \{1,2\} & , \quad \text{if } v \in X_3 \\ \{1,2,3\} & , \quad \text{otherwise} \end{cases}$$

Clearly, by renaming the colors, G has a 3-coloring if and only if (G, L) is colorable. The sets (S_1, S_2, S_3) are designed to "guess" information about certain types of colorings of G (type I–IV colorings defined later). Next we "trim" the collection $\mathcal{S}$, with the goal to only keep the sets that record legal colorings of each type.

For every $S = (S_1, S_2, S_3)$, proceed as follows. If $Q_3(S_i) = \emptyset$ and $Q_2(S_i) \neq \emptyset$, let $M(S_i)$ be the set of vertices of Y that are complete to $P(S_i)$ and anticomplete to $Q_1(S_i) \cup Q_2(S_i)$, otherwise let $M(S_i) = \emptyset$. If $Q_2(S_i) = Q_3(S_i) = \emptyset$, let $H(S_i)$ be the set of all i-caps, and otherwise let $H(S_i) = \emptyset$. If for some $i \neq j \in \{1, 2, 3\}$ $Q_1(S_i) = Q_1(S_j) = \emptyset$ and there is $y \in Y$ with both a neighbor in X_i and X_j, discard S.

Next suppose that for some $i \in \{1, 2, 3\}$, $Q_3(S_i) = \emptyset$, and $Q_1(S_i), Q_2(S_i) \neq \emptyset$. If there exist $x \in X_i$ and $y_1, y_2 \in Y$ such that

- x is adjacent to y_1 and not to y_2
- y_1 is adjacent to y_2,
- $M(S_i) \cup Q_2(S_i)$ is anticomplete to $\{y_1, y_2\}$, and
- x is complete to $M(S_i) \cup Q_2(S_i)$

then discard S.

Otherwise, let $E(S) = \bigcup_{i \in \{1,2,3\}}(E(S_i) \cup M(S_i) \cup H(S_i))$, and let c be a coloring of $G[E(S)]$ such that $c(v) = i$ for every $v \in M(S_i) \cup H(S_i)$. If for some $i \in \{1, 2, 3\}$, $Q_1(S_i) \neq \emptyset$ and the vertex of $Q_1(S_i)$ is colored i, discard c. If for some i, $Q_3(S_i) \neq \emptyset$ and the vertex of $Q_3(S_i)$ is colored i, discard c.

Otherwise, define the subpalette L_c^S of L as follows:

$$L_c^S(v) = \begin{cases} c(v) & , \quad \text{if } v \in E(S) \\ 1 & , \quad \text{if } Q_1(S_1) = \emptyset, \text{ and } v \in Y \text{ and } v \text{ has a neighbor in } X_1 \\ 2 & , \quad \text{if } Q_1(S_2) = \emptyset, \text{ and } v \in Y \text{ and } v \text{ has a neighbor in } X_2 \\ 3 & , \quad \text{if } Q_1(S_3) = \emptyset, \text{ and } v \in Y \text{ and } v \text{ has a neighbor in } X_3 \\ L(v) & , \quad \text{otherwise} \end{cases}$$

Fix $S = (S_1, S_2, S_3)$. Let $\ell \in \{1, 2, 3\}$ and let $X_\ell'^S$ be the set of vertices $x \in X_\ell$ with a neighbor w in $E(S_\ell)$ such that $c(w) \neq \ell$, and let

$$X'_S = X_1'^S \cup X_2'^S \cup X_3'^S \cup (E(S) \cap X).$$

Let Y_0^S be the set of vertices of Y that are complete to $P(S_1) \cup P(S_2) \cup P(S_3)$.

Let Y'_S be the set of vertices of $(Y \cup Z) \setminus (Y_0^s \cup E(S))$ with a neighbor in $Y_0^S \cup X'_S$. We now carry out three rounds of updating: first, for every $\ell \in \{1, 2, 3\}$, update X'_ℓ with respect to $E(S_\ell)$, then update Y'_S with respect to $Y_0^S \cup X'_S$ and finally update $Y \setminus Y'_S$ with respect to $Y_0^S \cup X'_S \cup E(S)$. This takes time $O(|V(G)|^2)$. Let $\mathcal{L}$ be the set of all the palettes L_c^S thus generated. Then $|\mathcal{L}| = O(|V(G)|^{12})$, and $\mathcal{L}$ can be constructed in time $O(|V(G)|^{15})$. Clearly, *(a)* holds.

We now define four different types of colorings of G that are needed to prove *(b).* Let c be a coloring of G and let $\ell \in \{1, 2, 3\}$. We say that *c is a type I coloring with respect to ℓ* if there exist vertices (p, q_1, q_2), where the following hold:

- $p \in X_\ell$
- $q_1, q_2 \in Y \cup Z$ such that p is adjacent to q_1 and not to q_2, and q_1 is adjacent to q_2
- $c(q_1) \neq \ell$, and $c(q_2) \neq \ell$.

We say that c *is a type II coloring with respect to* ℓ if c is not a type I coloring with respect to ℓ and there exist vertices (p, q_1, q_2, q_3), where the following hold:

- $p \in X_\ell$
- $q_1, q_2 \in Y \cup Z$ such that p is adjacent to q_1 and not to q_2, and q_1 is adjacent to q_2
- $q_3 \in Y$, and q_3 is adjacent to p and anticomplete to $\{q_1, q_2\}$.
- $c(q_1) \neq \ell$, $c(q_2) = \ell$ and $c(q_3) \neq \ell$.

We say that c *is a type III coloring with respect to* ℓ if c is not a type I or type II coloring with respect to ℓ and there exist vertices (p, q_1, q_2), where the following hold:

- $p \in X_\ell$
- $q_1, q_2 \in Y \cup Z$ such that p is adjacent to q_1 and not to q_2, and q_1 is adjacent to q_2.
- $c(q_1) \neq \ell$, and $c(q_2) = \ell$.

We say that c *is a type IV coloring with respect to* ℓ if c is not a type I, type II, or type III coloring with respect to ℓ and there exist vertices (p, q_1), where the following hold:

- $p \in X_\ell$
- $q_1 \in Y$ such that p is adjacent to q_1.
- q_1 is not an ℓ-cap.
- $c(q_1) \neq \ell$.
- if y is an ℓ-cap, then $c(y) = \ell$.

We claim that if c is a coloring of G that is not of type I,II,III or IV for some $i \in \{1, 2, 3\}$, then $c(y) = i$ for every $y \in Y$ with a neighbor in X_i. For suppose $c(y) \neq i$ for some $y \in Y$ with a neighbor $x \in X_i$. If y can be chosen to be an i-cap, then c is a type I,II or III coloring, and otherwise c is a type IV coloring. This proves the claim.

Next we prove *(b)*. Clearly if c is a coloring of (G, L) for some $L \in \mathcal{L}$, then c is a coloring of G. We show that if G is colorable, then (G, L) is colorable for some $L \in \mathcal{L}$.

Let c be a coloring of G. Suppose first that c is a type I,II, III or IV coloring with respect to 1. Then there exist p and possibly q_1, q_2, q_3 as in the definition of a type I,II, III or IV coloring. If c is a type III coloring, let M_1 be the set of all vertices in Y that are adjacent to p and anticomplete to $\{q_1, q_2\}$. If c is a type IV coloring, let H_1 be the set of all 1-caps. Moreover, if c is a type III coloring, we may assume that p, q_1, q_2 are chosen in such a way that M_1 is maximal, and so there do not exist $x \in X_i$ and $y_1, y_2 \in Y \cup Z$ such that

- x is adjacent to y_1 and not to y_2
- y_1 is adjacent to y_2
- $M_1 \cup \{q_2\}$ is anticomplete to $\{y_1, y_2\}$
- x is complete to $M_1 \cup \{q_2\}$.

Also, if c is a type IV coloring of G, then $c(y) = 1$ for every $y \in H_1$. Let $S_1 = (P, Q_1, Q_2, Q_3)$ such that $P = \{p\}$, and for $i \in \{1,2,3\}$ either $Q_i = \{q_i\}$ or $Q_i = \emptyset$ if q_i is not defined.

If c is not a type I, II, III or IV coloring with respect to 1, choose $p \in X_1$ and set $S_1 = (\{p\}, \emptyset, \emptyset, \emptyset)$.

Define S_2, M_2, H_2 and S_3, M_3, H_3 similarly, and let $S = (S_1, S_2, S_3)$. Recall that $E(S) = \bigcup_{i \in \{1,2,3\}} (E(S_i) \cup M_i \cup H_i)$. Now let d be the restriction of c to $G[E(S)]$. It is easy to see that $c(v) \in L_d^S(v)$ for every $v \in V(G)$, and so c is a coloring of (G, L_d^S). Thus *(b)* holds.

Fix $S \in \mathcal{S}$, c a coloring of $E(S)$ as described at the start of the proof, and $L_c^S \in \mathcal{L}$. For $i \in \{1,2,3\}$, let $P(S_i) = \{p_i\}$. Let $X''_S = X \setminus X'_S$. Let $T_S = (Y \cup Z) \setminus (Y'_S \cup Y_0^S \cup E(S))$. Now $|L_c^S(v)| = 1$ for every $v \in X'_S \cup E(S)$. Since at least two colors appear in $P(S_1) \cup P(S_2) \cup P(S_3)$, it follows that $|L_c^S(v)| = 1$ for every $v \in Y_0^s$. Setting $s_i = p_i$, we observe that *(g)* holds. Consequently, since every vertex of Y'_S has a neighbor in $X'_S \cup Y_0^S$, it follows that $|L_c^S(v)| \leq 2$ for every Y'_S and *(e)* holds. Next we show that *(f)* holds. Let $i \neq j \in \{1,2,3\}$, let $v \in X'_S \cap X_j$ and suppose that $L_c^S(v) = \{i\}$. If $v \in X'_j$, then $L_c^S(v)$ was changed in the first round of updating, and the assertion of *(f)* holds. Thus we may assume that $v \in P(S_j)$, and $Q_1(S_i) = \emptyset$. But then every $y \in Y$ with a neighbor in X_j has $L_c^S(y) = \{j\}$, and again *(f)* holds.

Next we prove a few structural statements about G, that will allow us to prove *(c)* and *(d)*.

(1) If $x \in X_i$ and $y_1, y_2, y_3 \in Y \cup Z$ are such that $x - y_1 - y_2 - y_3$ is a path, then every vertex of $X_j \cup X_k$ has a neighbor in $\{y_1, y_2, y_3\}$.

Proof: Suppose not. By symmetry, we may assume there exists a vertex $v \in X_j$ anticomplete to $\{y_1, y_2, y_3\}$. Suppose first that v is non-adjacent to x. Since by 2.2 $G[A_i \cup A_j]$ is connected, and since both x and v have neighbors in $A_i \cup A_j$, it follows that there exists a path P from x to v with interior in $A_i \cup A_j$. It follows that $V(P)$ is anticomplete to $\{y_1, y_2, y_3\}$ and so $v - P - x - y_1 - y_2 - y_3$ contains a P_7, a contradiction. Thus v is adjacent to x. Let $a \in N(v) \cap A_j$ and $b \in N(a) \cap A_k$, then $b - a - v - x - y_1 - y_2 - y_3$ is a P_7 in G, a contradiction. This proves *(1)*.

(2) If $x \in X_i$, $z \in Y$, and $y_1, y_2 \in Y''_S$ are such that $x - z - y_1 - y_2$ is a path, then $z \in Y_0^S$.

Proof: We may assume that $i = 1$. By *(1)*, each of p_2, p_3 has a neighbor in $\{y_1, y_2, z\}$. Since $y_1, y_2 \in Y''_S$, it follows that $\{y_1, y_2\}$ is anticomplete to $\{p_2, p_3\}$. This implies that z

is complete to $\{p_2, p_3\}$, and so $v - z - y_1 - y_2$ is a path for every $v \in \{p_2, p_3\}$. Now, by the same argument it follows that z is adjacent to p_1. Hence, $z \in Y_0^s$. This proves *(2)*.

Let P_L be the set of vertices $t \in T_S$ with $|L_c^S(t)| = 3$. From the definition of L_c^S, it follows that if $v \in T_S \setminus P_L$, then for some $i \in \{1, 2, 3\}$, v has a neighbor in $E(S) = (E(S_i) \setminus X) \cup H(S_i) \cup M(S_i)$.

(3) No vertex of X_S'' is mixed on an edge of P_L.

Proof: Suppose $x - y_1 - y_2$ is a path, where $x \in X_1 \cap X_S''$, and $y_1, y_2 \in P_L$. Then y_1 is an i-cap and $L_c^S(y_1) \neq \{1\}$. It follows that $Q_1(S_1) \neq \emptyset$, $Q_2(S_1) \neq \emptyset$, and $c(Q_1(S_1)) \neq 1$. Write $p = p_1$. Let $Q_1(S_1) = \{q_1\}$. Since $x \in X_S'' \cap X_\ell$, it follows that x is anticomplete to $P(S_1) \cup Q_1(S_1)$. Since $y_1, y_2 \in P_L$, it follows that $\{y_1, y_2\}$ is anticomplete to $E(S_1)$.

Let $Q_2(S_1) = \{q_2\}$. Suppose first that x is non-adjacent to q_2. Let P be a path from x to p with interior in $A_1 \cup A_2$ (such a path exists by 2.2). Now $y_2 - y_1 - x - P - p - q_1 - q_2$ is a path with at least seven vertices, a contradiction. This proves that x is adjacent to q_2, and since $x \in X_S''$, we deduce that $c(q_2) = 1$.

Next suppose $Q_3(S_1) \neq \emptyset$; let $Q_3(S_1) = \{q_3\}$. Then $c(q_3) \neq 1$, and so x is non-adjacent to q_3. Now $y_2 - y_1 - x - q_2 - q_1 - p - q_3$ is a P_7, a contradiction. This proves that $Q_3(S_1) = \emptyset$. Recall that when $Q_2(S_1) \neq \emptyset$ and $Q_3(S_1) \neq \emptyset$, $M(S_1)$ is defined to be the set of all vertices of Y that are adjacent to p and anticomplete to $\{q_1, q_2\}$. Then $L_c^S(v) = 1$ for every $m \in M(S_1)$, and $\{y_1, y_2\}$ is anticomplete to $M(S_1)$. Consequently, since $y_2 - y_1 - x - q_2 - q_1 - p - m$ is not a P_7 for any $m \in M(S_1)$, we deduce that x is complete to $M(S_1)$, and thus the quadruple S_1 was discarded during the construction of $\mathcal{L}$, a contradiction. This proves *(3)*.

(4) No vertex of Y_S' is mixed on an edge of P_L.

Proof: Suppose $y - y_1 - y_2$ is a path, where $y \in Y_S'$, and $y_1, y_2 \in P_L$. Then $y \notin E(S) \cup Y_0^S$. If y has a neighbor in $x \in X_S'$, then $x - y - y_1 - y_2$ is path, and so by *(2)* $y \in Y_0^S$, a contradiction. This proves that y is anticomplete to X_S', and so y has a neighbor in $y_0 \in Y_0^S$. By the definition of Y_0^S, y_0 is adjacent to p_1. Let $a_1 \in A_1$ be adjacent to p_1, and let $a_2 \in A_2$ be adjacent to a_1. Now $a_2 - a_1 - p_1 - y_0 - y - y_1 - y_2$ is P_7 in G, a contradiction. This proves *(4)*.

(5) If $T_S \setminus P_L$ is anticomplete to P_L.

Proof: Suppose $t \in T_S \setminus P_L$ has a neighbor $p \in P_L$. Then t has a neighbor $w \in E(S) \setminus X$, and since $|L_c^S(p)| = 3$, it follows that w is non-adjacent to p. Suppose first that $w \in Q_1(S_i) \cup Q_3(S_i) \cup M(S_i)$ for some $i \in \{1, 2, 3\}$. Then w has a neighbor $x \in X'$, and so $x - w - t - p$ is a path. Now *(2)* implies that $w \in Y_0$, and therefore $t \in Y'$, contrary to the fact that $t \in T_S$.

Next suppose that $w \in Q_2(S_i)$ for some $i \in \{1, 2, 3\}$. We may assume $i = 1$. Let $Q_1(S_1) = q_1$. Let $a \in A_1$ be a neighbor of p_1, and let $a' \in A_2$ be adjacent to a. Then $a' - a - p_1 - q_1 - w - t - p$ is a P_7 in G, a contradiction.

Consequently, $w \in H(S_i)$ for some $i \in \{1, 2, 3\}$. In particular, $H(S_i) \neq \emptyset$, and so $L^S_c(h) = \{i\}$ for every i-cap. Let $x \in X_i$ be adjacent to w. If x is anticomplete to $\{t, p\}$, then again by *(2)* $w \in Y^S_0$, a contradiction. So, since $t, p \notin H(S_i)$ it follows that x is complete to $\{t, p\}$, and in particular $p \in Y$. Therefore $N(p) \cap X \neq \emptyset$. Moreover, the fact that $p \notin H(S_i)$ implies that $N(p) \cap X$ is complete to $N(p) \setminus X$. Since $t \in N(p) \setminus X$, it follows that p is a connected vertex, contrary to the fact that G is (A_1, A_2, A_3)-clean. This proves *(5)*.

Now by *(3)*, *(4)* and *(5)*, for every connected component C of P_L, $V(C)$ is a homogeneous set. Since no vertex of P_L is connected, by 3.2 $|V(C)| = 1$, P_L is stable and *(c)* holds. Finally, setting $s_i = p_i$ for $i \in \{1, 2, 3\}$, $X' = X'_S$, and $Y_0 = Y'_S$, *(5)* implies that *(d)* holds. This completes the proof of 4.1. □

5 A Lemma

This section contains a lemma that captures the properties of the set P_L from 4.1 that makes it possible to reduce the size of the lists of the vertices in this set.

5.1. *Let L be an order 3 palette of a connected P_7-free graph G. Let Z be a set of subsets of $V(G)$. Suppose there exists disjoint non-empty subsets S_1, S_2, S_3 of $V(G)$ satisfying the following:*

- *$L(v) = \{1, 2, 3\} \setminus \{\ell\}$ for every $v \in S_\ell$ where $\ell \in \{1, 2, 3\}$.*
- *Let $i, j \in \{1, 2, 3\}$, and let $u_i, v_i \in S_i$ and $u_j, v_j \in S_j$, such that $\{u_i, v_i, u_j, v_j\}$ is a stable set. Then there exists a path P with ends $a, b \in \{u_i, v_j, u_j, v_j\}$ such that*
  1. *$\{a, b\} \neq \{u_i, u_j\}$ and $\{a, b\} \neq \{v_i, v_j\}$,*
  2. *$|L(w)| = 1$ for every interior vertex w of P, and*
  3. *$V(P) \setminus \{a, b\}$ is disjoint from and anticomplete to $\{u_i, v_j, u_j, v_j\} \setminus \{a, b\}$.*
- *For every distinct pair $i, j \in \{1, 2, 3\}$ and $u \in S_i$ there exist vertices v and w, such that $u - v - w$ is a path where both v and w are anticomplete to S_j with $|L(v)| = |L(w)| = 1$.*

Given a vertex $x \in V(G)$, define $N_\ell(x) = N(x) \cap S_\ell$ for $\ell = 1, 2, 3$. Let $X \subset V(G)$ be such that $N(x) \subseteq S_1 \cup S_2 \cup S_3$ for every $x \in X$, and no vertex of X is connected.
Then there exists a set $\mathcal{P}$ of $O(|V(G)|^9)$ restrictions of (G, L, Z) such that the following hold:

(a) For every $(G', L', Z') \in \mathcal{P}$, $|L'(v)| \leq 2$ for every $v \in X \cap V(G')$, and $|Z'| = O(|V(G)| + |Z|)$, and

(b) (G, L, Z) is colorable if and only if $\mathcal{P}$ is colorable.

Moreover, $\mathcal{L}$ can be constructed in time $O(|V(G)|^{10})$, and a 3-coloring of a restriction in $\mathcal{P}$ can be extended to a 3-coloring of G in $O(|V(G)|^2)$.

Proof. Let X' be the set of vertices $x \in X$ with $|L(x)| = 3$. If $X' = \emptyset$, let $\mathcal{P} = \{(G, L, Z)\}$.

By updating, we may assume that for every $x \in X'$ and y adjacent to x, $|L(y)| \geq 2$. If $N(x) \subseteq S_i$ for some $x \in X'$ and $i \in \{1, 2, 3\}$, then setting $L(x) = \{i\}$ does not change the colorability of (G, L, Z), so we may assume that for every $x \in X'$ at least two of the sets $N_1(x), N_2(x), N_3(x)$ are non-empty. Let X_1 to be the set of vertices $x \in X'$ for which $N_2(x)$ is not complete to $N_3(x)$; for every $x \in X_1$ fix $n_2^1(x) \in N_2(x)$ and $n_3^1(x) \in N_3(x)$ such that $n_2^1(x)$ is non-adjacent to $n_3^1(x)$. Define X_2 and $n_1^2(x), n_3^2(x)$ for every $x \in X_2$, and X_3 and $n_1^3(x), n_2^3(x)$ for every $x \in X_3$ similarly. Since no vertex of X' is connected, it follows that $X' = X_1 \cup X_2 \cup X_3$.

(1) Let $\{i, j, k\} = \{1, 2, 3\}$. There do not exist $x, y \in X_i$, $n_j \in N_j(x)$ and $n_k \in N_k(x)$ such that n_j is non-adjacent to n_k, and $\{x, n_j, n_k\}$ is anticomplete to $\{y, n_j^i(y)\}$, and $n_k^i(y)$ is anticomplete to $\{n_j, n_k\}$.

Proof: Write $n_j(y) = n_j^i(y)$, and $n_k(y) = n_k^i(y)$. By the third assumption of the theorem, there exist $a, b \in V(G)$ such that $n_j(y) - a - b$ is a path where both a and b are anticomplete to S_k with $|L(a)| = |L(b)| = 1$. Since $x, y \in X'$, it follows that $\{a, b\}$ is anticomplete to $\{x, y\}$. If x is adjacent to $n_k(y)$, then $n_k - x - n_k(y) - y - n_j(y) - a - b$ is a P_7 in G, a contradiction, so x is non-adjacent to $n_k(y)$. Now by the second assumption of the theorem there exists a path P with ends $a, b \in \{n_j, n_j(y), n_k, n_k(y)\}$, such that $\{a, b\} \neq \{n_j, n_k\}$, $\{a, b\} \neq \{n_j(y), n_k(y)\}$, every interior vertex w of P has $|L(w)| = 1$, and $V(P) \setminus \{a, b\}$ is disjoint from and anticomplete to $\{n_j, n_j(y), n_k, n_k(y)\} \setminus \{a, b\}$. Since $x, y \in X'$, it follows that $V(P) \setminus \{n_j, n_j(y), n_k, n_k(y)\}$ is anticomplete to $\{x, y\}$. But now $G[V(P) \cup \{x, y, n_j, n_j(y), n_k, n_k(y)\}]$ is a path of length at least 7, a contradiction. This proves *(1)*.

Let $\{i, j, k\} = \{1, 2, 3\}$. A coloring c of a restriction (G, L'', Z'') of (G, L, Z) is a *a type I coloring with respect to* i if there exists $x \in X_i$, $n_j \in N_j(x)$ and $n_k \in N_k(x)$ such that $c(n_j) = c(n_k) = i$.

(2) Let (G, L'', Z'') be a restriction of (G, L, Z). If (G, L'', Z'') admits a type I coloring with respect to i, then there exists a set $\mathcal{L}_i$ of $O(|V(G)|^3)$ subpalettes of L'' such that

(a) $|L'(v)| \leq 2$ for every $L' \in \mathcal{L}_i$ and $v \in X_i$, and

(b) (G, L'', Z'') *admits a type I coloring with respect to* i *if and only if* $(G, \mathcal{L}_i, Z'')$ *is colorable.*

Moreover, $\mathcal{L}_i$ *can be constructed in time* $O(|V(G)|^4)$.

Proof: For every $x \in X_i$, $n_j \in N_j(x)$, $n_k \in N_k(x)$ such that n_j is non-adjacent to n_k, and $c_1 \in \{j, k\}$ do the following.

Initialize the order 3 palette L_{x,n_j,n_k,c_1} of G:

- $L_{x,n_j,n_k,c_1}(x) = \{c_1\}$,
- $L_{x,n_j,n_k,c_1}(n_j) = L_{x,n_j,n_k,c_1}(n_k) = \{i\}$, and
- $L_{x,n_j,n_k,c_1}(v) = L''(v)$ for all $v \in V(G) \setminus \{x\}$.

Assume that $c_1 = j$; we perform a symmetric construction if $c_1 = k$. For every $y \in X_i \setminus \{x\}$ we modify L_{x,n_j,n_k,c_1} as follows:

$$L_{x,n_j,n_k,c_1}(y) = \begin{cases} L_{x,n_j,n_k,c_1}(y) \setminus \{i\} & , \text{ if } y \text{ is adjacent to one of } n_j, n_k, \text{ or } n_k^i(y) \text{ is adjacent to } x \\ L_{x,n_j,n_k,c_1}(y) \setminus \{j\} & , \text{ if } y \text{ is adjacent to } x, \text{ or } n_k^i(y) \text{ is adjacent to one of } n_j, n_k \\ L_{x,n_j,n_k,c_1}(v) \setminus \{k\} & , \text{ if } n_j^i(y) \text{ is adjacent to one of } n_j, n_k \end{cases}$$

Now *(1)* implies that $|L_{x,n_j,n_k,c_1}(y)| \leq 2$ for every $y \in X_i$. Let $\mathcal{L}_i$ be the set of all the $O(|V(G)|^3)$ palettes L_{x,n_j,n_k,c_1} thus constructed. By construction, if $(G, \mathcal{L}, Z'')$ is colorable then (G, L'', Z'') has a type I coloring with respect to i.

Now, suppose c is a type I coloring of (G, L'', Z'') with respect to i, and so for some $x \in X_i$, there exist $n_j \in N_j(x)$ and $n_k \in N_k(x)$ with $c(n_j) = c(n_k) = i$. Then n_j is non-adjacent to n_k. We may assume that $c(x) = j$. Then $c(x) \in L_{x,n_j,n_k,j}(x)$. Consider a vertex $y \in X_i \setminus \{x\}$. If y is adjacent to one of n_j, n_k, then $c(y) \neq i$. If $n_k^i(y)$ is adjacent to x, then, since $n_k^i(y) \in S_k$, it follows that $c(n_k^i(y)) = i$, and again $c(y) \neq i$. If y is adjacent to x, then $c(y) \neq j$. If $n_k^i(y)$ is adjacent to one of n_j, n_k, then, since $n_k^i(y) \in S_k$, it follows that $c(n_k^i(y)) = j$, and again $c(y) \neq j$. Finally, if $n_j^i(y)$ is adjacent to one of n_j, n_k, then, since $n_j^i(y) \in S_j$, it follows that $c(n_j^i(y)) = k$, and again $c(y) \neq k$. Thus, in all cases, $c(y) \in L_{x,n_j,n_k,c_1}(y)$, and *(2)* follows. This proves *(2)*.

(3) Let (G, L'', Z'') *be a restriction of* (G, L, Z). *If* (G, L'', Z'') *does not admit a type I coloring with respect to either of* i, j, *then there exists a subpalette* $M_{i,j}$ *of* L'' *such that*

(a) $|M_{i,j}(x)| \leq 2$ *for every* $x \in X_i \cap X_j$, *and*

(b) (G, L'', Z'') *is colorable if and only if* $(G, M_{i,j}, Z'')$ *is colorable.*

Moreover, $M_{i,j}$ *can be constructed in time* $O(|V(G)|^2)$.

Proof: For every $x \in X_i \cap X_j$, set $M_{i,j}(x) = \{i, j\}$. Clearly $|M_{i,j}(v)| \leq 2$ for every $x \in X_i \cap X_j$, and if $(G, M_{i,j}, Z'')$ is colorable, then (G, L'', Z'') is colorable. Suppose that (G, L'', Z'') is colorable, and let c be a coloring of (G, L'', Z''). Suppose that $c(x) \notin M_{i,j}(x)$ for some $v \in V(G)$. Then $x \in X_i \cap X_j$, and $c(x) = k$. Therefore $c(n_i^j(x)) = j$ and $c(n_j^i(x)) = i$. Since (G, L'', Z'') does not admit a type I coloring with respect to i, it follows that $c(n_k^i(x)) = j$, but then c is a type I coloring of (G, L'', Z'') with respect to j, a contradiction. This proves *(3)*.

(4) Let (G'', L'', Z'') be a restriction of (G, L, Z). Suppose (G'', L'', Z'') does not admit a type I coloring with respect to i. Let Y_i be the set of vertices $x \in X_i$ such that $N_i(x) = \emptyset$. Let $Z_i = \bigcup_{x \in Y_i} \{N_j(x), N_k(x)\}$. Then (G'', L'', Z'') is colorable if and only if $(G'' \setminus Y_i, L'', Z'' \cup Z_i)$ is colorable and a 3-coloring of $(G'' \setminus Y_i, L'', Z'' \cup Z_i)$ can be extended to a 3-coloring of (G'', L'', Z'') in time $O(|V(G)||Y_i|)$.

Proof: It is enough to prove that for every coloring c of (G, L, Z) and every $x \in X_i$ such that $N_i(x) = \emptyset$, the sets $N_j(x)$ and $N_k(x)$ are monochromatic with respect to c. Suppose not, we may assume for some coloring c there are vertices $u, v \in N_j(x)$ with $c(u) = i$ and $c(v) = k$. Since c is not a type I coloring of (G, L, Z), it follows that $c(w) = j$ for every $w \in N_k(x)$. But then x has neighbors of all three colors, contrary to the fact that c is a coloring. This proves *(4)*.

We now construct $\mathcal{P}$ as follows. We break the construction into four steps $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ and $\mathcal{P}_4$.

To construct $\mathcal{P}_1$, apply *(2)* to (G, L, Z) with $i = 1$, to construct $\mathcal{L}_1$. Now apply *(2)* to (G, L', Z) for every $L' \in \mathcal{L}_1$ with $i = 2$, to construct $\mathcal{L}_{12}$. Next apply *(2)* to (G, L', Z) for every $L' \in \mathcal{L}_{12}$ with $i = 3$, to construct $\mathcal{L}_{123}$. Then $|\mathcal{L}_{123}| = O(|V(G)|^9)$; by *(2)* this takes time $O(|V(G)|^{10})$. Let $\mathcal{P}_1$ consist of all (G, L', Z) with $L' \in \mathcal{L}_{123}$.

Next we construct $\mathcal{P}_2$. Apply *(4)* to (G, L', Z) for every $L' \in \mathcal{L}_{12}$ with $i = 3$; this creates a set $\mathcal{P}_2$ of $O(|V(G)|^6)$ triples $(G \setminus Y_3, L', Z \cup Z_3)$, and $|Z \cup Z_3| = |Z| + O(|V(G)|)$. This step can be performed in time $O(|(V(G)|^2)$ for every $L' \in \mathcal{L}_{12}$, and so takes time $O(|(V(G)|^8)$ in total.

Next we construct $\mathcal{P}_3$. Apply *(3)* to (G, L', Z) for every $L' \in \mathcal{L}_1$ with $i = 2$ and $j = 3$; this generates a set $\mathcal{P}_3'$ of $O(|V(G)|^3)$ triples (G, M', Z), and takes time $O(|(V(G)|^5)$. Now apply *(4)* to every $(G, M', Z) \in \mathcal{P}_3'$ with $i = 2$; this creates a set $\mathcal{P}_3''$ of $O(|V(G)|^3)$ triples $(G \setminus Y_2, M', Z \cup Z_2)$, and $|Z \cup Z_2| = |Z| + O(|V(G)|)$. This step can be performed in time $O(|(V(G)|^5)$. Now apply *(4)* to every $(G \setminus Y_2, M', Z \cup Z_2) \in \mathcal{P}_3''$ with $i = 3$; this creates a set $\mathcal{P}_3$ of $O(|V(G)|^3)$ triples $(G \setminus (Y_2 \cup Y_3), M', Z \cup Z_2 \cup Z_3)$, and $|Z \cup Z_2 \cup Z_3| = |Z| + O(|V(G)|)$. This step can be performed in time $O(|(V(G)|^5)$.

Finally, apply *(3)* to (G, L, Z) with $i = 1, j = 2$ to obtain (G, M_{12}, Z). Next apply *(3)* to (G, M_{12}, Z) with $i = 2, j = 3$ to obtain (G, M_{12}', Z). Next apply *(3)* to (G, M_{12}', Z) with $i = 1, j = 3$ to obtain (G, M_4, Z). Now apply *(4)* to with $i = 1, 2, 3$ to construct $\mathcal{P}_4 = \{(G \setminus (Y_1 \cup Y_2 \cup Y_3), M_4, Z \cup Z_1 \cup Z_2 \cup Z_3)\}$. This step takes time $O(|(V(G)|^2)$.

Let $\mathcal{P}' = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \cup \mathcal{P}_4$. Then $|\mathcal{P}'| = O(|V(G)|^9)$, and it can be constructed in time $O(|V(G)|^{10})$. Finally, repeat the construction described above for every permutation of the colors $\{1, 2, 3\}$ and let $\mathcal{P}$ be the union of the 3! sets of restrictions thus generated. It is still true that $|\mathcal{P}| = O(|V(G)|^9)$, and it can be constructed in time $O(|V(G)|^{10})$. Moreover, by the construction process and *(4)*, a 3-coloring of a restriction in $\mathcal{P}$ can be extended to a 3-coloring of G in time $O(|V(G)|^2)$.

(5) $\mathcal{P}$ *satisfies (a).*

Proof: It is enough to prove the result for $\mathcal{P}'$. By *(3)*, $|Z'| = |Z| + O(|V(G)|)$ for every $(G', L', Z') \in \mathcal{P}$. It remains to show that $|L'(x)| \leq 2$ for every $(G', L', Z') \in \mathcal{P}$ and $x \in X$.

Since $X = X_1 \cup X_2 \cup X_3$, *(2)* implies that $|L'(x)| \leq 2$ for every $x \in X$ and $(G, L', Z) \in \mathcal{P}_1$.

We now check the members of $\mathcal{P}_2$. Also by *(2)*, $|L'(x)| \leq 2$ for every $x \in X_1 \cup X_2$ and every $L' \in \mathcal{L}_{12}$. Since no vertex of X is connected, it follows that every $x \in X'$ with all three of $N_1(x), N_2(x), N_3(x)$ non-empty belongs to X_i for at least two values of i, and so if $x \in X' \setminus (X_1 \cup X_2)$, then $x \in Y_3$. Since $V(G') = V(G) \setminus Y_3$ for every $(G', L', Z') \in \mathcal{P}_2$, it follows that $|L'(x)| \leq 2$ for every $x \in X \cap V(G')$ and $(G', L', Z') \in \mathcal{P}_2$.

Next we check the members of $\mathcal{P}_3$. By (2), $|L'(x)| \leq 2$ for every $x \in X_1$ and every $L' \in \mathcal{L}_1$. By *(3)*, $|L'(x)| \leq 2$ for every $x \in X_1 \cup (X_2 \cap X_3)$ and every $(G, M', Z) \in \mathcal{P}_3'$. Since no vertex of X is connected, it follows that every $x \in X'$ with all three of $N_1(x), N_2(x), N_3(x)$ non-empty belongs to X_i for at least two values of i, and so if $x \in X' \setminus (X_1 \cup (X_2 \cap X_3))$, then $x \in Y_2 \cup Y_3$. Since $V(G') = V(G) \setminus (Y_2 \cup Y_3)$ for every $(G', M', Z') \in \mathcal{P}_3$, it follows that $|L'(x)| \leq 2$ for every $x \in X \cap V(G')$ and $(G', M', Z') \in \mathcal{P}_3$.

Finally, we check $(G \setminus (Y_1 \cup Y_2 \cup Y_3), M_4, Z \cup Z_1 \cup Z_2 \cup Z_3)$. By *(3)*, $|M_4(x)| \leq 2$ for every $x \in (X_1 \cap X_2) \cup (X_2 \cap X_3) \cup (X_1 \cap X_3)$. In particular $|M_4(x)| \leq 2$ for every $x \in X'$ with all three of $N_1(x), N_2(x), N_3(x)$ non-empty, and so if $x \notin (X_1 \cap X_2) \cup (X_2 \cap X_3) \cup (X_1 \cap X_3)$, then $x \in Y_1 \cup Y_2 \cup Y_3$. This proves *(5)*.

(6) $\mathcal{P}$ *satisfies (b).*

Proof: Suppose first that G admits a type I coloring with respect to each of $1, 2$ and 3. Then by *(2)*, some $(G', L', Z') \in \mathcal{P}_1$ is colorable.

Next suppose that G admits a type I coloring with respect to each each of $1, 2$ and not with respect to 3. By *(2)*, (G, L', Z) is colorable for some $L' \in \mathcal{L}_{12}$; now by *(4)* $(G \setminus Y_3, L', Z \cup Z_3) \in \mathcal{P}_2$ is colorable.

Next suppose that G admits a type I coloring with respect to 1, but not with respect to 2 or 3. By *(2)*, (G, L', Z) is colorable for some $L' \in \mathcal{L}_1$. By *(3)*, there is $(G, M', Z) \in \mathcal{P}_3'$ that is colorable. Now by *(4)* $(G \setminus (Y_2 \cup Y_3), M', Z \cup Z_3 \cup Z_3) \in \mathcal{P}_3$ is colorable.

Finally, suppose that G does not admit a type I coloring with respect to any of $1, 2, 3$. Now by (3) and *(4)* $(G \setminus (Y_1 \cup Y_2 \cup Y_3), M, Z \cup Z_1 \cup Z_2 \cup Z_3) \in \mathcal{P}_4$ is colorable. Since we performed the same construction for all permutation of colors $\{1, 2, 3\}$, this proves *(6)*.

Now 5.1 follows from *(5)* and *(6)*. □

6 Coloring Expansion

In this section, we show how to expand the set of palettes constructed in 4.1, yielding an equivalent polynomial sized collection of sub-problems all of which can be checked by applying 1.7.

6.1. *Let G be a connected P_7-free graph, and $A = (A_1, A_2, A_3)$ be a normal tripod in G, and assume that G is (A_1, A_2, A_3)-clean. Partition $V(G) = A \cup X \cup Y \cup Z$ as in 4.1. Let $\mathcal{L}$ be the set of palettes generated by 4.1 and consider a fixed palette $L \in \mathcal{L}$. Then there exists a set $\mathcal{P}_L$ of $O(|V(G)|^9)$ restrictions of $(G, L, \emptyset)$ such that the following hold:*

(a) For every $(G', L', S) \in \mathcal{P}_L$, $|L'(v)| \leq 2$ for every $v \in V(G')$ and $|S| = O(|V(G)|)$, and

(b) (G, L) is colorable if and only if $\mathcal{P}_L$ is colorable.

Moreover, $\mathcal{P}_L$ can be constructed in time $O(|V(G)|^{10})$, and a 3-coloring of a restriction in $\mathcal{P}_L$ can be extended to a 3-coloring of G in $O(|V(G)|^2)$.

Proof. We use the notation of 4.1. By 4.1, for every $x \in P_L$, $N(x) \subseteq (X \setminus X') \cup Y'$, and $|L(v)| \leq 2$ for every $v \in N(x)$.

Let $\{i, j, k\} = \{1, 2, 3\}$. We remind the reader that by 4.1

- *(a)* If $v \in Y'$ with $L(v) = \{i, j\}$, then there exists $u \in N(v) \cap (X' \cup Y_0^s)$ with $L(u) = \{k\}$
- *(b)* If $v \in X' \cap X_j$ with $L(v) = \{i\}$, then either there exists $u \in N(v)$ such that $L(u) = \{k\}$, or $L(y) = \{j\}$ for every $y \in Y$ with a neighbor in X_j, and
- *(c)* If $v \in Y_0^s$ with $L(v) = \{i\}$, then there exists $u, w \in N(v) \cap \{s_1, s_2, s_3\}$ such that $L(u) = \{k\}$ and $L(w) = \{j\}$.

Next we repeatedly update L until we perform a round of updating in which no list is changed. This requires at most $|V(G)|$ rounds of updating, and so takes time $O(|V(G)|^3)$. Now let P be the set of vertices $v \in P_L$ with $|L(v)| = 3$. By updating, we may assume that for every $v \in P$ and for every neighbor y of v, we have $|L(y)| = 2$. For $1 \leq i < j \leq 3$ and $k \in \{1, 2, 3\} \setminus \{i, j\}$, let S_k be the set of vertices $v \in (X \setminus X') \cup Y'$ such that v has a neighbor in P, and $L(v) = \{i, j\}$. Since we have updated, it follows that every vertex w with $L(w) \in \{\{i\}, \{j\}\}$ is anticomplete to S_k.

It is now enough to check that S_1, S_2, S_3, P satisfy the assumptions of 5.1 (where P plays the roles of X from 5.1). Since every vertex of P is anticomplete to $A_1 \cup A_2 \cup A_3$ it follows that no vertex of P is connected. By definition, the lists of S_1, S_2, S_3 satisfy the first condition.

Now we check the second condition. Let $1 \leq i < j \leq 3$ and let $u_i, v_i \in S_i$ and $u_j, v_j \in S_j$ such that $\{u_i, v_i, u_j, v_j\}$ is a stable set. We may assume $i = 1$ and $j = 2$. Then

$u_1, v_1 \in X_1 \cup Y'$ and $u_2, v_2 \in X_2 \cup Y'$. Suppose first that both $u_1, v_1 \in X_1$. By 2.2, there is a path P from u to v with interior in $A_1 \cup A_3$. Since $u_2, v_2 \in S_2$, it follows that the interior of P is anticomplete to and disjoint from $\{u_2, v_2\}$, as required.

Next suppose that $u_1 \in X_1$. Then $v_1 \in Y'$, and therefore v_1 is anticomplete to $A_1 \cup A_2 \cup A_3$. Assume first that $v_2 \in X_2$. Then $u_2 \in Y'$, and in particular, u_2 is anticomplete to $A_1 \cup A_2 \cup A_3$. Let P be a path from u_1 to v_2 with interior in $A_1 \cup A_2$ (which exists by 2.2); then P has the required properties. Thus we may assume that $v_2 \in Y'$. By *(a)*, there exists $w \in X \cup Y_0$ such that v_2 is adjacent to w, and $L(w) = \{2\}$. Then w is anticomplete to $\{u_1, v_1\}$. We may also assume w is anticomplete to $\{u_2\}$ since other wise $u_2 - w - v_2$ is the desired path. If $w \in X_1 \cup X_3$, then by 2.2 there is a path P from u_1 to w with interior in $A_1 \cup A_3$, and $u_1 - P - w - v_2$ is the desired path. So we may assume that $w \in Y_0$. Then $L(s_1) = \{3\}$, since s_3 is adjacent to w, $L(w) = \{2\}$ and $s_1 \in X_1$. Hence s_1 is anticomplete to $\{u_1, u_2, v_1, v_2\}$. By 2.2, there is a path P from s_3 to u_1 with interior in $A_1 \cup A_3$. But now $v_2 - w - s_1 - P - u_1$ is the required path.

Thus we may assume that $u_1, u_2, v_1, v_2 \in Y$. Let $a \in N(u_1) \cap (X' \cup Y_0^s)$ and $b \in N(v_1) \cap (X' \cup Y_0^s)$ with $L(a) = L(b) = \{1\}$. Such a, b exist by *(a)*. Then $\{a, b\}$ is anticomplete to $\{u_2, v_2\}$. If there is a path P from a to b with (possibly empty) interior in $A_1 \cup A_2 \cup A_3$, then $u_1 - a - P - b - v_1$ is the desired path, so we may assume no such path P exists. It follows that $a \neq b$, a is non-adjacent to b, and at least one of a, b belongs to Y_0. We may assume that $a \in Y_0$. Therefore $L(s_2) = \{3\}$, and so s_2 is anticomplete to $\{u_2, v_2\}$. If b is adjacent to some s_2, then $u_1 - a - s_2 - b - v_1$ is the desired path, so we may assume not. It follows that $b \in X$. By 2.2 there is a path from s_2 to b with interior in $A_1 \cup A_2 \cup A_3$, and now $u_1 - a - s_2 - P - b_1$ is the desired path. Thus the second condition holds.

Lastly, we verify that the third condition holds. Let $i, j \in \{1, 2, 3\}$ and let $k \in \{1, 2, 3\} \setminus \{i, j\}$. Consider $u \in S_i$.

We claim that u has a neighbor a with $L(a) = \{i\}$, and a has a neighbor b with $L(b) = \{k\}$, and $u - a - b$ is a path. Suppose first that $u \in X_i$. Then u has a neighbor $a \in A_i$, and a has a neighbor $b \in A_k$, as required. Thus we may assume that $u \in Y'$. Since $L(u) = \{j, k\}$, by *(a)*, there exists $a \in N(u) \cap (X' \cup Y_0^s)$ with list $\{i\}$. Since a has list $\{i\}$, it follows that $a \in X_j \cup X_k \cup Y_0$. By *(b)* and *(c)*, and since every vertex of X_k has a neighbor in A_k, it follows that a has a neighbor b with $L(b) = \{k\}$. Since $L(u) = \{j, k\}$ and we have updated, it follows that b is non-adjacent to u, and $u - a - b$ is a path. This proves the claim.

Since $L(v) = \{i, k\}$ for every $v \in S_j$, and since we have update, it follows that $\{a, b\}$ is anticomplete to S_j as required. Thus the third condition holds. This proves 6.1. □

7 Main Result

In this section we prove the main result of this paper 1.9, which we restate:

7.1. *There is an algorithm with the following specifications:*

Input: *A P_7-free graph G which contains a triangle.*

Output: *A 3-coloring of G, or a determination that none exists.*

Running time: $O(|V(G)|^{24})$.

Proof. We may also assume that G is connected (otherwise we run the following procedure for each connected component of G). By 2.4, at the expense of carrying out a time $O(|V(G)|^3)$ procedure we can determine that no 3-coloring of G exists (then we can stop), or obtain a connected graph G' satisfies the following:

- $|V(G')| \leq |V(G)|$,
- G' is connected,
- G' is 3-colorable if and only if G is 3-colorable,
- Any 3-coloring of G' can be extended to a 3-coloring of G in time $O(|V(G)|^2)$, and
- G' is either triangle-free or contains a normal tripod (A_1, A_2, A_3).

In the case that G' is triangle-free we can use the algorithm in [2] to either determine that no 3-coloring of G' exists or find a 3-coloring of G' in $O(|V(G)|^7)$. Thus we can either determine that no 3-coloring of G exists or use the 3-coloring of G' to find a 3-coloring of G in time $O(|V(G)|^2)$.

Thus we may assume that G' contains a normal tripod (A_1, A_2, A_3). By 3.1, at the expense of carrying out a time $O(|V(G)|^4)$ procedure, we can either determine that G is not 3-colorable (and stop), or may assume that G' is (A_1, A_2, A_3)-clean. By 4.1, in time $O(|V(G)|^{15})$ we can produce a set $\mathcal{L}$ of $O(|V(G)|^{12})$ order 3 palettes of G' such that G' has a 3-coloring if and only if $(G', \mathcal{L})$ is colorable. By 6.1 for a fixed $L \in \mathcal{L}$, in time $O(|V(G)|^{10})$ we can construct a set of $O(|V(G)|^9)$ restrictions $\mathcal{P}_L$ such that

- For every $(G'', L', X) \in \mathcal{P}_L$, $|L'(v)| \leq 2$ for every $v \in V(G'')$ and $|X| = O(|V(G)|)$,
- (G', L) is colorable if and only if $\mathcal{P}_L$ is colorable , and
- a 3-coloring of a restriction in $\mathcal{P}_L$ can be extended to a 3-coloring of G' in time $O(|V(G)|^2)$.

For every restriction in $\mathcal{P}_L$, by 1.7, in time $O(|V(G)^3|)$ we can either determine that it is not colorable, or find a coloring of it. Since $|\mathcal{L}| = O(|V(G)|^{12})$ and $|\mathcal{P}_L| = O(|V(G)|^9)$, we need to run 1.7 $O(|V(G)|^{21})$ times. Hence in time $O(|V(G)|^{24})$, we can either determine that no 3-coloring of G' exists, which means that no 3-coloring of G exists, or find a 3-coloring of G', which can be extended to a 3-coloring of G in time $O(|V(G)|^2)$. This proves 7.1. □

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References

[1] B. Aspvall, M. Plass, R. Tarjan, A Linear-Time Algorithm for Testing the Truth of Certain Quantified Boolean Formulas, Inf. Process. Lett. 8 (3) (1979), 121-123.

[2] F. Bonomo, M. Chudnovsky, P. Maceli, O. Schaudt, M. Stein and M. Zhong, Three-coloring graphs without triangles or induced paths on seven vertices, submitted for publication.

[3] M. Chudnovsky, P. Maceli and M. Zhong, Three-coloring graphs with no induced six-edge path I : the triangle-free case, manuscript.

[4] K. Edwards, The complexity of colouring problems on dense graphs, Theoret. Comput. Sci. 43 (1986), 337-343.

[5] C.T. Hoàng, M. Kamiński, V.V. Lozin, J. Sawada and X. Shu, Deciding k-colorability of P_5-free graphs in polynomial time, Algorithmica 57 (2010), 74-81.

[6] I. Holyer, The NP-completeness of edge coloring, SIAM J. Comput. 10 (1981), 718-720.

[7] S. Huang, Improved Complexity Results on k-Coloring P_t-Free Graphs, Proc. MFCS 2013, LNCS, to appear.

[8] M. Kamiński and V.V. Lozin, Coloring edges and vertices of graphs without short or long cycles. Contrib. Discrete Mah. 2 (2007), 61-66.

[9] R. M. Karp, Reducibility Among Combinatorial Problems, Complexity of Computer Computations, New York: Plenum., 85-103.

[10] D. Král, J. Kratochvíl, Z. Tuza, and G. Woeginger, Complexity of coloring graphs without forbidden induced subgraphs, Proceedings of WG 2001, LNCS 2204, 254-262.

[11] D. Leven and Z. Galil, NP-completeness of finding the chromatic index of regular graphs. J. Algorithm 4 (1983), 35-44.

[12] B. Randerath and I. Schiermeyer, 3-Colorability $\in P$ for P_6-free graphs, Discrete Appl. Math. 136 (2004), 299-313.

[13] B. Randerath, I. Schiermeyer and M.Tewes, Three-colorability and forbidden subgraphs. II: polynomial algorithms, Discrete Mathematics 251 (2002), 137153.

[14] J. Stacho, private communication.

[15] L. Stockmeyer, Planar 3-colorability is polynomial complete, SIGACT News (1973), 19-25.