Three-Dimensional Grid Drawings of Graphs

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Abstract. A three-dimensional *grid drawing* of a graph G is a placement of the vertices at distinct integer points so that the straight-line segments representing the edges of G are pairwise non-crossing. It is shown that for any fixed $r \geq 2$, every r-colorable graph of n vertices has a three-dimensional grid drawing that fits into a box of volume $O(n^2)$. The order of magnitude of this bound cannot be improved.

1 Introduction

In a *grid drawing* of a graph, the vertices are represented by distinct points with integer coordinates and the edges are represented by straight-line segments connecting the corresponding pairs of points. Grid drawings in the plane have a vast literature [BE]. In particular, it is known that every planar graph of n vertices has a two-dimensional grid drawing that fits into a rectangle of area $O(n^2)$, and this bound is asymptotically tight [FP],[S].

The possibility of three-dimensional representations of graphs was suggested by software engineers [MR]. The analysis of the volume requirement of such representations was initiated in [CE], where the following statement was proved. Every graph of n vertices has a three-dimensional grid drawing in a rectangular box of volume $O(n^3)$, and this bound cannot be improved. To establish the first half of the statement, it is sufficient to consider representations of *complete graphs*. Cohen et al. used a generalization of a well-known construction of Erdős showing that the vertices of a complete graph K_n can be placed at the points $(i, i^2 \bmod p, i^3 \bmod p)$, $1 \leq i \leq n$, where p is a prime between n and $2n$. Since no four of these points lie in the same plane, the resulting straight-line drawing of K_n has no crossing edges.

A complete r-partite graph is called *balanced*, if any two of its classes have the same number of points, or their sizes differ by one. Let $K_r(n)$ denote a balanced complete r-partite graph with $n \geq r$ vertices. That is, the vertex set of $K_r(n)$ splits into r disjoint classes, $V_1, V_2, \ldots, V_r$, such that $|V_i| = \lfloor n/r \rfloor$ or $\lceil n/r \rceil$, and two vertices are connected by an edge if and only if they belong to different V_i's.

It was pointed out by Cohen et al. [CE] that $K_2(n)$ has a three-dimensional grid drawing within a box of volume $O(n^2)$, and they asked whether this bound is optimal. T. Calamoneri and A. Sterbini [CS] proved that any such drawing

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requires a box whose volume is at least $\Omega(n^{3/2})$. Furthermore, they have shown that $K_3(n)$ and $K_4(n)$ also permit three-dimensional grid drawings of volume $O(n^2)$, and conjectured that the same is true for $K_r(n)$, for any fixed $r > 4$.

The aim of the present note is to answer the question of Cohen et al. and to verify the conjecture of Calamoneri and Sterbini. A graph is called *r-colorable* if its vertices can be colored by r colors so that no two adjacent vertices receive the same color. Equivalently, G is r-colorable if it is a subgraph of a complete r-partite graph.

Theorem. *For every $r \geq 2$ fixed, any r-colorable graph of n vertices has a three-dimensional grid drawing that fits into a rectangular box of volume $O(n^2)$. The order of magnitude of this bound cannot be improved.*

2 Proof of the Theorem

To prove the second assertion, it is enough to establish an $\Omega(n^2)$ lower bound on b, the number of integer points in a box B accommodating a three-dimensional grid drawing of the balanced complete bipartite graph $K_2(n)$ with vertex classes V_1, V_2. Clearly, $K_2(n)$ is r-colorable for any $r \geq 2$. Fix a grid drawing and consider the set of all vectors pointing from a vertex of V_1 to a vertex of V_2. The number of such vectors is $\lceil \frac{n}{2} \rceil \cdot \lfloor \frac{n}{2} \rfloor \geq \frac{n^2-1}{4}$, and no two of them can be identical, otherwise the corresponding four points would induce a parallelogram whose diagonals cross each other. (In fact, no two such vectors can point in the same direction.) On the other hand, the total number of vectors determined by two gridpoints in B is smaller than $8b$. Thus, $8b > \frac{n^2-1}{4}$, as required.

The proof of the upper bound is based on the following.

Lemma. *For any $r \geq 2$ and for any n divisible by r, the balanced complete r-partite graph $K_r(n)$ has a three-dimensional grid drawing that fits into a rectangular box of size $r \times 4n \times 4rn$.*

Proof. Let p be the smallest prime with $p \geq 2r - 1$ and set $N := p \cdot \frac{n}{r}$. Note that $p < 4r$ and thus $N < 4n$. For any $0 \leq i \leq r - 1$, let

$$V_i := \{(i, t, it) : 0 \leq t < N, t \equiv i^2 \pmod p\}.$$

These sets are pairwise disjoint, and each of them has precisely $\frac{N}{p} = \frac{n}{r}$ elements. Connect any two points belonging to different V_i's by a straight-line segment. The resulting drawing of $K_r(n)$ fits into a rectangular box of size $r \times 4n \times 4rn$, as desired.

It remains to show that no two edges of this drawing cross each other. Suppose, for contradiction, that there are two crossing edges, e and e'. We distinguish three different cases, according to the number of distinct classes V_i, the endpoints of e and e' belong to.

Case 1: The endpoints of e and e' are from four distinct classes, $V_{i_1}, V_{i_2}, V_{i_3}$, and V_{i_4}.

Let $(i_\alpha, t_\alpha, i_\alpha t_\alpha), 1 \leq \alpha \leq 4$, be the corresponding endpoints, so that $t_\alpha \equiv i_\alpha^2 \pmod{p}$. Then these points lie in a plane, and the determinant

$$D = \begin{vmatrix} 1 & i_1 & t_1 & i_1t_1 \\ 1 & i_2 & t_2 & i_2t_2 \\ 1 & i_3 & t_3 & i_3t_3 \\ 1 & i_4 & t_4 & i_4t_4 \end{vmatrix}$$

vanishes. Therefore, it must also vanish modulo p. However, modulo p this determinant reduces to the Vandermonde determinant. Thus,

$$D \equiv \begin{vmatrix} 1 & i_1 & i_1^2 & i_1^3 \\ 1 & i_2 & i_2^2 & i_2^3 \\ 1 & i_3 & i_3^2 & i_3^3 \\ 1 & i_4 & i_4^2 & i_4^3 \end{vmatrix} = \prod_{1 \leq \beta < \alpha \leq 4} (i_\alpha - i_\beta),$$

which is non-zero modulo p, a contradiction.

Case 2: The endpoints of e and e' are from three different classes, V_i, V_j, V_k.

Assume without loss of generality that two of these points, (i, t_1, it_1) and (i, t_2, it_2), belong to V_i, and the other two are $(j, s, js) \in V_j$ and $(k, u, ku) \in V_k$. These four points cannot be coplanar (hence e and e' cannot cross each other), because the corresponding test determinant

$$\begin{vmatrix} 1 & i & t_1 & it_1 \\ 1 & i & t_2 & it_2 \\ 1 & j & s & js \\ 1 & k & u & ku \end{vmatrix} = (j-i)(k-i)(t_2-t_1)(s-u)$$

is non-zero. To see this, observe that $s - u \equiv j^2 - k^2 \equiv (j-k)(j+k) \not\equiv 0 \pmod{p}$. This is the point where we use the assumption that $0 \leq j, k \leq r - 1 \leq (p-1)/2$.

Case 3: The endpoints of e and e' are from two different classes, V_i and V_j.

Let these points be $(i, t_1, it_1), (i, t_2, it_2) \in V_i$ and $(j, s_1, js_1), (j, s_2, js_2) \in V_j$. Now the corresponding test determinant

$$\begin{vmatrix} 1 & i & t_1 & it_1 \\ 1 & i & t_2 & it_2 \\ 1 & j & s_1 & js_1 \\ 1 & j & s_2 & js_2 \end{vmatrix} = (j-i)^2(t_2-t_1)(s_1-s_2)$$

does not vanish, therefore e and e' cannot cross each other.

This contradiction completes the proof of the Lemma. □

Now we return to the proof of the upper bound of the theorem. In fact, we can deduce a more precise statement.

Corollary. *There exists a $c > 0$ such that for any $r \geq 2$, any r-colorable graph of n vertices has a three-dimensional grid drawing that fits into a rectangular box of volume cr^2n^2.*

Proof. Fix an r-colorable graph G with n vertices. We split every color class into smaller parts such that all but one of them have *exactly* and the last one *at most* $\lceil \frac{n}{r} \rceil$ points. This defines a decomposition of the vertex set of G into at most $2r-1$ classes, whose sizes do not exceed $\lceil \frac{n}{r} \rceil$, and no two points belonging to the same class are connected by an edge. In other words, G is a subgraph of a balanced complete $(2r-1)$-partite graph K with $(2r-1)\lceil \frac{n}{r} \rceil < 2n+2r$ vertices. Applying the Lemma to K, the Corollary follows. □

3 Remarks and open problems

A The rectangular box used in the proof of the Theorem has two sides of size $O(n)$. We can use rectangular boxes of different shapes to represent $K_2(n)$.

Proposition. *There is a three dimensional grid drawing of $K_2(n)$ which fits into a rectangular box of size $O(n) \times O(\sqrt{n}) \times O(\sqrt{n})$.*

Proof. Let V_1 and V_2 be the vertex classes of $K_2(n)$ and let

$$V_1 = \{(i,0,0) \mid 0 \leq i \leq \lceil n/2 \rceil - 1\},$$

$$V_2 = \{(0,a,b) \mid 0 \leq a,b \leq k, (a,b) = 1\}$$

where $k \approx \pi\sqrt{n/6}$.

Then

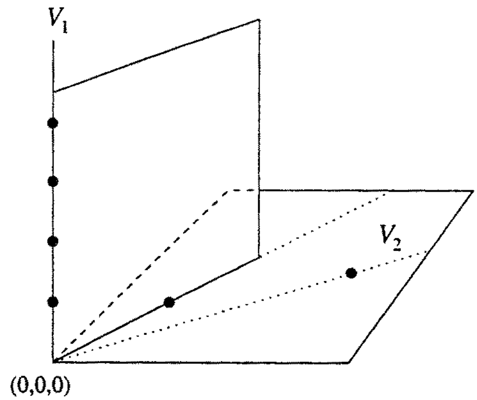
$$|V_1| = \lceil n/2 \rceil,$$

$$|V_2| = \sum_{i=1}^{k} \phi(i) \approx \frac{3k^2}{\pi^2} \approx n/2$$

(see [HW]). To see that there is no crossing, we observe that the points of V_2 lie on a horizontal plane and there are no two of these points on a line through $(0,0,0)$ (see Figure). □

B. It would be interesting to determine $S_r = S_r(n)$, the set of triples (s_1, s_2, s_3) for which every r-colorable graph of n vertices has a grid drawing that fits into a box of size $s_1 \times s_2 \times s_3$. In particular, what is the smallest $s = s(r)$ with $(s,s,s) \in S_r$? It is not hard to see that $s_i s_j \geq \lfloor n/2 \rfloor$ and $s_i > \frac{r}{5}$ holds for every $(s_1, s_2, s_3) \in S_r$ $(r \geq 2)$.

C. Since any graph of n vertices with fixed maximum degree $r-1$ is r-colorable, it follows from the Theorem that any such graph permits a grid drawing in a box of volume $O(n^2)$. It seems likely that for every fixed r, this bound can be substantially improved. We cannot even decide if every graph with maximum degree 3 has a grid drawing of volume $O(n)$.



Figure

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