

THREE EXAMPLES CONCERNING THE ORE CONDITION IN NOETHERIAN RINGS

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1. Introduction

A ring R is said to satisfy the right Ore condition with respect to a subset C of R if, given $a \in R$ and $c \in C$, there exist $b \in R$ and $D \in C$ such that $ad = cb$. It is well known that R has a classical right quotient ring if and only if R satisfies the right Ore condition with respect to C when C is the set of regular elements of R (a regular element of R being an element of R which is not a zero-divisor). It is also well known that not every ring has a classical right quotient ring. If we make the non-trivial assumption that R has a classical right quotient ring, it is natural to ask whether this property also holds in certain rings related to R such as the ring $M_n(R)$ of all n by n matrices over R . Some answers to this question are known when extra assumptions are made. For example, it was shown by L. W. Small in (5) that if R has a classical right quotient ring Q which is right Artinian then $M_n(Q)$ is the right quotient ring of $M_n(R)$ and eQe is the right quotient ring of eRe where e is an idempotent element of R . Also it was shown by C. R. Hajarnavis in (3) that if R is a Noetherian ring all of whose ideals satisfy the Artin-Rees property then R has a quotient ring Q and $M_n(Q)$ is the quotient ring of $M_n(R)$.

The first example we give is a right Noetherian ring R which has a right quotient ring but $M_2(R)$ does not. The second example is a Noetherian P.I. ring R which is its own quotient ring and which has an idempotent element e such that eRe has neither a right nor a left quotient ring. An ideal I of a ring R is said to satisfy the *AR*-property (short for Artin-Rees property) if, given a right ideal K of R , there is a positive integer n such that $K \cap I^n \subseteq KI$. The third example we give is a right Noetherian P.I. ring R which has a prime ideal P which satisfies the *AR*-property (in fact is nilpotent) but R does not satisfy the right Ore condition with respect to $C(P)$ where $C(P)$ denotes the set of elements of R whose images are regular elements of R/P . In (4) a method is indicated for constructing an example with similar properties, but that method must fail because the ring R which it produces is a principal right ideal ring whose nilpotent radical P is prime; in such a ring R the elements of $C(P)$ are precisely the right regular elements of R , and hence R satisfies the right Ore condition with respect to $C(P)$ because of the following result: If S is a right Noetherian ring with nilpotent radical N and if $a, c \in S$ with c right regular, then there exist $b \in S$ and $d \in C(N)$ such that $ad = cb$.

Conventions. $C(0)$ and $C(I)$ will denote respectively the set of regular elements of R and the set of elements of R which are regular mod (I) ; if there is a possibility of ambiguity

we shall write $C_R(0)$ and $C_R(I)$. By “the right quotient ring of R ” we shall mean the classical right quotient ring of R , i.e. the ring formed by inverting all the regular elements of R . In constructing matrix rings we shall use the following kind of notation:

$$\begin{pmatrix} S & M \\ 0 & T \end{pmatrix} = \left\{ \begin{pmatrix} s & m \\ 0 & t \end{pmatrix} : s \in S, m \in M, t \in T \right\}.$$

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2. The examples

Example 2.1. We shall show that there is a right Noetherian ring R such that R has a right quotient ring but $M_2(R)$ does not. The construction is in several stages and we start by taking T to be any right Noetherian integral domain which is not left Ore (i.e. there are non-zero left ideals A and B of T such that $A \cap B = 0$). Let u be an indeterminate which commutes with the elements of T and let C denote the set of all elements of the polynomial ring $T[u]$ which have non-zero constant term. We first show that $T[u]$ satisfies the right Ore condition with respect to C . We note that $T[u]$ is a right Noetherian domain so that any two non-zero right ideals of $T[u]$ have non-zero intersection. Let $a \in T[u]$ and $c \in C$. For the purposes of establishing the right Ore condition we may suppose that $a \neq 0$. We have $aT[u] \cap cT[u] \neq 0$ so that $af = cg$ for some non-zero elements f and g of $T[u]$. We can write $f = du^i$ for some $d \in C$ and non-negative integer i . Thus $adu^i = cg$ where c has non-zero constant term. Therefore $g = bu^i$ for some $b \in T[u]$. Hence $ad = cb$ with $d \in C$.

Now let S be the partial right quotient ring of $T[u]$ with respect to C and let D be the right quotient division ring of T . We can make D into a right S -module by identifying D with S/uS , i.e. by setting $Du = 0$. Set

$$R = \begin{pmatrix} T & T \\ 0 & T[u] \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} D & D \\ 0 & S \end{pmatrix}$$

with the usual matrix operations and using the right action of S on D defined above when calculating the $(1, 2)$ -entry of a product. We shall now show that Q is the right quotient ring of R . It is straightforward to check that the regular elements of R are given by

$$C_R(0) = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : 0 \neq a \in T, b \in T, c \in C \right\}.$$

Let

$$r = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in C_R(0)$$

then a has an inverse a^{-1} in D , c has an inverse c^{-1} in S , and

$$\begin{pmatrix} a^{-1} & -a^{-1}bc^{-1} \\ 0 & c^{-1} \end{pmatrix}$$

is an inverse for r in Q . Now let

$$q = \begin{pmatrix} f & g \\ 0 & h \end{pmatrix} \in Q;$$

we must show that $qr \in R$ for some $r \in C_R(0)$. Because D is the right quotient ring of T there is a non-zero element a of T such that $fa \in T$ and $ga \in T$. Also there exists $d \in C$ such that $hd \in T[u]$. Because both a and d are elements of C there exists $c \in C$ such that $c \in aT[u] \cap dT[u]$. Set

$$r = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$$

then $r \in C_R(0)$ and $qr \in R$. Thus Q is the right quotient ring of R . In future we shall omit the details of such verifications.

Finally we shall show that $M_2(R)$ does not have a right quotient ring. Because T is not left Ore there are non-zero elements p and q of T such that $Tp \cap Tq = 0$. Thus if s and t are elements of T such that $sp = tq$ then $s = t = 0$. Let x, y, z be the following elements of R :

$$x = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}, \quad y = \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix}, \quad \text{and} \quad z = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}.$$

It can easily be shown that x and y are regular elements of R and that $Rx \cap Ry = 0$ (so that $r_1x = r_2y$ implies $r_1 = r_2 = 0$), and z is right but not left regular (i.e. $zr = 0$ implies $r = 0$, but there is a non-zero element r of R such that $rz = 0$). Let a and c be the following elements of $M_2(R)$:

$$a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad c = \begin{pmatrix} z & x \\ 0 & y \end{pmatrix}.$$

It is easy to show that c is a regular element of $M_2(R)$ (using $Rx \cap Ry = 0$ to prove left regularity). Suppose that there are elements b and d of $M_2(R)$ such that $ad = cb$; we shall show that d is not regular. Let $b = [b_{ij}]$ and $d = [d_{ij}]$ with $b_{ij}, d_{ij} \in R$ for $1 \leq i, j \leq 2$. Comparing entries in the second row of ad and cb gives $0 = yb_{21} = yb_{22}$ so that $b_{21} = b_{22} = 0$. Now comparing the first rows of ad and cb gives $d_{11} = zb_{11}$ and $d_{12} = zb_{12}$. But there is a non-zero element r of R such that $rz = 0$. Let w be the element of $M_2(R)$ with r in the $(1, 1)$ -position and 0's elsewhere, then $w \neq 0$ and $wd = 0$. Therefore d is not a regular element of $M_2(R)$.

Remarks. (1) It is clear from the last paragraph that the ring of 2 by 2 upper triangular matrices over R does not have a right quotient ring, and therefore neither does the ring of 2 by 2 lower triangular matrices over R because these two rings are conjugate under an inner automorphism of $M_2(R)$.

(2) We do not know whether there is a ring R which is its own quotient ring and is such that $M_2(R)$ does not have a right quotient ring. Part of the difficulty with this sort of question is the problem of finding a convenient description for the regular elements of $M_2(R)$; for example, it is possible for a regular element of $M_2(R)$ to have entries which are all zero-divisors in R . However, the situation for triangular matrices is more manageable, and I am very grateful to H. Attarchi for communicating the following information to me.

Let R be a ring such that cR (Rc) is an essential right (left) ideal of R for each right (left) regular element c of R . Suppose that

$$w = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

is a regular element of the ring $U_2(R)$ of all 2 by 2 upper triangular matrices over R . Clearly a is a right regular element of R and c is left regular. Let $x \in 1(a) \cap Rcb^{-1}$ where $1(a) = \{r \in R: ra = 0\}$ and $Rcb^{-1} = \{r \in R: rb \in Rc\}$. There exists $y \in R$ such that $xb = -yc$. Set

$$z = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$$

then $zw = 0$, so that $z = 0$. Thus $1(a) \cap Rcb^{-1} = 0$ from which it follows that $1(a) = 0$ because Rcb^{-1} is an essential left ideal of R . Thus a is a regular element of R , and by symmetry so also is c . Thus, for such a ring R , a matrix w of the form given above is regular if and only if a and c are regular elements of R . This makes it easy to prove that, for example, if R is a left and right Noetherian ring which has a right quotient ring Q then $U_2(Q)$ is the right quotient ring of $U_2(R)$.

Example 2.2. We shall show that there is a left and right Noetherian P.I. ring R which is its own quotient ring but which has an idempotent element e such that eRe does not have a right or left quotient ring. Again the construction is in several stages. Let U be the ring of integers Z localised at the prime ideal $2Z$ and set

$$T = \begin{pmatrix} U & 2U \\ U & U \end{pmatrix},$$

then T is a left and right Noetherian prime P.I. ring. Set

$$M = \begin{pmatrix} 2U & 2U \\ U & U \end{pmatrix} \text{ and } A = \begin{pmatrix} 2U & 2U \\ U & 2U \end{pmatrix},$$

then M is a maximal ideal of T and A is an ideal of T (in fact A is the intersection of the two maximal ideals of T). We have

$$C_T(A) = \left\{ \begin{pmatrix} a & 2b \\ c & d \end{pmatrix} : a, b, c, d \in U, ad \notin 2U \right\}.$$

Thus if $t \in C_T(A)$ then $\det(t) \in U$ and $\det(t) \notin 2U$. Thus $\det(t)$ is a unit of U , so that the elements of $C_T(A)$ are units of T .

Set $F = T/M$ then $F \cong Z/2Z$. Set

$$S = \begin{pmatrix} F & F & F \\ 0 & T & F \\ 0 & 0 & F \end{pmatrix},$$

then S is a left and right Noetherian P.I. ring. Let S' be the ring which is constructed in the same way as S but using Z instead of U , then S' is the ring given in (2) and S is a partial

quotient ring of S' ; this is because T is the partial quotient ring of the ring

$$T' = \begin{pmatrix} Z & 2Z \\ Z & Z \end{pmatrix}$$

with respect to the set of elements of T' which have odd determinant. But it was shown in (2) that S' has neither a left nor a right quotient ring, and hence S has neither a left nor a right quotient ring; this assertion can be justified either by using the fact that S is a partial quotient ring of S' , or by modifying the argument given in (2) and using the fact that T does not satisfy either the right or left Ore condition with respect to $C_T(M)$.

We now aim to find suitable R and e with $eRe \cong S$. Set

$$I = \begin{pmatrix} 0 & F & F \\ 0 & A & F \\ 0 & 0 & 0 \end{pmatrix}$$

then I is an ideal of S . Set $C = C_S(0) \cap C_S(I)$ and let $s \in S$, then $s \in C$ if and only if s has 1 in the (1, 1)- and (3, 3)-positions and an element of $C_T(A)$ in the (2, 2)-position. It follows easily that the elements of C are units of S . Now set

$$R = \begin{pmatrix} S & S/I \\ 0 & S \end{pmatrix}$$

then R is a left and right Noetherian P.I. ring. Let e be the element of R which has 1 in the (1, 1)-position and 0's elsewhere, then e is idempotent and $eRe \cong S$. An element of R is regular if and only if its diagonal entries belong to $C_S(0) \cap C_S(I) = C$ (a right or left regular element of S is regular and similarly for the finite commutative ring S/I). But the elements of C are units of S , from which it follows that the regular elements of R are units modulo the nilpotent ideal consisting of all strictly upper triangular elements of R . Therefore the regular elements of R are units of R .

Example 2.3. We shall show that there is a right Noetherian P.I. ring R which has a prime ideal P such that P satisfies the AR -property but R does not satisfy the right Ore condition with respect to $C(P)$. Let F be a field and x an indeterminate. Let R be the ring of all matrices of the form

$$\begin{pmatrix} f(0) & g(x) \\ 0 & f(x) \end{pmatrix}$$

with $f(x), g(x) \in F[x]$. Let $t: R \rightarrow F[x]$ be the function which sends the matrix displayed above to $f(x)$ and set $P = \text{Ker}(t)$, then t is a surjective ring homomorphism and P consists of the strictly upper triangular elements of R . Thus P is a prime ideal of R and trivially satisfies the AR -property because it is nilpotent. Let e_{i2} denote the element of R with 1 in the (i, 2)-position and 0's elsewhere for $i = 1$ or 2. Set $a = e_{12}$ and $c = xe_{22}$ then $c \in C(P)$ and it is not possible to have $ad = cb$ with $b \in R$ and $d \in C(P)$.

Remarks. (1) The following are open questions: (a) If R is a left and right Noetherian ring which has a prime ideal P which satisfies the AR -property, does R satisfy the right Ore condition with respect to $C(P)$? This question is open even in the special case where P

is nilpotent. (b) If R is a right Noetherian prime ring which has a non-zero prime ideal P which satisfies the AR -property, does R satisfy the right Ore condition with respect to $C(P)$?

(2) It is shown in (1), Theorem 3.11, that if R is right Noetherian and P is a prime ideal of R which satisfies the AR -property then R satisfies the right Ore condition with respect to $C(P)$ if and only if P is weakly ideal invariant.

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