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Research Note

Three-Dimensional Storm Surge Computations

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Summary

The method for three-dimensional storm surge computation proposed by Heaps is modified by removing the bottom frictional stress from the vertical eigenfunction expansion. The stress is applied externally on each vertical column of fluid. Calculations on a simplified model show that the technique does not alter the response of the sea from that determined by Heaps' method. The modification allows the bottom frictional stress to be expressed as an arbitrary function of horizontal velocity rather than the linear function necessitated previously.

Introduction

There are not many numerical methods available in the literature for examining the internal structure of large scale sea motions. In the outstanding category there are the methods of Sarkisyan (see e.g. 1969), or Sarkisyan & Ivanov (1972), and Bryan & Cox (see e.g. 1967). These models are for very large seas and involve a very large time scale. Thus it was a considerable step forward when the method of Heaps (1971, 1973) was proposed for a shallow enclosed sea.

It was clear from the earlier models that computing time and space were very limiting factors. Heaps sought to economize on this by representing velocity variations through the depth with the aid of an eigenfunction expansion throughout each vertical column of fluid. By transforming the equation into two-dimensions, it was found only necessary to perform the calculations with the first few modes of the eigenfunction expansion. Hence both computing time and space were conserved.

A restriction of Heaps' scheme for general use is that his eigenfunction expansion included the friction law for the bottom stress as a boundary condition for the functions at the bottom. To retain orthogonal properties, this had to be a linear relation.

It is well known that for tidal motions in shallow seas a quadratic friction law gives closer agreement between numerical solutions and observations. With this in mind, an alternative eigenfunction expansion is proposed; this allows a quadratic friction law instead of a linear one.

Review of Heaps' model

The reader is referred to the original paper for complete details. The linearized

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equations of motion used for the calculation of storm surges by Heaps are

$$\frac{\partial u}{\partial t} - \gamma v = -g\frac{\partial \zeta}{\partial x} + \frac{\partial}{\partial z}\left(N\frac{\partial u}{\partial z}\right), \tag{1}$$

and

$$\frac{\partial v}{\partial t} + \gamma u = -g\frac{\partial \zeta}{\partial y} + \frac{\partial}{\partial z}\left(N\frac{\partial v}{\partial z}\right), \tag{2}$$

with standard notation, where γ is the Coriolis parameter, N is the coefficient of eddy viscosity, and z is positive down. The continuity equation appears in the form

$$\frac{\partial}{\partial x}\int_0^h u\,dz + \frac{\partial}{\partial y}\int_0^h v\,dz + \frac{\partial \zeta}{\partial t} = 0, \tag{3}$$

where h is the undisturbed water depth. The surface wind stress components, F_s and G_s, are expressed in terms of an eddy viscosity and vertical shear, i.e.

$$-\rho\left(N\frac{\partial u}{\partial z}\right)_0 = F_s, \quad -\rho\left(N\frac{\partial v}{\partial z}\right)_0 = G_s, \quad z = 0. \tag{4}$$

Similarly, the bottom conditions are written

$$-\rho\left(N\frac{\partial u}{\partial z}\right)_h = F_B, \quad -\rho\left(N\frac{\partial v}{\partial z}\right)_h = G_B, \quad z = h. \tag{5}$$

where F_B and G_B are the components of bottom friction, which is assumed to vary linearly with bottom current, i.e.

$$F_B = k\rho u_h, \qquad G_B = k\rho v_h, \tag{6}$$

where k is the friction constant.

Combining (5) and (6) yields the linear bottom condition

$$\left(N\frac{\partial u}{\partial z}\right)_h + ku_h = 0, \quad \left(N\frac{\partial v}{\partial z}\right)_h + kv_h = 0, \quad z = h. \tag{7}$$

Eigenfunctions, $f_r(z)$, are now found for the differential equation

$$\frac{d}{dz}\left[Nf_r'(z)\right] = -\lambda_r f_r(z), \tag{8}$$

subject to the boundary conditions

$$f_r'(0) = 0, \tag{9}$$

$$N_h f_r'(h) = -kf_r(h), \tag{10}$$

and the normalizing condition

$$f_r(0) = 1. \tag{11}$$

The horizontal velocity components are then expanded in terms of the eigenfunctions and the transform of equations (1)–(3) are taken with respect to the eigenfunctions, yielding

$$\frac{\partial u_r}{\partial t} + \lambda_r u_r - \gamma v_r = -ga_r\frac{\partial \zeta}{\partial x} + \frac{F_s}{\rho h}, \tag{12}$$

$$\frac{\partial v_r}{\partial t} + \lambda_r v_r + \gamma u_r = -g a_r \frac{\partial \zeta}{\partial y} + \frac{G_s}{\rho h}, \tag{13}$$

and

$$\frac{\partial \zeta}{\partial t} + \sum_{r=1}^{\infty} \left\{ \frac{\partial}{\partial x}(h a_r \phi_r u_r) + \frac{\partial}{\partial y}(h a_r \phi_r v_r) \right\} = 0, \tag{14}$$

where $r = 1, 2, 3, \ldots, \infty$, and

$$\left.\begin{aligned} u_r &= \frac{1}{h} \int_0^h u f_r(z)\,dz, & v_r &= \frac{1}{h} \int_0^h v f_r(z)\,dz, \\ a_r &= \frac{1}{h} \int_0^h f_r(z)\,dz, & \phi_r &= h \Big/ \int_0^h f_r^2(z)\,dz. \end{aligned}\right\} \tag{15}$$

Equations (12)–(14) are schematized into a difference scheme and numerical solutions generated for various problems.

Separation of bottom stress from the eigenfunction

It is obvious that the boundary condition (10) is linear and must necessarily be linear for the eigenfunctions to be orthogonal. Thus if (10) were modified to represent a quadratic friction law

$$\left.\begin{aligned} \left(N \frac{\partial u}{\partial z}\right)_h + \kappa u_h \sqrt{(u_h^2 + v_h^2)} &= 0, \\ \left(N \frac{\partial v}{\partial z}\right)_h + \kappa v_h \sqrt{(u_h^2 + v_h^2)} &= 0, \end{aligned}\right\} \tag{16}$$

the functions, $f_r(z)$, would no longer be orthogonal. However, if the bottom boundary condition of the d.e. (8) were modified so that the bottom stress was not included then the orthogonality property could be retained.

Hence a new set of eigenfunctions, $q_r(z)$, are defined for the equation (8), now subject to boundary conditions

$$q_r'(0) = 0, \tag{17}$$

$$q_r'(h) = 0, \tag{18}$$

and the normalizing condition

$$q_r(0) = 1. \tag{19}$$

Orthogonality of the q_r is preserved, however, the transform of equations (1)–(3) is altered. Equations (12) and (13) become

$$\frac{\partial u_r}{\partial t} + \lambda_r u_r - \gamma v_r = -g a_r \frac{\partial \zeta}{\partial x} + \frac{F_s}{\rho h} - \frac{F_B}{\rho h}, \tag{20}$$

and

$$\frac{\partial v_r}{\partial t} + \lambda_r v_r + \gamma u_r = -g a_r \frac{\partial \zeta}{\partial y} + \frac{G_s}{\rho h} - \frac{G_B}{\rho h}, \tag{21}$$

where the λ_r are determined from the system (8), (17), (18) and are different from those of the system (8), (9), (10). Equation (14) retains its form except that the functions u_r, v_r, a_r, ϕ_r are expressed as integrals of $q_r(z)$.

The removal of the bottom stress from the orthogonal functions is similar to the way in which surface stress is removed by Heaps. Although this method forms a discontinuity in the velocity gradient at the surface, Heaps has shown in his paper that the discontinuity has no radical effect on the storm surge calculation.

The bottom stress components, F_B and G_B, in (20) and (21) may now be defined in any suitable way, e.g. by (6), or by the equivalent of (16), i.e.

$$F_B = \kappa\rho u_h \sqrt{(u_h^2 + v_h^2)}, \quad G_B = \kappa\rho v_h \sqrt{(u_h^2 + v_h^2)}. \tag{22}$$

There is a minor penalty attached to the use of (20) and (21). These equations require the knowledge of u_h and v_h at each time step. They are given by

$$u_h = \sum_{r=1}^{\infty} \phi_r u_r q_r(h), \quad v_h = \sum_{r=1}^{\infty} \phi_r v_r q_r(h). \tag{23}$$

In the practical application of the method the summation is carried out over M terms and, since $\phi_r q_r(h)$ are known from the outset, (23) consists of an extra 2M multiplications and additions per time step.

Numerical verification

The technique required by equations (20), (21) and (23) is verified on a simple storm surge model. A two-dimensional model is assumed, having constant depth, no rotation, and $\partial/\partial y \equiv 0$. A constant wind stress, F_s, of 15 dyne cm^{-2} is applied in the x direction over an enclosed sea of length 420 km, represented by 14 grid spaces, and a depth of 65 m. Linear friction is assumed with $k = 0{\cdot}2$. The response of the sea surface is found for a wind stress lasting 20 hr, using time steps of 6 min. The results are compared with the corresponding Heaps' model. In view of the figures in Heaps' paper depicting the relevant amplitudes of each mode in the solution it was decided to use only the first six modes in each case.

No graphs are presented here because the two solutions are almost identical.

Discrepancies in elevations at each of 20 one-hour time intervals is at most 1 per cent. For the velocities the error is of the same order for all depths except the bottom where the error is at most 5 per cent. One possible source of error is the truncation of the modes after 6. The error term due to this truncation is unlikely to be the same at all points throughout the calculation. Nevertheless the correspondence of the two results is sufficiently established to show that the bottom stress may be removed from the eigenfunction expansion and in this way a quadratic law may be applied to the friction term.

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