Three dimensional canonical singularity and five dimensional $\mathcal{N} = 1$ SCFT

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ABSTRACT: We conjecture that every three dimensional canonical singularity defines a five dimensional $\mathcal{N} = 1$ SCFT. Flavor symmetry can be found from singularity structure: non-abelian flavor symmetry is read from the singularity type over one dimensional singular locus. The dimension of Coulomb branch is given by the number of compact crepant divisors from a crepant resolution of singularity. The detailed structure of Coulomb branch is described as follows: a) a chamber of Coulomb branch is described by a crepant resolution, and this chamber is given by its Nef cone and the prepotential is computed from triple intersection numbers; b) Crepant resolution is not unique and different resolutions are related by flops; Nef cones from crepant resolutions form a fan which is claimed to be the full Coulomb branch.

Contents

1 Introduction

One can define higher dimensional superconformal field theory (SCFT) in various ways. If our theory has a conformal manifold, it might be possible to find a weakly coupled gauge theory description: one can describe our theory by specifying matter contents and gauge groups, and the coordinates of conformal manifold are identified with gauge couplings. This method works for four dimensional $\mathcal{N} = 4$ Super Yang-Mills theory and four dimensional $\mathcal{N} = 2$ SCFTs. These gauge theory descriptions are often not unique and one can have very interesting S duality property.

Another way of defining a SCFT is as follows [1]: consider the Coulomb branch of a four dimensional $\mathcal{N} = 2$ theory, and there might be a locus where some extra massive BPS particles become massless, and one can get a SCFT if these extra massless particles are mutually non-local. Argyres-Douglas theory [2] was found in this way, and a very

general argument for the existence of such SCFT is given in [3]. These extra massless degree of freedoms make the description of physics singular at this particular point, and such singularities can often be made geometrical [4]. One can actually define a SCFT by simply specifying the geometrical singularity, and this idea has been used to engineer a large class of new four dimensional $\mathcal{N} = 2$ SCFTs [1, 5, 6].

Our focus in this paper is to use singularity approach to study five dimensional $\mathcal{N} = 1$ SCFT [7]. This type of SCFT has no SUSY preserving exact marginal deformations [8], therefore one can not write down a weakly coupled gauge theory description with conformal gauging. One can still define 5d $\mathcal{N} = 1$ SCFT [7] by going to a locus of Coulomb branch where one has extra massless particles including W bosons, instanton particles and tensionless strings.[1] Again, instead of specifying extra massless degree of freedoms, we may use a geometric singularity to define a 5d $\mathcal{N} = 1$ SCFT. Such method has been used in [9, 10] to study 5d $\mathcal{N} = 1$ SCFT and the main purpose of this paper is to provide a more systematic treatment.

The basic idea of engineering 5d $\mathcal{N} = 1$ SCFT is using M theory on a 3-fold singularity [9, 10]. Then the first question that we would like to answer is that what kind of singularity will lead to a five dimensional $\mathcal{N} = 1$ SCFT? The main conjecture of this paper is

Conjecture 1 *M theory on a 3d canonical singularity X defines a 5d $\mathcal{N} = 1$ SCFT.*

Familiar examples of 3d canonical singularity include toric Gorenstein singularity, quotient singularity C^3/G with G a finite subgroup of SL(3), and certain class of hypersurface singularities. One of basic argument for this conjecture is that this is the class of singularities that would appear in the degeneration limit of Calabi-Yau manifold [11]. Two dimensional canonical singularity has a ADE classification, and this leads to a remarkable ADE classification of six dimensional $(2, 0)$ SCFT [12].

The two very basic numeric invariants associated with a 5d SCFT are the rank of flavor symmetry f, and the rank of the Coulomb branch r. Finer information such as the enhancement of flavor symmetry, chamber structure and prepotential of Coulomb branch are also desired. One can derive many of these important physical results from the following geometric properties of singularity X:

- The non-abelian flavor symmetry can be read from the ADE type over one dimensional singular locus. The rank of other abelian flavor symmetry is read from the rank of local divisor class group of the singularity.

- The Coulomb branch is given by *crepant* resolution of the singularity X, and different crepant resolutions describe different chambers of the Coulomb branch. These different resolutions are related by flops, and the rank of Coulomb branch is constant across different chambers.

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[1]We want to emphasize that it is not sufficient to define a 5d $\mathcal{N} = 1$ SCFT by just specifying non-abelian gauge theory description on certain locus of Coulomb branch, and one need to also specify other massive degree frames such as instanton particles and strings which become massless or tensionless at the SCFT point.

- Given a crepant resolution, let's choose a basis D_i of generators of local divisor class group, and the preopotential is given by following formula [10]:

$$\mathcal{F} = \frac{1}{6}\left(\sum_i \phi_i D_i\right)^3 = \frac{1}{6}\sum (D_i \cdot D_j \cdot D_k)\phi_i \phi_j \phi_k. \qquad (1.1)$$

Here $(D_i \cdot D_j \cdot D_k)$ is the triple intersection number for divisors. If D_i is compact (resp. non-compact), the corresponding parameter ψ_i is regarded as Coulomb branch (mass) parameter. The range of real numbers ψ is determined by Nef cone. Inside Nef cone, one have an abelian gauge theory description in the IR. The co-dimensional one face of Nef cone describes the place where an extra massive particle becomes massless. At the intersection of these faces, more massive particles become massless, and sometimes one can have a non-abelian gauge theory description.

- For crepant resolution related by a flop, the corresponding Nef cones share a face. Nef cones from all crepant resolutions form a fan which is identified as the full Coulomb branch.

This paper is organized as follows: section 2 reviews some basic facts of 5d $\mathcal{N} = 1$ SCFT and 5d gauge theory; section 3 describes the classification of 3-fold canonical singularity. Section 4 discusses in detail the SCFT associated with toric canonical singularity; section 5 and 6 describe SCFTs associated with quotient and hypersurface singularity; finally a conclusion is given in section 7.

2 Generality of 5d $\mathcal{N} = 1$ SCFT

5d $\mathcal{N} = 1$ superconformal algera consists of conformal group SO(2,5), R symmetry group SU(2)$_R$, and possibly global symmetry group G. Representation theory of five dimensional $\mathcal{N} = 1$ superconformal algebra is studied in [8, 13], and a general multiplet is labeled as $[j_1, j_2]_\Delta^{(R)}$, here j_1, j_2, Δ are spins and scaling dimension, and R is SU(2)$_R$ quantum number. Short representations are classified in [8, 13], and one important class is called $C_R = [0,0]_\Delta^{(R)}$, with the relation $\Delta = \frac{3}{2}R$. Flavor currents is contained in multiplet C_1.

For 5d $\mathcal{N} = 1$ SCFT, the only SUSY preserving relevant deformation is the mass deformation, and there is *no* exact marginal deformation. One can have a different kind of SUSY preserving deformation by turning on expectation value of certain operators, and we could have continuous space of vacua called moduli space. The moduli space consists of various branches:

- There could be a Higgs branch which can be parameterized by the expectation value of operators C_R, and the SU(2)$_R$ symmetry acts non-trivially on this branch. We would like to determine the full chiral ring relation of the Higgs branch, from which we can read the non-abelian flavor symmetry G. The simplest data we'd like to determine is the rank f of the flavor group G.

- There could be a Coulomb branch which is parameterized by the real numbers ψ_j, $j = 1, \ldots r$. Unlike four dimensional $\mathcal{N} = 2$ theory, we can not parameterize

it by expectation value of BPS operators. The Coulomb branch is not lifted by turning on mass deformations with mass parameters m_i, $i = 1, \ldots, f$, but the low energy physics is changed. At generic point, the low energy theory could be described by abelian gauge theory with gauge group $U(1)^r$, and the important question is to determine the rank r and the prepotential which is a cubic function of $U(1)$ vector multiplet and mass parameter:

$$\mathcal{F}(m, \psi) = \sum_{i,j,k} a_{ijk} \phi_i \phi_j \phi_k \,. \tag{2.1}$$

We have $\phi_i = m_i$, $i = 1, \ldots, f$, and $\phi_{f+j} = \psi_j$, $j = 1, \ldots, r$. A particular intriguing feature of 5d $\mathcal{N} = 1$ SCFT is that there might be many chambers of Coulomb branch and each chamber has different prepotentials. The BPS particle has central charge $Z = n_e^{(i)} \psi_i + S_i m_i$, and one also has tensile strings whose tension is given by $Z = n_m^{(i)} \psi_i^D$ with ψ_i^D the dual coordinates.

- One could also have the mixed branch which is a direct product of a Coulomb and Higgs factor.

Five dimensional $\mathcal{N} = 1$ SCFT can be constructed as follows:

1. Particular UV completion of a non-abelian gauge theory [7, 9], and the typical example is SU(2) with $N_f \leq 7$ fundamental flavors and the corresponding UV SCFT is 5d E_{N_f+1} SCFT.

2. M theory on 3-fold singularities [9, 10, 14–17], and the cone over Del Pezzo k surfaces will give 5d E_k SCFT.

3. (p,q) five brane webs in type IIB string theory [18], and some of E_k SCFTs can be engineered in this way.

Many aspects of 5d $\mathcal{N} = 1$ SCFTs such as the verification of enhanced flavor symmetry, superconformal inex, etc have been recently studied in [19–54].

2.1 5d Gauge theory and enhanced flavor symmetry

Along some sub-locus of Coulomb branch, the low energy theory can be described by a non-abelian gauge theory. Part of enhanced flavor symmetry might be derived by analyzing the instanton operators [39–42, 48], and here we review the basic result. In this subsection, all the gauge groups are taken as SU(N) and we only consider bi-fundamental matter. Consider a single SU(N) gauge group coupled with n_f hypermultiplets in fundamental representation, one can turn on Chern-Simons term with level k satisfying the constraint

$$k + \frac{n_f}{2} \in Z \,. \tag{2.2}$$

We use $m_0 = \frac{1}{g_{cl}^2}$ to denote the classical gauge coupling. Part of coulomb branch can be parameterized by the real numbers $(a_1, \ldots, a_N)$ with the constraint $\sum a_i = 0$. Using Weyl

invariance, we can take $a_1 \geq a_2 \geq \ldots \geq a_N$. At $m_0 = 0$, the prepotential reads

$$\mathcal{F} = \frac{1}{6}\left(\sum_{i<j}^{N}(a_i - a_j)^3 + k\sum_{i=1}^{3}a_i^3 - \frac{n_f}{2}\sum_{i=1}^{N}|a_i|^3\right). \tag{2.3}$$

The manifest flavor symmetry is $U(n_f) \times U(1)_I$, where $U(1)_I$ is associated with the instanton current $*F \wedge F$. The SCFT point might be achieved by taking a_i, gauge couplings, and masses to be zero,[2] and the flavor symmetry at the SCFT point could be further enhanced.

We will also consider a linear quiver with $SU(N_i)$ gauge group $[n_i] - SU(N_1) - SU(N_2) - \ldots - SU(N_r) - [n_e]$. We also have the Chern-Simons level k_i on each gauge group $SU(N_i)$, and we put further constraints on k_i:

$$2|k_i| \leq 2N_i - N_f, \tag{2.4}$$

here N_f is the number of flavors on ith gauge group. The manifest flavor symmetry is $U(n_i) \times U(n_e) \times U(1)^r \times U(1)^{r-1}$, with $U(n_i)$ and $U(n_e)$ from the fundamental flavors on two ends of quiver, and $U(1)$s from instanton current and flavor of bi-fundamentals. The enhanced flavor symmetry can be found as follows. Consider two A_r Dynkin diagrams $\Gamma^\pm$, and for Γ^+, we color ith node of Γ^+ black if the following identity holds

$$2k_i = 2N_i - N_f. \tag{2.5}$$

Similarly, color ith node of Γ^- black if we have

$$-2k_i = 2N_i - N_f. \tag{2.6}$$

The flavor symmetry from Γ^+ is $G_{\Gamma^+} = \bigotimes_j A_{n_j} \times U(1)^{r-\sum n_i}$, here n_j is the length of jth connected sub-diagram whose vertices are black. Similarly, one can read the enhanced flavor symmetry G_{Γ^-} from Γ^-. If we have the quiver tail $2 - SU(2) - SU(3) - \ldots$, we replace the quiver as $SU(1) - SU(2) - SU(3) - \ldots$ and apply the above rule again. The overall flavor symmetry for the linear quiver is

$$G_f = G_{\Gamma^+} \times G_{\Gamma^-} \times U(n_i) \times U(n_e)/U(1). \tag{2.7}$$

The flavor symmetry might be further enhanced.

3 Canonical singularity and five dimensional $\mathcal{N} = 1$ SCFT

3.1 Three dimensional canonical singularity

We conjectured in the introduction that 3-fold canonical singularities would lead to 5d $\mathcal{N} = 1$ SCFT. In this section, we will review the relevant facts about 3-fold canonical singularity [55–57]. Let's start with the definition:

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[2]To completely specify the SCFT, we need to specify other BPS particles which become massless at SCFT point besides the W bosons.

Definition 1 *A variety X is said to have canonical singularity if its is normal and the following two conditions are satisfied*

- *the Weil divisor K_X[3] is Q-Cartier, i.e. rK_X is a Cartier divisor.[4] Here r is called index of the singularity.*

- *for any resolution[5] of singularities $f : Y \to X$, with exceptional divisors $E_i \in Y$, the rational numbers a_i satisfying*

$$K_Y = f^* K_X + \sum a_i E_i \,. \tag{3.1}$$

are nonnegative. The numbers a_i are called the discrepancies of f at E_i; if they are all positive X is said to have terminal *singularities. If $a_i = 0$, the corresponding prime exceptional divisor is called crepant divisor.*

Two dimensional canonical singularity is completely classified and is also called Du Val singularity. There is a ADE classification, and it can be characterized by the following three ways:

- They are described by the following hypersurface singularity:

$$
\begin{aligned}
A_n : & \quad x_1^2 + x_2^2 + x_3^{n+1} = 0 \,, \\
D_n : & \quad x_1^2 + x_2^{n-1} + x_2 x_3^2 = 0 \,, \\
E_6 : & \quad x_1^2 + x_2^3 + x_3^4 = 0 \,, \\
E_7 : & \quad x_1^2 + x_2^3 + x_2 x_3^3 = 0 \,, \\
E_8 : & \quad x_1^2 + x_2^3 + x_3^5 = 0 \,.
\end{aligned} \tag{3.2}
$$

- They are defined as the quotient singularity C^2/G with G a finite subgroup of SL(2).

- For A type canonical singularity, one has a toric description.

Let's first introduce the concept of crepant resolution. Let X be a variety with canonical singularities. A partial resolution of X is a proper birational morphism $\phi : Y \to X$ from a normal variety. The morphism ϕ is said to be *crepant* if

$$K_Y = \phi^* K_X \,. \tag{3.3}$$

Two dimensional canonical singularity actually has a crepant resolution where Y is smooth.

Three dimensional canonical singularity is not completely classified, but a lot of general properties have been studied by Reid [55–57]. There are some useful properties for 3-fold canonical singularities:

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[3] K_X is the canonical divisor associated with X.

[4] A Cartier divisor implies that it can be used to define a line bundle.

[5] Given an irreducible variety X, a resolution of singularities of X is a morphism $f : X' \to X$ such that:

a: X' is smooth and irreducible.

b: f is proper.

c: f induces an isomorphism of varieties $f^{-1}(X/X_{\text{sing}}) = X/X_{\text{sing}}$. Furthermore, $f : X' \to X$ is a projective resolution if f is a projective morphism.

1. Canonical singularity implies Du Val singularities in codimension 2. This implies that the singularity type over a one dimensional singular locus has to be of ADE type.

2. Every canonical singularity can be written as $X = \tilde{X}/\mu_r$ with $\tilde{X}$ an index one canonical singularity. The index one canonical singularity is also called rational Gorenstein singularity.

3. There exists partial crepant resolution of a canonical singularity:

Theorem 1 *Let X be an algebraic 3-fold with canonical singularities. Then there exists a crepant projective morphism $\phi : Y \to X$, which is an isomorphism in codimension 1, from a 3-fold Y with Q-factorial terminal singularities.*

A Q-factorial terminal singularity means that every Weil divisor is a Q-Cartier divisor. Such *crepant* resolution is not unique, and different resolutions are related by *flop*. The crucial fact is that the number of crepant divisors are independent of the choice of crepant resolution. In the following, by crepant resolution we always mean above crepant morphism.

3.1.1 Classification of rational Gorenstein singularity

Since an index r canonical singularity can be covered using an index one canonical singularity, we will focus on index one canonical singularity which is also called rational Gorenstein singularity. To a rational Gorenstein 3-fold singularity, the general hyperplane section H through the singularity is a two dimensional rational or elliptic Gorenstein singularity [55]. Two dimensional rational Gorenstein singularity is nothing but the ADE singularity, and elliptic Gorenstein singularity is next simplest surface singularity. The above two class of surface singularities are quite rigid which makes the classification much simpler. In fact, use the minimal resolution of surface singularity, one can attach a natural integer $k \geq 0$ for each 3-fold rational Gorenstein singuarlity. We have [55]:

- $k = 0$, the singularity is a cDV point, i.e. the singularity can be written as follows

$$f(x, y, z) + tg(x, y, z, t) = 0, \tag{3.4}$$

here $f(x, y, z)$ denotes the Du Val singularity.

$k \geq 1$, the general section H through the singularity has an elliptic Gorenstein singularity $p \in H$ with invariant k:

- If $k \geq 2$, then $k = \text{mult}_P X$.

- If $k \geq 3$, then $k + 1$ is equal to the embedding dimension $= \dim(m_p/m_p^2)$. if $k = 2$, then $P \in X$ is isomorphic to a hypersurface given by $x^2 + f(y, z, t)$ with f a sum of monomials with degree bigger than 4. If $k = 1$, then then $P \in X$ is isomorphic to a hypersurface given by $x^2 + y^3 + f(y, z, t) = 0$ with $f(y, z, t) = yf_1(z, t) + f_2(z, t)$ and f_1 (respectively f_2) is a sum of monomials $z^a t^b$ of degree ≥ 4 (respectively ≥ 6).

	r	Type	f	Conditions
(1)	any	$\frac{1}{r}(a,-a,1,0;0)$	$xy + g(z^r, t)$	$g \in m^2,\, a,r,\text{coprime}$
(2)	4	$\frac{1}{4}(1,1,3,2;2)$	$xy + z^2 + g(t)$	$g \in m^3$
			or $x^2 + z^2 + g(y,t)$	$g \in m^3$
(3)	2	$\frac{1}{2}(0,1,1,1;0)$	$x^2 + y^2 + g(z,t)$	$g \in m^4$
(4)	3	$\frac{1}{3}(0,2,1,1;0)$	$x^2 + y^3 + z^3 + t^3$	
			or $x^2 + y^3 + z^2 t + yg(z,t) + h(z,t)$	$g \in m^4,\, h \in m^6$
(5)	2	$\frac{1}{2}(1,0,1,1;0)$	or $x^2 + y^3 + yzt + g(z,t)$	$g \in m^4$
			or $x^2 + yzt + y^n + g(z,t)$	$g \in m^4,\, n \geq 4$
			or $x^2 + yz^2 + y^n + g(z,t)$	$g \in m^4,\, n \geq 3$
(6)	2	$\frac{1}{2}(1,0,1,1;0)$	$x^2 + y^3 + yg(z,t) + h(z,t)$	$g,h \in m^4,\, h_4 \neq 0$

Table 1. List of three-fold terminal singularity.

- If $k = 3$, the singularity P is given by a hypersurface.

- If $k = 4$, then P is given by a complete intersection defined by two polynomials (f_1, f_2).

- For $k \geq 5$, the classification is incomplete.

Once we classify the rational Gorenstein singularity, the next step is to find out which cyclic quotient μ_r would lead to a general canonical singularity.

Other class of familiar examples include the toric Gorenstein singularity which will be studied in next section, and the quotient singularity C^3/G with G a finite subgroup of SL(3).

3.1.2 Classification of terminal singularity

The 3-fold terminal singularity has been completely classified. Terminal singularity is important as they appear at the end of partial crepant resolution. A 3-fold singularity is called cDV singularity if it can be written as the following form

$$f(x_1, x_2, x_3) + tg(x_1, x_2, x_3, t) = 0\,. \tag{3.5}$$

Here $f(x_1, x_2, x_3)$ is the two dimensional Du Val singularity. Index one terminal singularity is classified by *isolated cDV* singularity.

Other terminal singularities are found as the cyclic quotient of the isolated cDV singularity, and is classified by Mori [58]. The major result is that terminal singularities belong to 6 families listed in table 1.

It is also proven in [59] that every isolated singularity in above list is terminal. If we start with a smooth point (case (1) in table 1), then the terminal singularity is classified as follows:

Theorem 2 *X is terminal if and only if (up to permutations of (x, y, z) and symmetries of μ_r) $X = C^3/\mu_r$ of type $\frac{1}{r}(a, -a, 1)$ with a coprime to r, where μ_r is the cyclic group of order r.*

3.2 Physics and geometry

Flavor symmetry. The flavor symmetry of SCFT can be read from the singularity structure. The non-abelian flavor symmetry can be read as follows: since our singularity is normal, the singular locus Γ is at most one dimensional, and one has Du Val ADE surface singularity over one dimensional singular locus. Using the fact that one get ADE gauge symmetry by putting M theory on ADE singularity, and since the singular locus is non-compact (affine), one get a corresponding ADE flavor group for each one dimensional component $\Gamma_i \subset \Gamma$. One also have abelian flavor symmetry associated with the local divisor class group, whose rank is denoted as d. The full flavor symmetry is then

$$G = \mathrm{U}(1)^d \prod_i G_i \,. \tag{3.6}$$

Here G_i is the corresponding ADE flavor symmetry associated with ith component of singular locus. The flavor symmetry might be further enhanced. We also use f to denote the rank of G.

Partial resolution and non-abelian gauge theory description. Let's start with a canonical singularity, and it might be possible to find a partial crepant resolution $f : Y \to X$ such that Y has one dimensional singular *compact* locus whose singular type is one of ADE type. According to the similar M theory argument we used for finding flavor symmetry, we get ADE gauge symmetry. It is also possible to get a non-abelian gauge theory description with Lagrangian description.

Coulomb branch and crepant resolution. Starting with canonical singularity, one can find crepant partial resolutions $f : Y \to X$, here Y has Q-factorial terminal singularity. For rational Gorenstein singularity, such partial resolution can be found by first using sequence of standard blow-up for $k \geq 3$. For $k = 0$ or $k = 1$, we use the following weighted blow up, and one affine piece of which is given by setting:

$$
\begin{aligned}
k = 2: &\qquad x = z^2 x_1 \,, &\qquad y = z y_1 \,, &\qquad z = z \,, \\
k = 1: &\qquad x = z^3 x_1 \,, &\qquad y = z^2 y_1 \,, &\qquad z = z \,.
\end{aligned}
\tag{3.7}
$$

After these blow-ups, we end-up with a variety Y with cDV singularity. We can further blow-up the one-dimensional Du Val singularity so that we get an isolated cDV singularity. One can further perform small resolution on non Q-factorial isolated cDV singularity, and the end result is that we get an variety Y with Q-factorial terminal singularity. Using the cyclic cover construction, one can construct such crepant resolution for every canonical singularity.

Such crepant resolutions are not unique, but these different resolutions are all related by a sequence flops! Let's briefly introduce the concept of flops using the conifold singularity X_0: $x^2 + y^2 + z^2 + w^2 = 0$. This singularity has two crepant resolutions X_1 and X_2 where the exceptional locus is a curve, we say that X_1 and X_2 are related by flop: first a curve C_1 in X_1 shrinks to zero size so we get singularity X_0, we then resolve X_0 in a different way to get another variety X_2 where the exceptional locus is another curve C_2.

Let's now fix a crepant resolution Y, and one can define the Mori cone and Nef cone. A divisor D is called Nef if

$$D \cdot C \geq 0,\tag{3.8}$$

for all complete curve C. All such Nef divisors form a cone and is called Nef-Cone. Its dual cone is called Mori cone, which can also be defined using the complete curve. Our major conjecture is that the range of Coulomb branch is just the Nef Cone.

Let's denote the compact crepant divisors as E_i, $i = 1, \ldots, r$, and non-compact crepant divisors as D_i, $i = 1, \ldots, f_1$, and further choose a basis D_j, $j = 1, \ldots, f - f_1$ of non-compact divisors which generates the local class group of original singularity. The Coulomb branch deformation is related to the compact divisors E_i, while the mass deformation is related to D_i. The prepotential takes the following form

$$F = \frac{1}{6} \left(\sum_{i=1}^{r} E_i \phi_i + \sum_{n=1}^{f} D_n m_n \right)^3 = \frac{1}{6} \sum E_i \cdot E_j \cdot E_k \phi_i \phi_j \phi_k .\tag{3.9}$$

Here $E_i \cdot E_j \cdot E_k$ is the triple intersection form (here we use Es to denote all the divisors).

One can do the similar computation for the Nef-cone and prepotential of different crepant partial resolutions. Since different crepant partial resolution is related by flop, and their Nef cones form a fan, which might be called Nef fan, our main conjecture in this paper is that

Conjecture 2 *The Coulomb branch of 5d $\mathcal{N} = 1$ SCFT is the Nef fan.*

The BPS spectrum can be described as follows: let's start with the generators C_i of Mori cone, and a M2 brane can wrap on those curves and form primitive BPS particles. On the other hand, M5 brane can wrap on the compact exceptional divisors and one get tensile strings. All these massive objects become massless in the SCFT limit.

4 Toric singularity

An important and useful class of 3-fold canonical singularity can be defined using toric method [60], and we call them toric singularity. Toric singularity can be defined in a combinatorial way and various of its properties such as crepant resolutions can be described explicitly, which makes many computations possible. We will describe these toric singularities which are also canonical singularity.

We first briefly review how to define a toric variety, for more details, see [60]. Let's start with a tree dimensional standard lattice N, and its dual lattice $M = \text{Hom}(N, Z)$. A rational convex polyhedral cone σ in $N_R = N \otimes R$ is defined by a set of lattice vectors $\{v_1, \ldots, v_n\}$ as follows:

$$\sigma = \{ r_1 v_1 + r_2 v_2 + \ldots + r_n v_n | r_i \geq 0 \} .\tag{4.1}$$

Here ray generator v_ρ of σ is a lattice vector such that it is not a multiple of another lattice vector. Its dual cone $\sigma^\vee$ is defined by the set in $M_R = M \otimes R$ such that

$$\sigma^\vee = \{ w \cdot v \geq 0 | w \in M_R, \, v \in \sigma \} ,\tag{4.2}$$

here we used standard pairing between lattices N and M. From $\sigma^\vee$, one can define a semigroup $S = \sigma^\vee \cap M$, and the affine variety associated with σ is

$$X_\sigma = \text{Spec}(\sigma^\vee \cap M). \tag{4.3}$$

We have the following further constraints on the cones:

- A strongly rational convex polyhedral cone (s.r.c.p.c.) is a convex cone σ such that $\sigma \cap (-\sigma) = 0$.

- A simplicial s.r.c.p.c. is a cone whose generators form a R basis of N_R.

A fan Σ with respect to a lattice N is a finite collection of of rational convex polyhedral cones in N_R such that

- any face τ of σ belongs to Σ;

- for $\sigma_1, \sigma_2 \in \Sigma$, the intersection $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.

A fan is called simplicial if all the cones are simplicial. By $|\Sigma| := \cup\{\sigma | \sigma \in \Sigma\}$ one denotes the support and $\Sigma(i)$ the set of all i dimensional cones of fan Σ.

There is an important cone-orbit correspondence for toric variety: for each k dimensional cone, one can associate a $3 - k$ dimensional toric invariant orbit in X_σ. In particular, the toric divisors are determined by one dimensional cone, which is also specified by the ray generator v_ρ.

We are interested in affine toric singularity which is defined by a single cone σ. The singular locus of affine toric singularity is given by the sub-cone σ_i whose generators do not form a Z-basis, and the singular locus corresponds to the orbit which is determined by σ_i. For 3-fold toric singularity, the maximal dimension of singular locus is one dimensional which is then formed by a two dimensional sub-cone σ_i in σ.

A toric singularity is Q-Gorenstein if one can find a vector m_σ in M_Q[6] such that $\langle m_\sigma, v_\rho \rangle = 1$ for all the one dimensional generator v_ρ of σ. A toric singularity is Gorenstein if one can find a vector H in M such that $\langle H, v_\rho \rangle = 1$ for all ρ. Equivalently, one can choose a hyperplane in N such that all the ray generators σ_ρ lie on it. For a 3-fold Q-Gorenstein singularity with index r, we take the hyperplane as $z = r$, and the ray generators form a two dimensional convex polygon P. Our 3-fold Q-Gorenstein toric singularity is then specified by a index r and a two dimensional convex polygon P at $z = r$ plane.

Let's start with a 3-fold Q-Gorenstein toric singularity. The classification of 3-fold canonical and terminal singularity is given as follows:

- The cone σ of 3-fold *canonical* singularity has the following property: there is no lattice points in σ between the origin and the polygon P. In particular, toric Gorenstein singularity is canonical.

- The cone σ of 3-fold *terminal* singularity is characterized as follows: there is no lattice points in σ between the origin and the polygon P, and furthermore there is no internal lattice points for P.

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[6]M_Q denotes points in M_R with rational coordinates.
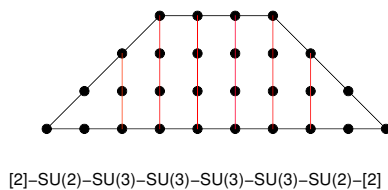
[2]–SU(2)–SU(3)–SU(3)–SU(3)–SU(3)–SU(2)–[2]

Figure 1. A toric diagram and one of its partial resolution, and we also write down a gauge theory description.

Among toric terminal singularities, the Q-factorial terminal singularity is specified by smooth point and the quotient singularity of the type C^3/μ_r, with $\mu_r = \frac{1}{r}(a, -a, 1)$, here r and a are co-prime. Smooth point is the only Q factorial Gorenstein terminal singularity. There is only one kind of non Q-factorial Gorenstein terminal singularity: the conifold singularity which is defined by the equation: $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0$.

4.1 Toric Gorenstein singularity

We focus on toric Gorenstein 3-fold singularity which is specified by a convex polygon P (it is often called toric diagram). The only terminal singularity is the smooth point whose associated convex polygon is a triangle with no internal lattice points and no boundary lattice points. We denote the number of lattice points on the boundary of P as N, and the number of internal lattice points as I.

4.1.1 Flavor symmetry

The enhancement of flavor symmetry is determined by the ADE type of one dimensional singular locus, which in the toric case, is associated with the two dimensional cone σ_i in σ. The two dimensional cone in σ is specified by boundary edges of P. For a one dimensional boundary in P, it is easy to determine the singularity type: the only toric ADE singularity is A type, and the corresponding singularity type for a boundary edge with d internal points is A_d, and the flavor symmetry has the following type

$$\prod_i A_{d_i} \times \mathrm{U}(1)^{N-3-\sum d_i}. \tag{4.4}$$

The local divisor class group has rank $N - 3 - \sum d_i$, which is equal to the number of vertices of P minus three. So the rank of the mass parameter is given by the following formula

$$f = N - 3. \tag{4.5}$$

Example. Let's see figure 1 for a toric diagram, and the flavor symmetry is

$$G = \mathrm{SU}(9) \times \mathrm{SU}(3) \times \mathrm{SU}(3) \times \mathrm{SU}(3) \times \mathrm{U}(1) \tag{4.6}$$

We also write down a gauge theory description for it. Using the instanton method reviewed in section II (see [42] and use the fact that the two SU(3) gauge groups connected with SU(2) gauge groups have CS level $k = \frac{1}{2}$), we see that field theory result and geometric result agree.
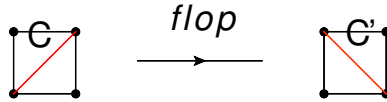
Figure 2. Two different resolutions of a lattice polygon can be related by sequence of flops shown above.

4.1.2 Partial resolution and gauge theory description

One can consider partial resolution which is formed by adding edges connecting the vertices of P. Those internal edges represent non-abelian gauge group of A_{d_i} type if there are d_i internal points on extra edges. Those internal edges will cut P into various smaller convex polygons P_i. The physical interpretation is following: one have a non-abelian gauge theory description: each P_i represents a matter system, and each added internal edge connecting two smaller polygons P_A, P_B represents a gauge group coupled to two matter systems represented by P_A and P_B. Sometimes we also have non-trivial CS term too.

For some convex polygon P, it is possible to find above type of subdivision of P such that each subsystem P_i carries no internal lattice points. Those P_i could be interpreted as free matter, and one would have a non-abelian Lagrangian description.

Often one could find more than one type of such decompositions of P, and one have different Lagrangian descriptions. It might be tempting to call these different descriptions as some kind of duality. However, we would like to avoid that because the description of non-abelian gauge theory itself does not define a SCFT.

4.1.3 Crepant resolution and Coulomb branch solution

The resolution of toric singularity can be described in following explicit way: one construct a new fan Σ from the cone σ by adding rays inside σ. The *crepant* resolution is given by the uni-modular triangulation of the lattice polygon P. Such triangulation is easily found by using the lattice points in polygon P.[7] The final toric variety is smooth as the only Q-factorial toric Gorenstein singularity is smooth. Obviously, such triangulations are not unique, but they can all be related by the *flops*, see figure 2.

Exceptional divisors. Let's choose a crepant triangulation P' of P. The exceptional divisors corresponding to the internal and boundary lattice points of P which are not vertices. The number of such divisors is a constant and is independent of choice of crepant resolution. The local divisor class group of P' is generated by the ray generators v_ρ subject to following relations

$$\sum_\rho \langle v_\rho, e_i \rangle D_\rho = 0, \qquad i = 1, 2, 3. \tag{4.7}$$

Here D_ρ is the corresponding divisor from v_ρ, and e_i is the standard basis of M and we used the bracket to denote the standard paring between N and M. So there are three

---

[7]For a two dimensional lattice polygon, its area is given by the formula $A = \frac{B}{2} + I - 1$, here B is the boundary lattice points, and I is the internal points. Uni-modular triangulation implies that all the triangles in the triangulation has area $\frac{1}{2}$. This is achieved by a triangle with no internal lattice points and no other boundary points except three vertices.
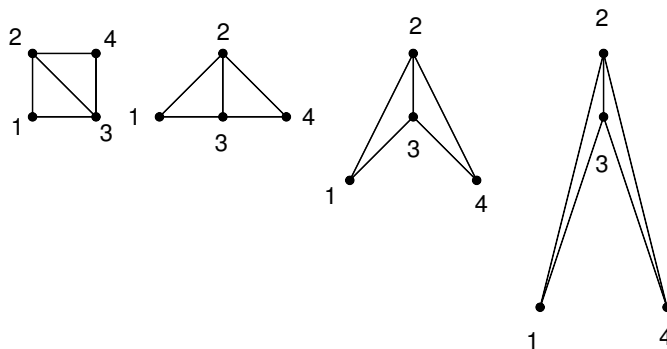
Figure 3. Quadrilaterals in a triangulation of a polygon P.

equations and the number of independent divisors are $N + I - 3$. The above relation is independent of the triangulation and only depend on polygon P. The topology of these divisors are follows: a) the divisors from internal lattice points are compact; b) the divisors from vertices of the lattice polygon are non-compact, i.e. of the form C^2; c) the divisors from other boundary lattice points are semi-compact, i.e. of the form $C \times P^1$.

Triple intersection number. Consider a quadrilateral in the triangulation whose vertices have coordinates u_i, $i = 1, \ldots, 4$. We assume that the diagonal C connects u_2 and u_3. The coordinates u_i of quadrilateral's vertices satisfy the relation:

$$u_1 + au_2 + bu_3 + u_4 = 0. \tag{4.8}$$

Since the z coordinate of u_is are all 1, we have $a + b = -2$. The only non-trivial triple intersection numbers involving the curve $C = E_2 \cdot E_3$ are

$$C \cdot E_1 = 1, \qquad C \cdot E_4 = 1, \qquad C \cdot E_2 = a, \qquad C \cdot E_3 = b. \tag{4.9}$$

We have following solutions (see figure 3):

$$
\begin{aligned}
\text{Case 1}: &\quad a = -1, &\quad b = -1 \\
\text{Case 2}: &\quad a = 0, &\quad b = -2 \\
\text{Case 3}: &\quad a = 1, &\quad b = -3 \\
&\quad \ldots \\
\text{Case } i: &\quad a = i - 2, &\quad b = -i
\end{aligned} \tag{4.10}
$$

Notice that only case 1 allows the flop. Using above formula, we can compute all the triple intersection number involving at least one compact divisors E. The self-intersection number for a compact divisor E can be computed as follows: we always have a relation between divisors

$$E + \sum D_i = 0 \;\rightarrow\; E^3 = -\sum E^2 D_i, \tag{4.11}$$

and one can get E^3 using the known triple intersection number. In fact only the divisors connected to E contribute to E^3. Other intersection numbers involving no compact divisors

can be computed using three relations between the divisors and above known intersection numbers involving at least one compact divisor.[8]

Mori cone. Mori cone is an important structure of a algebraic variety. It is described in the toric case as the following simple set:

$$\sum R_{>0}[V(\tau)],\tag{4.12}$$

here $V(\tau)$ is the complete curve corresponding to a wall of a crepant resolution P'. These walls are represented by the internal edges of P'. The generators of Mori cone can be found as follows. For each internal edge C_i which is a diagonal edge of a quadrilateral, we have the following relations for the four generators corresponding to the vertices of quadrilateral:

$$u_1^{(i)} + au_2^{(i)} + bu_3^{(i)} + u_4^{(i)} = 0\,.\tag{4.13}$$

Not all of these relations are independent. Now a curve is the generator of Mori cone if the corresponding relation can not be written as the sum of other relations with *positive* coefficients. There are a total of $N + I - 3$ generators, and the Mori cone is just R_{N+I-3}^+ once we identify those generators.

Nef cone. A divisor $D = \sum a_i D_i$ is called Nef if it satisfies the following condition

$$D \cdot C \geq 0\,,\tag{4.14}$$

for all irreducible complete curve C. The space of Nef divisors form a cone and called Nef cone. This cone is dual to the Mori cone, and is defined as

$$\left(\sum a_i D_i\right) \cdot C_k \geq 0\,.\tag{4.15}$$

Here D_i is the basis of the divisors, C_k is the generator of Mori cone. So each generator of Mori cone defines a hypersurface in the space of divisors, and this hypersurface gives a co-dimension one surface of the Nef cone.

Prepotential. Each crepant resolution and its Nef cone gives a chamber of Coulomb branch, and the prepotential in this chamber is given by

$$\mathcal{F} = \frac{1}{6}\vec{D} \cdot \vec{D} \cdot \vec{D} = \frac{1}{6}\left(\sum D_i\phi_i\right)^3.\tag{4.16}$$

Here $\vec{D}$ is a general point in Nef cone and D_i denote the basis of the divisors. To compare our result with the formula from the gauge theory, we need to use following expression from

---

[8]Those numbers are not really the usual intersection number: usually one consider the restriction of a Cartier divisor D on a complete curve C and then compute the degree of the corresponding line bundle on C which is denoted as $D \cdot C$. In our case, C is defined as the intersection of two divisors $D_i \cdot D_j$ so we have a triple intersection number. We do not have above interpretation of triple intersection number involving three non-compact divisor, and these numbers are coming from the consistent condition on the divisor relations.
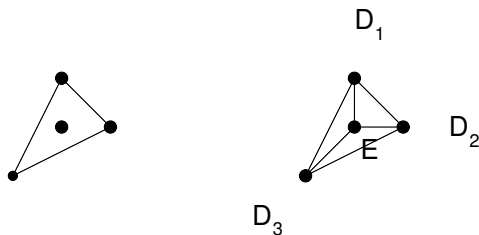
Figure 4. One toric diagram and its crepant resolution.

SU(N) gauge theory. Let's denote $a_j = \phi_j - \phi_{j-1}$ with $\phi_N = \phi_0 = 0$. Consider SU(N) SYM theory with CS term k and n_f fundamental flavors, we have (assume gauge coupling is zero):

$$
\begin{aligned}
\mathcal{F} &= \frac{1}{6}\left[\sum_{i<j}(a_i - a_j)^3 + k\sum_{i=1}^{N} a_i^3 - \frac{n_f}{2}\sum_{i=1}^{N}|a_i|^3\right] \\
&= \frac{1}{6}\left(8\sum_{j=1}^{N-1}\phi_j^3 - 3\sum_{j=1}^{N-1}(\phi_j^2\phi_{j+1} + \phi_j\phi_{j+1}^2) + 3\sum_{j=1}^{N-1}(k+N-2j-1)(\phi_j^2\phi_{j+1} - \phi_j\phi_{j+1}^2)\right. \\
&\left.\quad - \frac{n_f}{2}\sum_{i=1}^{N}|a_i|^3\right).
\end{aligned}
\tag{4.17}
$$

Flop. Different crepant resolutions are related by flops, see figure 2. Now consider a quadrilateral and the corresponding flop, we have two relations for two curves:

$$
\begin{aligned}
C: & \qquad u_1 - u_2 - u_3 + u_4 = 0 \\
C': & \qquad -u_1 + u_2 + u_3 - u_4 = 0
\end{aligned}
\tag{4.18}
$$

The constraint on the dual Nef cone is

$$
\begin{aligned}
\sum a_i D_i \cdot C \geq 0 \;\; &\rightarrow \;\; a_1 - a_2 - a_3 + a_4 \geq 0\,, \\
\sum a_i D_i \cdot C' \geq 0 \;\; &\rightarrow \;\; -a_1 + a_2 + a_3 - a_4 \geq 0\,.
\end{aligned}
\tag{4.19}
$$

So they define the same hypersurface in the space of divisors, but the two chamber is living on different sides of this hypersurface. In fact, this hypersurface is shared by two Nef cones corresponding to these two crepant resolutions related by above flop!

Example 1. Consider the toric diagram and its unique triangulation shown in figure 4. The divisor class group is generated by torus invariant divisors E, D_1, D_2, D_3 associated with the ray generators of the fan. These divisors satisfy the relations:

$$
E + D_1 + D_2 + D_3 = 0\,, \qquad D_2 - D_3 = 0\,, \qquad D_1 - D_3 = 0\,,
\tag{4.20}
$$

and we have:

$$
E = -3D_1\,, \qquad D_2 = D_3 = D_1\,.
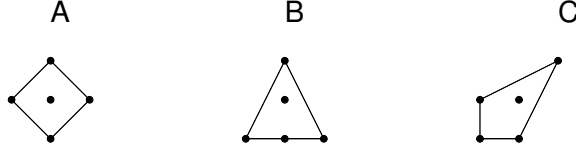\tag{4.21}
$$

Figure 5. Three toric diagrams which have pure SU(2) gauge theory description along certain locus of Coulomb branch.

The triple intersection numbers are

$$ED_1^2 = 1, \qquad E^2 D_1 = -3, \qquad E^3 = 9. \tag{4.22}$$

There are three complete curve $C_i = E \cdot D_i$, and they are equal due to the divisor relations, i.e. $C_1 = C_2 = C_3 = C$. The Mori cone is generated by C and is just the positive real line. The Nef cone is computed as follows

$$a D_1 \cdot C = a \geq 0, \; \rightarrow \; a \geq 0. \tag{4.23}$$

Now for a point in Nef cone, the prepotential is

$$\mathcal{F} = \frac{1}{6}(\phi E)^3 = \frac{3}{2}(\phi^3), \qquad -a = 3\phi. \tag{4.24}$$

The effective coupling constant is

$$\frac{1}{g^2} = \frac{1}{2}\frac{\partial^2 \mathcal{F}}{\partial^2 \phi} = \frac{9}{2}\phi. \tag{4.25}$$

Example 2. We will compute the Coulomb branch of three toric diagrams listed in figure 5.

A: consider toric diagram A in figure 5 and its unique crepant resolution shown in figure 6. The relations between the divisors are:

$$E = -2D_1 - 2D_2, \qquad D_1 = D_3, \qquad D_2 = D_4. \tag{4.26}$$

The triple intersection numbers are $E^3 = 8$, $E^2 D_1 = E^2 D_2 = -2$.

The complete curves are $C_1 = E \cdot D_1$, $C_2 = E \cdot D_2$, $C_3 = E \cdot D_3$, $C_4 = E \cdot D_4$. They have the relations $C_1 = C_3$ and $C_2 = C_4$. So the Mori cone is generated by C_1 and C_2. Let's take D_2 and D_1 as the basis of divisor class. The Nef cone is defined as

$$\begin{aligned}
\big(a[D_1] + b[D_2]\big) \cdot C_1 \geq 0 \; &\rightarrow \; a \geq 0, \\
\big(a[D_1] + b[D_2]\big) \cdot C_2 \geq 0 \; &\rightarrow \; b \geq 0.
\end{aligned} \tag{4.27}$$

The prepotential is then

$$\begin{aligned}
\mathcal{F} = \frac{1}{6}(aD_1 + bD_2)^3 &= \frac{1}{6}(E\phi - D_1 m_0)^3 = \frac{1}{6}(8\phi^3 + 6m_0\phi^2), \\
-a = 2\phi + m_0 \leq 0, \qquad &-b = 2\phi \leq 0.
\end{aligned} \tag{4.28}$$

We change the coordinate to make the contact with the gauge theory result. The effective gauge coupling is

$$g = \frac{1}{2}\frac{\partial^2 \mathcal{F}}{\partial^2 \phi} = 4\phi + m_0. \tag{4.29}$$
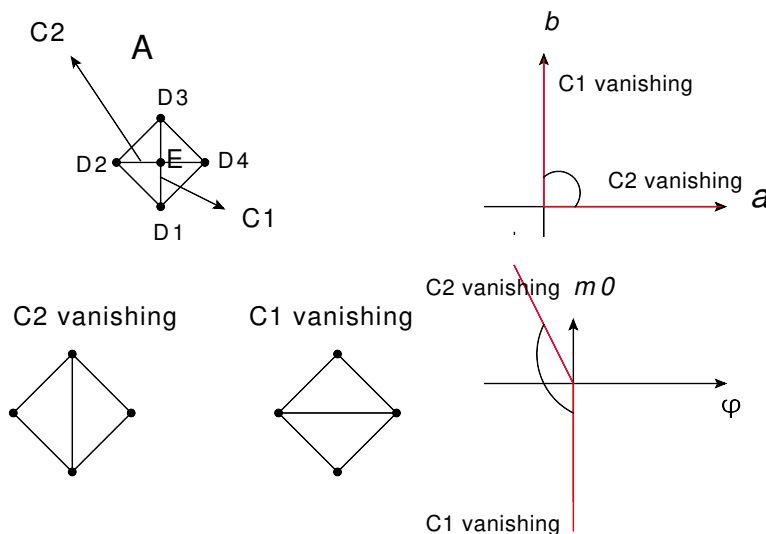
– 17 –

Figure 6. Left: the crepant resolution of toric diagram A in figure 5 and two partial resolutions where we have a gauge theory description. Right: up is the Nef cone with basis of divisors D_1 and D_2; lower is the cone using the basis E and $-D_1$.

Remark 1. It is interesting to note that the range of m_0. One SU(2) gauge theory description is valid in the range $\phi \leq 0$ and $m_0 \leq 0$ (we choose a different Weyl chamber from the usual one where m_0 and ϕ are both non-negative). In the other region $m_0 > 0$ we have a different gauge theory theory description.

Remark 2. The boundary of the Coulomb branch corresponds to the locus where there is extra massless particle. In our case, we know that it corresponds to shrink one of the cycle C_1 or C_2, and we get a pure SU(2) gauge theory description. We interpret the boundary defined by the equation 2ϕ ($a = 0$ and C_1 is vanishing) as the indication that a W boson become massless, and the boundary defined by the equation $2\phi + m_0 = 0$ ($b = 0$ and C_2 is vanishing) indicates that a instanton particle carry 2 electric charge and one instanton charge become massless.

B: next let's consider toric diagram B in figure 5 and its crepant resolution shown in figure 7. We have the relation for the divisors:

$$E = -2D_1 - 4D_2, \qquad D_3 = D_1 + 2D_2, \qquad D_4 = D_2. \tag{4.30}$$

The basis of the divisors are D_1 and D_2. We have the triple intersection number $E^3 = 8$, $ED_1^2 = -2$, $E^2D_2 = -2$. The complete curves are:

$$C_1 = E \cdot D_1, \qquad C_2 = E \cdot D_2, \qquad C_3 = E \cdot D_3 = C_1 + 2C_2. \tag{4.31}$$

The Mori cone is generated by C_1 and C_2. The Nef cone is defined as

$$\begin{aligned}
\big(a[D_1] + b[D_2]\big) \cdot C_1 &= -2a + b \geq 0, \\
\big(a[D_1] + b[D_2]\big) \cdot C_2 &= a \geq 0.
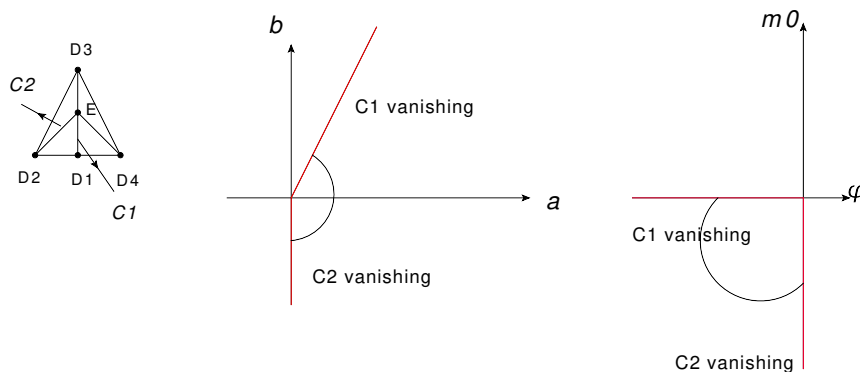\end{aligned} \tag{4.32}$$

– 18 –

Figure 7. The crepant resolution of a toric diagram B in figure 5 and its Coulomb branch.

The prepotential can be written as

$$\mathcal{F} = \frac{1}{6}\big(a[D_1] + b[D_2]\big)^3 = \frac{1}{6}(\phi E - m_0 D_2)^3 = \frac{1}{6}(8\phi^3 + 6m_0\phi^2)\,. \tag{4.33}$$

Here $-a = 2\phi$, $-b = 4\phi + m_0$. The range of coordinates is

$$\phi \leq 0\,, \qquad m_0 \leq 0\,. \tag{4.34}$$

The coupling constant is $g = \frac{1}{2}\frac{\partial^2 \mathcal{F}}{\partial^2 \phi} = 4\phi + m_0$.

Remark. The prepotential of this example takes the exact same form as the above case. However, there are two important differences. First, the range of the Coulomb branch is different. Second, we still have a massless W boson at the boundary $2\phi = 0$ ($a = 0$ with C_2 vanishing), but on the other boundary we have a particle with quantum number m_0 ($2a - b = 0$ with C_1 vanishing) which carries no electric charge and one instanton charge to become massless. We do not see a SU(2) gauge theory description on the other boundary. In the literature [30], it is argued that this SCFT and the theory studied in last case describe the same SCFT. Our analysis of Coulomb branch indicates that they are different theories.

C: next let's consider toric diagram C shown in figure 5 and its crepant resolution shown in left of figure 8. We have the following relation of the divisors

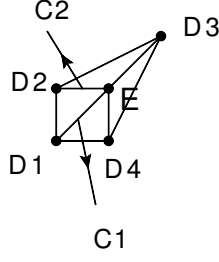$$E = -2D_1 - 3D_2\,, \qquad D_4 = D_2\,, \qquad D_3 = D_1 + D_2\,. \tag{4.35}$$

The basis of the divisor can be taken as D_1 and D_2. The triple intersection numbers are $E^3 = 8$, $E^2 D_2 = -2$, $E^2 D_1 = -1$, $E D_1^2 = -1$. The complete curves are

$$C_1 = E{\cdot}D_1\,, \qquad C_2 = E{\cdot}D_2\,, \qquad C_3 = E{\cdot}D_3 = C_1{+}C_2\,, \qquad C_4 = E{\cdot}D_4 = C_2\,. \tag{4.36}$$

C_1 and C_2 are the generators for the Mori cone. The Nef cone is defined as:

$$\begin{aligned}
\big(a[D_1] + b[D_2]\big) \cdot C_1 &= -a + b \geq 0\,, \\
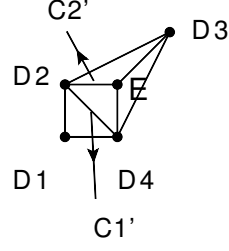\big(a[D_1] + b[D_2]\big) \cdot C_2 &= a \geq 0\,.
\end{aligned} \tag{4.37}$$

– 19 –

Figure 8. Two resolutions of toric diagram C in figure 5.

The prepotential is

$$\mathcal{F} = \frac{1}{6}(aD_1 + bD_2)^3 = \frac{1}{6}(E\phi - D_2 m_0)^3 = \frac{1}{6}(8\phi^3 + 6m_0\phi^2)\,,$$
$$-a = 2\phi\,, \qquad -b = 3\phi + m_0\,. \tag{4.38}$$

For this chamber at the boundary ($a = 0$ or $\phi = 0$), we have C_2 vanishing, and we have a SU(2) gauge theory description. At other boundary defined by $-a+b = 0$, we have another particle with BPS mass formula $\phi + m_0$ to become massless. This is clearly different from the case A and B.

For the other resolution shown in the right of figure 8, the triple intersection numbers are $E^3 = 9$, $E^2 D_2 = -3$, $ED_2^2 = 1$. We have complete curves

$$C_1' = D_4 \cdot D_2\,, \qquad C_2' = E \cdot D_2\,, \qquad C_3' = E \cdot D_3 = C_2'\,, \qquad C_4' = E \cdot D_4 = C_2'\,. \tag{4.39}$$

The Nef-cone is defined as:

$$\big(a[D_1] + b[D_2]\big) \cdot C_1' = a - b \geq 0\,,$$
$$\big(a[D_1] + b[D_2]\big) \cdot C_2' = b \geq 0\,. \tag{4.40}$$

The prepotential in this chamber is

$$\mathcal{F} = \frac{1}{6}(aD_1 + bD_2)^3 = \frac{1}{6}(E\phi - D_2 m_0)^3 = \frac{1}{6}(9\phi^3 + 9m_0\phi^2 + 3m_0^2\phi)$$
$$-a = 2\phi\,, \qquad -b = 3\phi + m_0 \tag{4.41}$$

One boundary of this chamber is defined as $a - b = 0$ which gives the same BPS particle from the first chamber. There is another boundary defined by $b = 0$ (corresponds to C_2' vanishing). The full Coulomb branch is shown in figure 9.

Example 3. Consider four toric diagrams and their resolutions shown in figure 10, and let's consider only compact divisors. The prepotential is

$$\mathcal{F} = \frac{1}{6}(\phi_1^3 E_1^3 + 3\phi_1^2\phi_2 E_1^2 E_2 + 3\phi_1\phi_2^2 E_1 E_2^2 + \phi_2^3 E_2^3)\,. \tag{4.42}$$

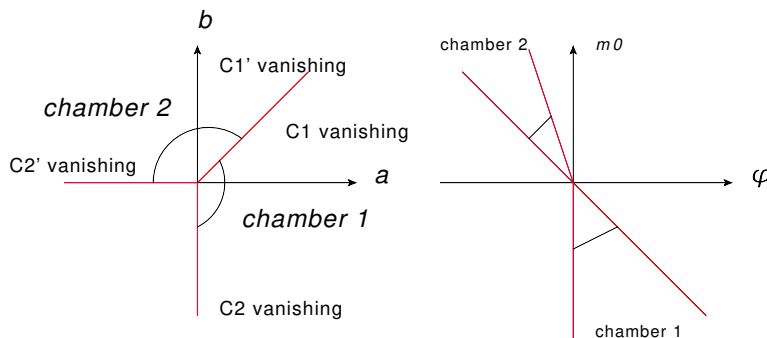The data for intersection numbers is shown in table 2.

Figure 9. The Nef fan for the toric diagram C in figure 5, see figure 8 for the resolutions and notation for the curves.
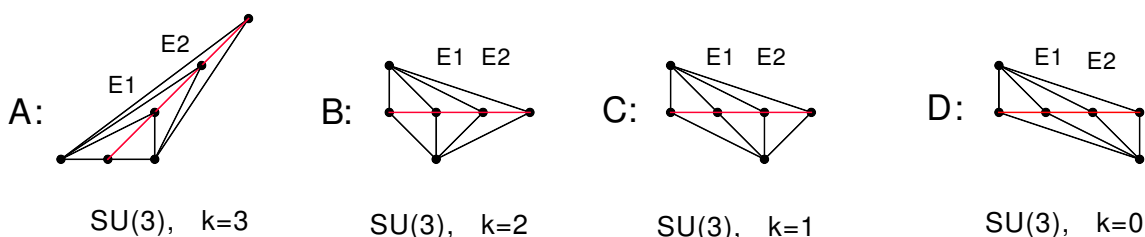

Figure 10. Four toric diagrams with their partial crepant resolutions. We also write down the gauge theory descriptions.

	E_1^3	E_2^3	$E_1^2 E_2$	$E_1 E_2^2$
A	8	8	-4	2
B	8	8	-3	1
C	8	8	-2	0
D	8	8	-1	-1

Table 2. The triple intersection number of the compact divisors for the toric diagrams in figure 10.

For the prepotential for SU(3) gauge theory with $n_f = 0$ and CS level k, we get

$$\mathcal{F} = 8\phi_1^3 + 8\phi_2^3 + 3(k-1)\phi_1^2\phi_2 + 3(-k-1)\phi_1\phi_2^2 \,. \tag{4.43}$$

If $n_f = 0$, the prepotential is invariant under the change $k \to -k$, $\phi_1 \to \phi_2$. Comparing above formula and the formula (4.42), we can write down the CS level for the corresponding toric diagrams in figure 10. We leave the detailed study of the Coulomb branch to the interested reader.
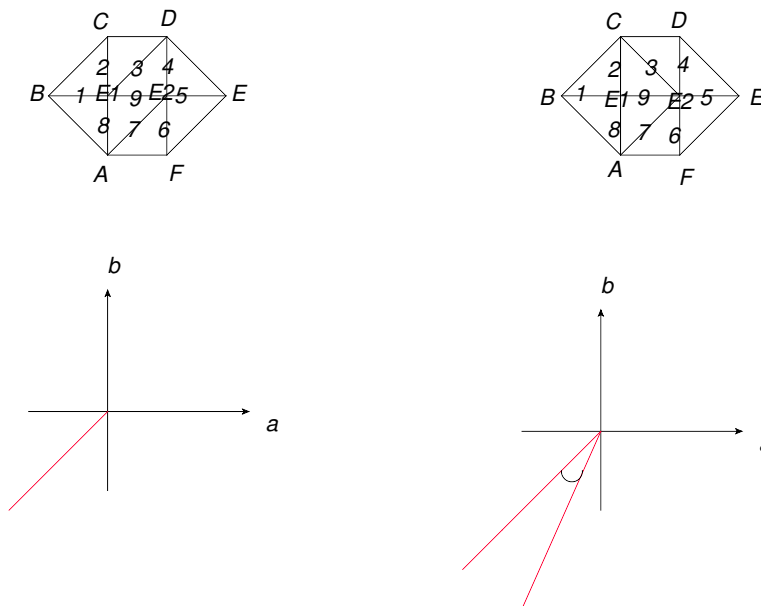
Figure 11. Two resolutions of a toric diagram and its Nef-cone.

Example 4. Let's consider the toric diagram and one of its crepant resolution shown in left of figure 11. The relations[9] for the complete curves are

$$
\begin{aligned}
C_1: && -2E_1 + A + C &= 0\,, & C_5: && -2E_2 + D + F &= 0\,, \\
\bullet\, C_2: && -C - E_1 + B + D &= 0\,, & \bullet\, C_6: && A + E - E_2 - F &= 0\,, \\
\bullet\, C_3: && E_2 + C - D - E_1 &= 0\,, & \bullet\, C_7: && E_1 + F - A - E_2 &= 0\,, \\
C_4: && -2E_2 + E_1 + E &= 0\,, & C_8: && -2E_1 + B + E_2 &= 0\,, \\
\bullet\, C_9: && A + D - E_1 - E_2 &= 0\,. &&&&
\end{aligned}
\tag{4.44}
$$

We take C_2, C_3, C_6, C_7, C_9 as the generator for the Mori cones, and other curves are expressed as:

$$
C_1 = C_3 + C_9\,, \qquad C_4 = C_6 + C_7\,, \qquad C_5 = C_7 + C_9\,, \qquad C_8 = C_2 + C_3\,.
\tag{4.45}
$$

We get a $\mathrm{SU}(2) - \mathrm{SU}(2)$ quiver gauge theory if C_3, C_7, C_9 are vanishing. Now let's compute the Nef cone (let's take the mass parameters to be zero), we have

$$
(aE_1 + bE_2) \cdot C_i \geq 0\,, \qquad i = 2, 3, 6, 7, 9
\tag{4.46}
$$

and we get the constraints:

$$
C_2: \ -a \geq 0\,, \quad C_3: \ -a + b \geq 0\,, \quad C_6: \ -b \geq 0\,, \quad C_7: \ a - b \geq 0\,, \quad C_9: \ -a - b \geq 0\,.
\tag{4.47}
$$

---

[9]Here the notation of divisors indicate their coordinates, one should not confuse it with the relations between divisors.

So this chamber is defined by the equations $a = b$ and $a \leq 0$, $b \leq 0$. It is interesting to note that not all of the Coulomb branch of the $SU(2) - SU(2)$ gauge theory can be probed! The prepotential in this chamber is

$$\mathcal{F} = \frac{1}{6}(aE_1 + bE_2)^3 = \frac{1}{6}(7a^3 + 7b^3 - 3a^2 b - 3ab^2). \tag{4.48}$$

Next let's compute the Mori cone and Nef cone for the resolution shown on the right of figure 11. The relation for the complete curves are

$$
\begin{aligned}
C_1 : &\quad -2E_1 + A + C = 0, &\qquad C_5 : &\quad -2E_2 + D + F = 0, \\
\bullet\, C_2 : &\quad -2E_1 + B + E_2 = 0, &\qquad \bullet\, C_6 : &\quad A + E - E_2 - F = 0, \\
\bullet\, C_3 : &\quad -E_2 - C + D + E_1 = 0, &\qquad \bullet\, C_7 : &\quad E_1 + F - A - E_2 = 0, \\
C_4 : &\quad -2E_2 + E_1 + E = 0, &\qquad C_8 : &\quad -2E_1 + B + E_2 = 0, \\
\bullet\, C_9 : &\quad -2E - 1 + A + C = 0.
\end{aligned}
\tag{4.49}
$$

The generators for the Mori cone are C_2, C_3, C_6, C_7, C_9, and the other curves are expressed as

$$C_1 = C_9, \qquad C_8 = C_2, \qquad C_4 = C_6 + C_7, \qquad C_5 = C_3 + C_9. \tag{4.50}$$

One also has the $SU(2) - SU(2)$ quiver gauge theory description when C_3, C_7, C_9 vanish. The Nef cone is again defined as

$$(aE_1 + bE_2) \cdot C_i \geq 0, \qquad i = 2, 3, 6, 7, 9 \tag{4.51}$$

and we have the constraints:

$$C_2 : \ -2a + b \geq 0, \quad C_3 : \ a - b \geq 0, \quad C_6 : \ -b \geq 0, \quad C_7 : \ a - b \geq 0, \quad C_9 : \ -2a \geq 0. \tag{4.52}$$

The corresponding cone is shown in figure 11, and one of the boundary corresponds to vanishing of C_2 (one find a $SU(2)$ gauge group here), while the other boundary corresponds to vanishing of C_3 and C_7. The prepotential in this chamber is

$$\mathcal{F} = \frac{1}{6}(aE_1 + bE_2)^3 = 8a^3 + 6b^3 - 6a^2 b. \tag{4.53}$$

Remark. It was argued in [9, 10] that the UV limit of a quiver gauge theory does not define a SCFT since the prepotential can not be convex in the whole Coulomb branch of gauge theory which is identified with the Weyl Chamber. Our computation above shows that actually the range of Coulomb branch is not the naive Weyl chamber of the gauge theory.

4.1.4 Relation to (p, q) web and gauge theory construction

Given a crepant resolution of a toric Gorenstein singularity, one can construct a dual diagram: put a trivalent vertex inside a triangle with edges perpendicular to the boundaries

of triangle, and one get a (p,q) web by connecting those trivalent vertices. So a (p,q) web is completely equivalent to a toric Gorenstein singularity.

As we see from above example, given a crepant resolution and it is possible to find a non-abelian gauge theory at certain sub-locus of the Coulomb branch. However, the SCFT is not completely characterized by the non-abelian gauge theory description as SCFT has other massless particles from the boundary of the Coulomb branch which can not be determined by the gauge theory.

4.1.5 Classification: lower rank theory

For toric Gorenstein singularity, the classification is reduced to the classification of two dimensional convex lattice polygone up to following integral unimodular affine transformation on the plane:

$$
\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \tag{4.54}
$$

here (a,b,c,d,e,f) are integers and $ad - bc = 1$. It is easy to check the number of boundary points, internal points and the area of the polygon is not changed under above transformation. There are also some interesting facts about the convex lattice polygon, and here we collect two of the useful ones:

1. The area of a lattice polygon is given by the Pick's formula

$$
A = \frac{B}{2} + I - 1. \tag{4.55}
$$

Here B is the number of boundary points and I is the number of interior points.

2. The boundary points are constrained by the interior points, i.e.

$$
B \leq 2I + 6, \qquad I > 1, \tag{4.56}
$$

and $B \leq 2I + 7$ for $I = 1$.

Rank zero theory. The rank zero theory is classified by the convex polygon without any interior point. They were classified in [61]. By TRIANG(p,h) we denote the triangle whose vertices are $(0,0)$, $(p,0)$, and $(0,h)$. By TRAP(p,q,h) we denote the trapezoid whose vertices are $(0,0)$, $(p,0)$, $(0,h)$ and (q,h). Then we have the following classification: if K is a convex lattice polygon with $g = 0$ (g denotes the number of interior point), then K is lattice equivalent to one of the following polygons

1. TRIANG$(p,1)$, where p is any positive integer. The flavor symmetry is SU(p). We interpret this theory as "zero" flavor of SU(p) group.

2. TRIANG$(2,2)$. The flavor symmetry is SU$(2) \times$ SU$(2) \times$ SU(2). We interpret this theory as the tri-fundamental of SU(2) groups.

3. TRAP$(p,q,1)$, where p and q are any positive integers. The flavor symmetry is SU$(p) \times$ SU$(q) \times$ U(1). This is the bi-fundamental for SU$(p) \times$ SU(q).
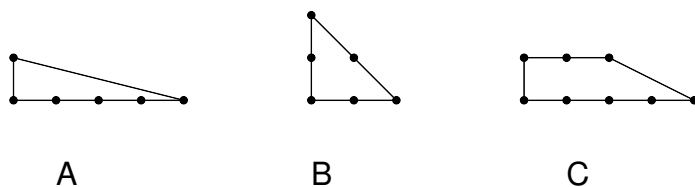
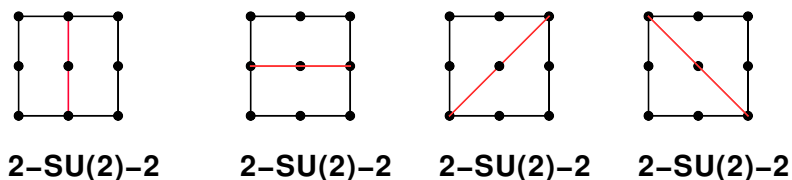**Figure 12**. Toric diagram of rank zero theory.

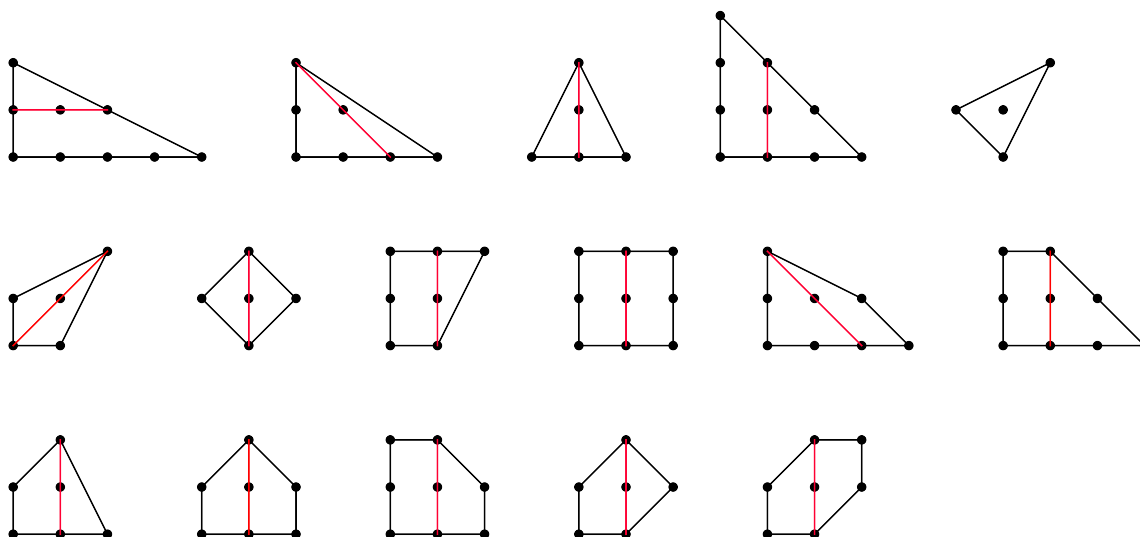**Figure 13**. Non-abelian gauge theory descriptions for a rank one toric diagram.

Figure 14. Toric diagram of rank one theory. The red lines indicate the partial resolution which give a gauge theory description.

See figure 12 for some examples. With the field theory interpretation of these rank one zero toric diagram, one can easily write down the gauge theory descriptions from the partial resolution of the general toric Gorenstein singularity.

Example. See figure 13 for an example, and we list four gauge theory descriptions.

Rank one theory. The rank one theory is classified by the convex polygon with only one interior point, see figure 14. We also show the partial resolution and the interested reader can write down the gauge theory description.

Rank two theory. The rank two theory is classified by the convex polygon with two interior points [62], see figure 15 and figure 16. We also write down one partial resolutions for them.
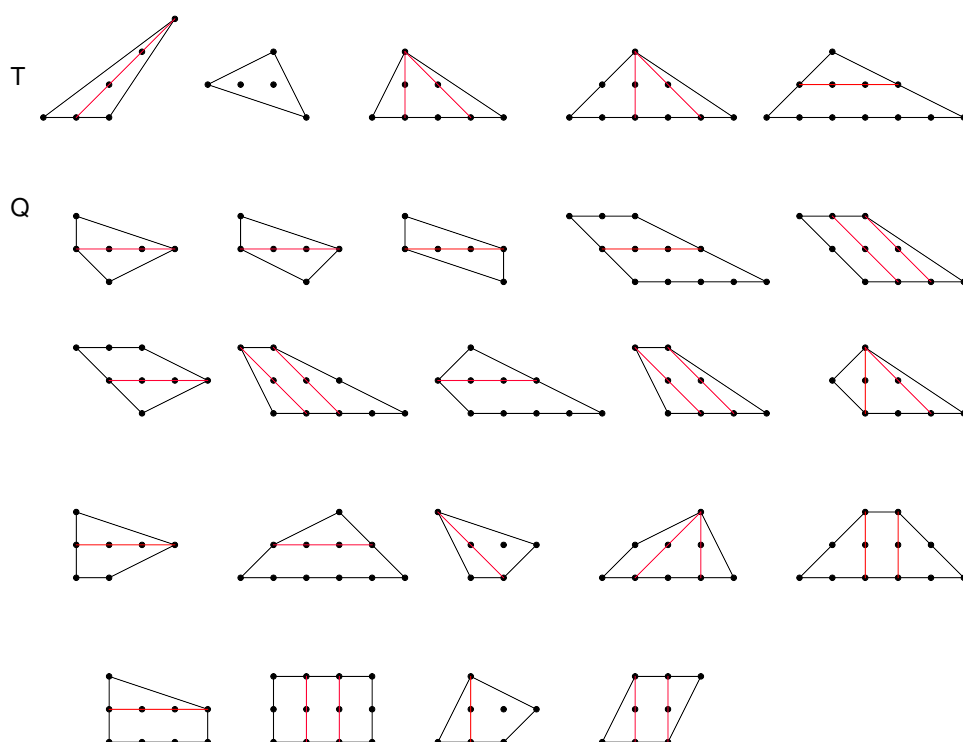
Figure 15. Part A of toric diagram of rank two theory. The red lines indicate the partial resolution which give a gauge theory description.
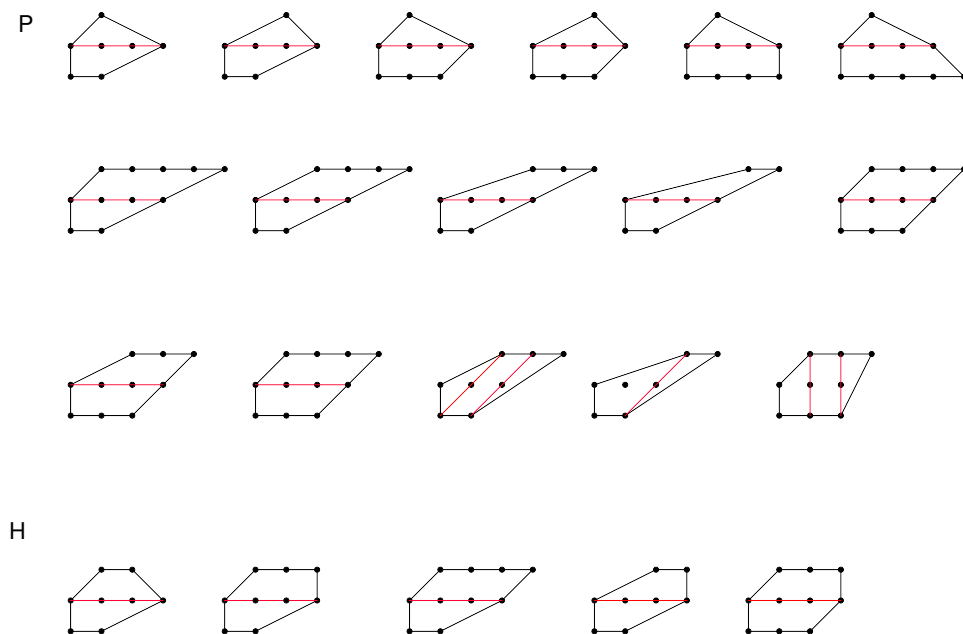

Figure 16. Part B of toric diagram of rank two theory. The red lines indicate the partial resolution which will give a gauge theory description.

4.2 Toric Q-Gorenstein singularity

One can also consider Q-Gorenstein toric singularity, and the computation of Mori and Nef cones are similar. The major interesting new feature is that the generic point of Coulomb branch has an interacting part described by the Q-factorial terminal toric singularity. It would be interesting to further study the theories defined by those singularities.

5 Quotient singularity

Another important class of 3-fold canonical singularity is quotient singularity C^3/G where G is a finite subgroup of $SL(3, C)$. All such finite subgroups have been listed in [63]. If G is abelian, such singularity is also toric, and we can use the toric method to study them. For more general class of singularities, We would like to point out some important properties associated with such singularities:

- *Flavor symmetry.* Let $G \in GL(n, C)$ be a small subgroup, let $S = \{z \in C^3 : g(z) = z\}$. Then the singular locus of V_G is S/G. Since we are interested in finite subgroup of $SL(3, C)$, the singular locus is at least co-dimension two. The singularity over such one dimensional singular locus is two dimensional ADE singularity, and they give the corresponding non-abelian flavor symmetries.

  In fact, three dimensional quotient singularity is isolated if and only if G is an abelian subgroup and 1 is not an eigenvalue of g for every nontrivial element of g in G.

- *Crepant resolution.* There exists crepant resolution for the quotient singularity [64], and the number of crepant divisors are related to the representation theory of finite group G.

6 Hypersurface singularity

A third class of 3-fold canonical singularity can be defined by a single equation and we call them hypersurface singularity. Let's consider a hypersurface singularity $f : C^4 \to C$, and further impose the condition that f is isolated, i.e. equations $f = 0$ and $\frac{\partial f}{\partial z_i} = 0, i = 0, \ldots, 3$ have a unique solution at the origin. We also impose the condition that f has a C^* action

$$f(\lambda^{q_i} z_i) = f(z_i), \qquad q_i > 0. \tag{6.1}$$

The rational condition imposes the constraint on q_i:

$$\sum q_i > 1. \tag{6.2}$$

Since hypersurface singularity is Gorenstein, so the above rational condition also implies that it is canonical! The partial crepant resolution for these singularities can be found using the blow-up method. We'd like to compute the local divisor class group and the number of crepant divisors.

We could consider more general isolated hypersurface singularity. Let's consider an isolated hypersurface singularity $f : C^{n+1} \to C$. Isolated singularity implies that equations

$f = 0$ and $\frac{\partial f}{\partial z_i} = 0$, $i = 1, \ldots, n+1$ have a unique solution at the origin. We may represent such an isolated singularity by a polynomial

$$f = \sum_{\nu \in N^{n+1}} a_\nu z^\nu. \tag{6.3}$$

We set

$$\begin{aligned}
&\text{supp}\, f = \{\nu \in N^{n+1} | a_\nu \neq 0\} \\
&\Gamma_+(f) : \text{convex hull of } \cup_{\nu \in \text{supp}\, f} (\nu + R_+^{n+1}) \\
&\Gamma(f) : \text{union of compact faces of } \Gamma_+(f)
\end{aligned} \tag{6.4}$$

Let σ be a face of $\Gamma(f)$, set $f_\sigma = \sum_{\nu \in \sigma} a_\nu x^\nu$. A polynomial f is called *Newton non-degenerate* if the critical points of f_σ do not consist of points in C^{*n+1} (points with all coordinates non-zero) for all the faces in $\Gamma(f)$. We may further assume that f is *convenient* in the sense that its support intersects with the coordinate axis, and this assumption does not change any generality.

Given a *Newton non-degenerate* singularity f, one can define a Newton order on the polynomial ring $C[z_1, \ldots, z_{n+1}]$. For ith n dimensional face σ_i of $\Gamma(f)$, one can define a C^* action l_i such that $l_i(f_{\sigma_i}) = 1$. For a monomial $\nu = \sum_{j=1}^{n+1} z_j^{m_j}$, we define its weight as

$$\alpha(\nu) = \min_i \left(l_i(\nu) \right). \tag{6.5}$$

We notice that $w(\nu) = l_i(\nu)$ if ν is in cone $(0, \sigma_i)$. For a polynomial g in $C[z_1, \ldots, z_{n+1}]$, we define

$$\alpha(g) = \min\{w(\nu) | \nu \in \text{supp}\, g\} . \tag{6.6}$$

For such an singularity, we can define two algebras called Milnor algebra J_f and Tjurina algebra T_f:

$$\begin{aligned}
J_f &= \frac{C[z_1, \ldots, z_{n+1}]}{\left\{ \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_{n+1}} \right\}} , \\
T_f &= \frac{C[z_1, \ldots, z_{n+1}]}{\left\{ f, \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_{n+1}} \right\}} .
\end{aligned} \tag{6.7}$$

These two algebras are finite dimensional since f defines an isolated singularity. $\mu = \dim(J_f)$ is called Milnor number and $\tau = \dim(T_f)$ is called Tjurina number. Obviously $\mu \geq \tau$, and $\mu = \tau$ if and only if the singularity is *quasi-homogeneous* (or semi-quasihomogeneous).

Example. Given an isolated singularity $f = x^4 + y^7 + x^2 y^3$, the newton polyheron is shown in figure 17. The Milnor number and Tjurina number are $\mu = 16$, $\tau = 14$. The C^* actions from f_1 and f_2 are $l_1(x) = 1/4$, $l_1(y) = 1/6$ and $l_1(x) = \frac{2}{7}$, $l_1(y) = \frac{1}{7}$. For a monomial xy, we have $l_1(xy) = \frac{3}{7}$, $l_2(xy) = \frac{5}{12}$, so $\alpha(xy) = \min\left(\frac{3}{7}, \frac{5}{12}\right) = \frac{5}{12}$.

Let's focus on 3 dimensional singularity from now on, i.e. $n = 3$. Given a Newton non-degenerate singularity, one can define a singularity spectrum $S(f)$ using the Newton
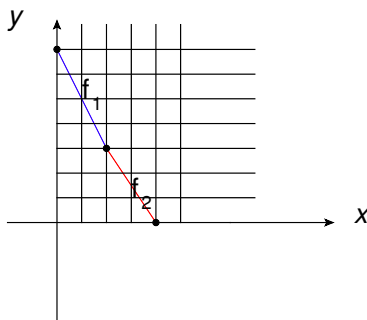
Figure 17. Newton polyhedron for singularity $f = x^4 + y^7 + x^2y^3$.

filtration defined above. For the quasi-homogeneous singularity, the singularity spectrum can be found easily. Let's take a monomial basis ϕ_i, $i = 1, \ldots, \mu$ of the Jacobian algebra J_f, the spectrum is given by the following formula

$$S(f) = \alpha(\phi_i) + \sum q_i - 1, \qquad \phi_i \in J_f \tag{6.8}$$

We can denote the spectrum $S(f)$ as an ordered set $(\alpha_1, \ldots, \alpha_\mu)$, and rational condition implies $\alpha_1 > 0$, which implies $\sum q_i - 1 > 0$.[10]

For general Newton non-degenerate singularity, it is possible also to find a regular monomial basis B of J_f such that the singularity spectrum is given by

$$S(f) = \alpha(m + \mathbf{1}) - 1, \qquad m \in B \tag{6.9}$$

Here m denotes the exponent of the monomial basis, and $\mathbf{1}$ denotes the vector $(1, 1, 1, 1)$. Rational condition implies that $\alpha_1 > 1$. The divisor class group $\rho(X)$ can be expressed [65] as follows

$$\rho(X) = \{\#\alpha_i = 1, \ \alpha_i \in S(f)\}. \tag{6.10}$$

namely we count the number of ones in singularity spectrum $S(f)$.

Remark. These 3d canonical quasi-homogeneous singularity has been used in [1] to study four dimensional $\mathcal{N} = 2$ SCFT. In that context, the weights with value 1 in singularity spectrum give the mass parameters.

Let's define a crepant weighting as the weights $w(\mathbf{1}) = w(f) + 1$ (here the weighting has positive integral weights on coordinates). We denote all such weighting as $W(f)$. The number of crepant divisors are found as [65]:

$$c(X) = \sum_{\alpha \in W(f), \dim \Gamma_\alpha = 1} \text{length} \, \Gamma_\alpha + \#\{\alpha \in W(f) : \dim \Gamma_\alpha \geq 2\}, \tag{6.11}$$

where Γ_α denotes the face of $\Gamma(f)$ corresponding to α.

More generally, one could consider isolated complete intersection canonical singularity which is defined by two polynomials (f_1, f_2). Those complete intersection singularities with C^* action has been classified in [5]. The divisor class group for such quasi-homogeneous singularity has been computed in [66].

---

[10]Since 1 is always the generator of J_f with minimal weight, and we have $\alpha_1 = \sum q_i - 1$.

7 Conclusion

We argue that every 3-fold canonical singularity defines a five dimensional $\mathcal{N} = 1$ SCFT. We can read off the non-abelian flavor symmetry from the singularity structure, and the Coulomb branch is described by the crepant resolutions of the singularity. For each crepant resolution, one can compute Nef cone which describes a chamber of Coulomb branch. The faces of Nef cone correspond to the locus where an extra massless particle appears. Different resolutions are related by flops, and these Nef cones form a fan which is claimed to be the full Coulomb branch.

We have given a description for the Coulomb branch of several simple theories defined by toric singularity. Detailed computations for prepotential and Coulomb branch structure for other interesting toric theories will appear elsewhere. The non-toric example would be also interesting to study too, i.e. complex cone over Fano orbifolds should be interesting to study.

It would be interesting to study other properties of these theories such as superconformal index and Seiberg-Witten curve for 5d theory on circle.

References

[1] D. Xie and S.-T. Yau, *4d $\mathcal{N} = 2$ SCFT and singularity theory. Part I: Classification*, [arXiv:1510.01324](https://arxiv.org/abs/1510.01324) [INSPIRE].

[2] P.C. Argyres and M.R. Douglas, *New phenomena in* SU(3) *supersymmetric gauge theory*, [*Nucl. Phys.* **B 448** (1995) 93](https://doi.org/) [hep-th/9505062] [INSPIRE].

[3] P.C. Argyres, M.R. Plesser, N. Seiberg and E. Witten, *New N = 2 superconformal field theories in four dimensions*, [*Nucl. Phys.* **B 461** (1996) 71](https://doi.org/) [hep-th/9511154] [INSPIRE].

[4] S. Gukov, C. Vafa and E. Witten, *CFT's from Calabi-Yau four-folds*, [*Nucl. Phys.* **B 584** (2000) 69](https://doi.org/) [*Erratum ibid.* **B 608** (2001) 477] [hep-th/9906070] [INSPIRE].

[5] B. Chen, D. Xie, S.-T. Yau, S.S.T. Yau and H. Zuo, *4d $\mathcal{N} = 2$ SCFT and singularity theory. Part II: Complete intersection*, [*Adv. Theor. Math. Phys.* **21** (2017) 121](https://doi.org/) [arXiv:1604.07843] [INSPIRE].

[6] Y. Wang, D. Xie, S.S.T. Yau and S.-T. Yau, *4d $\mathcal{N} = 2$ SCFT from complete intersection singularity*, arXiv:1606.06306 [InSPIRE].

[7] N. Seiberg, *Five-dimensional SUSY field theories, nontrivial fixed points and string dynamics*, *Phys. Lett.* **B 388** (1996) 753 [hep-th/9608111] [InSPIRE].

[8] C. Cordova, T.T. Dumitrescu and K. Intriligator, *Deformations of superconformal theories*, *JHEP* **11** (2016) 135 [arXiv:1602.01217] [InSPIRE].

[9] D.R. Morrison and N. Seiberg, *Extremal transitions and five-dimensional supersymmetric field theories*, *Nucl. Phys.* **B 483** (1997) 229 [hep-th/9609070] [InSPIRE].

[10] K.A. Intriligator, D.R. Morrison and N. Seiberg, *Five-dimensional supersymmetric gauge theories and degenerations of Calabi-Yau spaces*, *Nucl. Phys.* **B 497** (1997) 56 [hep-th/9702198] [InSPIRE].

[11] C.-L. Wang, *On the incompleteness of the Weil-Petersson metric along degenerations of Calabi-Yau manifolds*, *Math. Res. Lett.* **4** (1997) 157.

[12] E. Witten, *Some comments on string dynamics*, in Proceedings of *Future Perspectives in String Theory (Strings'95)*, Los Angeles U.S.A., 13–18 Mar 1995, pp. 501–523 [hep-th/9507121] [InSPIRE].

[13] M. Buican, J. Hayling and C. Papageorgakis, *Aspects of superconformal multiplets in $D > 4$*, *JHEP* **11** (2016) 091 [arXiv:1606.00810] [InSPIRE].

[14] E. Witten, *Phase transitions in M-theory and F-theory*, *Nucl. Phys.* **B 471** (1996) 195 [hep-th/9603150] [InSPIRE].

[15] M.R. Douglas, S.H. Katz and C. Vafa, *Small instantons, Del Pezzo surfaces and type-I-prime theory*, *Nucl. Phys.* **B 497** (1997) 155 [hep-th/9609071] [InSPIRE].

[16] N.C. Leung and C. Vafa, *Branes and toric geometry*, *Adv. Theor. Math. Phys.* **2** (1998) 91 [hep-th/9711013] [InSPIRE].

[17] S. Katz, P. Mayr and C. Vafa, *Mirror symmetry and exact solution of 4D N = 2 gauge theories I*, *Adv. Theor. Math. Phys.* **1** (1998) 53 [hep-th/9706110] [InSPIRE].

[18] O. Aharony, A. Hanany and B. Kol, *Webs of (p, q) 5-branes, five dimensional field theories and grid diagrams*, *JHEP* **01** (1998) 002 [hep-th/9710116] [InSPIRE].

[19] A. Iqbal and C. Vafa, *BPS degeneracies and superconformal index in diverse dimensions*, *Phys. Rev.* **D 90** (2014) 105031 [arXiv:1210.3605] [InSPIRE].

[20] H.-C. Kim, J. Kim and S. Kim, *Instantons on the 5-sphere and M5-branes*, arXiv:1211.0144 [InSPIRE].

[21] B. Assel, J. Estes and M. Yamazaki, *Wilson loops in 5d $\mathcal{N} = 1$ SCFTs and AdS/CFT*, *Ann. Henri Poincaré* **15** (2014) 589 [arXiv:1212.1202] [InSPIRE].

[22] D. Rodríguez-Gómez and G. Zafrir, *On the 5d instanton index as a Hilbert series*, *Nucl. Phys.* **B 878** (2014) 1 [arXiv:1305.5684] [InSPIRE].

[23] O. Bergman, D. Rodríguez-Gómez and G. Zafrir, *5d superconformal indices at large N and holography*, *JHEP* **08** (2013) 081 [arXiv:1305.6870] [InSPIRE].

[24] H.-C. Kim, S. Kim, S.-S. Kim and K. Lee, *The general M5-brane superconformal index*, arXiv:1307.7660 [InSPIRE].

[25] L. Bao, V. Mitev, E. Pomoni, M. Taki and F. Yagi, *Non-Lagrangian theories from brane junctions*, *JHEP* **01** (2014) 175 [arXiv:1310.3841] [ɪɴSPIRE].

[26] O. Bergman, D. Rodríguez-Gómez and G. Zafrir, *Discrete θ and the 5d superconformal index*, *JHEP* **01** (2014) 079 [arXiv:1310.2150] [ɪɴSPIRE].

[27] H. Hayashi, H.-C. Kim and T. Nishinaka, *Topological strings and 5d T_N partition functions*, *JHEP* **06** (2014) 014 [arXiv:1310.3854] [ɪɴSPIRE].

[28] M. Taki, *Notes on enhancement of flavor symmetry and 5d superconformal index*, arXiv:1310.7509 [ɪɴSPIRE].

[29] O. Bergman, D. Rodríguez-Gómez and G. Zafrir, *5-brane webs, symmetry enhancement and duality in 5d supersymmetric gauge theory*, *JHEP* **03** (2014) 112 [arXiv:1311.4199] [ɪɴSPIRE].

[30] M. Taki, *Seiberg duality, 5d SCFTs and Nekrasov partition functions*, arXiv:1401.7200 [ɪɴSPIRE].

[31] C. Hwang, J. Kim, S. Kim and J. Park, *General instanton counting and 5d SCFT*, *JHEP* **07** (2015) 063 [*Addendum ibid.* **04** (2016) 094] [arXiv:1406.6793] [ɪɴSPIRE].

[32] G. Zafrir, *Duality and enhancement of symmetry in 5d gauge theories*, *JHEP* **12** (2014) 116 [arXiv:1408.4040] [ɪɴSPIRE].

[33] M. Esole, S.-H. Shao and S.-T. Yau, *Singularities and gauge theory phases*, *Adv. Theor. Math. Phys.* **19** (2015) 1183 [arXiv:1402.6331] [ɪɴSPIRE].

[34] M. Esole, S.-H. Shao and S.-T. Yau, *Singularities and gauge theory phases II*, *Adv. Theor. Math. Phys.* **20** (2016) 683 [arXiv:1407.1867] [ɪɴSPIRE].

[35] H. Hayashi and G. Zoccarato, *Exact partition functions of Higgsed 5d T_N theories*, *JHEP* **01** (2015) 093 [arXiv:1409.0571] [ɪɴSPIRE].

[36] O. Bergman and G. Zafrir, *Lifting 4d dualities to 5d*, *JHEP* **04** (2015) 141 [arXiv:1410.2806] [ɪɴSPIRE].

[37] V. Mitev, E. Pomoni, M. Taki and F. Yagi, *Fiber-base duality and global symmetry enhancement*, *JHEP* **04** (2015) 052 [arXiv:1411.2450] [ɪɴSPIRE].

[38] D. Gaiotto and H.-C. Kim, *Surface defects and instanton partition functions*, *JHEP* **10** (2016) 012 [arXiv:1412.2781] [ɪɴSPIRE].

[39] N. Lambert, C. Papageorgakis and M. Schmidt-Sommerfeld, *Instanton operators in five-dimensional gauge theories*, *JHEP* **03** (2015) 019 [arXiv:1412.2789] [ɪɴSPIRE].

[40] Y. Tachikawa, *Instanton operators and symmetry enhancement in 5d supersymmetric gauge theories*, *Prog. Theor. Exp. Phys.* **2015** (2015) 043B06 [arXiv:1501.01031] [ɪɴSPIRE].

[41] G. Zafrir, *Instanton operators and symmetry enhancement in 5d supersymmetric* USp*,* SO *and exceptional gauge theories*, *JHEP* **07** (2015) 087 [arXiv:1503.08136] [ɪɴSPIRE].

[42] K. Yonekura, *Instanton operators and symmetry enhancement in 5d supersymmetric quiver gauge theories*, *JHEP* **07** (2015) 167 [arXiv:1505.04743] [ɪɴSPIRE].

[43] H. Hayashi and G. Zoccarato, *Topological vertex for Higgsed 5d T_N theories*, *JHEP* **09** (2015) 023 [arXiv:1505.00260] [ɪɴSPIRE].

[44] S.-S. Kim, M. Taki and F. Yagi, *Tao probing the end of the world*, *Prog. Theor. Exp. Phys.* **2015** (2015) 083B02 [arXiv:1504.03672] [ɪɴSPIRE].

[45] G. Zafir, *Brane webs and O5-planes*, *JHEP* **03** (2016) 109 [arXiv:1512.08114] [INSPIRE].

[46] O. Bergman and G. Zafir, *5d fixed points from brane webs and O7-planes*, *JHEP* **12** (2015) 163 [arXiv:1507.03860] [INSPIRE].

[47] D. Gaiotto and H.-C. Kim, *Duality walls and defects in 5d $\mathcal{N} = 1$ theories*, *JHEP* **01** (2017) 019 [arXiv:1506.03871] [INSPIRE].

[48] O. Bergman and D. Rodríguez-Gómez, *A note on instanton operators, instanton particles and supersymmetry*, *JHEP* **05** (2016) 068 [arXiv:1601.00752] [INSPIRE].

[49] J. Qiu and M. Zabzine, *Review of localization for 5d supersymmetric gauge theories*, arXiv:1608.02966 [INSPIRE].

[50] C.-M. Chang, O. Ganor and J. Oh, *An index for ray operators in 5d E_n SCFTs*, *JHEP* **02** (2017) 018 [arXiv:1608.06284] [INSPIRE].

[51] E. D'Hoker, M. Gutperle and C.F. Uhlemann, *Holographic duals for five-dimensional superconformal quantum field theories*, *Phys. Rev. Lett.* **118** (2017) 101601 [arXiv:1611.09411] [INSPIRE].

[52] E. D'Hoker, M. Gutperle and C.F. Uhlemann, *Warped $AdS_6 \times S^2$ in type IIB supergravity II: global solutions and five-brane webs*, *JHEP* **05** (2017) 131 [arXiv:1703.08186] [INSPIRE].

[53] M. Esole, P. Jefferson and M.J. Kang, *Euler characteristics of crepant resolutions of Weierstrass models*, arXiv:1703.00905 [INSPIRE].

[54] H. Hayashi and K. Ohmori, *5d/6d DE instantons from trivalent gluing of web diagrams*, arXiv:1702.07263 [INSPIRE].

[55] M. Reid, *Canonical 3-folds*, WMI Preprint, University of Warwick (1979).

[56] M. Reid, *Minimal models of canonical 3-folds*, *Adv. Stud. Pure Math.* **1** (1983) 131.

[57] M. Reid, *Young person's guide to canonical singularities*, *Alg. Geom.* **46** (1987) 345.

[58] S. Mori, *On 3-dimensional terminal singularities*, *Nagoya Math. J.* **98** (1985) 43.

[59] J. Kollár and N.I. Shepherd-Barron, *Threefolds and deformations of surface singularities*, *Invent. Math.* **91** (1988) 299.

[60] D.A. Cox, J.B. Little and H.K. Schenck, *Toric varieties*, American Mathematical Society (2011).

[61] S. Rabinowitz, *A census of convex lattice polygons with at most one interior lattice point*, *Ars Combinatoria* **28** (1989) 83.

[62] X. Wei and R. Ding, *Lattice polygons with two interior lattice points*, *Math. Notes* **91** (2012) 868.

[63] S.S.-T. Yau and Y. Yu, *Gorenstein quotient singularities in dimension three*, American Mathematical Society (1993).

[64] S.-S. Roan, *Minimal resolutions of gorenstein orbifolds in dimension three*, *Topology* **35** (1996) 489.

[65] M. Caibăr, *On the divisor class group of 3-fold singularities*, *Int. J. Math.* **14** (2003) 105.

[66] H. Flenner, *Divisorenklassengruppen quasihomogener Singularitäten*, *J. Reine Angew. Math.* **328** (1981) 128.