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Three Dimensional Propagation of Planetary Waves

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Abstract

The ideas of ray tracing from geometrical optics and wave propagation in a slowly varying medium are used to study the propagation of plaentary waves in the atmosphere. Kinematic wave theory is applied to wave solutions of the linearised quasi-geostrophic poten-tial vorticity equation on the sphere. An index is defined for planetary wave propagation in the vertical-meridional plane and it is shown that wave activity is refracted towards larger values of this index.
Ray solutions for stationary planetary waves are calculated for simple basic states and
for basic states representative of Northern Hemisphere summer, autumn and winter condi-tions. The results agree with those from observational and numerical model studies of stationary planetary waves in the atmosphere. It is shown that the sphericity of the Earth
and the curvature of the zonal flow are important factors determining the propagation of planetary waves.

1. Introduction

There have been many theoretical studies of
the vertical propagation of planetary waves in
the atmosphere since the classic paper by
Charney and Drazin (1961). These can be divided into two broad groups. The first includes analytical studies (e.g. Charney and Drazin, 1961; Dickinson, 1968a) where a simple basic state is assumed so that the wave equation is separable. The horizontal and vertical wave structures can then be considered separately. The second group includes numerical model studies (e.g. Matsuno, 1970; Schoeberl and Geller, 1977) where realistic basic states make the wave equation non-separable and wave solu-tions are found using numerical models.
In Hoskins and Karoly (1981, hereafter HK), the theory of waves propagating in a slowly varying medium was applied to Rossby waves propagating in a barotropic atmosphere. Ray solutions for horizontal propagation in the tropo-sphere agreed well with observational and numerical model results. In the present paper, this theory is extended to study vertical as well

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as horizontal propagation of planetary waves in a baroclinic atmosphere.
In section two, wave solutions of the linearized quasi-geostrophic potential vorticity equation are found. Kinematic wave theory is used to deter-mine the variation of the group velocity and wavenumber for an arbitrary zonally-symmetric basic state and tests for the validity of this wave solution are suggested. An index is defined which determines the propagation of planetary waves in the vertical-meridional plane. A general result is proved showing that the direction of propagation of wave activity is refracted towards the gradient of this index. The study is then limited to the consideration of stationary plane-tary waves only. In this case the notion of refractive index has been introduced by Matsuno (1970).
Ray solutions are calculated for a constant angular velocity flow and a simple jet model of the stratosphere in section three. The sphe-ricity of the Earth is shown to be important in refracting rays towards the horizontal as they approach the equator. However, for a sufficiently strong jet, a vertical waveguide can exist which prevents this equatorward propagation.
In section four, ray solutions are calculated for realistic representations of Northern Hemi-sphere summer, autumn and winter conditions.

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The rays, the vertical group velocities and the
vertical phase tilts are in broad agreement with
observations of stationary planetary waves. The
curvature of the mean zonal flow profile is
shown to be important in determining the
refractive index.

2. Wave theory

a. Quasi-geostrophie wave equation
We take as our starting point a set of quasi-geostrophic equations of motion on a sphere using pressure as vertical coordinate. When linearized about a basic state with zonal mean zonal wind u (*, p) at latitude * and pressure p, they may be reduced to a single equation for the streamfunction when two approximations consistent with quasi-geostrophic scaling are made (see Dickinson, 1968b; Hollingsworth et al., 1976). This equation for the perturbation stream-function * in spherical coordinates (*, *, p) may
be written:

and to scale the streamfunetion:

Then cost2* times (1) gives

where fM = 2* cos* sin* is the Coriolis parame-ter in Mercator coordinates,

and

if variables are non-dimensionalised using as horizontal length scale the radius of the Earth a, vertical length scale H0 and time scale *-1 where * is the angular velocity of the Earth then
(6) becomes

where

Here the buoyancy frequency N, defined by and

is T/T0 times the Brunt-Vaisala frequency, T(p) being a standard temperature distribution with
T0 a characteristic value and H0 = RT0/g.
It is convenient, as in HK, to use a Mercator projection of the sphere:

Then

and

is cos* times the meridional gradient of quasi-geostrophic potential vorticity on the sphere. The wave equation (7) is the extension of the barotropic wave equation (5.9) in HK to a baro-clinic atmosphere. It is also the Mercator pro-jection analogue of the wave equations used by Matsuno (1970) and Schoeberl and Geller (1977) in numerical models of planetary wave propa-gation.
Assuming that uM and *M are slowly varying, the dispersion equation for plane wave solutions

The Mercator basic zonal flow uM=u/cos* is

proportional to the angular velocity. It is also convenient to use as vertical coordinate

of (7) is

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As described by Whitham (1960), for example,
the activity of almost-plane waves moves with
the group velocity

cg = (ug, vg, wg),
where

and prevents the use of the simple analysis often used in determining the validity of this WKB type of solution (e.g. Dingle, 1973).
However it is possible to define an improved solution to (7) for varying l and m using the phase-integral form

and

where the "total wavenumber" K = (k2 +l2+

* 2m2)1/2. Defining a ray to be everywhere in the direction of the local group velocity, wave activity propagates along a ray with speed equal to the group velocity. Hayashi (1980) obtained expressions for the zonal and vertical group velocity for stationary Rossby waves on a *

plane for constant zonal flow which are a simplified form of (10).
Kinematic wave theory (Whitham, 1960) is used to give the variation of the wave frequency and wavenumber along a ray. Since the dis-persion equation has no explicit dependence on x and t, k and * are constant along a ray. The variation of the meridional and vertical wave-

number are given by

Some idea of the validity of this approximate solution can be obtained by considering the meridional variation of the coefficients in (13) for constant m and z and the vertical variation for constant l and y. These analyses suggest criteria for the validity of the phase-integral solution, as in HK, that

i.e. that the wavelength is slowly varying. It is not possible to define these criteria as a function of y and z for a given frequency and basic state. However it is possible to define two parameters
which are related to these criteria and which
can be determined once the basic state is speci-
fied. These can be regarded as WKB parame-ters for the meridional and vertical wavenumbers,

where

where dg/dt=*/*t+cg** is the advective deri-vative moving with the group velocity. Equations
(10) and (11) form a Hamiltonian set of equa-
tions for the group velocity and wavenumber
which may be integrated to give the wave rays
for a given frequency and basic state.

b. Phase-integral solution
The plane wave expression (8) is an approxi-mate solution to the wave equation (7) with varying coefficients. Since we have shown that k and w are constant along rays, a solution is posed of the form

Substituting (12) into (7) gives the elliptic equation

where the wavenumber term l2+*2m2 [defined by (9)] and *2 are functions of y and z. This functional dependence makes (13) non-separable

assuming that the variation of * is small. The ratios K/l and K/*m are large only close to turning points of a ray in the meridional and
vertical directions respectively, where l*0 or
m*0 and the phase-integral solution is not valid. Elsewhere the magnitudes of Wl and Wm provide an indication of the validity of the phase-integral solution for a given basic state.

c. Wave amplitude
Bretherton and Garrett (1968), see also Stur-rock (1962) and Whitham (1965), showed that the conservation of wave action can be written in the useful form

where E/*' is the wave action density, E the
wave energy density and *' = * - uMk the
Doppler shifted frequency. Andrews and Mc-Intyre (1978) have discussed the derivation of a general form of this equation and its simplifi-cation to (16) for small amplitude waves in a

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conservative, slowly varying medium.
For stationary solutions, the divergence theo-
rem can be applied to the wave action flux along
a ray tube, following Lighthill (1978), to give

where dA is the cross section of the ray tube.
For a horizontal section,

where *x and *y are representative of the magnitude of the section in the zonal and meri-dional directions. Since the ray equations are independent of x, *x is constant and (17) be-

comes

Then

Using dgl/dt=-*F/*y and the z component of (22), (24) gives

Similarly

From (23), the rate of change of * following a ray is given by

Using (23), (25) and (26), we have
The wave energy density

per unit volume

where * is the horizontal mean density and A the amplitude of the scaled streamfunction, Sub-stituting (19) into (18) gives the expression for the variation of the streamfunction amplitude along a ray

This expression allows the amplitude variation to be determined along a ray in, regions where the phase integral solution is valid. It does not make it possible to determine the wave amplitude as a single-valued function of y and z for a given basic state, as was possible in HK.

d. Ray refraction: a general result
Consider a dispersion equation for any waves in the two dimensional space (y, z) which is per-fectly general except that it is isotropic in the y and z wavenumbers l and m in the sense that

where K2 = k2 +l2+ m2, k being constant. The group velocity in the y-z plane is

The angle * of a ray to the horizontal is given by

Now consider waves of a particular frequency
* and define K* such that F (y, z, K*) = *.

where i is a unit vector in the "x" direction.

Thus rays are always refracted towards the direction of *K*.

To be more specific, if r is the radius of cur-vature of a ray and x is the angle from cg' to
* K* then dg*/dt=cg'/r and

Referring to Fig. la, this is the distance AB along the normal to the K*, contour at A. (28) states that the centre of curvature must lie on the line CB. The actual position on this line is implied by the direction of the ray at A.
For a given basic state and frequency *, the wavenumber K*, can be found as a function of y and z only and the refractive index for wave propagation is fixed. From (28), ray refraction is large for large gradients in K*, near turning points (K*, *k) and for rays nearly tangent to K* contours. It is clear, as shown in Fig. 1b, that a ridge in the K* field will act as a wave guide for rays propagating sufficiently parallel to the ridge. The erect is similar to that of an underwater ridge or sandbank on ocean surface
waves.

We note also that with a dispersion relation of the form (21), the problem is time-reversible and rays can represent propagation in either direction.

e. Ray refraction: planetary waves
Apart from the variation of *, the dispersion equation for planetary waves (9) may be put in the standard form (21) by using a stretched vertical coordinate z=z/a. Neglecting the varia-

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Near a turning point, the first term always domi-
nates. Elsewhere, the relative variation of K*2
usually exceeds that of *2 and the above results
on ray refraction hold qualitatively.
For stationary planetary waves, *=c=0 and
(29) becomes

We note that

Thus where the phase speed of a wave is much smaller than the zonal flow, its rays are almost the same as those for stationary waves. Neglect-ing the last term in (32), from maps of Ks it is easy to determine those for K*. For small propagation speeds in a westerly flow, i.e. 0< *c * <uM, K*, is larger for eastward moving waves which are thus able to propagate into regions in which stationary waves of the same zonal wave-number cannot exist. The propagation of west-ward moving waves is inhibited.
From (10) the three dimensional group velocity for stationary planetary waves may be written

In isotropic coordinates, the magnitude of the group velocity is

Fig. 1(a) Diagrammatic representation of the curva-ture of a ray at point A in a medium of refrac-
tive index K* , with gradient *K*. The distance
AB along the normal to the K* contour at A
is given by (28) and depends only on k and K*. The line BC on which the centre of curvature
lies is parallel to the tangent at A. The actual position of C on this line for any ray is given
by the direction of the ray at A, i.e. the inter-section with BC of the normal to the local cg. Fig. 1(b) Diagrammatic representation of the wave-guide effect of a ridge in the K* field. Rays propagating sufficiently parallel to the ridge are focused into the ridge.

tion of *, the results (27) and (28) are then exact in the y-z plane and

where c is the zonal phase speed */k. If the variation of * is included, it can easily be shown that (27) is modified to

For Ks2* *2n2, this simplifies to an extension of the barotropic result in HK that wave activity propagates along a ray at a speed double that of the component of the basic flow in the direc-tion of the three dimensional ray.
For the remainder of this study, only stationary waves will be considered.

3. Simple basic states

The ray theory of the preceding section is applied to two simple basic states, a constant angular velocity flow and a simple model of the stratosphere, so that the importance of different terms in determining the variation of the station-ary wavenumber can be investigated.

a. Constant angular velocity flow
The simplest basic state on the sphere is a constant angular velocity flow (super-rotation) with uM=*, a constant. In addition, we assume that the temperature and static stability are constant. This leads to the simplifications

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and

The stationary wavenumber is independent of
height and longitude and its variation with lati-
tude is due solely to the sphericity of the Earth.
Since Ks is independent of height dgm/dt=0
and m is constant along a ray. Thus l varies along a ray to satisfy l2=Ks2 - k2 - *2m2. Consider a wave propagating upward, pole-ward and eastward. The vertical wavenumber

is constant so the vertical group velocity will always be positive. Since Ks is a decreasing function of latitude, l decreases toward the pole and the ray becomes more vertically and zonally oriented. The latitude at which Ks2 = k2 + *2m2

is the turning point for the ray in the meridional direction. Near this latitude, the phase-integral solution is not valid but the group velocity is unchanged when a higher order approximate solution is considered. After reflection at the

turning point, the wave propagates equatorward, the ray becomes more meridionally oriented as l increases and the phase-integral solution be-comes valid again. Since *Ks is horizontal and directed towards the equator, the ray is refracted towards the direction of *Ks and becomes more

meridionally oriented as the equator is ap-proached.
For actual numbers, we refer to the super-rotation used in HK with * =1/30.875 and the

speed at the equator is *15ms-1 We choose values representative of the stratosphere for the scale height *6.5km and the Brunt-Vaisala frequency *4.5 * 10-4 s-2.
To calculate a wave ray for this basic state, the equations (10) and (11) for the group velocity and wavenumber are numerically integrated with respect to time using a fourth order Runge-Kutta scheme for given initial position and wavenumber. The frequency is calculated in-dependently using the dispersion equation to maintain a check on the accuracy of the numeri-cal procedure. The phase-integral expression for the wave phase in (14) is used to give the phase variation along the ray. An initial position is chosen and the stationary wavenumber calcu-lated. For each value of integer zonal wave-number less than Ks, there is an infinite number of wavenumber combinations (l, m) which satisfy
the dispersion equation and an infinite number
of directions for wave propagation in the y-z plane. A number of combinations representa-
tive of the range are used with each zonal wave-

number to give the initial wavenumbers for the rays.

To separate the dependence of Ks on the variation of *M from that due to the variation of *, * is kept constant with latitude at its value at 45*. Thus the refraction of the rays is due only to the *Ks term in (30). The rays for zonal wavenumber one from a source at 45* in this basic state are shown in Fig. 2(a) with crosses marking points at 1 day time intervals along each ray, indicating the speed of wave propa-gation. The analytic expression for these rays is found by integrating the ray slope in the y-z plane to give

where cos*=k [*/2(1+*)]1/2 and k2=k2+

Fig. 2 Rays and propagation times marked by crosses every day for zonal wavenumber one from a source at 45* and z=0.7 in the super-rotation flow. Solutions for (a) * kept constant
at its value at 45* and (b) full variation of *.

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*2(m2+1/4). The maximum meridional propa-
gation is to the latitude * where l=0 and Ks2 =
k2 + *2m2.

Fig. 2(b) shows the rays for zonal wavenumber
one calculated using the correct variation of *.*
Comparison of Fig. 2(b) with 2(a) and considera-tion of the equation for ray refraction (30) shows that, for * >45*, */*<0 and the ray curva-ture is decreased whereas for *<45*, */*> 0 and the ray curvature is increased. The tendency of the rays to the horizontal is more pronounced in low latitudes in Fig. 2(b) than in Fig. 2(a) and this is due to the variation of *.

This refraction of the rays towards the hori-zontal is similar to that shown by the Eliassen-Palm (EP) flux in observational and modelling studies of planetary waves in the atmosphere (Edmon et al., 1980; Sato, 1980; Dunkerton et al., 1981; Palmer, 1981). The EP flux is parallel to the meridional projection of the group velocity for small amplitude planetary waves in a slowly varying zonal wind (McIntyre, 1980). Our study shows that the refraction of the planetary wave group velocity towards the equator is a funda-mental property of sphericity of the Earth and is not due to variations in the zonal flow. The

presence of critical lines is not necessary.
b. Uniform vertical shear

Simmons (1974) used a simple representation of the stratosphere in which the zonal wind is assumed to have a simple meridional variation between 40* and the pole with a maximum at 65* and a uniform vertical shear. The tempera-ture and static stability are again constant. The zonal flow, shown in Fig. 3a, is zero at the sur-face and at 40* so that critical lines exist at the equatorward and lower boundaries of the flow region. The phase-integral solution breaks down close to a critical line but a linear analysis performed in the region shows that wave activity is absorbed by the critical line (Dickinson, 1968a). The non-linear behaviour of critical lines in the

atmosphere is uncertain (Tung, 1979) but in this study, they are treated as total wave absorbers.

* The rays shown in Fig. 2b are complete provided that there is a wave absorber in the southern hemisphere. As pointed out to the authors by M. Schoeberl, if the constant angular velocity flow holds there as well, then the rays will be reflected back into the northern hemisphere. There is then vertical propagation in the wave-guide between the polar regions. This corresponds to the separable analytic problem in this case which implies zero horizontal phase tilt.

Fig. 3 The basic state for the simple model of the stratosphere. Shown are: (a) zonal wind (con-tour interval 20ms-1), (b) vorticity gradient,
* M (contour interval 1) and (c) stationary wave-number, Ks. (Solid contours at interval 1 for
0 < Ks <5. Other intermediate contours are dash-
ed and labelled.)

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The vorticity gradient *M and stationary wave-
number Ks are shown in Fig. 3b and c. For a
flow with uniform vertical shear, Ks asymptotes
to a function of latitude only as z increases and
there are only small variations of Ks above z
* 8. There is a maximum in *M at 62*, near the jet maximum, indicating that the dominant term in the vorticity gradient is the meridional curvature of the zonal flow and not spherical effects. Close to the pole above z=1 there is a region where *M is negative. The stationary wavenumber is undefined in this region, placing a limit on the poleward propagation since wave solutions are possible only for *M > 0. There is a local maximum in Ks near 54* above z=8, on the equatorward side of the jet maximum, but elsewhere there is a general decrease in Ks with increasing latitude and height. The WKB parameters for this basic state are both less than 0.3 except where the stationary wavenumber is small.

Wave rays are calculated in the same way as before except that, in this case, m is not con-stant with height. In the solution as a critical line is approached, the ray becomes normal to the critical line and the group velocity tends to zero. The ray is terminated close to the critical line.

The rays for tonal wavenumber one from a source at 60*, z=0.77 are shown in Fig. 4(a) with circles marking points at * phase intervals along the rays. Comparison of Fig. 4(a) with Fig. 3(c) shows that rays are refracted towards
* Ks, indicating the dominance of the first term in (30). The general pattern of the rays is very similar to that for the super-rotation except that m varies along each ray and may change sign, giving a turning point in the vertical.
The vertical separation of the phase circles is used to calculate the vertical phase tilt of
* 8*km-1. This does not agree with the funda-
mental solutions of Simmons (1974) which had no phase tilt with height in this basic state. The difference is due to the absence of dissipation and the fixed meridional structure in Simmons' solu-

tion and the dissipation inherently assumed in this model to give wave absorption at critical lines. If this were not present the rays would be reflected back from the equatorward boundary. The phase tilt with height is in broad agreement with observations of stationary wavenumber one in the winter stratosphere (Hirota and Barnett, 1977; van Loon et al., 1973).
The rays for zonal wavenumber one from a

Fig. 4 Rays for zonal wavenumber one in the sim-ple stratospheric basic state. (a) Source at 60* and z=0.77 with phases marked by circles at * intervals along each ray. (b) Source at 55* with propagation times marked by crosses every day.

source at 55*, z=0.77 are shown in Fig. 4(b) with crosses at 1 day time intervals. Note the decrease in the group velocity and that the rays become normal to the critical line as the critical

line is approached. These rays have many similarities to those from the source at 60* with the single major difference that one ray enters the region of the local maximum in Ks and is trapped in this region, propagating to great height. This shows the importance that strong westerly winds can have in determining the vorticity gradient and leading to increased verti-cal propagation. This is in agreement with the idea of a strong westerly wind waveguide sug-gested by Simmons (1974) rather than a weak westerly waveguide suggested by Dickinson

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(1968a).
4. More realistic basic states

Several realistic polynomial representations of
the variation of the zonal mean zonal flow and
horizontal mean temperature have been obtained
based on the data from Oort and Rasmusson

(1971) and Newell et al. (1972) for the tropo-sphere and lower stratosphere and from the COSPAR (1972) International Reference Atmos-phere for the stratosphere and mesosphere. These representations must be continuous to at least the third derivative since they are used as the basic states for ray solutions. Polynomial expres-sions are used for both the zonal flow and temperature.
A single hemisphere only is considered and propagation of wave activity out of the hemi-sphere is prevented by changing the flow at low latitudes so that easterly wind exists at the equator at all heights. This does not affect the flow at middle latitudes.

a. Northern hemisphere winter
The polynomial representation of the zonal flow in the Northern Hemisphere winter is shown in Fig. 5(a). It has a tropospheric maximum of 26ms-1 at 33* and a polar night jet maximum of 75ms-1 at 43*, z*9. The vorticity gradient and stationary wavenumber are also shown in Fig. 5. The critical line in the Ks profile is shown by the multiple lines close together at low latitudes.

There are maxima in *M near the jet maxima, as for the simple jet model, and a local minimum in *M at 55*, z*2.5, near the tropopause. There are regions of negative *M at the surface at high latitudes and at upper levels at high latitudes. Charney and Stern (1962) showed that if the meridional gradient of the quasi-geostrophic potential vorticity changes sign in a region, the flow may be unstable to wave disturbances. Simmons (1978) investigated this instability mechanism for typical stratospheric conditions and found very small growth rates for the un-stable modes. Leovy and Webster (1976) have found that this instability criterion is satisfied in their observations of mean winter upper stratospheric conditions. The possibility of any instability is avoided in this study by considering stationary waves only.
The stationary wavenumber profile shows many similarities to that for the simple stratosphere model, with decreasing Ks with increasing lati-tude and height. The major differences are the

Fig. 5 Northern Hemisphere winter basic state. Shown are: (a) zonal wind (contour interval 10ms-1). (b) *M (interval 1) and (c) Ks (inter-
val 1 for 5<Ks<0, intermediate contours are dashed). The lines close together at low latitudes indicate the critical line.

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large meridional gradient in Ks on the poleward
side of the tropospheric jet, a large vertical
gradient of Ks at the tropopause at mid-latitudes
and regions of undefined Ks in regions of easterly
wind and negative *M. There is a local minimum
in Ks in the same region as that in *M.
The Ks profile is similar to that for the square refractive index Q0 of Matsuno (1970) apart from in the troposphere and at high latitudes. Q0 is related to Ks2/cost2* but was calculated by Matsuno assuming constant temperature and static stability. The flow used by Matsuno was adjusted so that there were no regions of nega-tive meridional vorticity gradient. Matsuno noted the tendency of wave energy to avoid the region of the local minimum in Q0 in his numerical model solutions. Palmer (1981) has noted a tendency for the EP flux to be focussed into regions of large Q0 in observations of planetary waves in the winter stratosphere. This tendency and that noted by Matsuno is in agreement with the refraction of wave activity towards *Ks in accordance with (30).
The WKB parameters for the wavenumber calculated for this flow are less than 0.5 away from the critical line or regions of small Ks. However there are localised maxima of Wl*0.7 at 40*, z=2.2 and Wm*1.2 at 50*, z=2.0 near the minimum in Ks at the tropopause. This places the validity of the phase-integral solution in doubt in this region. However the large gradients of Ks in this region lead to refraction of the rays away from the region. This will be discussed again later.
The ray solutions are calculated in the same way as before. Downward wave propagation and initial negative vertical wavenumbers are con-sidered and, where necessary, rays are reflected from the surface z=0 as from a solid boundary. However, it should be noted that the WKB parameters suggest that the small velocities near the surface render the solution invalid there.

The rays and phases for zonal wavenumber one from a source at 60*, z=0.4 are shown in Fig. 6(a). These rays have many similarities to those for the simple model of the stratosphere, with general upward and equatorward propaga-tion, but the differences in Ks lead to some important differences in the rays. In general, the rays are trapped in the vertical below z
* 2.5 by the decrease in Ks above the tropopause except at high latitudes. The local maximum in Ks near 65* causes focussing of the rays with an effect like a vertical waveguide from the

Fig. 6 Rays for zonal wavenumber one in the win-ter basic state. (a) Rays and phases for a source
at 60* and z=0.4. It should be recalled that this wave cannot occur in the region in Fig. 5 where Ks<l. (b) Rays and propagation times for a source at 45* and z=0.4.

troposphere into the stratosphere between the pole and the minimum in Ks.
The maximum vertical propagation is to z
* 7 or about 50km, a typical height for satellite observations of large amplitude stationary wave-number one. The phase tilt with height in the troposphere is difficult to determine because the rays are trapped between the surface and the tropopause. The phase tilt in the stratosphere is approximately 8*km-1, in agreement with the simple stratosphere model but greater than observed phase tilts in the Northern Hemisphere winter stratosphere (Hirota and Barnett, 1977; van Loon et al., 1973).
The latitude of the source determines the

number of rays which can have significant

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vertical propagation in this basic state. The
rays and propagation times for zonal wave-
number one from a source at 45*, z=0.4 are
shown in Fig. 6(b). The rays are similar to those
for the 60* source except that fewer rays can
enter the vertical waveguide at high latitudes and propagate into the stratosphere. For this source there is greater vertical propagation, to z *8 or about 55km, than for the 60* source. Since uM is small, the speed of wave propagation is small close to the surface and realistic dissipation would appear likely to be important for rays which reflect from the surface. The vertical

group velocity is approximately 5km day-1 in the stratosphere, in agreement with observations of the vertical propagation of planetary waves by Hirota and Sato (1969). The vertical group velocity and phase tilt of the rays in the strato-sphere are not affected by the latitude of the

source.

For sources at low latitudes, no rays have significant vertical propagation without reflection from the surface first. If such reflection is as-sociated with significant damping, this would suggest that low latitude sources are less likely to give a large response in the stratosphere com-pared to high latitude sources. The relevance of this model to meridional wave propagation in the troposphere will be discussed later.
The extent of the vertical and meridional

propagation is reduced by increasing the zonal wavenumber. As shown in Fig. 7, wavenumber
two has maximum vertical propagation to z
* 5. Wavenumber three is trapped in the tropo-sphere. This reduction of the vertical propagation is in agreement with observations and other theoretical studies of planetary waves. The
reduction of the meridional propagation is in
agreement with the barotropic ray solutions of
HK. For zonal wavenumber two in the strato-
sphere, the vertical group velocity is about 7km day-1 and the vertical phase tilt about 12*km-1, both larger than for wavenumber one.
The variation of the streamfunction amplitude may be calculated using (20). The horizontal section of the ray tube is determined by calcu-lating two rays for initial positions with small meridional separation and the same initial direc-tion i.e. the same ratio l/m for both rays. The meridional separation *y between the rays is used in (20). This equation has singularities at ray turning points in the meridional and vertical directions where *y*0 and wg*0 respectively. The phase-integral solution is not valid close to

Fig. 7 Rays and propagation times for zonal wave-number two in the winter basic state from a source at 45* and z=0.4. It should be recalled
that this wave cannot occur in the region in
Fig. 5 where Ks<2.

these singularities but it has not been possible to find a higher order approximate solution to correct the large amplitudes given by (20) close to these singularities. The amplitude is also not a single-valued function of y and z since neither the wavenumber (l, m) or cg are single-valued functions. Thus it is not possible to produce a graphical representation of the streamfunction amplitude variation as a function of y and z. It is possible to calculate the amplitude along each ray and this shows a general increase with latitude and a large increase with height.
Ray solutions have also been obtained for a zonal flow with the same shape as in Fig. 5(a) but with stronger westerly winds, having a tropo-spheric jet maximum of 36ms-1 and a polar night jet maximum of 94 ms-1. For this stronger flow, there is a reduction in the number of rays propagating from the troposphere into the strato-sphere. The vertical group velocity in the strato-sphere is greatly increased and the vertical phase tilt is reduced so that their values agree better with observations. The magnitude of the group velocity is such that wave activity can propagate from the tropopause to the stratopause in less than 4 days.

b. Northern hemisphere summer
A basic state representative of Northern Hemi-sphere summer conditions has been generated. Vertical propagation is prevented by the critical line below 10 mb at all latitudes. The zonal flow

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has a westerly jet maximum of 14ms-1 at 45*,
z=1.5 with easterly winds in the stratosphere
and in equatorial regions. The vorticity gradient
is negative near the surface between 30* and
70* and at polar latitudes above z=1. Thus Ks
is real only in a region around the tropospheric jet and stationary waves are possible in this region only.
The rays for zonal wavenumber one from a source in the middle troposphere propagate out-ward and terminate at the critical lines or close

to regions of negative *M, where the numerical procedure breaks down. The restriction of large amplitude stationary planetary waves to the troposphere in the summer and the lack of vertical propagation is in good agreement with observations (van Loon et al., 1973). The east-ward phase tilt with height for wavenumber one in the Northern Hemisphere summer found by van Loon is consistent with downward wave

propagation from a source in the upper tropo-sphere.

c. Northern hemisphere autumn
A basic state representative of the Northern Hemisphere autumnal equinox has been gen-erated. The zonal flow, shown in Fig. 8(a), is much weaker than in the winter but it is stronger than in the summer and westerly winds extend into the middle latitude stratosphere. There is a tropospheric jet maximum of 17ms-1 and a stratospheric jet maximum of 31ms-1. Unlike the winter case, the vorticity gradient does not have a maximum at the stratospheric jet because of the reduced amplitude and curvature of the jet. The stationary wavenumber is shown in Fig. 8(b) and it has a more uniform decrease with latitude and height than in winter. There is no local maximum in Ks at the tropopause and thus no vertical waveguide from the troposphere to the stratosphere.
The rays from a source at 60*, z=1, shown in Fig. 8(c), have smaller vertical propagation than for the winter case, reaching a maximum height below z=6 or about 40km. The vertical group velocity is considerably reduced and the vertical phase tilt is slightly increased. The reduced vertical propagation during the autumn is in agreement with observations but it is in contrast to the suggestion from the model of Charney and Drazin (1961) that stationary plane-tary waves have increased vertical propagation in weak westerly winds. This is also in contrast to the suggestion of Dickinson (1969) that

Fig. 8 NH autumn basic state. Shown are: (a) zonal wind (intervals 5ms-1), (b) Ks (contours
for 5 <Ks <0 at interval 1) and (c) rays for zonal wavenumber one from a source at 60*
and z=1.

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dissipation in the atmosphere leads to small
amplitudes for stationary wavenumbers one and
two in the stratosphere during the autumn. Dis-
sipation is likely to be more important in the
autumn than in winter because of the smaller

group velocities but the reduced vertical propa-gation occurs without dissipation being con-sidered.

5. Discussion

In this paper it has been shown how kinematic wave theory can be applied to the propagation of planetary waves in a zonally-symmetric atmos-phere. As a special case of a general result, it has been shown that almost all the necessary information on the propagation of a wave of particular frequency is contained in a map of the total wavenumber K*. This shows the direc-

tion of the refraction of rays and the regions into which they can propagate. We have con-centrated on waves with zero frequency for which the crucial parameter is the stationary wavenumber Ks.

It has been shown that the tendency for planetary waves to propagate towards the equator is a property of the spherical domain and not necessarily due to the presence of critical lines. The magnitude and curvature of the zonal flow is crucial in causing significant variations of Ks. A region of local maximum of Ks causes focussing of rays and acts as a waveguide. It has been shown that such waveguides can be associated with strong stratospheric jets allowing propagation to great heights. In the Northern Hemisphere winter basic state, a vertical wave-guide exists at high latitudes between the tropo-sphere and stratosphere. As a check on this, Ks has been determined from the data of Oort

and Rasmussen (1971). Again there is a local maximum near the tropopause at 65*N, near the latitude where the largest amplitude for stationary wavenumbers one and two is observed at this

height.
Using the concept of stationary wavenumber, it is possible to understand what flow conditions lead to the possibility of waveguides and con-sequent enhanced propagation of stationary planetary waves.
The validity of the phase-integral solution for realistic basic states is open to question as varia-

1 It is of interest that the data also indicates a region of negative potential vorticity gradient close to the surface poleward of 45*N. Ks is undefined in this region.

tions of the basic state are not small. Charney and Drazin (1961) have suggested that the WKB technique is not suitable for investigating the vertical propagation of stationary planetary waves. However, the excellent agreement of the results obtained using this technique with those from observational and modelling studies suggests the validity of this method. In general, the criteria for the validity of the solution are satisfied throughout much of the flow region. These criteria are related to *K*K*-1, which also determines the radius of curvature of a wave ray. The greatest ray refraction occurs where

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プラネタリー波の3次元的伝播
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 幾何光学およびゆっくり変化する媒質中の波動伝播論における光路追跡(ray tracing)の考え方を大気中のプ
ラネタリー波伝播の研究に応用した。線形化した球面上の準地衡風ポテンシャル渦度方程式の解法に波動の運動
学理論を適用する。子午断面内のプラネタリー波の伝播を支配する指数を定義し,波動活性量(wave activity:
エネルギーに類似し,平均流変化のある場合にも保存する量)が,この指数の大きくなる方に向けて屈折される
ことを示す。
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