Published for SISSA by Springer

Received: June 8, 2019
Accepted: July 15, 2019
Published: July 26, 2019

Three-dimensional (higher-spin) gravities with extended Schrödinger and l-conformal Galilean symmetries

Dmitry Chernyavsky[a,b] **and Dmitri Sorokin**[c,d]

[a] School of Physics, Tomsk Polytechnic University, Lenin Ave. 30, 634050 Tomsk, Russia
[b] Tomsk State University of Control Systems and Radioelectronics, Lenin Ave. 40, 634050 Tomsk, Russia
[c] I.N.F.N., Padova Section, Via F. Marzolo 8, 35131 Padova, Italy
[d] Dipartimento di Fisica e Astronomia "Galileo Galilei", Università degli Studi di Padova, Via F. Marzolo 8, 35131 Padova, Italy

E-mail: recnart@list.ru, sorokin@pd.infn.it

Abstract: We show that an extended $3D$ Schrödinger algebra introduced in [1] can be reformulated as a $3D$ Poincaré algebra extended with an SO(2) R-symmetry generator and an SO(2) doublet of bosonic spin-1/2 generators whose commutator closes on $3D$ translations and a central element. As such, a non-relativistic Chern-Simons theory based on the extended Schrödinger algebra studied in [1] can be reinterpreted as a relativistic Chern-Simons theory. The latter can be obtained by a contraction of the $\mathrm{SU}(1,2) \times \mathrm{SU}(1,2)$ Chern-Simons theory with a non principal embedding of $\mathrm{SL}(2,\mathbb{R})$ into $\mathrm{SU}(1,2)$. The non-relativisic Schrödinger gravity of [1] and its extended Poincaré gravity counterpart are obtained by choosing different asymptotic (boundary) conditions in the Chern-Simons theory. We also consider extensions of a class of so-called l-conformal Galilean algebras, which includes the Schrödinger algebra as its member with $l = 1/2$, and construct Chern-Simons higher-spin gravities based on these algebras.

Keywords: Conformal and W Symmetry, Higher Spin Gravity, Conformal Field Theory

ArXiv ePrint: 1905.13154

https://doi.org/10.1007/JHEP07(2019)156

JHEP07(2019)156

Contents

1 Introduction

Three-dimensional theories of gravity and their supersymmetric and higher-spin extensions have been under extensive study for several decades. A characteristic feature of a majority of these models is that they describe massless gauge fields of spin $s > 0$ which do not propagate in the three-dimensional bulk. As a manifestation of this feature, these theories admit a description in terms of Chern-Simons actions for gauge fields valued in the adjoint representation of corresponding symmetry groups, as was first observed for the case of 3D supergravity [2]. In spite of having on-shell zero field strengths (or curvatures), these theories exhibit a rich structure on the boundary of 3D manifolds and give rise to a variety of holographic dualities.

A main activity has been in studying relativistic (higher-spin) gravity models in AdS_3 and Minkovski space backgrounds. However, non-relativistic gravity theories based on different extensions of the $3D$ Galilean group have also attracted attention, in particular, in relation to non-AdS holography and its condensed matter applications (see e.g. [3] for a review and references). A Chern-Simons (CS) formulation of Galilean gravity was put forward in [4] and further generalized to a wider class of models in [1] including a conformal non-projectable Hořava-Lifshitz gravity associated with an extended Schrödinger algebra.

In the latter theory (dubbed Schrödinger gravity) the authors of [1] found solutions with $z = 2$ Lifshitz geometries. On the other hand, $z = 2$ Lifshitz solutions were also found within relativistic higher-spin CS theories based on $\mathrm{SL}(N, \mathbb{R}) \times \mathrm{SL}(N, \mathbb{R})$ gauge groups [5, 6]. As was shown in [7] these solutions of the Chern-Simons theory (build of connections) are not equivalent to Lifshitz solutions in a metric-like theory. In the case of [1] it was shown that their Lifshitz solutions do not have this problem, since the Newton-Cartan Chern-Simons theory is not a Lorentzian metric theory.

One of the aims of this note is to show that the CS theory based on the extended Schrödinger algebra can actually be reinterpreted in terms of a relativistic CS theory. The reason is that the extended Schrödinger algebra has an $sl(2, \mathbb{R}) \sim so(1, 2)$ subalgebra. Other generators of the extended Schrödinger algebra transform under a vector or a spinor representation of $sl(2, \mathbb{R})$, or are $sl(2, \mathbb{R})$ singlets. Thus, the algebra acquires a form similar to a centrally extended $\mathcal{N} = 2$, $D = 3$ Poincaré superalgebra, but with a doublet of commuting spinor generators. The $\mathrm{SO}(2) \sim \mathrm{U}(1)$ generator of $2d$ Galilean rotations becomes the R-symmetry generator of this "bosonic supersymmetry" algebra.

Upon having rewritten the extended Schrödinger algebra in the relativistic form, one finds that it can be obtained by a contraction of an $su(1, 2) \oplus sl(2, \mathbb{R}) \oplus so(2)$ algebra or as a contraction and truncation of an $su(1, 2) \oplus su(1, 2)$ algebra. The latter is one of the real forms of $sl(3) \oplus sl(3)$. Its difference with respect to the conventional real form $sl(3, \mathbb{R}) \oplus sl(3, \mathbb{R})$ has been discussed in the context of Chern-Simons spin-3 gravity e.g. in [8, 9].

The above observations point at a relation of Schrödinger gravity to Chern-Simons constructions of $3D$ higher-spin theories in the following sense. It is well known (see e.g. [9–11]) that the physical content and asymptotic behavior of a theory described by an $\mathrm{SL}(N) \times \mathrm{SL}(N)$ Chern-Simons action depends on the choice of particular vacuum boundary conditions which in the relativistic case are related to the choice of the embedding of $\mathrm{SL}(2, \mathbb{R})$ into $\mathrm{SL}(N)$. In other words, one and the same Chern-Simons action may describe physically different theories. In this respect, the Schrӧdinger gravity can be regarded as a specific choice of a non-relativistic vacuum associated with an embedding of the Galilean group into the extended Schrӧdinger group or its expansion to the $\mathrm{SU}(1, 2) \times \mathrm{SU}(1, 2)$, or even higher-rank group underlying a certain Chern-Simons action.

In the second part of this paper we will consider extensions of a class of so-called l-conformal Galilean algebras [12, 13] which includes the Schrödinger algebra as its member with $l = 1/2$, construct Chern-Simons higher-spin gravities based on these algebras (which turn out to be a subclass of so-called Hietarinta algebras [14]) and discuss asymptotic symmetries in these theories.

2 Extended Schrödinger as extended Poincaré

In this section we will show that the extended Schrödinger algebra associated with a Galilean $d = 2$ space can be recast in a relativistic form as an extended $D = 2+1$ Poincaré algebra. Our staring point is the $d = 2$ Schrödinger algebra written in the standard basis

$$\begin{aligned}
&[I, P^i] = \epsilon^{ij} P^j\,, && [I, G^i] = \epsilon^{ij} G^j, && [H, G^i] = P^i, \\
&[D, H] = -2H && [H, K] = D\,, && [D, K] = 2K, \\
&[H, G^i] = P^i && [D, P^i] = -P^i, && \\
&[D, G^i] = G^i && [K, P^i] = -G^i, &&
\end{aligned} \tag{2.1}$$

where H, K and D are, respectively, the generators of time translations, special conformal transformations and dilatations forming the one-dimensional conformal algebra isomorphic to $sl(2, \mathbb{R})$. P^i and G^i $(i = 1, 2)$ generate spatial translations and Galilei boosts, while I generates the SO(2) rotations in the $2d$ Galilean space. It is known that the commutator of translations and Galilean boosts can be centrally extended $[P^i, G^j] = N\delta^{ij}$ and, when one considers the Galilean algebra only, the result is the so called Bargmann algebra. In [1] it was proposed to extend the Scrödinger algebra further by adding three new elements which appear in the commutators of the Galilean boosts and translations

$$[G^i, G^j] = S\epsilon^{ij}, \qquad [P^i, P^j] = Z\epsilon^{ij}, \qquad [P^i, G^j] = N\delta^{ij} - Y\epsilon^{ij}. \tag{2.2}$$

The new elements S, Y and Z are central with respect to the Galilean subalgebra, but have nontrivial commutation relations with the conformal subalgebra generators

$$\begin{aligned}
&[H, Y] = -Z\,, && [H, S] = -2Y\,, && [K, Y] = S, \\
&[K, Z] = 2Y\,, && [D, S] = 2S\,, && [D, Z] = -2Z.
\end{aligned} \tag{2.3}$$

It turns out that the above commutation relations form a $D = 2 + 1$ Poincaré algebra. In order to see this, let us redefine the generators as follows

$$\begin{aligned}
&Z = M_{-1}, && S = M_{+1}, && Y = M_0, \\
&H = L_{-1}, && K = L_{+1}, && D = -2L_0.
\end{aligned} \tag{2.4}$$

Upon this redefinition the algebra (2.3) take the form of the $D = 2, 1$ Poincaré algebra written in the BMS_3 basis

$$[L_m, L_n] = (m - n)L_{m+n}, \qquad [L_m, M_n] = (m - n)M_{m+n}, \qquad m, n = \pm1, 0. \tag{2.5}$$

As the next step let us combine the Galilean translations and boosts into a single set of generators Z^i_α $(\alpha = \mp\frac{1}{2})$ such that

$$P^i = -\sqrt{2}Z^i_{-\frac{1}{2}}, \qquad G^i = \sqrt{2}Z^i_{+\frac{1}{2}} \tag{2.6}$$

Now we can rewrite the commutation relations (2.1) and (2.2) as follows

$$\begin{aligned}
&[L_m, Z^i_\alpha] = \left(\frac{m}{2} - \alpha\right) Z^i_{m+\alpha}, && [I, Z^i_\alpha] = \epsilon^{ij} Z^j_\alpha, && \\
&[Z^i_\alpha, Z^j_\beta] = \frac{1}{2}\epsilon^{ij} M_{\alpha+\beta} + \frac{1}{2}N(\alpha - \beta)\delta^{ij}, && m, n = \pm1, 0, && \alpha, \beta = \pm\frac{1}{2}.
\end{aligned} \tag{2.7}$$

Curiously, the structure of these relations resembles the form of a centrally extended $\mathcal{N}=2$, $D=3$ Poincaré superalgebra, but with the commuting (bosonic) spinor generators Z^i_α instead of anti-commuting ones. To make this similarity more explicit, let us now rewrite the extended Schrödinger algebra in a manifestly $D=2+1$ Lorentz invariant form. To this end let us perform the following redefinition

$$\begin{aligned}\sqrt{2}J^- &= -L_{-1} = H\,, & \sqrt{2}\mathcal{P}^- &= -M_{-1} = -Z\,,\\ \sqrt{2}J^+ &= L_{+1} = K\,, & \sqrt{2}\mathcal{P}^+ &= M_{+1} = S\,,\\ J^2 &= L_0 = -\frac{1}{2}D\,, & \mathcal{P}^2 &= M_0 = Y.\end{aligned} \tag{2.8}$$

Then the algebra (2.5) and (2.7) take the form of a $3D$ relativistic algebra[1]

$$\begin{aligned}&[J^a,J^b]=\epsilon^{abc}J_c, \qquad [J^a,\mathcal{P}^b]=\epsilon^{abc}\mathcal{P}_c, \qquad [I,Z^i_\alpha]=\epsilon^{ij}Z^j_\alpha,\\ &[J^a,Z^i_\beta]=\frac{1}{2}Z^i_\beta(\gamma^a)^\beta{}_\alpha, \quad [Z^i_\alpha,Z^j_\beta]=-\frac{1}{2}\epsilon^{ij}(C\gamma_a)_{\alpha\beta}\mathcal{P}^a+NC_{\alpha\beta}\delta^{ij},\end{aligned} \tag{2.9}$$

where we have also rescaled $N\to -2N$. From the structure of (2.9) we see that J^a generate $\mathrm{SO}(1,2)$ rotations and $\mathcal{P}^a$ generate $3D$ translations thus forming the $3D$ Poincaré algebra, while Z^i_α plays the role of the $\mathrm{SO}(2)$ doublet of $\mathrm{SL}(2,\mathbb{R})$-spinors generating a "bosonic supersymmetry". The generator I of the $2d$ Galilean rotations is now traded for the $\mathrm{SO}(2)$ R-symmetry generator. We have thus shown that the extended Schrödinger algebra is isomorphic to an extended Poincaré one. As one could notice, upon passing from one form to another, the geometrical meaning of the generators change. In particular, the generator I of the $2d$ Galilean rotations is now traded for the $\mathrm{SO}(2)$ R-symmetry generator, the generators of the $1d$ conformal algebra become that of the $\mathrm{SO}(1,2)$ rotations, while the Galilean translations and boosts form the doublet of $\mathrm{SL}(2,\mathbb{R})$ spinors according to (2.6) and (2.8).

2.1 Gravity theory with extended Schrödinger symmetry

The extended Schrödinger algebra has a nonsingular bilinear form. In [1] it was used to construct a Chern-Simons action with the gauge group generated by (2.1) and (2.2) and to derive in this way a novel version of the non-projectable conformal Hořava-Lifshitz gravity. The isomorphism of the extended Schrödinger algebra and the extended Poincaré algebra established in the previous section can be used to reformulate the Chern-Simons action of [1] as a relativistic model. In order to do so, let us present the non-degenerate symmetric bilinear form of the extended Schrödinger algebra in the relativistic basis (2.9)

$$\langle J^a,\mathcal{P}^b\rangle=\eta^{ab}, \qquad \langle Z^i_\alpha,Z^j_\beta\rangle=C_{\alpha\beta}\epsilon^{ij}, \qquad \langle I,N\rangle=-1. \tag{2.10}$$

The Chern-Simons action

$$S=\frac{k}{4\pi}\int_{\mathcal{M}_3}\langle \mathbf{A}d\mathbf{A}+\frac{2}{3}\mathbf{A}^3\rangle \tag{2.11}$$

is constructed with the use of a one-form gauge connection $\mathbf{A}=dx^\mu A_\mu(x)$ taking values in the algebra (2.9) and having the following components

$$\mathbf{A}=e^a\mathcal{P}_a+\omega^a J_a+\lambda^{i\alpha}Z^i_\alpha+Nv+Ib. \tag{2.12}$$

[1] See appendix A for our notation and conventions.

In (2.11) the wedge-product of the differential forms is implied. To have a contact with the Einstein gravity the value of the level k is set to be $k = 1/(4G)$ with G being the Newton's constant.

Using the expression for the bilinear form (2.10), the action (2.11) can be rewritten (up to a boundary term) as follows

$$S = \frac{k}{4\pi} \int_{\mathcal{M}_3} \left(2R_a\, e^a - \epsilon^{ij}(\bar{\lambda}^i \nabla \lambda^j) - 2v db\right), \tag{2.13}$$

where the covariant derivative is defined as

$$\nabla \lambda^i = d\lambda^i + \frac{1}{2}\omega^a \gamma_a \lambda^i - b\epsilon^{ij}\lambda^j \tag{2.14}$$

and

$$R^a = d\omega^a + \frac{1}{2}\epsilon^{abc}\omega_b \omega_c \tag{2.15}$$

is the curvature associated with the $\mathrm{SL}(2,\mathbb{R})$ connection ω^a.

The first term in (2.13) is the action for Einstein gravity written in the first-order formalism with $e^a(x)$ being associated with the gravitational dreibein field. As usual, the equations of motion of the Chern-Simons theory imply that the curvature $\mathbf{F}_2 = d\mathbf{A} + \mathbf{A}^2$ vanishes and locally $\mathbf{A}$ is a pure gauge, which implies that all the fields are non-dynamical in the $3D$ bulk and the non-trivial properties of the theory are determined by their behaviour on the $2d$ boundary. In this respect, the fact that the spin-3/2 fields λ^i_α have the bosonic statistics is not as troublesome as in higher dimensional theories, but still may indicate that unitarity may be lost on the boundary. We will study this issue for the asymptotic symmetry group of this theory in section 2.4.

As usually, we can construct a relativistic metric tensor with the use of the fields e^a and an affine connection (associated with ω^a), whose antisymmetric part is defined by the torsion constructed with the fields λ^i_α. A vacuum solution of the field equations for such a system is clearly the flat $3D$ Minkowski space. On the other hand, as was considered in [1], the Chern-Simons action based on the extended Schrödinger algebra allows one to define another metric and affine connection which correspond to a non-relativistic $3D$ geometry. In our notation, the metric of the Galilean geometry was constructed in [1] with the one-forms ω^0 and $\lambda^i_{+\frac{1}{2}}$ (associated, respectively, with the generators H and P^i in (2.8) and (2.6)) which play the role of the Galilean dreibein. Evidently, this corresponds to a different choice of geometry and boundary conditions for the Chern-Simons field equations of motion. These alternative choices result in physically different theories. The situation is analogous to different choices of the embedding of the $\mathrm{SL}(2,\mathbb{R})$ group into $\mathrm{SL}(N) \times \mathrm{SL}(N)$ Chern-Simons theories which (together with asymptotic boundary conditions) lead to different $3D$ higher-spin gravity models (see e.g. [9–11, 15, 16] and references therein).

Another curious fact about the action (2.13) is that it can be obtained in a limit of zero cosmological constant from the $\mathrm{SU}(1,2) \times \mathrm{SU}(1,2)$ Chern-Simons theory. Or putting it differently, the extended Schrödinger gravity can be expanded to the latter.

2.2 The extended Schrödinger gravity by contraction and truncation of $\mathbf{SU(1,2)\times SU(1,2)}$ Chern-Simons theory

As in the case of its $sl(3,\mathbb{R})$ counterpart [10], the $su(1,2)$ algebra allows for two $sl(2,\mathbb{R})$ embeddings, the principle embedding and a non-principle one (see appendix B). It turns out that the extended Schrodinger algebra is related to the non-principle embedding for which the $su(1,2)$ algebra takes the following form

$$\begin{aligned}
&[\mathcal{J}^a,\mathcal{J}^b]=\epsilon^{abc}\mathcal{J}_c, && [\mathcal{I},\mathcal{Z}^i_\alpha]=\epsilon^{ij}\mathcal{Z}^j_\alpha,\\
&[\mathcal{J}^a,\mathcal{Z}^i_\alpha]=\frac{1}{2}(\gamma^a)^\beta{}_\alpha\mathcal{Z}^i_\beta, && [\mathcal{Z}^i_\alpha,\mathcal{Z}^j_\beta]=-\frac{1}{2}\epsilon^{ij}(C\gamma_a)_{\alpha\beta}\mathcal{J}^a+\frac{3}{4}\delta^{ij}C_{\alpha\beta}\mathcal{I}\,,
\end{aligned} \tag{2.16}$$

where J^a form the $sl(2,\mathbb{R})$ subalgebra.

Let us now consider two copies of (2.16) which form the $su(1,2)\oplus su(1,2)$ algebra, and take the following linear combination of their generators which are distinguished by 'tilde'

$$\begin{aligned}
&\mathcal{P}^a=\frac{1}{\rho}(\mathcal{J}^a-\tilde{\mathcal{J}}^a), && J^a=\mathcal{J}^a+\tilde{\mathcal{J}}^a, && Z^i_\alpha=\sqrt{\frac{2}{\rho}}\mathcal{Z}^i_\alpha, && \tilde{Z}^i_\alpha=\sqrt{\frac{2}{\rho}}\tilde{\mathcal{Z}}^i_\alpha\,,\\
&I=\mathcal{I}-\tilde{\mathcal{I}}, && N=\frac{3}{4\rho}(\mathcal{I}+\tilde{\mathcal{I}}).
\end{aligned} \tag{2.17}$$

The generators $\mathcal{P}^a$ and J^a form the $so(2,2)$ algebra of AdS_3 isometry, and ρ can be viewed as the AdS_3 radius. When $\rho\to\infty$, the $so(2,2)$ algebra gets contracted to the $3D$ Poincaré algebra. Taking also into consideration the generators Z^i_α, $\tilde{Z}^i_\alpha$, I and N, in the limit $\rho\to\infty$ one recovers the extended Schrödinger algebra in the form (2.9) but with the extra doublet $\tilde{Z}^i_\alpha$ of the spinor generators

$$[\tilde{Z}^i_\alpha,\tilde{Z}^j_\beta]=\frac{1}{2}\epsilon^{ij}(C\gamma_a)_{\alpha\beta}\mathcal{P}^a+\delta^{ij}C_{\alpha\beta}N\,. \tag{2.18}$$

We see that the generators $\tilde{Z}^i_\alpha$ further extend the algebra (2.9). In the non-relativistic setting, these generators correspond to an extra copy of the Galilei-like translations and boosts $\tilde{Z}^i_\alpha=(\tilde{P}^i,\tilde{G}^i)$.[2] The extended Schrödinger algebra is obtained upont trancation of these additional generators. An alternative possibility, which does not require the truncation of extra spinor generators, is to obtain the extended Schrödinger algebra directly by contraction of $su(1,2)\oplus sl(2,\mathbb{R})\oplus so(2)$.[3]

The above observation of the relation between the algebras allows us to view the gravity model (2.13) as a contraction and truncation of the $\mathrm{SU}(1,2)\times\mathrm{SU}(1,2)$ CS theory [8]. For

[2]It might be of interest to see whether the extended Schrödinger algebra (2.9) with the addition of (2.18) can be alternatively viewed as a certain algebra expansion, a technique considered e.g. in [17–19] and references there in.

[3]Note that, instead of the contraction of $su(1,2)\oplus su(1,2)$ we could also consider the contraction of $sl(3,R)\oplus sl(3,R)$ by simply assuming that the vector indices i,j in (2.21) be transformed under the $\mathrm{SO}(1,1)$ group instead of $\mathrm{SO}(2)$ (see appendix B). However, in that case, because of non-compactness of $\mathrm{SO}(1,1)$ we would arrive at an algebra which would not have an interpretation as an extended Schrödinger (or Galilean) algebra. From the point of view of a non-relativistic gravity interpretation a somewhat similar case in which one deals with a different (non-compact) real form is dubbed pseudo-Newton-Cartan geometry [20]. We are thankful to a referee for indicating this and the point of footnote 2.

JHEP07(2019)156

our case of the non-principle embedding of $sl(2,\mathbb{R})$, it is natural to define the $\mathrm{SU}(1,2)\times\mathrm{SU}(1,2)$ gauge field one-form $\mathbf{A}$ in the basis (2.17)

$$\mathbf{A} = \mathbf{A} + \tilde{\mathbf{A}} = e^a P_a + \omega^a J_a + \lambda^{i,\alpha} Z^i_\alpha + \tilde{\lambda}^{i,\alpha}\tilde{Z}^i_\alpha + Nv + Ib. \tag{2.19}$$

Then, using the invariant bilinear forms[4]

$$\langle J^a, P^b\rangle = \eta^{ab}, \quad \langle I, N\rangle = -1, \quad \langle Z^i_\alpha, Z^j_\beta\rangle = C_{\alpha\beta}\epsilon^{ij}, \quad \langle \tilde{Z}^i_\alpha, \tilde{Z}^j_\beta\rangle = -C_{\alpha\beta}\epsilon^{ij}, \tag{2.20}$$

one gets the $\mathrm{SU}(1,2)\times\mathrm{SU}(1,2)$ CS action in the form

$$S = \frac{k}{4\pi}\int_{\mathcal{M}_3}\left(2e^a R_a + \frac{1}{3\rho^2}\epsilon_{abc}e^a e^b e^c - \epsilon^{ij}\bar{\lambda}^i\nabla\lambda^j + \epsilon^{ij}\bar{\tilde{\lambda}}^i\tilde{\nabla}\tilde{\lambda}^j - 2vdb\right), \tag{2.21}$$

where

$$\nabla\lambda^i = d\lambda^i + \frac{1}{2}\omega^a\gamma_a\lambda^i + \frac{1}{2\rho}e^a\gamma_a\lambda^i - \frac{3}{4\rho}\epsilon^{ik}v\lambda^k - \epsilon^{ik}b\lambda^k \tag{2.22}$$

and

$$\tilde{\nabla}\tilde{\lambda}^i = d\tilde{\lambda}^i + \frac{1}{2}\omega^a\gamma_a\tilde{\lambda}^i - \frac{1}{2\rho}e^a\gamma_a\tilde{\lambda}^i - \frac{3}{4\rho}\epsilon^{ik}v\tilde{\lambda}^k + \epsilon^{ik}b\tilde{\lambda}^k.$$

We see that in the form (2.21) the CS action describes gravity (associated with the dreibein e^a and spin connection ω^a) plus bosonic spin-3/2 fields λ^i_α and $\tilde{\lambda}^i_\alpha$ coupled to gravity and a pair of spin-1 fields v and b, which is known to be the consequence of the choice of the non-principle embedding of $\mathrm{SL}(2,\mathbb{R})$ into $\mathrm{SU}(1,2)$. The action (2.13) is obtained from (2.21) by taking the limit $\rho\to\infty$ and setting the fields $\tilde{\lambda}^i$ to zero. This is consistent with the field variations under the local symmetries generated by the extended Schrödinger algebra (2.9). As one might expect, the theory obtained in this limit is different from the usual asymptotically flat spin-3 gravity discussed e.g. in [21–23].

2.3 Asymptotic symmetry

Let us now analyze asymptotic symmetry of the gravity theory (2.13) based on the extended Schrödinger group assuming that the $3D$ geometry is relativistic and described by the dreibein e^a and the connection ω^a.

As we have shown in the previous section this theory can be obtained by contraction and truncation of the $\mathrm{SU}(1,2)\times\mathrm{SU}(1,2)$ CS theory with a non-principle embedding of $\mathrm{SL}(2,\mathbb{R})$ into $\mathrm{SU}(1,2)$. One can thus expect that the asymptotic symmetry of (2.13) can be recovered by a contraction of the asymptotic symmetry of the $\mathrm{SU}(1,2)\times\mathrm{SU}(1,2)$ CS theory. In the case of the principle embedding the asymptotic symmetry of the latter was identified with a $W_3\times W_3$ algebra in [8]. In the case of the non-principle embedding of $\mathrm{SL}(2,\mathbb{R})$ in the $\mathrm{SL}(3,\mathbb{R})\times\mathrm{SL}(3,\mathbb{R})$ theory it was shown [10] that the asymptotic algebra is the direct product of two copies of a $W_3^{(2)}$ algebra (also known as the Bershadsky-Polyakov algebra [24, 25]). In the $\mathrm{SU}(1,2)\times\mathrm{SU}(1,2)$ case the asymptotic algebra is a different real form of $W_3^{(2)}$, which we will call $W_{1,2}^{(2)}$.

---

[4]Using the redefinitions (2.17), (B.3) and (B.6) one can see that (up to a normalization constant) the bilinear forms (2.20), correspond to the difference between the two copies of the bilinears (B.2) of the $su(1,2)$ algebra. In other words, the CS action (2.21) is equal to $\frac{k}{4\pi}\int_{\mathcal{M}_3}\Big(\langle\mathbf{A}d\mathbf{A}+\frac{2}{3}\mathbf{A}^3\rangle - \langle\tilde{\mathbf{A}}d\tilde{\mathbf{A}}+\frac{2}{3}\tilde{\mathbf{A}}^3\rangle\Big)$.

JHEP07(2019)156

In the appendix D we will obtain the asymptotic symmetry algebra of the theory (2.13) by contraction and truncation of the $W^{(2)}_{1,2}\oplus W^{(2)}_{1,2}$ algebra, while in this section we derive it directly.

We assume that the boundary of the $3D$ manifold $\mathcal{M}_3$ has a cylindrical topology with the compact directions parameterized by the coordinate ϕ and the non-compact one is t. The radial coordinate r measures how far we are from the boundary. As usually, we assume that at the boundary the gauge field behaves as

$$\mathbf{A}=h^{-1}(d+\mathfrak{a})h, \tag{2.23}$$

where a group element h depends on the radial coordinate $h=h(r)$ only. When all the fields, except the ones defining Einstein gravity are set to zero, one assumes to have the BMS_3 boundary conditions (see e.g. [26–28])

$$\begin{aligned}\mathfrak{a}^0_\phi&=-\mathcal{L}M_{-1}-\mathcal{M}L_{-1}+L_{+1},\\ \mathfrak{a}^0_t&=-\mathcal{M}M_{-1}+M_{+1},\end{aligned} \tag{2.24}$$

where we used the basis (2.5) and (2.7) of the generators of the extended Schrödinger algebra. As an extension of (2.24) we define the following boundary conditions for the connection $\mathfrak{a}$

$$\begin{aligned}\mathfrak{a}_\phi&=\mathfrak{a}^0_\phi+\mathcal{N}\mathcal{I}M_{-1}+\mathcal{N}^2L_{-1}+\sqrt{2}\mathcal{C}^iZ^i_{-\frac{1}{2}}+\mathcal{I}N+\mathcal{N}I,\\ \mathfrak{a}_t&=\mathfrak{a}^0_t+\mathcal{N}^2M_{-1}+2\mathcal{N}N,\end{aligned} \tag{2.25}$$

where $\mathcal{M}$, $\mathcal{N}$, $\mathcal{I}$, $\mathcal{L}$ and $\mathcal{C}$ are functions of ϕ and t describing asymptotic dynamics of the fields of the model. Specifying appropriately the element h, these boundary conditions include physically interesting solutions, such as cosmological horizons [29, 30]. One can notice a close relation of these boundary conditions to the ones proposed in [31] for asymptotically flat $\mathcal{N}=2$, $D=3$ supergravity. It can be verified that the gauge field (2.25) satisfies the equations of motion $d\mathbf{A}+\mathbf{A}^2=0$, provided

$$\dot{\mathcal{L}}=\mathcal{M}',\qquad \dot{\mathcal{I}}=2\mathcal{N}',\qquad \dot{\mathcal{N}}=\dot{\mathcal{M}}=\dot{\mathcal{C}}^i=0, \tag{2.26}$$

where, hereinafter, prime and dot denote, respectively, the derivative with respect to ϕ and t. The boundary conditions (2.25) thus ensure a well defined variation principal of the CS action (2.11) for getting the equations of motion such that

$$\delta S|_{EOM}=\frac{k}{4\pi}\int_{\partial\mathcal{M}}\langle\delta\mathbf{A},\mathbf{A}\rangle=0. \tag{2.27}$$

Indeed, taking into account the bilinear form (2.10) written in the basis (2.5) and (2.7)

$$\langle L_{+1},M_{-1}\rangle=\langle L_{-1},M_{+1}\rangle=-\langle I,N\rangle=-2,\quad \langle L_0,M_0\rangle=1,\quad \langle Z^i_{-\frac{1}{2}},Z^j_{+\frac{1}{2}}\rangle=\epsilon^{ij}, \tag{2.28}$$

one can explicitly check that the integrand in the expression above vanishes for the boundary conditions (2.25).

JHEP07(2019)156

We now look for the transformations

$$\delta\mathbf{A} = d\boldsymbol{\lambda} + [\mathbf{A}, \boldsymbol{\lambda}], \tag{2.29}$$

that preserve (2.25), i.e. the transformations which map the boundary conditions to the same class. We find that the algebra-valued parameter $\boldsymbol{\lambda}$ of these transformations should be of the following form

$$\begin{aligned}
\boldsymbol{\lambda} = & \left(\frac{1}{2}\varepsilon_L'' - \varepsilon_L(\mathcal{M} - \mathcal{N}^2)\right) L_{-1} \\
& + \left(\frac{1}{2}\varepsilon_M'' - \varepsilon_L(\mathcal{L} - \mathcal{N}\mathcal{I}) - \varepsilon_M(\mathcal{M} - \mathcal{N}^2) - \frac{1}{2}\mathcal{C}^i\varepsilon^i\right) M_{-1} \\
& + \sqrt{2}(\varepsilon'^j\epsilon^{ij} + \varepsilon_L\mathcal{C}^i + \mathcal{N}\varepsilon^i) Z^i_{-\frac{1}{2}} + (\varepsilon_I + \mathcal{N}\varepsilon_L) I + (\varepsilon_N + 2\varepsilon_M\mathcal{N} + \varepsilon_L\mathcal{I}) N \\
& - \varepsilon_L' L_0 + \varepsilon_L L_{+1} - \varepsilon_M' M_0 + \varepsilon_M M_{+1} - \sqrt{2}\epsilon^{ij}\varepsilon^j Z^i_{+\frac{1}{2}},
\end{aligned} \tag{2.30}$$

where the parameters ε are subject to the constraints

$$\dot{\varepsilon}_M = \varepsilon_L', \qquad \dot{\varepsilon}_N = 2\varepsilon_I', \qquad \dot{\varepsilon}_L = \dot{\varepsilon}^i = \dot{\varepsilon}_I = 0, \tag{2.31}$$

and depend arbitrarily on ϕ. The transformations generated by $\boldsymbol{\lambda}$ imply the following transformation rules for the functions describing boundary dynamics

$$\begin{aligned}
\delta\mathcal{L} &= 2\varepsilon_L'\mathcal{L} + \varepsilon_L\mathcal{L}' + 2\varepsilon_M'\mathcal{M} + \varepsilon_M\mathcal{M}' + \frac{3}{2}\mathcal{C}^i\varepsilon'^i + \frac{1}{2}\mathcal{C}'^i\varepsilon^i + \mathcal{I}\varepsilon_I' + \mathcal{N}\varepsilon_N' - \frac{1}{2}\varepsilon_M''', \\
\delta\mathcal{M} &= 2\varepsilon_L'\mathcal{M} + \varepsilon_L\mathcal{M}' + 2\varepsilon_I'\mathcal{N} - \frac{1}{2}\varepsilon_L''', \\
\delta\mathcal{C}^i &= \frac{3}{2}\varepsilon_L'\mathcal{C}^i + \varepsilon_L\mathcal{C}'^i - \mathcal{M}\epsilon^{ij}\varepsilon^j + 2\varepsilon'^i\mathcal{N} + \varepsilon^i\mathcal{N}' + \epsilon^{ij}\mathcal{C}^j\varepsilon_I + \epsilon^{ij}\varepsilon''^j, \\
\delta\mathcal{N} &= \varepsilon_I' + \mathcal{N}'\varepsilon_L + \mathcal{N}\varepsilon_L', \\
\delta\mathcal{I} &= \varepsilon_N' + 2(\varepsilon_M'\mathcal{N} + \varepsilon_M\mathcal{N}') + \varepsilon_L'\mathcal{I} + \varepsilon_L\mathcal{I}' + \epsilon^{ij}\mathcal{C}^i\varepsilon^j.
\end{aligned} \tag{2.32}$$

For each symmetry transformation there is an associated conserved charge $Q[\boldsymbol{\lambda}]$ whose field variation in the CS theory is (see e.g. [32] for details)

$$\delta Q[\boldsymbol{\lambda}] = -\frac{k}{2\pi}\int_0^{2\pi} \langle\boldsymbol{\lambda}, \delta\mathfrak{a}_\phi\rangle d\phi. \tag{2.33}$$

Taking into account the bilinear form (2.28) and the expression for $\boldsymbol{\lambda}$ (2.30), one finds

$$\langle\boldsymbol{\lambda}, \delta\mathfrak{a}_\phi\rangle = -2\left(\varepsilon_I\delta\mathcal{I} + \varepsilon_N\delta\mathcal{N} + \varepsilon_M\delta\mathcal{M} + \varepsilon_L\delta\mathcal{L} + \varepsilon^i\delta\mathcal{C}^i\right). \tag{2.34}$$

The variation (2.33) defines the Poisson bracket of the charges

$$\delta_{\boldsymbol{\lambda}_2} Q[\boldsymbol{\lambda}_1] = \Big[Q[\boldsymbol{\lambda}_1], Q[\boldsymbol{\lambda}_2]\Big],$$

which form the classical algebra of the asymptotic symmetries. To get an explicit form of this algebra, it is common to expand the fields and the transformation parameters in Fourier modes. In view of (2.33) and (2.34), the Fourier modes of the conserved charges are

JHEP07(2019)156

given by $Q_n = \frac{k}{\pi}\int_0^{2\pi} d\phi e^{-in\phi}X$, where X stands for the functions, describing asymptotic dynamics. Using this expression together with the transformation rules (2.32), one finds the asymptotic symmetry algebra

$$\begin{aligned}
&[L_m, L_n] = i(m-n)L_{m+n}, && [L_m, M_n] = i(m-n)M_{m+n} - ikn^3\delta_{m+n,0},\\
&[L_m, I_n] = -inI_{m+n}, && [L_m, N_n] = -inN_{m+n},\\
&[M_m, I_n] = -2inN_{m+n}, && [I_m, N_n] = -2ink\delta_{m+n,0},\\
&[L_m, C^i_p] = i\left(\frac{m}{2}-p\right)C^i_{p+m}, && [I_m, C^i_p] = -\epsilon^{ij}C^j_{m+p},\\
&[C^i_p, C^j_q] = -M_{p+q}\epsilon^{ij} + i(p-q)N_{p+q}\delta^{ij} - 2q^2k\delta_{p+q,0}\epsilon^{ij}, &&
\end{aligned} \tag{2.35}$$

where $m, n \in \mathbb{Z}$ and $p, q \in \mathbb{Z} + \frac{1}{2}$. The generators are associated to the asymptotic fields as follows $(L, M, I, N, C) \sim (\mathcal{L}, \mathcal{M}, \mathcal{I}, \mathcal{N}, \mathcal{C})$. The first line represents the BMS_3 algebra with the standard central charge $c = 12k$ [27].[5] As one might expect, the algebra has a very similar form to that of the asymptotic symmetry superalgebra in an $\mathcal{N} = 2$, $D = 3$ supergravity theory [31].

2.4 Unitarity issue

The theory at the boundary is governed by representations of the asymptotic symmetry algebra (2.35). An important question is whether the algebra under consideration has unitary representations. Note that though the unitarity issues with the $W_3^{(2)} \oplus W_3^{(2)}$ algebra are known [11, 15], the existence of unitary representations of the contraction of these algebra may not be excluded *a priori*.

Unitarity implies that one must have a positive semi-definite Kac matrix, which is a matrix constructed out of inner products between descendents at level N. We will define the notion of level and descendents for the algebra below. Here we aim to explore the Kac matrix at the first two levels and derive constraints imposed by the positivity condition.

To proceed, let us redefine the generator $M_m \to M_m - \frac{k}{2}\delta_{m,0}$ and replace the Dirac brackets with the quantum commutators $[,] \to i[,]$ to bring the algebra (2.35) to the following (quantum) form

$$\begin{aligned}
&[L_m, L_n] = (m-n)L_{m+n}, && [L_m, M_n] = (m-n)M_{m+n} - k(n^3-n)\delta_{m+n,0},\\
&[L_m, I_n] = -nI_{m+n}, && [L_m, N_n] = -nN_{m+n},\\
&[M_m, I_n] = -2nN_{m+n}, && [I_m, N_n] = -2nk\delta_{m+n,0},\\
&[L_m, C^i_p] = \left(\frac{m}{2}-p\right)C^i_{p+m}, && [I_m, C^i_p] = i\epsilon^{ij}C^j_{m+p},\\
&[C^i_p, C^j_q] = iM_{p+q}\epsilon^{ij} + (p-q)N_{p+q}\delta^{ij} + 2ik\left(q^2-\frac{1}{4}\right)\delta_{p+q,0}\epsilon^{ij}. &&
\end{aligned} \tag{2.36}$$

[5] Infinite dimensional extensions of the extended Schrödinger algebra and their connection with the BMS_3 algebra were earlier noticed by Yang Lei (unpublished). We thank Jelle Hartnog for pointing this out to us.

As usual, we consider the generators of this algebra as operators acting on a vector space of quantum states. The operators have the hermiticity relations

$$
\begin{aligned}
&L_m^\dagger = L_{-m}, && M_m^\dagger = M_{-m}, && (C_p^i)^\dagger = C_{-p}^i,\\
&I_m^\dagger = I_{-m}, && N_m^\dagger = N_{-m}.
\end{aligned}
\tag{2.37}
$$

Let us define a primary state $|\psi\rangle$ by

$$
\begin{aligned}
&L_0|\psi\rangle = l_0|\psi\rangle, && M_0|\psi\rangle = m_0|\psi\rangle,\\
&I_0|\psi\rangle = i_0|\psi\rangle, && N_0|\psi\rangle = n_0|\psi\rangle,
\end{aligned}
\tag{2.38}
$$

with $L_m|\psi\rangle = M_m|\psi\rangle = I_m|\psi\rangle = N_m|\psi\rangle = C_p^i|\psi\rangle = 0$ for $m > 0$ and $p > -\frac{1}{2}$, where m_0, n_0, i_0 and l_0 are positive numbers. Descendants are supposed to be generated by the integer and half-integer spin operators with negative m and p, respectively. The level of a state is defined with respect to the operator L_0 (see [33–36] for the analysis of representations of the BMS_3 algebra). There are seven states at the level one, generated by the operators $L_{-1}, M_{-1}, I_{-1}, N_{-1}, C^1_{-\frac{1}{2}}C^2_{-\frac{1}{2}}, C^1_{-\frac{1}{2}}C^1_{-\frac{1}{2}}$, $C^2_{-\frac{1}{2}}C^2_{-\frac{1}{2}}$. Let us first consider the vacuum state defined by $m_0 = n_0 = l_0 = i_0 = 0$. At the level one the only states which give a nonzero contribution to the Kac matrix $K^{(1)}$ are $I_{-1}|\psi\rangle$ and $N_{-1}|\psi\rangle$. In this case the Kac matrix is

$$
K^{(1)} = \begin{pmatrix} 0 & 2k \\ 2k & 0 \end{pmatrix}. \tag{2.39}
$$

Clearly, to satisfy the semi-positivity condition one needs the vanishing central charge $k = 0$. It can also be shown that the vanishing of the central charge is required for a non-vacuum state with nonzero eigenvalues m_0, n_0, l_0, and i_0 to have a non-negative norm. The same issue was encountered in [11] for the $W_3^{(2)}$ algebra.

To satisfy the unitarity condition with the nonzero central charge, we can truncate the algebra by requiring that the asymptotic fields $\mathcal{N}$ and $\mathcal{I}$ in (2.25) are zero, and hence the generators I_m and N_m are removed from the algebra (2.36). This also requires to remove the generators N and I from the initial algebra (2.9). As a result we have the theory with the vanishing fields $b = v = 0$ in (2.13). To write down the Kac matrix at the level one for the algebra (2.36) in which I_m and N_m are absent, let us define a vector of states $|\phi\rangle = (L_{-1}, M_{-1}, C^1_{-\frac{1}{2}}C^2_{-\frac{1}{2}}, C^1_{-\frac{1}{2}}C^1_{-\frac{1}{2}}, C^2_{-\frac{1}{2}}C^2_{-\frac{1}{2}})|\psi\rangle$. Using it, the Kac matrix at the level one can be written as $K^{(1)} = |\phi\rangle^\dagger \otimes |\phi\rangle$, where the tensor product is for the vector, not for the states. Using the commutation relations (2.36), one finds

$$
K^{(1)} = \begin{pmatrix}
2l_0 & 2m_0 & im_0 & 0 & 0 \\
2m_0 & 0 & 0 & 0 & 0 \\
-im_0 & 0 & m_0^2 & 0 & 0 \\
0 & 0 & 0 & 0 & -2m_0^2 \\
0 & 0 & 0 & -2m_0^2 & 0
\end{pmatrix}. \tag{2.40}
$$

This matrix is positive semi-definite only if the eigenvalue of the operator M_0 is vanishing $m_0 = 0$. The same condition for unitarity was found for the BMS_3 algebra in [34]. It can be checked that the positivity condition for the Kac matrix at the $\frac{1}{2}$-level does not lead to any new restrictions.

To summarize, we have shown that the unitarity condition can only be satisfied for the algebra with the truncated operators I_m and N_m. To answer the question whether there are unitary representations for the truncated algebra one needs to analyze the Kac matrix for every level. We leave it for a future study.

3 Extending l-conformal Galilean algebra

The Schrödinger algebra is a particular instance in the family of nonrelativistic conformal algebras dubbed l-conformal Galilean algebras [12, 13]. These algebras and their realizations in physical models have been under extensive study (see e.g. [37–48].

l-conformal Galilean algebras are parameterized by an integer or a half-integer l and $l = \frac{1}{2}$ corresponds to the Schrödinger algebra. We will show that, by analogy with the extension of the Schrödinger algebra, one can also extended the l-conformal Galilei algebra with an arbitrary l. In what follows we will restrict ourselves to the cases of $d = 2$ and $d = 1$, where d is the dimension of the Galilean space on which l-conformal Galilean algebra naturally acts. For our purposes it is convenient to deal with the l-conformal Galilean algebra in the basis considered e.g. in [37, 38]. In the case $d = 2$ its nonvanishing commutation relations are

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n}, & [L_m, C^i_p] &= (lm-p)C^i_{m+p}, \\ [I, C^i_p] &= \epsilon^{ij}C^j_p, \qquad m,n = \pm 1, 0, & p &= -l, \dots, l, \qquad i = 1,2. \end{aligned} \tag{3.1}$$

Clearly, the case $d = 1$ can be extracted from (3.1) by discarding the vector index on the generator C and dropping out the generator I. As previously, the generators L_m span the conformal subalgebra and I produces rotations in the Galilean space. C^i_{-l} generate translations, C^i_{-l+1} Galilean boosts, while all the remaining C^i_m, $m = -l+2, \dots, l$ are acceleration generators. In the next two subsections we aim to extend the algebra (3.1).[6]

3.1 Half-integer l

We first construct an extension of the l-conformal Galilean algebra for an arbitrary half-integer l in $d = 2$. Similar to the structure of the extended Schrödinger algebra (2.5) and (2.7), we introduce new generators which appear in the non-zero commutators of the generators C^i_p as follows

$$[C^i_p, C^j_q] = \epsilon^{ij} f^{(l)}_{p,q} M_{p+q} + N^{(l)}_{p,q}\delta^{ij}, \tag{3.2}$$

where $f^{(l)}_{p,q}$ are symmetric and $N^{(l)}_{p,q}$ are antisymmetric structure constants. We assume that for an arbitrary l there is a Poincaré subalgebra (2.5) in the extended l-conformal Galilei algebra. Hence, the commutation relations above imply that the only nonzero structure constants $f^{(l)}_{p,q}$ are those with $|p+q| \leq 1$. $N^{(l)}_{p,q}$ define a central extension. Their form was

[6] Note that supersymmetric extensions of the l-conformal Galilean algebra were constructed earlier in [49, 50].

found earlier in [42] in a slightly different notation. The form of $N^{(l)}_{p,q}$ is fixed by the by the Jacobi identity for (L_m, C^i_p, C^j_q)

$$(lm-p)N_{p+m,q}-(lm-q)N_{q+m,p}=0, \tag{3.3}$$

which yields

$$N^{(l)}_{-p,p}=(-1)^{(p+1/2)^2}\frac{(2l-2p)}{(2l+1)}\prod_{s=\frac{1}{2}}^{p}\frac{(2l+2s)}{(2l-2s)}N, \qquad p>0, \tag{3.4}$$

while all the other components of $N^{(l)}_{p,q}$ vanish. By N we denote the only independent element. The Jacobi identities for (L_m, C^i_p, C^j_q) also require the structure constants $f^{(l)}_{p,q}$ to satisfy the following relation

$$(m-p-q)f^{(l)}_{p,q}-(lm-p)f^{(l)}_{q,p+m}-(lm-q)f^{(l)}_{p,q+m}=0. \tag{3.5}$$

Curiously, this is exactly the condition on the structure constants in the odd sector of a hyper-Poincaré algebra [51], a higher-spin generalization of the conventional Poincaré supersymmetry algebra first introduced in [14]. This restriction implies that the structure constants should satisfy a recurrence relation

$$f^{(l)}_{p,-p-1}=f^{(l)}_{-p,p+1}=-\frac{p+l+1}{2p}f^{(l)}_{p,-p}, \qquad p>0, \tag{3.6}$$

and all the $f^{(l)}_{p,-p}$ can be expressed via $f^{(l)}_{-\frac{1}{2},\frac{1}{2}}$ [51]

$$f^{(l)}_{-p,p}=2p\prod_{s=\frac{1}{2}}^{p-1}\frac{2l+2s+2}{2s-2l}f^{(l)}_{-\frac{1}{2},\frac{1}{2}}, \tag{3.7}$$

where we normalize the first element in the recurrence as $f^{(l)}_{-\frac{1}{2},\frac{1}{2}}=1$. There are also two other nontrivial elements of the structure constants which one is not able to identify from (3.6). These are $f^{(l)}_{\frac{1}{2},\frac{1}{2}}=f^{(l)}_{-\frac{1}{2},-\frac{1}{2}}=\frac{1+2l}{2}$.

In [51] it was pointed out that the structure constants $f^{(l)}_{p,q}$ can be expressed via homogeneous polynomials. In the same manner $N_{p,q}$ can also be presented as polynomials. For instance, the extended $l=\frac{3}{2}$ Galilean algebra has the following form

$$\begin{aligned}
&[L_m,L_n]=(m-n)L_{m+n}, \qquad [L_m,M_n]=M_{m+n},\\
&[L_m,C^i_p]=\left(\frac{3m}{2}-p\right)C^i_{m+p}, \qquad [I,C^i_p]=\epsilon^{ij}C^j_p, \qquad m,n=\pm1,0, \quad p,q=\pm\frac{3}{2},\pm\frac{1}{2},\\
&[C^i_p,C^j_q]=\frac{1}{4}\left(9+8pq-6p^2-6q^2\right)\epsilon^{ij}M_{p+q}-\frac{1}{2}(p-q)\left(p^2+q^2-\frac{5}{2}\right)\delta^{ij}N
\end{aligned} \tag{3.8}$$

The structure constantsin the commutator $[C^i_p, C^j_q]$ are in agreement with the relations (3.6) and (3.7). As in the case $l=\frac{1}{2}$ of the extended Schrödinger algebra, we can rewrite the algebra (3.8) in the $3d$ Lorentz-invariant form (see appendix B)

$$
\begin{aligned}
&[J^a, J^b] = \epsilon^{abc} J_c, && [J^a, \mathcal{P}^b] = \epsilon^{abc}\mathcal{P}_c,\\
&[J^a, Z^{b,i}_\alpha] = \frac{3}{2}(Z^{b,i}\gamma^a)_\alpha - (Z^{a,i}\gamma^b)_\alpha, && [I, Z^{a,i}_\alpha] = \epsilon^{ij} Z^{a,j}_\alpha,\\
&[Z^{a,i}_\alpha, Z^{b,j}_\beta] = \epsilon^{ij}\left(-2(C\gamma^c)_{\alpha\beta}\mathcal{P}_c\eta^{ab} + \frac{5}{2}\epsilon^{abc}C_{\alpha\beta}\mathcal{P}_c + \frac{1}{2}(C\gamma^{(a})_{\alpha\beta}\mathcal{P}^{b)}\right)\\
&\qquad + \delta^{ij}\left(\epsilon^{abc}(C\gamma_c)_{\alpha\beta} - 2\eta^{ab}C_{\alpha\beta}\right)N, && \qquad (3.9)
\end{aligned}
$$

where the higher-spin generator Z^{ai}_α is gamma-traceless $Z^{a,i}\gamma_a = 0$.

In the case of a generic half-integer l the number of the generators C^i_p are equal to the number of independent components of a symmetric gamma-traceless tensor $Z^{a_1\ldots a_n i}_\alpha$ with $n = l - \frac{1}{2}$. Hence, by analogy with the hyper-Poincaré algebras [51], we can present the extended l-conformal Galilean algebra in the following form:

$$
\begin{aligned}
&[J^a, J^b] = \epsilon^{abc} J_c, \qquad [J^a, \mathcal{P}^b] = \epsilon^{abc}\mathcal{P}_c, \qquad [I, Z^{a_1\ldots a_n,i}] = \epsilon^{ij} Z^{a_1\ldots a_n,j}\\
&[J^a, Z^{b_1\ldots b_n,i}] = \left(n + \frac{1}{2}\right) Z^{b_1\ldots b_n,i}\gamma^a - Z^{a(b_2\ldots b_n|,i}\gamma^{|b_1)},\\
&[Z^{a_1\ldots a_n,i}_\alpha, Z^{b_1\ldots b_n,j}_\beta] = \epsilon^{ij} f^{a_1\ldots a_n b_1\ldots b_n c}_{\alpha\beta}\mathcal{P}_c + \delta^{ij} N^{a_1\ldots a_n b_1\ldots b_n}_{\alpha\beta} N, \qquad (3.10)
\end{aligned}
$$

where the structure constants are SO(1, 2) invariant tensors constructed with the use of the gamma-matrices, Minkowski metric, Levi-Cevita tensor and the charge conjugation matrix.

3.2 Integer l

For integer l there is no solution for the recurrence relation (3.5). To resolve this issue we should change the commutation relations of C^i_p in (3.8) as follows

$$
[C^i_p, C^j_q] = \delta^{ij} f^{(l)}_{p,q} M_{p+q}.
$$

W will further restrict ourselves to the case $d = 1$ because it is related to $3D$ relativistic systems which is the main topic of this paper. Then the commutation relations for the generators C_p have the following form

$$
[C_p, C_q] = f^{(l)}_{p,q} M_{p+q}. \qquad (3.11)
$$

All the other commutation relations have the same form as in (3.8), while the generator I is dropped out. The nonzero structure constants $f^{(l)}_{p,q}$ are those with $|p + q| \leq 1$. From the Jacobi identities for the set of generators (L_m, C_p, C_q) we find the following constraint[7]

$$
(m - p - q) f^{(l)}_{p,q} + (lm - p) f^{(l)}_{q,p+m} - (lm - q) f^{(l)}_{p,q+m} = 0. \qquad (3.12)
$$

It implies that non-zero structure constants should be related as in (3.6). Explicitly, they are $f^{(l)}_{0,1} = f^{(l)}_{-1,0} = \frac{l}{2} f_{-1,1}$ and

$$
f^{(l)}_{-p,p} = p \prod_{s=1}^{p-1} \frac{l + s + 1}{s - l} f^{(l)}_{-1,1}. \qquad (3.13)
$$

[7]Note that for the half-integer l there is no nontrivial solution of (3.12).

JHEP07(2019)156

In what follows, we normalize $f^{(l)}_{-1,1} = 1$. Again, the structure constants have the form of polynomials.

Let us consider two simple examples of the extended l-conformal Galilean algebra.

Case $l = 1$.

$$\begin{aligned}
&[L_m, L_n] = (m-n)L_{m+n}, && [L_m, M_n] = (m-n)M_{m+n},\\
&[L_m, C_n] = (m-n)C_{m+n}, && \\
&[C_m, C_n] = (m-n)M_{m+n}, && m,n = \pm 1, 0,
\end{aligned} \tag{3.14}$$

where we have also redefined $M_m \to -M_m$ and $C_m \to 2C_m$. This is the Maxwell algebra in three dimensions written in the BMS_3-like basis (see e.g. [52]). In order to present it in the standard Lorentz-invariant form one makes the redefinition as in eq. (C.2) and gets

$$\begin{aligned}
&[J^a, J^b] = \epsilon^{abc} J_c, && [J^a, \mathcal{P}^b] = \epsilon^{abc}\mathcal{P}_c.\\
&[J^a, Z^b] = \epsilon^{abc} Z_c, && [\mathcal{P}^a, \mathcal{P}^b] = \epsilon^{abc} Z_c,
\end{aligned} \tag{3.15}$$

which is a conventional form of the $3D$ Maxwell algebra [53, 54]. The generator $Z_{ab} = \epsilon_{abc}Z^c$ of this albebra is associated with a constant electro-magnetic field strength. Note that the role of Z^a and of the translation generator $\mathcal{P}^a$ can be interchanged, and the algebra takes the form of the simplest Hietarinta algebra [14] used in [55]

$$\begin{aligned}
&[J^a, J^b] = \epsilon^{abc} J_c, && [J^a, \mathcal{P}^b] = \epsilon^{abc}\mathcal{P}_c.\\
&[J^a, Z^b] = \epsilon^{abc} Z_c, && [Z^a, Z^b] = \epsilon^{abc}\mathcal{P}_c.
\end{aligned} \tag{3.16}$$

Case $l = 2$.

$$\begin{aligned}
&[L_m, L_n] = (m-n)L_{m+n}, && [L_m, M_n] = (m-n)L_{m+n},\\
&[L_m, C_p] = (2m-p)C_{m+p} && \\
&[C_p, C_q] = \frac{1}{6}(p-q)(2p^2+2q^2-pq-8)M_{p+q}, && \\
&m,n = \pm 1, 0, && p,q = \pm 2, \pm 1, 0.
\end{aligned} \tag{3.17}$$

In a similar fashion we can rewrite $l = 2$ commutation relations (3.17) in the Lorentz invariant form by redefining the generators as in (C.4)

$$\begin{aligned}
&[J^a, J^b] = \epsilon^{abc} J_c, && [J^a, \mathcal{P}^b] = \epsilon^{abc}\mathcal{P}_c,\\
&[J^a, Z^{bc}] = \epsilon^{da(b} Z^{c)}{}_d, && [Z^{ab}, Z^{cd}] = \mathcal{P}_e \epsilon^{ec(a}\eta^{b)d} + \mathcal{P}_e \epsilon^{ed(b}\eta^{a)c},
\end{aligned} \tag{3.18}$$

where the generator Z^{ab} is symmetric and traceless $Z^{ab}\eta_{ab} = 0$.

Generic integer l. As in the case of the half-integer l one can represent the extended integer l-conformal Galilean algebra in a $3D$ relativistic form by introducing a higher spin generator $Z^{a_1 \dots a_l}$, which is symmetric and traceless. Indeed, the number of generators C_n for a given integer l is equal to $2l+1$, which is exactly the number of independent components of a traceless symmetric tensor of rank l in three dimensions. We thus get the following algebra which is a subclass of the Hietarinta algebras [14]

$$\begin{aligned} [J^a, J^b] &= \epsilon^{abc} J_c, & [J^a, \mathcal{P}^b] &= \epsilon^{abc}\mathcal{P}_c, \\ [J^a, Z^{a_1\dots a_l}] &= \epsilon^{ab(a_1} Z^{a_2\dots a_l)}{}_b, & [Z^{a_1\dots a_l}, Z^{b_1\dots b_l}] &= f^{a_1\dots a_l b_1\dots b_l c}\mathcal{P}_c, \end{aligned} \tag{3.19}$$

where the structure constants are SO$(1,2)$ invariant tensors respecting the tracelessness of $Z^{a_1\dots a_l}$. These algebras can be further extended by relaxing the traceless condition allowing $Z^{a_1\dots a_l}$ be an arbitrary mixed-symmetry tensor.

4 Relativistic gravity models with extended *l*-conformal Galilean symmetry

We shall now construct higher-spin gravity theories which are invariant under local extended l-conformal Galilean symmetry. It can be shown that the l-conformal Galilean algebra has a non-degenerate SO$(1,2)$-invariant bilinear form for any l. However, instead of exploiting the standard Chern-Simons construction requiring the explicit use of the bilinear form, we will write down directly the final action and check its symmetry properties starting from the case of the half-integer l.

4.1 Half-integer *l*

The higher-spin $3D$ gravity action invariant under the local transformations generated by the algebra (3.10) is[8]

$$S = \frac{k}{4\pi}\int_{\mathcal{M}_3}\left(2e^a R_a - \epsilon^{ij}\bar{\lambda}^i_{a_1\dots a_n}\nabla\lambda^{a_1\dots a_n,j} - 2vdb\right), \tag{4.1}$$

where the covariant derivative is defined by

$$\nabla\lambda^{a_1\dots a_n,i} = d\lambda^{a_1\dots a_n,i} + \left(n+\frac{1}{2}\right)\omega^b\gamma^b\lambda^{a_1\dots a_n,i} - \omega^b\gamma^{(a_1}\lambda^{a_2\dots a_n)b,i} - b\epsilon^{ij}\lambda^{a_1\dots a_n,j}, \tag{4.2}$$

and $n = l - \frac{1}{2}$.

For generality, one could also add to the action (4.1) a Chern-Simons term constructed with the spin connection ω^a (see e.g. [57])

$$S_{\mathtt{m}} = \frac{k}{4\pi\,\mathtt{m}}\int_{\mathcal{M}_3}\left(\omega^a d\omega_a + \frac{1}{3}\epsilon_{abc}\omega^a\omega^b\omega^c\right), \tag{4.3}$$

where $\mathtt{m}$ is the parameter of mass dimension.

Note that the addition of (4.3) to (4.1) does not change the non-dynamical nature of the fields in the bulk, in particular $R^a = 0$ on the mass shell, because e^a and ω^a are considered as independent fields. This is in contrast to topologically massive gravity [58, 59] in which the spin connection is *a priori* constructed from the dreibein.

[8]Its form can be read off from the $l = \frac{1}{2}$ action (2.13) and also from the hyper-gravity action in [51, 56].

By construction, in addition to local Poincaré symmetry this theory enjoys gauge symmetry associated to the generators $Z_\alpha^{a_1\dots a_n,i}$, I and N (3.10). Local Poincaré transformations read

$$\delta e^a = d\alpha^a + \epsilon^{abc}(e_b\beta_c + \omega_b\alpha_c), \qquad \delta\omega^a = d\beta^a + \epsilon^{abc}\omega_b\beta_c,$$
$$\delta\lambda^{a_1\dots a_n,i} = -\left(n+\frac{1}{2}\right)\beta^a\gamma_a\lambda^{a_1\dots a_n,i} + \beta_a\gamma^{(a_1}\lambda^{a_2\dots a_n)a,i}, \tag{4.4}$$

where β^a is the gauge parameter, corresponding to the Lorentz rotations J^a, while α^a is the parameter of local translations $\mathcal{P}^a$. Gauge symmetry transformations generated by Z are given by

$$\delta e^a = \left(n+\frac{1}{2}\right)\epsilon^{ij}\bar{\lambda}^{a_1\dots a_n,i}\gamma^a\varepsilon^j_{a_1\dots a_n},$$
$$\delta\lambda^{a_1\dots a_n,i} = \nabla\varepsilon^{a_1\dots a_n,i}, \qquad\qquad \delta v = -\bar{\lambda}^{a_1\dots a_n,i}\varepsilon^i_{a_1\dots a_n}, \tag{4.5}$$

and the gauge parameter is totally symmetric and gamma-traceless. For checkin the invariance of the action under these transformations the following identity is useful

$$\nabla^2\varepsilon^{a_1\dots a_n,i} = \left(n+\frac{1}{2}\right)R^a\gamma_a\varepsilon^{a_1\dots a_n,i} - R_a\gamma^{(a_1}\varepsilon^{a_2\dots a_n)a,i} - \epsilon^{ij}db\varepsilon^{a_1\dots a_n,j}. \tag{4.6}$$

The gauge transformations associated to the generators I and N are

$$\delta\lambda^{a_1\dots a_n,i} = \kappa\epsilon^{ij}\lambda^{a_1\dots a_n,j}, \qquad \delta b = d\kappa, \qquad \delta v = d\varphi, \tag{4.7}$$

where κ and φ are the gauge parameters. The structure of the action (4.1) is very similar to hypergravity theory [51, 56, 60, 61], but it also includes the coupling of the higher-spin fields to the R-Symmetry gauge field b.

4.2 Integer l

Let us now turn to the case of integer l. Again, one may see that there exists a bilinear form for the algebra (3.19), but we found it simpler to construct the higher-spin gravity action without using it explicitly. The action (to which one can also add the CS term (4.3)) has the following form

$$S = \frac{k}{4\pi}\int_{\mathcal{M}_3}\left(2e^aR_a + \lambda^{a_1\dots a_l}\nabla\lambda_{a_1\dots a_l}\right), \tag{4.8}$$

where the covariant derivative is given by

$$\nabla\lambda^{a_1\dots a_l} = d\lambda^{a_1\dots a_l} + \epsilon^{bc(a_1}\lambda_b{}^{a_2\dots a_l)}\omega_c. \tag{4.9}$$

In addition to the local Poincaré symmetry, which is given by the first row in (4.4) and

$$\delta\lambda^{a_1\dots a_l} = -\epsilon^{ab(a_1}\lambda^{a_2\dots a_l)}{}_a\beta_b, \tag{4.10}$$

this action is invariant under the gauge transformations

$$\delta e^a = \epsilon^{abc}\lambda_{bb_2\dots b_l}\varepsilon_c{}^{b_2\dots b_l}, \qquad \delta\lambda^{a_1\dots a_l} = \nabla\varepsilon^{a_1\dots a_l}, \tag{4.11}$$

associated to the generators $Z^{a_1\cdots a_l}$. In the case $l=1$ the action is invariant under local symmetry generated by (3.16) which is 'dual' to the Maxwell algebra (3.15). A 3D gravity model based on the Hietarinta algebra (3.16) and its higher-spin extensions describing 3D gravity coupled to mixed symmetry fields $\lambda^{a_1\cdots a_n}$ were constructed in [55]. Earlier, the 3D gravity model based on the conventional Maxwell algebra (3.15) was constructed and studied in [57, 62, 63].[9]

The most general 'bi-gravity' action based on the algebra (3.16) has the following form

$$S=\frac{k}{4\pi}\int_{\mathcal{M}_3}\left[(2e^aR_a+\lambda^a\nabla\lambda_a)+2\mathtt{a}\,\lambda^aR_a+\frac{1}{\mathtt{m}}\left(\omega^a d\omega_a+\frac{1}{3}\epsilon_{abc}\omega^a\omega^b\omega^c\right)\right], \tag{4.12}$$

where $T^a=De^a$ and $\mathtt{a}$ is a coupling constant parameter in addition to k and $\frac{1}{\mathtt{m}}$.

This action is similar to the Maxwell Chern-Simons gravity action of [57] based on the algebra (3.15) and can be constructed by using the bilinear form

$$\langle J_a,P_b\rangle=\eta_{ab},\qquad \langle Z_a,Z_b\rangle=\eta_{ab},\qquad \langle J_a,J_b\rangle=\frac{1}{\mathtt{m}}\eta_{ab},\qquad \langle J_a,Z_b\rangle=\mathtt{a}\eta_{ab}. \tag{4.13}$$

To pass from one action to another one should swap the one-form fields e^a with λ^a.

4.3 Asymptotic symmetries in $l=\frac{3}{2}$, $l=1$ and $l=2$ cases

In this section we will study the asymptotic symmetry of the extended gravity theories described by the actions (4.1) and (4.8) for the cases $l=\frac{3}{2}$ and $l=1,2$.

Case $l=\frac{3}{2}$. As in the case $l=\frac{1}{2}$ of the extended Schrödinger algebra discussed in section 2.3, we may relax the boundary conditions and allow for the additional fields to have nonzero excitations, defining the boundary conditions in such a way that they include the BMS_3 ones. For simplicity, for the $l=\frac{3}{2}$ case we assume that the central charge in the algebra (3.8) is zero and take the boundary conditions in the form

$$\mathfrak{a}_\phi=\mathfrak{a}^0_\phi+\frac{1}{3}\mathcal{C}^iC^i_{-\frac{3}{2}},\qquad \mathfrak{a}_t=\mathfrak{a}^0_t, \tag{4.14}$$

where $\mathfrak{a}^0_\phi$ and $\mathfrak{a}^0_t$ are given in (2.24). In order to satisfy the equations of motion we still need the functions $\mathcal{L}$ and $\mathcal{M}$ in (2.24) to be related as in (2.26), while $\mathcal{C}$ should be a function of ϕ only. The same restrictions are imposed on the functions describing asymptotic dynamics in the cases $l=1,2$, which we will study below. One may notice a close similarity between these boundary conditions and the ones in the $N=1$ supergravity [28] or in the hyper-gravity theories [51]. The algebra-valued parameter $\boldsymbol{\lambda}$, which generates the transformation preserving these boundary conditions, is given by

$$\begin{aligned}\boldsymbol{\lambda}=&\,L_{+1}\varepsilon_L-L_0\varepsilon'_L+\left(\frac{1}{2}\varepsilon''_L-\varepsilon_L\mathcal{M}\right)L_{-1}+\left(\frac{1}{2}\varepsilon''_M-\varepsilon_L\mathcal{L}-\varepsilon_M\mathcal{M}-\frac{3}{2}\mathcal{C}^i\varepsilon^i\right)M_{-1}\\&+M_{+1}\varepsilon_M-M_0\varepsilon'_M+\epsilon^{ij}\left(C^i_{+\frac{3}{2}}\varepsilon^j-C^i_{+\frac{1}{2}}\varepsilon'^j\right)+\epsilon^{ij}C^i_{-\frac{1}{2}}\left(\frac{1}{2}\varepsilon''^j-\frac{3}{2}\mathcal{M}\varepsilon^j\right)\\&+\epsilon^{ij}C^i_{-\frac{3}{2}}\left(\frac{1}{3}\mathcal{C}^i\varepsilon_L+\frac{1}{2}\mathcal{M}'\varepsilon^j+\frac{7}{6}\mathcal{M}\varepsilon'^j-\frac{1}{6}\varepsilon'''^j\right).\end{aligned} \tag{4.15}$$

[9] Higher-spin extensions of the Maxwell algebra and corresponding gravity models were considered in [64]. See also [65] for a detailed study of the 3D Maxwell group, its infinite-dimensional extensions, applications and additional references.

The requirement that the components of the gauge field $\mathfrak{a}_t$ along the time direction be preserved by the same transformation implies that the parameters ε_L and ε_M are related as in (2.31), while ε^i is a time independent function. Following the steps of section 2.3, one finds the asymptotic symmetry algebra

$$
\begin{aligned}
[L_m, L_n] &= i(m-n)L_{m+n}, \qquad\qquad [L_m, C_p^i] = i\left(\frac{3m}{2}-p\right)C^i_{m+p},\\
[L_m, M_n] &= i(m-n)M_{m+n} - ikn^3\delta_{m+n,0},\\
[C_p^i, C_q^j] &= \left(2pq - \frac{3}{2}p^2 - \frac{3}{2}q^2\right)\epsilon^{ij}M_{p+q} - \frac{9}{4k}\sum_s M_{p+q-s}M_s\epsilon^{ij} - \epsilon^{ij}kp^4\delta_{p+q,0},
\end{aligned}
\tag{4.16}
$$

where we have only wrote the non-zero commutators.

In contrast to the case $l=\frac{1}{2}$, the algebra involves a nonlinear term, which is common for asymptotic symmetry algebras of higher-spin gravity theories (see [8] and references therein).

Case $l = 1$. The structure of the asymptotic symmetry of gravity with the gauged Maxwell symmetry (3.15) was studied in [52]. As we mentioned above, in this case the roles of the generator Z_a the translation generator P_a, and of the corresponding spin-2 fields get interchanged in comparison to the $l=1$ algebra (3.16) and the gravity action (4.12). As such, in the latter case we have the different definition of the 3D space-time and different boundary conditions (see (C.2) for the redefinition of the generators of (3.16))

$$
\mathfrak{a}_\phi = \mathfrak{a}^0_\phi - \mathcal{C}(\phi)C_{-1}, \qquad \mathfrak{a}_t = \mathfrak{a}^0_t,
\tag{4.17}
$$

where $\mathfrak{a}^0_\phi$ and $\mathfrak{a}^0_t$ are given in (2.24).

The corresponding parameter of the transformations compatible with these boundary conditions are

$$
\begin{aligned}
\boldsymbol{\lambda} = &\left(\frac{1}{2}\varepsilon''_L - \varepsilon_L\mathcal{M}\right)L_{-1} + \left(\frac{1}{2}\varepsilon''_M - \varepsilon_L\mathcal{L} - \varepsilon_M\mathcal{M} - \mathcal{C}\varepsilon\right)M_{-1}\\
&+\left(\frac{1}{2}\varepsilon'' - \mathcal{M}\varepsilon - \mathcal{C}\varepsilon_L\right) + L_{+1}\varepsilon_L - L_0\varepsilon'_L + M_{+1}\varepsilon_M - M_0\varepsilon'_M + C_{+1}\varepsilon - C_0\varepsilon',
\end{aligned}
\tag{4.18}
$$

where the parameters ε_L and ε_M are related as in (2.31) and ε is time independent. As a result, we get the asymptotic symmetry algebra similar to that in [52] but with the interchanged role of the generators M_n and C_n

$$
\begin{aligned}
[L_m, L_n] &= i(m-n)L_{m+n} - \frac{ik}{\mathtt{m}}n^3\delta_{m+n,0},\\
[L_m, C_n] &= i\,(m-n)\,C_{m+n} - i\mathtt{a}kn^3\delta_{m+n,0},\\
[L_m, M_n] &= i(m-n)M_{m+n} - ikn^3\delta_{m+n,0},\\
[C_m, C_n] &= i(m-n)M_{m+n} - ikn^3\delta_{m+n,0},\\
[M_m, M_n] &= 0 = [M_m, C_n]\,,
\end{aligned}
\tag{4.19}
$$

where the central charge in the first line depends on the mass parameter $\mathtt{m}$ of the CS spin-connection term and the central charge in the second line is proportional to the coupling constant $\mathtt{a}$ associated with the second Einstein-like term in the action (4.12).

JHEP07(2019)156

Case $l = 2$. Though the theories with integer and half-integer l have different properties and field content, as we have seen previously, they have boundary conditions of a very similar form. For the case $l = 2$ we have

$$\mathfrak{a}_\phi = \mathfrak{a}_\phi^0 + \mathcal{C}(\phi) C_{-2}, \qquad \mathfrak{a}_t = \mathfrak{a}_t^0. \tag{4.20}$$

The parameter of the transformations (2.29) preserving these boundary conditions has the following form

$$\begin{aligned}\boldsymbol{\lambda} = {} & L_{+1}\varepsilon_L - L_0\varepsilon_L' + \left(\frac{1}{2}\varepsilon_L'' - \varepsilon_L\mathcal{M}\right) L_{-1} + \left(\frac{1}{2}\varepsilon_M'' - \varepsilon_L\mathcal{L} - \varepsilon_M\mathcal{M} + 4\mathcal{C}\varepsilon\right) M_{-1} \\ & + M_{+1}\varepsilon_M - M_0\varepsilon_M' + C_{+2}\varepsilon - C_{+1}\varepsilon' + C_0\left(\frac{1}{2}\varepsilon'' - 2\mathcal{M}\varepsilon\right) \\ & + C_{-1}\left(-\frac{1}{6}\varepsilon''' + \frac{2}{3}\mathcal{M}'\varepsilon + \frac{5}{3}\mathcal{M}\varepsilon'\right) \\ & + C_{-2}\left(\mathcal{C}\varepsilon_L - \frac{1}{6}\mathcal{M}''\varepsilon - \frac{2}{3}\mathcal{M}\varepsilon'' - \frac{7}{12}\mathcal{M}'\varepsilon' + \mathcal{M}^2\varepsilon\right),\end{aligned} \tag{4.21}$$

where, again, the parameters ε_L and ε_M are related as in (2.31) and $\varepsilon = \varepsilon(\phi)$. The corresponding asymptotic symmetry algebra is

$$\begin{aligned} &[L_m, L_n] = i(m-n)L_{m+n}, \qquad\qquad [L_m, C_p] = i\,(2m-p)\,C_{m+p}, \\ &[L_m, M_n] = i(m-n)M_{m+n} - ikn^3\delta_{m+n,0}, \\ &[C_p, C_q] = (p-q)\left(pq - 2p^2 - 2q^2\right) M_{p+q} - \frac{8}{k}(p-q)\sum_s M_{p+q-s}M_s + kq^5\delta_{p+q,0}, \end{aligned} \tag{4.22}$$

which also has the nonlinear term.

As in the case $l = \frac{3}{2}$, our boundary conditions for $l = 1, 2$ are similar to those in supergravity theories [28, 51], but with a fermionic generator term replaced by the bosonic one associated to the generator C_{-l}, as in (4.20). The above consideration can be extended to the case of arbitrary l for which a suitable choice of boundary conditions should lead to asymptotic symmetries whose algebra is a generalization of (4.16), (4.19) and (4.22).

5 Conclusion

We have shown that the extended Schrödinger algebra and the corresponding Chern-Simons action describing the conformal non-projectable Hořava-Lifshitz gravity constructed in [1], can be viewed as an extended $3D$ Poincaré algebra allowing one to rewrite the Chern-Simons action of [1] in a manifestly $3D$ Lorentz-invariant form. So with a different (relativistic) choice of boundary conditions the same Chern-Simons action describes a relativistic $3D$ theory coupled to two spin-1 gauge fields and a doublet of bosonic spin-3/2 fields. We have shown that the above theory can be regarded as an asymptotic flat-space contraction (and truncation) of the $\mathrm{SU}(1,2) \times \mathrm{SU}(1,2)$ Chern-Simons theory with a non principle embedding of $\mathrm{SL}(2,\mathbb{R})$ into $\mathrm{SU}(1,2)$. The asymptotic symmetry algebra of this theory has the form (2.35). Because of the spin-statistics correspondence for the spin-3/2 fields

JHEP07(2019)156

and the corresponding generators of the gauge symmetry, the asymptotic states on the $2d$ boundary are, in general, not unitary, unless the algebra and the spectrum of states are further truncated. It would be of interest to analyze a similar issue for the non-relativistic choice of the metric and corresponding boundary conditions associated with the conformal non-projectable Hořava-Lifshitz gravity of [1]. This study can be carried out following the lines of [66, 67] which considered the most general boundary conditions in $3D$ gravity. In this way one may expect to obtaine a centrally extended affine version of the extended Schrödinger algebra at the boundary and a corresponding field spectrum describing excitations around the $z = 2$ Lifshitz geometries found in [1].

We have also constructed extensions of l-conformal Galiean algebras (with $l = 1/2$ referring to the Schrödinger algebra) and corresponding relativistic higher-spin gravity theories, and derived their asymptotic symmetries for the cases of $l = \frac{3}{2}$ and $l = 2$. In this regard, it would be of interest to study whether and how these theories can be obtained by an asymptotic flat-space contraction of conventional Chern-Simons higher-spin gravities and their asymptotic W-algebras. These issues will be considered elsewhere.

Acknowledgments

The authors are grateful to Andrea Campoleoni, Daniel Grumiller, Jelle Hartong, Radoslav Rashkov, Patricio Salgado-Rebolledo and Evgeny Skvortsov for interest to this work and useful discussions. D.C. would like to thank INFN, Sezione di Padova, for hospitality during initial stages of this work. D.S. acknowledges support and hospitality extended to him at the ESI (Vienna) Program "Higher spins and holography" (March 11-22, 2019) and at the School of Physics and Astrophysics, University of Western Australia where part of this work was done. Work of D.C. was supported by the Russian Science Foundation, grant No 19-11-00005. Work of D.S. was supported in part by the Australian Research Council project No. DP160103633.

A Conventions

Our conventions are such that the Minkowski metric is given in null coordinates, in which the only nontrivial components of the metric are $\eta^{+-} = \eta^{-+} = \eta^{22} = 1$. Accordingly, the gamma-matrices are given by

$$\gamma^- = \sqrt{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \gamma^+ = \sqrt{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A.1}$$

and satisfy the identities

$$\gamma^a\gamma^b = \eta^{ab} + \epsilon^{abc}\gamma_c, \qquad (\gamma_a)^\alpha{}_\beta(\gamma^a)^\rho{}_\sigma = 2\delta^\alpha_\sigma\delta^\rho_\beta - \delta^\alpha_\beta\delta^\rho_\sigma, \tag{A.2}$$

where $\epsilon^{-+2} = 1$. We define the conjugate spinor as $\bar{\lambda}_\alpha = C_{\alpha\beta}\lambda^\beta$, where the conjugation matrix is given by $C_{\alpha\beta} = \epsilon_{\alpha\beta}$ with $\epsilon_{12} = 1$. Hence, the conjugation matrix is real and antisymmetric, while its product with a gamma-matrix is symmetric $(C\gamma^a)_{\alpha\beta} = (C\gamma^a)_{\beta\alpha}$.

Throughout the text round brackets denote symmetrization of the indices enclosed by them without a normalization factor, e.g.

$$\lambda^{(ab)} = \lambda^{ab} + \lambda^{ba}. \tag{A.3}$$

B $su(1,2)$ and $sl(3,R)$ algebras

The commutation relations

$$\begin{aligned} &[L_m, L_n] = (m-n)L_{m+n}, \qquad [L_m, W_p] = (2m-p)W_{m+p}, \\ &[W_p, W_q] = \frac{\sigma}{3}(p-q)(2p^2+2q^2-pq-8)L_{p+q}, \end{aligned} \tag{B.1}$$

with $m,n=\pm1,0$ and $p,q=\pm2,\pm1,0$, represent the $su(1,2)$ algebra for $\sigma=1$ and $sl(3,R)$ one for $\sigma=-1$. The corresponding bilinear form is

$$\begin{aligned} &\langle L_{-1}, L_{+1}\rangle = -1, \qquad \langle L_0, L_0\rangle = \frac{1}{2}, \\ &\langle W_{-1}, W_{+1}\rangle = \sigma, \qquad \langle W_{-2}, W_{+2}\rangle = -4\sigma, \qquad \langle W_0, W_0\rangle = -\frac{2\sigma}{3} \end{aligned} \tag{B.2}$$

The $sl(2,R)$ subalgebra is generated by $(L_{\pm1}, L_0)$ and this embedding of $sl(2,R)$ algebra into $su(1,2)$ is known as principal. The non-principle embedding is obtained by the following choice of the $sl(2,R)$ generators

$$\mathcal{L}_{-1} = \frac{\sigma}{4}W_{-2}, \qquad \mathcal{L}_0 = \frac{1}{2}L_0, \qquad \mathcal{L}_{+1} = \frac{1}{4}W_{+2}. \tag{B.3}$$

And upon the following redefinition of the rest of the generators

$$\mathcal{I} = -\frac{1}{2}W_0, \quad \mathcal{C}^1_{+\frac{1}{2}} = \frac{\sigma}{2}L_{+1}, \quad \mathcal{C}^2_{+\frac{1}{2}} = \frac{\sigma}{2}W_{+1}, \quad \mathcal{C}^1_{-\frac{1}{2}} = \frac{1}{2}W_{-1}, \quad \mathcal{C}^2_{-\frac{1}{2}} = \frac{\sigma}{2}L_{-1}, \tag{B.4}$$

one transforms the algebra (B.1) to the form

$$\begin{aligned} &[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n}, \qquad [\mathcal{L}_m, \mathcal{C}^i_p] = \left(\frac{m}{2}-p\right)\mathcal{C}^i_{m+p} \\ &[\mathcal{C}^i_p, \mathcal{C}^j_q] = \epsilon^{ij}\mathcal{L}_{p+q} - \frac{3}{2}\eta^{ij}(p-q)\mathcal{I}, \qquad [\mathcal{I}, \mathcal{C}^i_p] = \epsilon^{ij}\mathcal{C}^j_p, \end{aligned} \tag{B.5}$$

Here $\eta^{ij} = diag(\sigma, 1)$ and the summation over the indices (i,j) is performed with respect to this metric. The difference between the $sl(3,R)$ and $su(1,2)$ algebra is that in $sl(3,R)$ the generator $\mathcal{I}$ is associated with a non-compact $so(1,1)$ subalgebra, while in $su(1,2)$ it generates $so(2)$ rotations. For $\sigma=1$ the commutation relations (B.5) defining $su(1,2)$ can be written in the form (2.16) upon the redefinition

$$\begin{aligned} &\mathcal{J}^- = -\frac{1}{\sqrt{2}}\mathcal{L}_{-1}, \qquad \mathcal{J}^+ = \frac{1}{\sqrt{2}}\mathcal{L}_{+1}, \qquad \mathcal{J}^2 = \mathcal{L}_0, \\ &\mathcal{Z}^i_1 = \frac{1}{\sqrt{2}}\mathcal{C}^i_{-\frac{1}{2}}, \qquad \mathcal{Z}^i_2 = \frac{1}{\sqrt{2}}\mathcal{C}^i_{+\frac{1}{2}}. \end{aligned} \tag{B.6}$$

JHEP07(2019)156

C 3D Lorentz-covariant form of the extended l-conformal Galilean algebra

Here we list the redefinition of the generators which bring the commutation relations of the extended l-conformal Galilean algebra (for $l = 1, \frac{3}{2}$ and 2) to the Lorentz-covariant form. For the each case the redefinition of the generators of the conformal subalgebra is

$$\sqrt{2}J^- = -L_{-1}, \quad \sqrt{2}J^+ = L_{+1}, \quad J^2 = L_0, \tag{C.1}$$

Redefinitions of all the other generators are:

- $l = 1$, from (3.14) to (3.15)

$$\begin{aligned} &\mathcal{P}^- = -\sqrt{2}M_{-1}, && \mathcal{P}^+ = \sqrt{2}M_{+1}, && \mathcal{P}^2 = 2M_0, \\ &Z^- = C_{-1}, && Z^+ = -C_{+1}, && Z^2 = -\sqrt{2}C_0. \end{aligned} \tag{C.2}$$

- $l = \frac{3}{2}$, from (3.8) to (3.9)

$$\begin{aligned} &\mathcal{P}^- = -\sqrt{2}M_{-1}, \quad \mathcal{P}^+ = \sqrt{2}M_{+1}, \quad \mathcal{P}^2 = 2M_0, \\ &Z_1^{-,i} = C^i_{-\frac{3}{2}}, \quad Z_2^{-,i} = C^i_{-\frac{1}{2}}, \quad Z_1^{+,i} = -C_{+\frac{1}{2}}, \quad Z_2^{+,i} = -C_{+\frac{3}{2}}. \end{aligned} \tag{C.3}$$

  Note also that the condition $(Z^{a,i}\gamma_a)^\alpha = 0$ implies that $\sqrt{2}Z_1^{+,i} = Z_2^{2,i}$ and $-\sqrt{2}Z_2^{-,i} = Z_1^{2,i}$.

- $l = 2$, from (3.17) to (3.18)

$$\begin{aligned} &Z^{--} = C_{-2}, \quad Z^{-+} = -C_0, \quad Z^{-2} = -\sqrt{2}C_{-1}, \quad Z^{++} = C_2, \quad Z^{+2} = \sqrt{2}C_1, \\ &\mathcal{P}^- = -\sqrt{2}M_{-1}, \quad \mathcal{P}^+ = \sqrt{2}M_{+1}, \quad \mathcal{P}^2 = 2M_0. \end{aligned} \tag{C.4}$$

  The traceless condition $Z^{ab}\eta_{ab} = 0$ implies $Z^{22} = -2Z^{-+}$.

D Contraction of $\boldsymbol{W^{(2)}_{1,2} \oplus W^{(2)}_{1,2}}$ algebra

The contraction of the $W_3^{(2)} \oplus W_3^{(2)}$ was considered in [68]. It is reasonable to expect that the contraction of the direct product of two $W^{(2)}_{1,2}$ algebras with the finite-dimensional subalgebra $su(1,2)$ leads to the asymptotic symmetry algebra (2.35). The $W^{(2)}_{1,2}$ has the following form

$$\begin{aligned} [\mathcal{L}_m, \mathcal{L}_n] &= (m-n)\mathcal{L}_{m+n} - \frac{\rho}{2}kn^3\delta_{m+n,0}, \\ [\mathcal{L}_m, \mathcal{I}_n] &= -n\mathcal{I}_{m+n}, \\ [\mathcal{I}_m, \mathcal{I}_n] &= \frac{2}{3}k\rho m\delta_{m+n,0}, \\ [\mathcal{L}_m, \mathcal{C}^i_p] &= \left(\frac{m}{2} - p\right)\mathcal{C}^i_{m+p}, \\ [\mathcal{I}_m, \mathcal{C}^i_p] &= -\epsilon^{ij}\mathcal{C}^j_{m+p}, \\ [\mathcal{C}^i_p, \mathcal{C}^j_q] &= -\epsilon^{ij}\left(\mathcal{L}_{p+q} - \frac{3}{k\rho}\sum_s \mathcal{I}_{p+q-s}\mathcal{I}_s + k\rho p^2\delta_{p+q,0}\right) - \frac{3}{2}\delta^{ij}(p-q)\mathcal{I}_{p+q}, \end{aligned} \tag{D.1}$$

where ρ is a parameter proportional to the AdS_3 radius.

Let us take two copies of the algebra differed by $\pm$ superscript and define

$$L_m = i(\mathcal{L}^+_m - \mathcal{L}^-_{-m}), \qquad M_m = \frac{1}{\rho}(\mathcal{L}^+_m + \mathcal{L}^-_{-m}), \qquad C^i_p = \sqrt{\frac{2}{\rho}}\mathcal{C}^{+,i}_p,$$
$$I_m = \mathcal{I}^+_m - \mathcal{I}^-_{-m}, \qquad N_m = \frac{2}{3}\frac{i}{\rho}(\mathcal{I}^+_m + \mathcal{I}^-_{-m}). \tag{D.2}$$

Taking the limit $\rho \to \infty$ and truncating the generators $C^{-,i}_p$ one finds the algebra of the form (2.35) except for the commutators

$$[M_m, I_n] = \frac{2}{3} in N_{m+n}, \tag{D.3}$$
$$[C^i_p, C^j_q] = -\epsilon^{ij}\left(M_{p+q} + \frac{3}{2k}\sum_s N_{p+q-s}N_s + 2kq^2\delta_{p+q,0}\right) + i(p-q)N_{p+q}\delta^{ij}.$$

Making the redefinition $M_m \to M_m - \frac{3}{2k}\sum_s N_{m-s}N_s$ these commutation relations take the form of (2.35). In particular, note that this redefinition keeps intact the form of the commutator $[L_m, M_n]$ in (2.35).

References

[1] J. Hartong, Y. Lei and N.A. Obers, *Nonrelativistic Chern-Simons theories and three-dimensional Hořava-Lifshitz gravity*, *Phys. Rev.* **D 94** (2016) 065027 [arXiv:1604.08054] [INSPIRE].

[2] A. Achucarro and P.K. Townsend, *A Chern-Simons action for three-dimensional anti-de Sitter supergravity theories*, *Phys. Lett.* **B 180** (1986) 89 [INSPIRE].

[3] S. Prohazka, *Chern-Simons holography: boundary conditions, contractions and double extensions for a journey beyond anti-de Sitter*, Ph.D. thesis, Vienna Tech. U., Vienna, Austria (2017) [arXiv:1710.11110] [INSPIRE].

[4] G. Papageorgiou and B.J. Schroers, *A Chern-Simons approach to Galilean quantum gravity in* 2 + 1 *dimensions*, *JHEP* **11** (2009) 009 [arXiv:0907.2880] [INSPIRE].

[5] M. Gary, D. Grumiller and R. Rashkov, *Towards non-AdS holography in* 3*-dimensional higher spin gravity*, *JHEP* **03** (2012) 022 [arXiv:1201.0013] [INSPIRE].

[6] M. Gary, D. Grumiller, S. Prohazka and S.-J. Rey, *Lifshitz holography with isotropic scale invariance*, *JHEP* **08** (2014) 001 [arXiv:1406.1468] [INSPIRE].

[7] Y. Lei and S.F. Ross, *Connection versus metric description for non-AdS solutions in higher-spin theories*, *Class. Quant. Grav.* **32** (2015) 185005 [arXiv:1504.07252] [INSPIRE].

[8] A. Campoleoni, S. Fredenhagen, S. Pfenninger and S. Theisen, *Asymptotic symmetries of three-dimensional gravity coupled to higher-spin fields*, *JHEP* **11** (2010) 007 [arXiv:1008.4744] [INSPIRE].

JHEP07(2019)156

[9] A. Campoleoni, S. Fredenhagen and S. Pfenninger, *Asymptotic W-symmetries in three-dimensional higher-spin gauge theories*, JHEP **09** (2011) 113 [arXiv:1107.0290] [INSPIRE].

[10] M. Ammon, M. Gutperle, P. Kraus and E. Perlmutter, *Spacetime geometry in higher spin gravity*, JHEP **10** (2011) 053 [arXiv:1106.4788] [INSPIRE].

[11] A. Castro, E. Hijano and A. Lepage-Jutier, *Unitarity bounds in AdS_3 higher spin gravity*, JHEP **06** (2012) 001 [arXiv:1202.4467] [INSPIRE].

[12] J. Negro, M.A. del Olmo and A. Rodríguez-Marco, *Nonrelativistic conformal groups*, J. Math. Phys. **38** (1997) 3786.

[13] M. Henkel, *Local scale invariance and strongly anisotropic equilibrium critical systems*, Phys. Rev. Lett. **78** (1997) 1940 [cond-mat/9610174] [INSPIRE].

[14] J. Hietarinta, *Supersymmetry generators of arbitrary spin*, Phys. Rev. **D 13** (1976) 838 [INSPIRE].

[15] H. Afshar, M. Gary, D. Grumiller, R. Rashkov and M. Riegler, *Semi-classical unitarity in 3-dimensional higher-spin gravity for non-principal embeddings*, Class. Quant. Grav. **30** (2013) 104004 [arXiv:1211.4454] [INSPIRE].

[16] C. Bunster, M. Henneaux, A. Perez, D. Tempo and R. Troncoso, *Generalized black holes in three-dimensional spacetime*, JHEP **05** (2014) 031 [arXiv:1404.3305] [INSPIRE].

[17] O. Khasanov and S. Kuperstein, *(In)finite extensions of algebras from their Inonu-Wigner contractions*, J. Phys. **A 44** (2011) 475202 [arXiv:1103.3447] [INSPIRE].

[18] D. Hansen, J. Hartong and N.A. Obers, *Gravity between Newton and Einstein*, arXiv:1904.05706 [INSPIRE].

[19] E. Bergshoeff, J.M. Izquierdo, T. Ortín and L. Romano, *Lie algebra expansions and actions for non-relativistic gravity*, arXiv:1904.08304 [INSPIRE].

[20] J. Hartong, Y. Lei, N.A. Obers and G. Oling, *Zooming in on AdS_3/CFT_2 near a BPS bound*, JHEP **05** (2018) 016 [arXiv:1712.05794] [INSPIRE].

[21] H. Afshar, A. Bagchi, R. Fareghbal, D. Grumiller and J. Rosseel, *Spin-3 gravity in three-dimensional flat space*, Phys. Rev. Lett. **111** (2013) 121603 [arXiv:1307.4768] [INSPIRE].

[22] H.A. Gonzalez, J. Matulich, M. Pino and R. Troncoso, *Asymptotically flat spacetimes in three-dimensional higher spin gravity*, JHEP **09** (2013) 016 [arXiv:1307.5651] [INSPIRE].

[23] J. Matulich, A. Perez, D. Tempo and R. Troncoso, *Higher spin extension of cosmological spacetimes in 3D: asymptotically flat behaviour with chemical potentials and thermodynamics*, JHEP **05** (2015) 025 [arXiv:1412.1464] [INSPIRE].

[24] M. Bershadsky, *Conformal field theories via Hamiltonian reduction*, Commun. Math. Phys. **139** (1991) 71 [INSPIRE].

[25] A.M. Polyakov, *Gauge transformations and diffeomorphisms*, Int. J. Mod. Phys. **A 5** (1990) 833 [INSPIRE].

[26] A. Ashtekar, J. Bicak and B.G. Schmidt, *Asymptotic structure of symmetry reduced general relativity*, Phys. Rev. **D 55** (1997) 669 [gr-qc/9608042] [INSPIRE].

JHEP07(2019)156

[27] G. Barnich and G. Compere, *Classical central extension for asymptotic symmetries at null infinity in three spacetime dimensions*, Class. Quant. Grav. **24** (2007) F15 [gr-qc/0610130] [INSPIRE].

[28] G. Barnich, L. Donnay, J. Matulich and R. Troncoso, *Asymptotic symmetries and dynamics of three-dimensional flat supergravity*, JHEP **08** (2014) 071 [arXiv:1407.4275] [INSPIRE].

[29] A. Bagchi, S. Detournay, R. Fareghbal and J. Simón, *Holography of* 3*D flat cosmological horizons*, Phys. Rev. Lett. **110** (2013) 141302 [arXiv:1208.4372] [INSPIRE].

[30] G. Barnich, A. Gomberoff and H.A. Gonzalez, *The flat limit of three dimensional asymptotically anti-de Sitter spacetimes*, Phys. Rev. **D 86** (2012) 024020 [arXiv:1204.3288] [INSPIRE].

[31] O. Fuentealba, J. Matulich and R. Troncoso, *Asymptotic structure of* $N = 2$ *supergravity in* 3*D: extended super-BMS*$_3$ *and nonlinear energy bounds*, JHEP **09** (2017) 030 [arXiv:1706.07542] [INSPIRE].

[32] M. Bañados, *Global charges in Chern-Simons field theory and the* $(2+1)$ *black hole*, Phys. Rev. **D 52** (1996) 5816 [hep-th/9405171] [INSPIRE].

[33] A. Bagchi, R. Gopakumar, I. Mandal and A. Miwa, *GCA in* 2*d*, JHEP **08** (2010) 004 [arXiv:0912.1090] [INSPIRE].

[34] A. Bagchi and R. Fareghbal, *BMS/GCA redux: towards flatspace holography from non-relativistic symmetries*, JHEP **10** (2012) 092 [arXiv:1203.5795] [INSPIRE].

[35] A. Campoleoni, H.A. Gonzalez, B. Oblak and M. Riegler, *BMS modules in three dimensions*, Int. J. Mod. Phys. **A 31** (2016) 1650068 [arXiv:1603.03812] [INSPIRE].

[36] A. Bagchi, A. Saha and Zodinmawia, *BMS characters and modular invariance*, arXiv:1902.07066 [INSPIRE].

[37] D. Martelli and Y. Tachikawa, *Comments on Galilean conformal field theories and their geometric realization*, JHEP **05** (2010) 091 [arXiv:0903.5184] [INSPIRE].

[38] A. Bagchi and R. Gopakumar, *Galilean conformal algebras and AdS/CFT*, JHEP **07** (2009) 037 [arXiv:0902.1385] [INSPIRE].

[39] C. Duval and P.A. Horvathy, *Non-relativistic conformal symmetries and Newton-Cartan structures*, J. Phys. **A 42** (2009) 465206 [arXiv:0904.0531] [INSPIRE].

[40] C. Duval and P. Horvathy, *Conformal Galilei groups, Veronese curves and Newton-Hooke spacetimes*, J. Phys. **A 44** (2011) 335203 [arXiv:1104.1502] [INSPIRE].

[41] S. Fedoruk, E. Ivanov and J. Lukierski, *Galilean conformal mechanics from nonlinear realizations*, Phys. Rev. **D 83** (2011) 085013 [arXiv:1101.1658] [INSPIRE].

[42] A. Galajinsky and I. Masterov, *Remarks on* l*-conformal extension of the Newton-Hooke algebra*, Phys. Lett. **B 702** (2011) 265 [arXiv:1104.5115] [INSPIRE].

[43] A. Galajinsky and I. Masterov, *Dynamical realization of* l*-conformal Galilei algebra and oscillators*, Nucl. Phys. **B 866** (2013) 212 [arXiv:1208.1403] [INSPIRE].

[44] K. Andrzejewski, J. Gonera, P. Kosiński and P. Maślanka, *On dynamical realizations of* l*-conformal Galilei groups*, Nucl. Phys. **B 876** (2013) 309 [arXiv:1305.6805] [INSPIRE].

[45] D. Chernyavsky and A. Galajinsky, *Ricci-flat spacetimes with* l*-conformal Galilei symmetry*, Phys. Lett. **B 754** (2016) 249 [arXiv:1512.06226] [INSPIRE].

[46] D. Chernyavsky, *Coset spaces and Einstein manifolds with l-conformal Galilei symmetry*, *Nucl. Phys.* **B 911** (2016) 471 [arXiv:1606.08224] [INSPIRE].

[47] A. Fahad, S.I. Phillip and M. Ian, *On Casimir operators of conformal Galilei algebras*, *J. Math. Phys.* **60** (2019) 013509 [arXiv:1809.06666] [INSPIRE].

[48] A. Galajinsky and I. Masterov, *Casimir operators of centrally extended l-conformal Galilei algebra*, *Nucl. Phys.* **B 943** (2019) 114618 [arXiv:1902.08012] [INSPIRE].

[49] A. Galajinsky and S. Krivonos, *$N = 4$ l-conformal Galilei superalgebras inspired by $D(2,1;\alpha)$ supermultiplets*, *JHEP* **09** (2017) 131 [arXiv:1706.08300] [INSPIRE].

[50] A. Galajinsky and I. Masterov, *$N = 4$ l-conformal Galilei superalgebra*, *Phys. Lett.* **B 771** (2017) 401 [arXiv:1705.02814] [INSPIRE].

[51] O. Fuentealba, J. Matulich and R. Troncoso, *Asymptotically flat structure of hypergravity in three spacetime dimensions*, *JHEP* **10** (2015) 009 [arXiv:1508.04663] [INSPIRE].

[52] P. Concha, N. Merino, O. Mišković, E. Rodríguez, P. Salgado-ReboLledó and O. Valdivia, *Asymptotic symmetries of three-dimensional Chern-Simons gravity for the Maxwell algebra*, *JHEP* **10** (2018) 079 [arXiv:1805.08834] [INSPIRE].

[53] H. Bacry, P. Combe and J.L. Richard, *Group-theoretical analysis of elementary particles in an external electromagnetic field. 1. The relativistic particle in a constant and uniform field*, *Nuovo Cim.* **A 67** (1970) 267 [INSPIRE].

[54] R. Schrader, *The Maxwell group and the quantum theory of particles in classical homogeneous electromagnetic fields*, *Fortsch. Phys.* **20** (1972) 701 [INSPIRE].

[55] S. Bansal and D. Sorokin, *Can Chern-Simons or Rarita-Schwinger be a Volkov-Akulov Goldstone?*, *JHEP* **07** (2018) 106 [arXiv:1806.05945] [INSPIRE].

[56] O. Fuentealba, J. Matulich and R. Troncoso, *Extension of the Poincaré group with half-integer spin generators: hypergravity and beyond*, *JHEP* **09** (2015) 003 [arXiv:1505.06173] [INSPIRE].

[57] P. Salgado, R.J. Szabo and O. Valdivia, *Topological gravity and transgression holography*, *Phys. Rev.* **D 89** (2014) 084077 [arXiv:1401.3653] [INSPIRE].

[58] S. Deser, R. Jackiw and S. Templeton, *Three-dimensional massive gauge theories*, *Phys. Rev. Lett.* **48** (1982) 975 [INSPIRE].

[59] S. Deser, R. Jackiw and S. Templeton, *Topologically massive gauge theories*, *Annals Phys.* **140** (1982) 372 [*Erratum ibid.* **185** (1988) 406] [*Annals Phys.* **281** (2000) 409] [INSPIRE].

[60] C. Aragone and S. Deser, *Higher spin vierbein gauge fermions and hypergravities*, *Nucl. Phys.* **B 170** (1980) 329 [INSPIRE].

[61] R. Rahman, *The uniqueness of hypergravity*, arXiv:1905.04109 [INSPIRE].

[62] S. Hoseinzadeh and A. Rezaei-Aghdam, *$(2+1)$-dimensional gravity from Maxwell and semisimple extension of the Poincaré gauge symmetric models*, *Phys. Rev.* **D 90** (2014) 084008 [arXiv:1402.0320] [INSPIRE].

[63] L. Avilés, E. Frodden, J. Gomis, D. Hidalgo and J. Zanelli, *Non-relativistic Maxwell Chern-Simons gravity*, *JHEP* **05** (2018) 047 [arXiv:1802.08453] [INSPIRE].

JHEP07(2019)156

[64] R. Caroca, P. Concha, O. Fierro, E. Rodríguez and P. Salgado-ReboLledó, *Generalized Chern-Simons higher-spin gravity theories in three dimensions*, *Nucl. Phys.* **B 934** (2018) 240 [arXiv:1712.09975] [INSPIRE].

[65] P. Salgado-ReboLledó, *The Maxwell group in* $2+1$ *dimensions and its infinite-dimensional enhancements*, arXiv:1905.09421 [INSPIRE].

[66] D. Grumiller and M. Riegler, *Most general* AdS_3 *boundary conditions*, *JHEP* **10** (2016) 023 [arXiv:1608.01308] [INSPIRE].

[67] D. Grumiller, W. Merbis and M. Riegler, *Most general flat space boundary conditions in three-dimensional Einstein gravity*, *Class. Quant. Grav.* **34** (2017) 184001 [arXiv:1704.07419] [INSPIRE].

[68] M. Riegler, *How general is holography?*, Ph.D. thesis, Vienna Tech. U., Vienna, Austria (2016) [arXiv:1609.02733] [INSPIRE].

JHEP07(2019)156