Three-Dimensional Instability of Finite-Amplitude Water Waves

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Computations based on the full water-wave equations reveal that there are two distinct types of instabilities for gravity waves of finite amplitude on deep water. One is predominantly two dimensional and is related to all the known results for special cases. The other is predominantly three dimensional and becomes dominant when the wave steepness is sufficiently large.

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The set of equations governing surface gravity waves on irrotational, inviscid, incompressible fluid of great depth is

$$\nabla^2\varphi = 0, \quad -\infty < z < \eta(x, y, t); \tag{1}$$

$$\varphi_t + \tfrac{1}{2}(\nabla\varphi)^2 + gz = 0, \quad \eta_t + \nabla\varphi\cdot\nabla\eta - \varphi_z = 0, \quad \text{on } z = \eta(x, y, t), \tag{2}$$

where g is the gravitational acceleration, $\varphi(x, y, z, t)$ is the velocity potential, and $\eta(x, y, t)$ is the free surface. These equations admit two-dimensional (y-independent), steady, periodic solutions in which η takes the form

$$\eta = \eta_s = \sum_0^\infty A_n \cos[2n\pi(x - Ct)/\lambda], \tag{3}$$

where the Fourier coefficients A_n and the wave speed C are functions of the wave steepness h/λ, where h is the peak-to-trough height and λ the wavelength. The first few terms in the expansion in powers of h/λ were calculated by Stokes and Rayleigh.[1] Recent calculations by various authors have given η_s up to the limiting wave steepness of $h/\lambda \approx 0.141$.[2]

We consider the stability of these two-dimensional steady waves to arbitrary infinitesimal three-dimensional perturbations in which the free-surface disturbance is of the form

$$\eta' = \exp\{i[p(x - Ct) + qy - \sigma t]\} \times \sum_{-\infty}^{\infty} a_n \exp[in(x - Ct)], \tag{4}$$

where without loss of generality we have taken $\lambda = 2\pi$ and $g = 1$. The perturbation wave numbers p and q are arbitrary real numbers. It is obvious that (4) is an eigenvector of the infinitesimal perturbations to (3) with σ the eigenvalue. Instability corresponds to $\mathrm{Im}\sigma \neq 0$, since σ occurs in complex conjugate pairs.

The problem is to determine σ and the corresponding a_n. This was accomplished numerically by truncating the infinite sum in (4) to $2N+1$ modes, substituting $\eta_s + \eta'$ and the corresponding $\varphi_s + \varphi'$ into (2), and satisfying the boundary conditions at $2N+1$ points. The resulting homogeneous linear system of order $4N+2$ was solved as an eigenvalue problem for σ by standard methods. The accuracy of the solutions was improved by Newton's method when necessary. For small values of h/λ (less than 0.1), $N = 20$ sufficed to give σ reliably to three significant figures. As h/λ was increased, larger values of N were needed, and for the steepest wave studied ($h/\lambda = 0.131$), $N = 50$ was used.

Two distinct regions of instability were identified, denoted as I and II. Plots of instability regions in the p-q plane for various values of h/λ are shown in Fig. 1.

The eigenvectors corresponding to instability region I have dominant components $n = 1$ and $n = -1$ for $h/\lambda \to 0$. For very small values of h/λ, the instability band is very narrow and lies near the curve defined by

$$p - 1 + [q^2 + (p-1)^2]^{1/4} = p + 1 - [q^2 + (p+1)^2]^{1/4}. \tag{5}$$

The band is symmetrical about $q = 0$ and $p = 0$ (with $a_n \to a_{-n}$). Near the origin, the instability bandwidth along the p axis is proportional to h/λ. Near $p = \frac{5}{4}$, the bandwidth is of order $(h/\lambda)^4$. For sufficiently large values of h/λ, the band diminishes in size. At $h/\lambda \approx 0.108$, the instability band detaches from the origin ($p = q = 0$), indicating that the system is no longer unstable to infinitely long wave perturbations. At $h/\lambda \approx 0.124$, this type of instability disappears.

The eigenvectors corresponding to instability region II have dominant components $n = 1$ and $n = -2$ for $h/\lambda \to 0$. For small values of h/λ, the instability band lies near the curve

$$p - 2 + [q^2 + (p-2)^2]^{1/4} = p + 1 - [q^2 + (p+1)^2]^{1/4}. \tag{6}$$

The band is symmetrical about $q = 0$ and $p = 0.5$

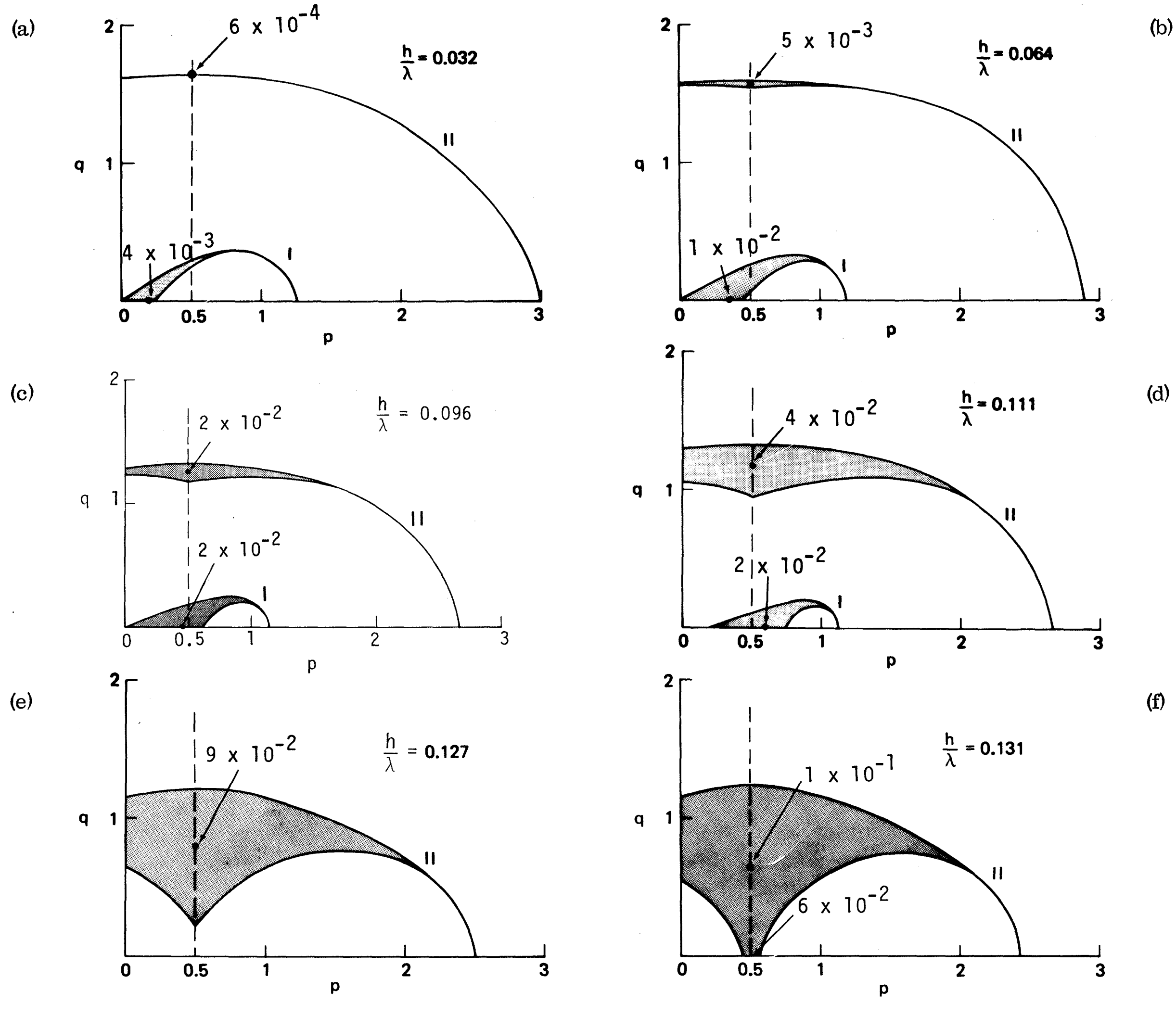

FIG. 1. Instability regions in p-q plane for various values of wave steepness: (a) $h/\lambda = 0.032$; (b) $h/\lambda = 0.064$; (c) $h/\lambda = 0.096$; (d) $h/\lambda = 0.111$; (e) $h/\lambda = 0.127$; and (f) $h/\lambda = 0.131$. Shaded regions denote instability. Points of maximum instability are marked by dots, with the approximate growth rates shown.

(with $a_n \to a_{-n}$). The bandwidth along the p axis is proportional to $(h/\lambda)^5$. Unlike in region I, this instability band continues to grow with increasing h/λ, being widest at $p = 0.5$. At $h/\lambda \approx 0.13$, the instability region touches the p axis at $p = 0.5$, indicating the onset of two-dimensional instability of this type.

The maximum growth rate of the type-I instability is proportional to $(h/\lambda)^2$ for small h/λ. For each value of h/λ, the maximum instability occurs when $q = 0$, so that type-I instability is predominantly two dimensional. The maximum growth rate of the type-II instability is of order $(h/\lambda)^3$. The maximum instability occurs at $p = 0.5$ and $q \neq 0$. Thus, type-II instability is predominantly three dimensional.

For values of $h/\lambda \leqslant 0.09$, type-I instability has the larger maximum growth rate. For larger values of h/λ, type-II instability dominates. As discussed above, only the type-II instability exists for sufficiently large h/λ. This changeover from two-dimensional to three-dimensional may be partly responsible for the observation that relatively calm ocean surfaces exhibit two-dimensional features, whereas a choppy sea is strongly three dimensional.

The two instability regions discussed above appear to be the first members of two classes of instabilities. The first class of instability occurs (for small h/λ) near curves given by

$$p-m+[q^2+(p-m)^2]^{1/4} = p+m-[q^2+(p+m)^2]^{1/4}, \qquad (7)$$

where m is a positive integer. Equation (5), corresponding to the type-I instability, is the case $m=1$. This class of instability has eigenvectors with dominant components $n=\pm m$ (for small h/λ). The instability regions are symmetrical about $q=0$ and $p=0$. We denote the entire family of instabilities of this type as class-I instabilities.

The second class of instabilities lies near curves given by

$$p-(m+1)+\{q^2+[p-(m+1)]^2\}^{1/4} = p+m-[q^2+(p+m)^2]^{1/4}, \qquad (8)$$

so that Eq. (6), corresponding to the type-II instability, is the case $m=1$. The associated eigenvectors have dominant components with $n=m$ and $n=-(m+1)$. The instability regions are symmetrical about $q=0$ and $p=0.5$. We denote this second family of instabilities as class-II instabilities (see Fig. 2).

Equations (7) and (8) express the condition that the eigenvalue problem is degenerate as $h/\lambda \to 0$, which produces a band of parametric instability. Physically, this can be interpreted as a resonance between the modes m and $-m$, and m and $-(m+1)$, respectively, which have equal frequencies as $h/\lambda \to 0$ when (7) and (8) are satisfied.[3]

For large values of m, the values of p and q are large and the eigenvectors represent short waves on a nonuniform flow (the undisturbed Stokes wave). It is noteworthy that the instability is not recovered by a standard WKB-type method.

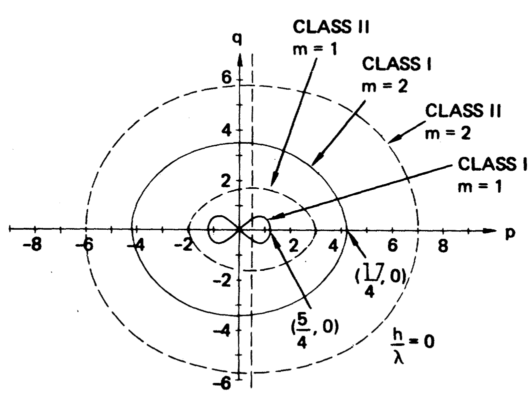

FIG. 2. Resonance curves in p-q plane for class-I (solid lines) and class-II (dashed lines) instabilities [given by Eqs. (7) and (8)] for $m=1$ and $m=2$.

For $m=1$, the instability bandwidth and maximum growth rate are proportional to powers of h/λ. It is expected that these powers increase with increasing m, so that the higher-m instabilities are hardly detectable when h/λ is small. However, the importance of these instabilities for large values of h/λ cannot be *a priori* determined.

The possibility of further instability regions which do not exist for small h/λ also cannot be excluded.

We now discuss the relationship of these instabilities to known results. We first consider class I. Phillips's figure 8 resonant diagram[4] is precisely the curve given by (5), i.e., the first member ($m=1$) of the class-I instability. For small but nonzero values of h/λ, the Phillips diagram develops into a narrow band of instability. In the region $|p| \ll 1$, $|q| \ll 1$, it corresponds to the sideband instability obtainable from the three-dimensional nonlinear Schrödinger equation.[5] The two-dimensional special case ($q=0$, $|p| \ll 1$) is the so-called Benjamin-Feir instability.[6] For larger values of h/λ, or for regions where $|p| \ll 1$, $|q| \ll 1$ no longer hold, the instability band can be obtained from a better approximation: the Zakharov equation.[7] The Zakharov equation loses quantitative accuracy when h/λ exceeds about 0.06, although it qualitatively reproduces the eventual detachment of the instability region from $p=q=0$[8] and the subsequent disappearance of the type-I instability.[9] The special case of $q=0$ and p rational has been studied numerically by Longuet-Higgins using the full equations.[10]

The class-II instability can be related to a high-order parametric instability, whose possible existence was pointed out by Zakharov.[5] Recently, Hasselmann[11] gave a plausibility argument for its presence in the special case of $q=0$.

The instability diagrams show that the three-dimensional instability band broadens along the line $p=0.5$ as h/λ increases. The band touches the p axis at $p=0.5$ when $h/\lambda \approx 0.13$. This two-dimensional instability of steep waves was first discovered by Longuet-Higgins.[10] Maximum instability for class II occurs for $p=0.5$, $q \neq 0$, and is fully three dimensional for all our calculated cases.

For $p=0.5$, the instability is co-propagating with the unperturbed wave (i.e., $\mathrm{Re}\,\sigma=0$). This implies that a new class of steady three-dimensional waves will bifurcate from the point of stability exchange with $p=0.5$.[8] We suggest that this three-dimensional instability and bifurcation is responsible for the striking three-dimensional patterns observed by Su[12] in an experiment.

The results here show that gravity waves are unstable to short-wave perturbations. In particular, there exist narrow bands of wave number p for which two-dimensional, short-wave disturbances are unstable. There is no contradiction with Longuet-Higgins's[10] results that two-dimensional superharmonic perturbations are stable for $h/\lambda < 0.139$, as he only calculated pure or low-order rational harmonics which lie outside the narrow instability bands.

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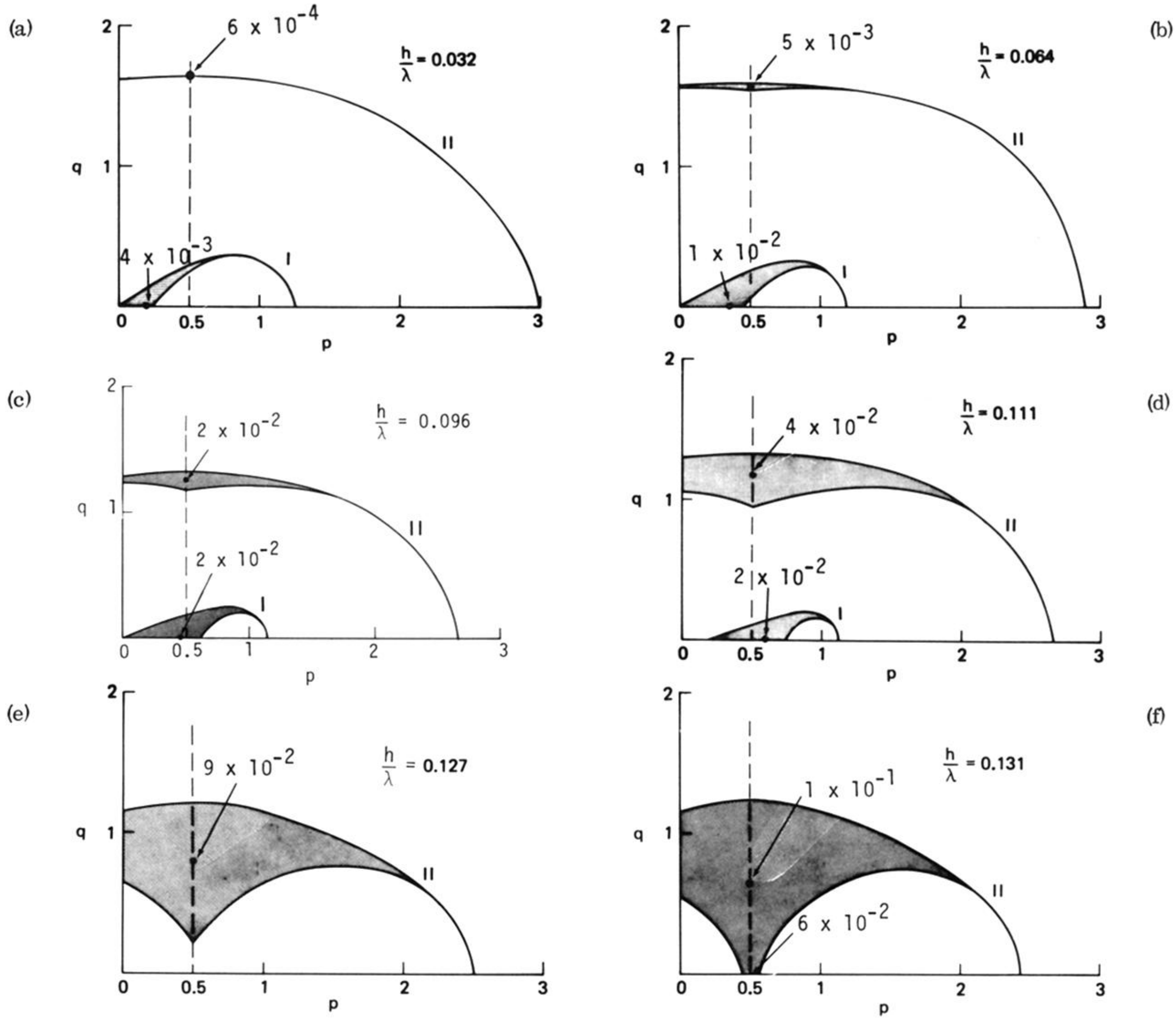

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