

Three flavor neutrino oscillations in matter

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Abstract

We derive analytic expressions for three flavor neutrino oscillations in the presence of matter in the plane wave approximation using the Cayley–Hamilton formalism. Especially, we calculate the time evolution operator in both flavor and mass bases. Furthermore, we find the transition probabilities, matter mass squared differences, and matter mixing angles all expressed in terms of the vacuum mass squared differences, the vacuum mixing angles, and the matter density. The conditions for resonance in the presence of matter are also studied in some examples.

I. INTRODUCTION

There are at present essentially four different types of experiments looking for neutrino oscillations [1]: solar neutrino experiments, atmospheric neutrino experiments, accelerator neutrino experiments, and reactor neutrino experiments. Among the accelerator and reactor experiments, the accelerator long baseline experiments [2] are just beginning to become operative. Also the Sudbury Neutrino Observatory (SNO) [3] has started to take data on solar neutrinos. An accumulating amount of data on neutrino oscillations is therefore becoming accessible. Yet, at present there is no general agreement on the mixing angles or the mass squared differences of the neutrinos.

In a previous paper [4], we described a global analysis of neutrino oscillation data in a three flavor neutrino scenario. In that analysis, we deliberately considered mass ranges for the solution that avoided the region where the solar and atmospheric neutrinos could be affected by the so-called Mikheyev–Smirnov–Wolfenstein (MSW) effect [5].

Here we would like to report on the second stage of a global analysis of all the neutrino oscillation data in the three flavor neutrino model and present analytic expressions for the oscillation probabilities in the presence of matter expressed in the Cabibbo–Kobayashi–Maskawa (CKM) mixing matrix elements (or [vacuum] mixing angles) and the neutrino energies or masses, i.e., incorporating the MSW effect.

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We will assume as before that CP nonconservation is negligible at the present level of experimental accuracy [6]. Thus, the CKM matrix for the neutrinos is real.

Previous work on models for three flavor neutrino oscillations in matter includes works by Barger *et al.* [7], Kim and Sze [8], and Zaglauer and Schwarzer [9]. Our method is different from all these approaches and their parametrizations also differ slightly from ours. In particular, we calculate the time evolution operator and do not introduce the auxiliary (effective) matter mixing angles.

Approximative solutions for three flavor neutrino oscillations in matter have been presented by Kuo and Pantaleone [10] and Joshipura and Murthy [11]. Approximative treatments have also been done by Toshev and Petcov [12]. D’Olivo and Oteo have made contributions by using an approximative Magnus expansion for the time evolution operator [13] and Aquino, Bellandi, and Guzzo have found the auxiliary matter mixing angles in terms of the vacuum mixing angles [14]. Extensive numerical investigations of matter enhanced three flavor oscillations have been made by Fogli *et al.* [15].

Below we derive the expressions for the transition probabilities in the case of three flavor neutrino oscillations when the neutrinos pass through matter (the MSW effect) using Cayley–Hamilton’s theorem. This allows us to give rather explicit analytic formulas in terms of elementary functions for the quite involved expressions for the transition probabilities. We derive the time evolution operator for a constant matter (electron) density. The most straightforward way to use this result is to approximate any density profile with step functions and to obtain the total time evolution as a product of the evolution in each step. This can easily be handled numerically [16].

The three auxiliary angles of rotation from the matter mass eigenstate basis to the flavor state basis can be identified from these expressions and we show their relation to the ordinary formalism.

The application of this formalism to the analysis of the neutrino oscillation data will be presented in a forthcoming publication. Here we will only illustrate the qualitative behavior of the transition probabilities and the auxiliary matter mixing angles as functions of the matter density for different choices of the vacuum mixing angles and the mass squared differences.

It is possible to adapt the formalism presented here to the case of a scenario with an arbitrary number of neutrinos, but the eigenvalues have to be solved for numerically for the case of five or more neutrino flavors.

The outline of our paper is as follows. We first go through some formalism in Sec. II and then set out to find the solution for the time evolution operator and the transition probabilities in the presence of matter. In Sec. III, the main result of our analysis is given by the time evolution operator for the neutrinos when passing through matter with constant matter density expressed as a finite sum of elementary functions in the matrix elements of the Hamiltonian, and in Sec. IV, the transition probabilities in presence of matter are given. In Sec. V, we determine the auxiliary matter mixing angles. Finally, in Sec. VI, we present a discussion of our main results and the illustrations.

II. FORMALISM

Neutrinos are produced in flavor “eigenstates” $|\nu_\alpha\rangle$ ($\alpha = e, \mu, \tau$) created by the interaction of the weak gauge bosons with the charged leptons. Between the source, the production point of the neutrinos, and the detector, the neutrinos evolve as mass eigenstates $|\nu_a\rangle$ ($a = 1, 2, 3$), i.e., as states with definite mass. At the detector, the neutrino flavors can again be identified by charge current interactions. It is also possible to detect the neutrinos by weak neutral current interactions. In this case, however, the weak neutral currents do not distinguish the flavors on the detector side, and the total neutrino flux will be measured.

Let the flavor state basis and the mass eigenstate basis be given by $\mathcal{H}_f \equiv \{|\nu_\alpha\rangle\}_{\alpha=e,\mu,\tau}$ and $\mathcal{H}_m \equiv \{|\nu_a\rangle\}_{a=1}^3$, respectively. Then the flavor states $|\nu_\alpha\rangle \in \mathcal{H}_f$ can be obtained as superpositions of the mass eigenstates $|\nu_a\rangle \in \mathcal{H}_m$, or vice versa. Observe that the bases \mathcal{H}_f and \mathcal{H}_m are just two different representations of the same Hilbert space \mathcal{H} .

In the present analysis, we will use the plane wave approximation to describe neutrino oscillations. In this approximation, a neutrino field ν_α with flavor α is a linear combination of neutrino mass fields ν_a such that [17]

$$\nu_\alpha = \sum_{a=1}^3 U_{\alpha a} \nu_a, \quad \alpha = e, \mu, \tau, \quad (1)$$

where the $U_{\alpha a}$ ’s are entries in a 3×3 unitary matrix U . Taking the Hermitian conjugate of Eq. (1) yields

$$\nu_\alpha^\dagger = \sum_{a=1}^3 U_{\alpha a}^* \nu_a^\dagger, \quad \alpha = e, \mu, \tau. \quad (2)$$

Acting with Eq. (2) on the vacuum state $|0\rangle$ gives

$$|\nu_\alpha\rangle = \sum_{a=1}^3 U_{\alpha a}^* |\nu_a\rangle, \quad \alpha = e, \mu, \tau, \quad (3)$$

since the states are defined as $|\nu_\alpha\rangle \equiv \nu_\alpha^\dagger |0\rangle$, where $\alpha = e, \mu, \tau$, and $|\nu_a\rangle \equiv \nu_a^\dagger |0\rangle$, where $a = 1, 2, 3$. In what follows, we will use the short-hand notations $|\alpha\rangle \equiv |\nu_\alpha\rangle$ and $|a\rangle \equiv |\nu_a\rangle$ for the flavor states and the mass eigenstates, respectively. An arbitrary neutrino state $\psi \in \mathcal{H}$ can, of course, be expressed in both the flavor and mass bases as

$$\psi \equiv \sum_{\alpha=e,\mu,\tau} \psi_\alpha |\alpha\rangle = \sum_{\alpha=e,\mu,\tau} \psi_\alpha \sum_{a=1}^3 U_{\alpha a}^* |a\rangle = \sum_{a=1}^3 \left(\sum_{\alpha=e,\mu,\tau} \psi_\alpha U_{\alpha a}^* \right) |a\rangle \equiv \sum_{a=1}^3 \psi_a |a\rangle, \quad (4)$$

where ψ_α , $\alpha = e, \mu, \tau$, and ψ_a , $a = 1, 2, 3$, are the components of the state ψ in the flavor state basis and the mass eigenstate basis, respectively, and they are related to each other by

$$\psi_a = \sum_{\alpha=e,\mu,\tau} U_{\alpha a}^* \psi_\alpha, \quad a = 1, 2, 3. \quad (5)$$

In matrix form, one has

$$\psi_f = U\psi_m, \quad (6)$$

where

$$\psi_f = (\psi_\alpha) = \begin{pmatrix} \psi_e \\ \psi_\mu \\ \psi_\tau \end{pmatrix} \in \mathcal{H}_f \quad \text{and} \quad \psi_m = (\psi_a) = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \in \mathcal{H}_m.$$

Here the subscripts f and m denote the flavor state basis and the mass eigenstate basis, respectively.

The unitary matrix U is given by

$$U = (U_{\alpha a}) = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix}. \quad (7)$$

A convenient parametrization for $U = U(\theta_1, \theta_2, \theta_3)$ is given by [18]

$$U = \begin{pmatrix} C_2 C_3 & S_3 C_2 & S_2 \\ -S_3 C_1 - S_1 S_2 C_3 & C_1 C_3 - S_1 S_2 S_3 & S_1 C_2 \\ S_1 S_3 - S_2 C_1 C_3 & -S_1 C_3 - S_2 S_3 C_1 & C_1 C_2 \end{pmatrix}, \quad (8)$$

where $S_i \equiv \sin \theta_i$ and $C_i \equiv \cos \theta_i$ for $i = 1, 2, 3$. This is the so-called standard representation of the CKM mixing matrix. The quantities θ_i , where $i = 1, 2, 3$, are the so-called (vacuum) mixing angles. We have here put the CP phase equal to zero in the CKM matrix. This means that $U_{\alpha a}^* = U_{\alpha a}$ for $\alpha = e, \mu, \tau$ and $a = 1, 2, 3$.

In the mass eigenstate basis \mathcal{H}_m , the Hamiltonian \mathcal{H} for the propagation of the neutrinos in vacuum is diagonal and given by

$$H_m = \begin{pmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{pmatrix}, \quad (9)$$

where $E_a = \sqrt{m_a^2 + \mathbf{p}^2}$, $a = 1, 2, 3$, are the energies of the neutrino mass eigenstates $|a\rangle$, $a = 1, 2, 3$ with masses m_a , $a = 1, 2, 3$. Note that we assume the momentum \mathbf{p} to be the same for all mass eigenstates.

When neutrinos propagate in matter, there is an additional term coming from the presence of electrons in the matter [5]. This term is diagonal in the flavor state basis \mathcal{H}_f and is given by

$$V_f = A \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (10)$$

where $A = \pm\sqrt{2}G_F N_e$ is the matter density. Here G_F is the Fermi weak coupling constant and N_e is the electron density. The sign depends on whether we deal with neutrinos (+) or antineutrinos (-). We will assume that the electron density N_e is constant throughout the

matter in which the neutrinos are propagating. In the mass representation, this piece of the Hamiltonian is $V_m = U^{-1}V_fU$, where again U is the CKM matrix.

We are looking for the unitary transformation that leads from the initial state $\psi_f(0)$ in flavor basis at time $t = 0$ of production of the neutrino, to the state of the same neutrino $\psi_f(t)$ at the detector at time t . This transformation is given by the operator $U_f(t) \equiv U_f(t, 0)$, where $U_f(t_2, t_1)$ is the time evolution operator from time t_1 to time t_2 in flavor basis. This operator can be formally written as $U_f(t) = e^{-iH_ft}$. When the neutrinos are propagating through vacuum, the Hamiltonian in flavor basis is $H_f = UH_mU^{-1}$. In this case, the exponentiation of H_f can be effectuated easily: $U_f(t) = e^{-iH_ft} = Ue^{-iH_mt}U^{-1}$, and the result can be expressed in closed form. In the case when the neutrinos propagate through matter, the Hamiltonian is not diagonal in either the mass eigenstate or the flavor state basis, and we have to calculate the operator $U_f(t)$. This is the task we now set out for.

III. CALCULATING THE TIME EVOLUTION OPERATOR

The Schrödinger equation in the mass eigenstate basis is

$$i\frac{d}{dt}\psi_m(t) = \mathcal{H}_m\psi_m(t), \quad (11)$$

where

$$\mathcal{H}_m = H_m + V_m$$

is the Hamiltonian.

Similarly, in the flavor state basis, one has

$$i\frac{d}{dt}\psi_f(t) = \mathcal{H}_f\psi_f(t), \quad (12)$$

where

$$\mathcal{H}_f = H_f + V_f.$$

In the mass eigenstate basis, the matter term (the potential matrix) is

$$V_m = U^{-1}V_fU. \quad (13)$$

Thus, the total Hamiltonian in the mass eigenstate basis is given by

$$\mathcal{H}_m = H_m + U^{-1}V_fU. \quad (14)$$

Equation (11) has the solution

$$\psi_m(t) = e^{-i\mathcal{H}_mt}\psi_m(0). \quad (15)$$

Inserting $t = L$ into Eq. (15), one finds the solution in the mass eigenstate basis to be

$$\psi_m(L) = e^{-i\mathcal{H}_mL}\psi_m(0) \equiv U_m(L)\psi_m(0) \quad (16)$$

and in the flavor state basis

$$\begin{aligned}\psi_f(L) &= U\psi_m(L) = Ue^{-i\mathcal{H}_m L}\psi_m(0) = Ue^{-i\mathcal{H}_m L}U^{-1}U\psi_m(0) = Ue^{-i\mathcal{H}_m L}U^{-1}\psi_f(0) \\ &\equiv U_f(L)\psi_f(0).\end{aligned}\tag{17}$$

Inserting the potential matrix in the flavor basis and the CKM matrix into Eq. (13) yields

$$V_m = A \begin{pmatrix} U_{e1}^2 & U_{e1}U_{e2} & U_{e1}U_{e3} \\ U_{e2}U_{e1} & U_{e2}^2 & U_{e2}U_{e3} \\ U_{e3}U_{e1} & U_{e3}U_{e2} & U_{e3}^2 \end{pmatrix}, \tag{18}$$

which is a (real) symmetric matrix. Then, the Hamiltonian in the mass eigenstate basis is

$$\mathcal{H}_m = \begin{pmatrix} E_1 + AU_{e1}^2 & AU_{e1}U_{e2} & AU_{e1}U_{e3} \\ AU_{e2}U_{e1} & E_2 + AU_{e2}^2 & AU_{e2}U_{e3} \\ AU_{e3}U_{e1} & AU_{e3}U_{e2} & E_3 + AU_{e3}^2 \end{pmatrix}. \tag{19}$$

Of course, we also have $U_f(L) = e^{-i\mathcal{H}_f L}$, but we prefer to first work out $U_m(L) = e^{-i\mathcal{H}_m L}$, since the structure of \mathcal{H}_m is much simpler than that of \mathcal{H}_f , and afterwards perform the transformation to $U_f(L)$.

Next, we need to find the exponential of the matrix $-i\mathcal{H}_m L$ in order to obtain $U_m(L)$. Generally, when one wants to find the exponential of an $N \times N$ matrix M , one has to calculate

$$e^M = \sum_{n=0}^{\infty} \frac{1}{n!} M^n. \tag{20}$$

The characteristic equation of the matrix M is given by

$$\chi(\lambda) \equiv \det(M - \lambda I) = \lambda^N + c_{N-1}\lambda^{N-1} + \cdots + c_1\lambda + c_0 = 0, \tag{21}$$

where c_i , $i = 0, 1, \dots, N-1$, are some coefficients. Cayley–Hamilton’s theorem implies that λ in the characteristic equation can be replaced with the matrix M itself, giving

$$M^N + c_{N-1}M^{N-1} + \cdots + c_1M + c_0I = 0, \tag{22}$$

where I is the $N \times N$ identity matrix. Hence, one has

$$M^N = -c_{N-1}M^{N-1} - \cdots - c_1M - c_0I, \tag{23}$$

which means that

$$M^p = c_{N-1}^{(p)}M^{N-1} + \cdots + c_1^{(p)}M + c_0^{(p)}I, \quad \text{for any } p \geq N, \tag{24}$$

where $c_i^{(p)}$, $i = 0, 1, \dots, N-1$, are some coefficients. The exponential of the matrix M can then be written as

$$e^M = a_0I + a_1M + \cdots + a_{N-1}M^{N-1} = \sum_{n=0}^{N-1} a_n M^n, \tag{25}$$

where a_i , $i = 0, 1, \dots, N-1$, are some coefficients. Thus, the infinite series in Eq. (20) is reduced to only N terms, where N is the dimension of the matrix M .

An arbitrary $N \times N$ matrix M can always be written as

$$M = M_0 + \frac{1}{N}(\text{tr } M)I, \quad (26)$$

where M_0 is an $N \times N$ traceless matrix, i.e., $\text{tr } M_0 = 0$. Splitting up M as in Eq. (26) is useful, since the identity matrix I is commuting with all other matrices.

Now using Eqs. (25) and (26), one can write the exponential of the matrix $-i\mathcal{H}_m L$ as

$$e^{-i\mathcal{H}_m L} = \phi e^{-iLT} = \phi [a_0 I + a_1(-iLT) + a_2(-iLT)^2] = \phi (a_0 I - iLT a_1 - L^2 T^2 a_2), \quad (27)$$

where $\phi \equiv e^{-iL \text{tr } \mathcal{H}_m / 3}$ is a complex phase factor, $T \equiv \mathcal{H}_m - (\text{tr } \mathcal{H}_m)I/3$ is a traceless matrix, and a_0 , a_1 , and a_2 are coefficients to be determined. The infinite series in Eq. (20) is here reduced to only three terms. The trace of the matrix \mathcal{H}_m is

$$\text{tr } \mathcal{H}_m = E_1 + E_2 + E_3 + A. \quad (28)$$

The matrix T can now be written as

$$T = (T_{ab}) = \begin{pmatrix} AU_{e1}^2 - \frac{1}{3}A + \frac{1}{3}(E_{12} + E_{13}) & AU_{e1}U_{e2} & AU_{e1}U_{e3} \\ AU_{e1}U_{e2} & AU_{e2}^2 - \frac{1}{3}A + \frac{1}{3}(E_{21} + E_{23}) & AU_{e2}U_{e3} \\ AU_{e1}U_{e3} & AU_{e2}U_{e3} & AU_{e3}^2 - \frac{1}{3}A + \frac{1}{3}(E_{31} + E_{32}) \end{pmatrix}, \quad (29)$$

where $E_{ab} \equiv E_a - E_b$. The six quantities E_{ab} , where $a, b = 1, 2, 3$ and $a \neq b$, are not linearly independent, since they obey the following relations:

$$E_{ba} = -E_{ab}, \quad (30)$$

$$E_{12} + E_{23} + E_{31} = 0. \quad (31)$$

This means that only two of the E_{ab} 's are linearly independent, e.g., E_{21} and E_{32} .

The following linear system of equations will determine the coefficients a_0 , a_1 , and a_2 introduced in Eq. (27) above,

$$\begin{aligned} e^{-iL\lambda_1} &= a_0 - iL\lambda_1 a_1 - L^2 \lambda_1^2 a_2 \\ e^{-iL\lambda_2} &= a_0 - iL\lambda_2 a_1 - L^2 \lambda_2^2 a_2, \\ e^{-iL\lambda_3} &= a_0 - iL\lambda_3 a_1 - L^2 \lambda_3^2 a_2 \end{aligned} \quad (32)$$

where λ_a , $a = 1, 2, 3$, are the eigenvalues of the matrix T , i.e., solutions to the characteristic equation

$$\lambda^3 + c_2 \lambda^2 + c_1 \lambda + c_0 = 0$$

with

$$c_2 = -\text{tr } T = 0,$$

$$\begin{aligned}
c_1 &= T_{11}T_{22} - T_{12}T_{21} + T_{11}T_{33} - T_{13}T_{31} + T_{22}T_{33} - T_{23}T_{32} \\
&= -\frac{1}{3}A^2 + \frac{1}{3}A \left[U_{e1}^2 (E_{21} + E_{31}) + U_{e2}^2 (E_{12} + E_{32}) + U_{e3}^2 (E_{13} + E_{23}) \right] \\
&\quad - \frac{1}{9} (E_{12}^2 + E_{13}^2 + E_{23}^2 + E_{12}E_{13} + E_{21}E_{23} + E_{31}E_{32}),
\end{aligned}$$

$$\begin{aligned}
c_0 &= -\det T = -\frac{2}{27}A^3 + \frac{1}{9}A^2 \left[U_{e1}^2 (E_{21} + E_{31}) + U_{e2}^2 (E_{12} + E_{32}) + U_{e3}^2 (E_{13} + E_{23}) \right] \\
&\quad - \frac{1}{9}A \left[U_{e1}^2 (E_{21} + E_{23}) (E_{31} + E_{32}) + U_{e2}^2 (E_{12} + E_{13}) (E_{31} + E_{32}) \right. \\
&\quad \left. + U_{e3}^2 (E_{12} + E_{13}) (E_{21} + E_{23}) + \frac{1}{3} (E_{12}^2 + E_{13}^2 + E_{23}^2 + E_{12}E_{13} + E_{21}E_{23} + E_{31}E_{32}) \right] \\
&\quad - \frac{1}{27} (E_{12} + E_{13}) (E_{21} + E_{23}) (E_{31} + E_{32}).
\end{aligned}$$

Note that the coefficients c_0 , c_1 , and c_2 are all real. Introducing the relativistic limit (see Appendix A), E_{21} and E_{32} can be written in terms of the mass squared differences Δm^2 ($= \Delta m_{21}^2$) and ΔM^2 ($= \Delta m_{32}^2$) as

$$E_{21} = \frac{\Delta m^2}{2E} \quad \text{and} \quad E_{32} \simeq \frac{\Delta M^2}{2E},$$

where E is the neutrino energy.

The solutions to the characteristic equation are [19]

$$\lambda_1 = -\frac{1}{2}(s_1 + s_2) - \frac{1}{3}c_2 + i\frac{\sqrt{3}}{2}(s_1 - s_2) = -\frac{1}{2}(s_1 + s_2) + i\frac{\sqrt{3}}{2}(s_1 - s_2), \quad (33)$$

$$\lambda_2 = -\frac{1}{2}(s_1 + s_2) - \frac{1}{3}c_2 - i\frac{\sqrt{3}}{2}(s_1 - s_2) = -\frac{1}{2}(s_1 + s_2) - i\frac{\sqrt{3}}{2}(s_1 - s_2), \quad (34)$$

$$\lambda_3 = s_1 + s_2 - \frac{1}{3}c_2 = s_1 + s_2, \quad (35)$$

where

$$s_1 = [r + (q^3 + r^2)^{1/2}]^{1/3}, \quad s_2 = [r - (q^3 + r^2)^{1/2}]^{1/3},$$

and

$$\begin{aligned}
q &= \frac{1}{3}c_1 - \frac{1}{9}c_2^2 = \frac{1}{3}c_1, \\
r &= \frac{1}{6}(c_1c_2 - 3c_0) - \frac{1}{27}c_2^3 = -\frac{1}{2}c_0.
\end{aligned}$$

Since T is a Hermitian matrix (T is a real symmetric matrix), it holds that $q^3 + r^2 \leq 0$ and all eigenvalues λ_a , where $a = 1, 2, 3$, are real. Furthermore, inserting $q = c_1/3$ and $r = -c_0/2$ into the inequality $q^3 + r^2 \leq 0$, one finds $c_1^3/27 + c_0^2/4 \leq 0$ from which one can conclude that $c_1 \leq 0$ and $|c_1| \geq 3|c_0|^{2/3}/\sqrt[3]{4}$, since $c_0, c_1 \in \mathbb{R}$. Thus, the roots (eigenvalues) can be written as

$$\lambda_1 = -\sqrt{-\frac{1}{3}c_1} \cos \left[\frac{1}{3} \arctan \left(\frac{1}{c_0} \sqrt{-c_0^2 - \frac{4}{27}c_1^3} \right) \right] + \sqrt{-c_1} \sin \left[\frac{1}{3} \arctan \left(\frac{1}{c_0} \sqrt{-c_0^2 - \frac{4}{27}c_1^3} \right) \right], \quad (36)$$

$$\lambda_2 = -\sqrt{-\frac{1}{3}c_1} \cos \left[\frac{1}{3} \arctan \left(\frac{1}{c_0} \sqrt{-c_0^2 - \frac{4}{27}c_1^3} \right) \right] - \sqrt{-c_1} \sin \left[\frac{1}{3} \arctan \left(\frac{1}{c_0} \sqrt{-c_0^2 - \frac{4}{27}c_1^3} \right) \right], \quad (37)$$

$$\lambda_3 = 2\sqrt{-\frac{1}{3}c_1} \cos \left[\frac{1}{3} \arctan \left(\frac{1}{c_0} \sqrt{-c_0^2 - \frac{4}{27}c_1^3} \right) \right]. \quad (38)$$

Observe that $\lambda_1 + \lambda_2 + \lambda_3 = 0$. This relation is satisfied because of the fact that the trace of T is zero. In addition, the relations $\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = c_1 \leq 0$ and $\lambda_1\lambda_2\lambda_3 = -c_0$ are fulfilled.

The λ_a 's are, up to a constant, the energy eigenvalues of the neutrinos in the presence of matter. The measurable quantities are the energy differences $|\lambda_a - \lambda_b|$, which in the relativistic limit are related to the effective matter mass squared differences $\Delta\tilde{m}_{ab}^2$ as

$$|\lambda_a - \lambda_b| = \frac{|\Delta\tilde{m}_{ab}^2|}{2E}, \quad a, b = 1, 2, 3, \quad a \neq b, \quad (39)$$

where E is again the neutrino energy.

Introducing the following matrix form representation for the system of equations in Eq. (32),

$$\mathbf{e} = \Lambda \mathbf{a}, \quad (40)$$

where

$$\mathbf{e} = \begin{pmatrix} e^{-iL\lambda_1} \\ e^{-iL\lambda_2} \\ e^{-iL\lambda_3} \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 1 & -iL\lambda_1 & -L^2\lambda_1^2 \\ 1 & -iL\lambda_2 & -L^2\lambda_2^2 \\ 1 & -iL\lambda_3 & -L^2\lambda_3^2 \end{pmatrix}, \quad \text{and} \quad \mathbf{a} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix},$$

one obtains the solution

$$\begin{aligned} \mathbf{a} &= \Lambda^{-1} \mathbf{e} \\ &= \frac{1}{D} \begin{pmatrix} iL^3(\lambda_2\lambda_3^2 - \lambda_2^2\lambda_3) & iL^3(\lambda_1^2\lambda_3 - \lambda_1\lambda_3^2) & iL^3(\lambda_1\lambda_2^2 - \lambda_1^2\lambda_2) \\ -L^2(\lambda_2^2 - \lambda_3^2) & -L^2(\lambda_3^2 - \lambda_1^2) & -L^2(\lambda_1^2 - \lambda_2^2) \\ -iL(\lambda_3 - \lambda_2) & -iL(\lambda_1 - \lambda_3) & -iL(\lambda_2 - \lambda_1) \end{pmatrix} \begin{pmatrix} e^{-iL\lambda_1} \\ e^{-iL\lambda_2} \\ e^{-iL\lambda_3} \end{pmatrix} \\ &= \frac{1}{D} \begin{pmatrix} iL^3 [e^{-iL\lambda_1}(\lambda_2\lambda_3^2 - \lambda_2^2\lambda_3) + e^{-iL\lambda_2}(\lambda_1^2\lambda_3 - \lambda_1\lambda_3^2) + e^{-iL\lambda_3}(\lambda_1\lambda_2^2 - \lambda_1^2\lambda_2)] \\ -L^2 [e^{-iL\lambda_1}(\lambda_2^2 - \lambda_3^2) + e^{-iL\lambda_2}(\lambda_3^2 - \lambda_1^2) + e^{-iL\lambda_3}(\lambda_1^2 - \lambda_2^2)] \\ -iL [e^{-iL\lambda_1}(\lambda_3 - \lambda_2) + e^{-iL\lambda_2}(\lambda_1 - \lambda_3) + e^{-iL\lambda_3}(\lambda_2 - \lambda_1)] \end{pmatrix}, \end{aligned} \quad (41)$$

where

$$D \equiv \det \Lambda = iL^3 (\lambda_1\lambda_2^2 - \lambda_1^2\lambda_2 + \lambda_2\lambda_3^2 - \lambda_2^2\lambda_3 + \lambda_3\lambda_1^2 - \lambda_3^2\lambda_1).$$

Thus, the exponential of the matrix $-i\mathcal{H}_m L$ in Eq. (27) can be written as

$$\begin{aligned}
e^{-i\mathcal{H}_m L} = \phi e^{-iLT} = -i\phi \frac{L^3}{D} & \left\{ [e^{-iL\lambda_1} (\lambda_2^2 \lambda_3 - \lambda_2 \lambda_3^2) + e^{-iL\lambda_2} (\lambda_1 \lambda_3^2 - \lambda_1^2 \lambda_3) \right. \\
& + e^{-iL\lambda_3} (\lambda_1^2 \lambda_2 - \lambda_1 \lambda_2^2)] I + [e^{-iL\lambda_1} (\lambda_3^2 - \lambda_2^2) + e^{-iL\lambda_2} (\lambda_1^2 - \lambda_3^2) \\
& + e^{-iL\lambda_3} (\lambda_2^2 - \lambda_1^2)] T + [e^{-iL\lambda_1} (\lambda_3 - \lambda_2) + e^{-iL\lambda_2} (\lambda_3 - \lambda_1) \\
& \left. + e^{-iL\lambda_3} (\lambda_1 - \lambda_2)] T^2 \right\}. \tag{42}
\end{aligned}$$

Inserting the expression for D into Eq. (42) gives

$$\begin{aligned}
e^{-i\mathcal{H}_m L} = & \frac{1}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} \phi e^{-iL\lambda_1} [\lambda_2 \lambda_3 I - (\lambda_2 + \lambda_3)T + T^2] \\
& + \frac{1}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} \phi e^{-iL\lambda_2} [\lambda_1 \lambda_3 I - (\lambda_1 + \lambda_3)T + T^2] \\
& + \frac{1}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \phi e^{-iL\lambda_3} [\lambda_1 \lambda_2 I - (\lambda_1 + \lambda_2)T + T^2]. \tag{43}
\end{aligned}$$

Using the various relations for the eigenvalues finally yields

$$\begin{aligned}
e^{-i\mathcal{H}_m L} = \phi e^{-iL\lambda_1} \frac{(\lambda_1^2 + c_1)I + \lambda_1 T + T^2}{3\lambda_1^2 + c_1} & + \phi e^{-iL\lambda_2} \frac{(\lambda_2^2 + c_1)I + \lambda_2 T + T^2}{3\lambda_2^2 + c_1} \\
& + \phi e^{-iL\lambda_3} \frac{(\lambda_3^2 + c_1)I + \lambda_3 T + T^2}{3\lambda_3^2 + c_1}, \tag{44}
\end{aligned}$$

which can be written as

$$U_m(L) = e^{-i\mathcal{H}_m L} = \phi \sum_{a=1}^3 e^{-iL\lambda_a} \frac{1}{3\lambda_a^2 + c_1} [(\lambda_a^2 + c_1)I + \lambda_a T + T^2]. \tag{45}$$

The evolution operator for the neutrinos in the flavor basis is thus given by

$$U_f(L) = e^{-i\mathcal{H}_f L} = U e^{-i\mathcal{H}_m L} U^{-1} = \phi \sum_{a=1}^3 e^{-iL\lambda_a} \frac{1}{3\lambda_a^2 + c_1} [(\lambda_a^2 + c_1)I + \lambda_a \tilde{T} + \tilde{T}^2], \tag{46}$$

where $\tilde{T} \equiv U T U^{-1}$. Equation (46) is our final expression for $U_f(L)$.

Let us pause for a moment to contemplate Eqs. (45) and (46). Since $\mathcal{H}_f = U \mathcal{H}_m U^{-1}$, it is clear that $\tilde{T} = \mathcal{H}_f - (\text{tr } \mathcal{H}_f)I/3 = \mathcal{H}_f - (\text{tr } \mathcal{H}_m)I/3$ due to the invariance of the trace under transformation of U . In fact, the characteristic equation is also invariant under U and therefore so are the coefficients c_0, c_1, c_2 , and the eigenvalues $\lambda_1, \lambda_2, \lambda_3$. However, the expression for \mathcal{H}_f is much more complicated than that for \mathcal{H}_m , which is the reason why we work with \mathcal{H}_m instead of \mathcal{H}_f .

The formula (46) is our main result for the evolution operator. It expresses the time (or L) evolution directly in terms of the mass squared differences and the vacuum mixing angles without introducing any auxiliary matter mixing angles.

IV. PROBABILITY AMPLITUDES AND TRANSITION PROBABILITIES

The probability amplitude is defined as

$$A_{\alpha\beta} \equiv \langle \beta | U_f(L) | \alpha \rangle, \quad \alpha, \beta = e, \mu, \tau. \quad (47)$$

Inserting Eq. (46) into Eq. (47) gives

$$A_{\alpha\beta} = \phi \sum_{a=1}^3 e^{-iL\lambda_a} \frac{(\lambda_a^2 + c_1)\delta_{\alpha\beta} + \lambda_a \tilde{T}_{\alpha\beta} + (\tilde{T}^2)_{\alpha\beta}}{3\lambda_a^2 + c_1}, \quad (48)$$

where $\delta_{\alpha\beta}$ is Kronecker's delta and the entries $\tilde{T}_{\alpha\beta}$ and $(\tilde{T}^2)_{\alpha\beta}$, $\alpha, \beta = e, \mu, \tau$, are given in Appendix B. Here $\tilde{T}_{\alpha\beta} = \tilde{T}_{\beta\alpha}$ and $(\tilde{T}^2)_{\alpha\beta} = (\tilde{T}^2)_{\beta\alpha}$.

The probability of transition from a neutrino flavor α to a neutrino flavor β is thus defined by the expression

$$P_{\alpha\beta} \equiv |A_{\alpha\beta}|^2 = A_{\alpha\beta}^* A_{\alpha\beta}, \quad \alpha, \beta = e, \mu, \tau. \quad (49)$$

Inserting the expression for the probability amplitude into the definition of the transition probability, one finds

$$P_{\alpha\beta} = \sum_{a=1}^3 \sum_{b=1}^3 e^{-iL(\lambda_a - \lambda_b)} \frac{(\lambda_a^2 + c_1)\delta_{\alpha\beta} + \lambda_a \tilde{T}_{\alpha\beta} + (\tilde{T}^2)_{\alpha\beta}}{3\lambda_a^2 + c_1} \frac{(\lambda_b^2 + c_1)\delta_{\alpha\beta} + \lambda_b \tilde{T}_{\alpha\beta} + (\tilde{T}^2)_{\alpha\beta}}{3\lambda_b^2 + c_1}. \quad (50)$$

Since $\delta_{\alpha\beta}$, $\tilde{T}_{\alpha\beta}$, and $(\tilde{T}^2)_{\alpha\beta}$ all are symmetric, it holds that $P_{\alpha\beta} = P_{\beta\alpha}$.

Setting $A = 0$ gives

$$\langle b | U_m(L) | a \rangle = \langle b | e^{-iH_m L} | a \rangle = e^{-iE_a L} \delta_{ab}, \quad a, b = 1, 2, 3, \quad (51)$$

$$A_{\alpha\beta} = \langle \beta | U_f(L) | \alpha \rangle = \sum_{a=1}^3 U_{\alpha a} U_{\beta a} e^{-iE_a L}, \quad \alpha, \beta = e, \mu, \tau, \quad (52)$$

and

$$P_{\alpha\beta} = \delta_{\alpha\beta} - 4 \sum_{a=1}^3 \sum_{\substack{b=1 \\ a < b}}^3 U_{\alpha a} U_{\beta a} U_{\alpha b} U_{\beta b} \sin^2 x_{ab}, \quad \alpha, \beta = e, \mu, \tau, \quad (53)$$

where $x_{ab} \equiv E_{ab}L/2$, which are the well-known transition probabilities for vacuum oscillations (see, e.g., Ref. [4]).

We can write the probabilities for oscillations in matter in a form analogous to the ones for the vacuum probabilities, as

$$\begin{aligned} P_{\alpha\beta} = & \left(\sum_{a=1}^3 \frac{(\lambda_a^2 + c_1)\delta_{\alpha\beta} + \lambda_a \tilde{T}_{\alpha\beta} + (\tilde{T}^2)_{\alpha\beta}}{3\lambda_a^2 + c_1} \right)^2 \\ & - 4 \sum_{a=1}^3 \sum_{\substack{b=1 \\ a < b}}^3 \frac{(\lambda_a^2 + c_1)\delta_{\alpha\beta} + \lambda_a \tilde{T}_{\alpha\beta} + (\tilde{T}^2)_{\alpha\beta}}{3\lambda_a^2 + c_1} \frac{(\lambda_b^2 + c_1)\delta_{\alpha\beta} + \lambda_b \tilde{T}_{\alpha\beta} + (\tilde{T}^2)_{\alpha\beta}}{3\lambda_b^2 + c_1} \sin^2 \tilde{x}_{ab}, \\ & \alpha, \beta = e, \mu, \tau, \end{aligned} \quad (54)$$

where $\tilde{x}_{ab} \equiv (\lambda_a - \lambda_b)L/2$. For $L = 0$ Eq. (46) reads

$$I = \sum_{a=1}^3 \frac{1}{3\lambda_a^2 + c_1} [(\lambda_a^2 + c_1)I + \lambda_a \tilde{T} + \tilde{T}^2]. \quad (55)$$

This means that the corresponding matrix element version of the above formula is given by

$$\delta_{\alpha\beta} = \sum_{a=1}^3 \frac{(\lambda_a^2 + c_1)\delta_{\alpha\beta} + \lambda_a \tilde{T}_{\alpha\beta} + (\tilde{T}^2)_{\alpha\beta}}{3\lambda_a^2 + c_1}, \quad \alpha, \beta = e, \mu, \tau. \quad (56)$$

Hence, the transition probabilities in matter are

$$P_{\alpha\beta} = \delta_{\alpha\beta} - 4 \sum_{a=1}^3 \sum_{\substack{b=1 \\ a < b}}^3 \frac{(\lambda_a^2 + c_1)\delta_{\alpha\beta} + \lambda_a \tilde{T}_{\alpha\beta} + (\tilde{T}^2)_{\alpha\beta}}{3\lambda_a^2 + c_1} \frac{(\lambda_b^2 + c_1)\delta_{\alpha\beta} + \lambda_b \tilde{T}_{\alpha\beta} + (\tilde{T}^2)_{\alpha\beta}}{3\lambda_b^2 + c_1} \sin^2 \tilde{x}_{ab},$$

$$\alpha, \beta = e, \mu, \tau. \quad (57)$$

From unitarity, there are only three independent transition probabilities, since the other three can be obtained from them, i.e., from the equations

$$P_{ee} + P_{e\mu} + P_{e\tau} = 1, \quad (58)$$

$$P_{\mu e} + P_{\mu\mu} + P_{\mu\tau} = 1, \quad (59)$$

$$P_{\tau e} + P_{\tau\mu} + P_{\tau\tau} = 1, \quad (60)$$

where $P_{e\mu} = P_{\mu e}$, $P_{e\tau} = P_{\tau e}$, and $P_{\mu\tau} = P_{\tau\mu}$. We will choose $P_{e\mu}$, $P_{e\tau}$, and $P_{\mu\tau}$ as the three independent ones. The expression for the transition probabilities in Eq. (57) thus simplifies to

$$P_{\alpha\beta} = -4 \sum_{a=1}^3 \sum_{\substack{b=1 \\ a < b}}^3 \frac{\lambda_a \tilde{T}_{\alpha\beta} + (\tilde{T}^2)_{\alpha\beta}}{3\lambda_a^2 + c_1} \frac{\lambda_b \tilde{T}_{\alpha\beta} + (\tilde{T}^2)_{\alpha\beta}}{3\lambda_b^2 + c_1} \sin^2 \tilde{x}_{ab}, \quad (61)$$

which is valid only for $\alpha \neq \beta$.

V. DETERMINING THE AUXILIARY MATTER MIXING ANGLES

The flavor states can be expressed as linear combinations of either the vacuum mass eigenstates ($A = 0$) in the basis \mathcal{H}_m or the matter mass eigenstates ($A \neq 0$) in the basis \mathcal{H}_M . The corresponding components are related to each other as follows:

$$\psi_f = U\psi_m, \quad (62)$$

$$\psi_f = U^M\psi_M, \quad (63)$$

where $U = U(\theta_1, \theta_2, \theta_3)$ and $U^M = U^M(\theta_1^M, \theta_2^M, \theta_3^M)$ are the unitary mixing matrices for vacuum and matter, respectively. Here θ_i^M , $i = 1, 2, 3$, are the auxiliary (effective) matter mixing angles.

Combining these expressions for the flavor components, one obtains the following relation between the two different sets of mass eigenstate components:

$$\psi_M = R\psi_m, \quad (64)$$

where

$$R \equiv (U^M)^{-1}U.$$

The matrix R is, of course, a unitary matrix (even orthogonal, since U and U^M are real). This means that the matter mixing matrix can be expressed in the vacuum mixing matrix as

$$U^M(\theta_1^M, \theta_2^M, \theta_3^M) = U(\theta_1, \theta_2, \theta_3)R^{-1}. \quad (65)$$

The relations between the different bases can be depicted as in the following commutative diagram:

$$\begin{array}{ccc} \psi_m \in \mathcal{H}_m & \xrightarrow{U} & \mathcal{H}_f \ni \psi_f \\ R \downarrow & \nearrow U^M & \\ \psi_M \in \mathcal{H}_M & & \end{array}$$

From this diagram one readily obtains the Hamiltonian in presence of matter \mathcal{H}_M as

$$\mathcal{H}_M \equiv R\mathcal{H}_m R^{-1} \quad \text{or} \quad \mathcal{H}_M \equiv RTR^{-1} + \frac{1}{3}(\text{tr } \mathcal{H}_m)I,$$

which is diagonal in the mass eigenstate basis in matter, \mathcal{H}_M .

Due to the invariance of the trace, we have

$$T_M \equiv \mathcal{H}_M - \frac{1}{3}(\text{tr } \mathcal{H}_M)I = RTR^{-1}, \quad (66)$$

where T_M is a traceless diagonal matrix. This implies that

$$e^{-iLT_M} = Re^{-iLT}R^{-1} = \sum_{a=1}^3 e^{-iL\lambda_a} \frac{1}{3\lambda_a^2 + c_1} [(\lambda_a^2 + c_1)I + \lambda_a T_M + T_M^2], \quad (67)$$

which is actually equivalent to the system of equations (32).

Inverting the first relation above, we obtain

$$U_f(L) = \phi U e^{-iLT} U^{-1} = \phi U R^{-1} e^{-iLT_M} R U^{-1} = \phi U^M e^{-iLT_M} (U^M)^{-1}. \quad (68)$$

From this one immediately obtains the transition amplitude

$$A_{\alpha\beta} = \langle \beta | U_f(L) | \alpha \rangle = \phi \sum_{a=1}^3 U_{\alpha a}^M U_{\beta a}^M e^{-iL\lambda_a}, \quad \alpha, \beta = e, \mu, \tau, \quad (69)$$

where the $U_{\alpha a}^M$'s are the entries in the matrix U^M . This allows us to identify the U^M matrix elements with those in Eq. (48) as

$$U_{\alpha a}^M U_{\beta a}^M = \frac{(\lambda_a^2 + c_1)\delta_{\alpha\beta} + \lambda_a \tilde{T}_{\alpha\beta} + (\tilde{T}^2)_{\alpha\beta}}{3\lambda_a^2 + c_1}, \quad \alpha, \beta = e, \mu, \tau, \quad a = 1, 2, 3. \quad (70)$$

The matter mixing angles can therefore be expressed as follows

$$\theta_1^M \equiv \arctan \frac{U_{\mu 3}^M}{U_{\tau 3}^M} = \arctan \frac{\lambda_3 \tilde{T}_{e\mu} + (\tilde{T}^2)_{e\mu}}{\lambda_3 \tilde{T}_{e\tau} + (\tilde{T}^2)_{e\tau}}, \quad (71)$$

$$\theta_2^M \equiv \arcsin U_{e3}^M = \arcsin \sqrt{\frac{\lambda_3^2 + c_1 + \lambda_3 \tilde{T}_{ee} + (\tilde{T}^2)_{ee}}{3\lambda_3^2 + c_1}}, \quad (72)$$

$$\theta_3^M \equiv \arctan \frac{U_{e2}^M}{U_{e1}^M} = \arctan \frac{\lambda_2 \tilde{T}_{e\alpha} + (\tilde{T}^2)_{e\alpha}}{\lambda_1 \tilde{T}_{e\alpha} + (\tilde{T}^2)_{e\alpha}} \frac{3\lambda_1^2 + c_1}{3\lambda_2^2 + c_1}, \quad \alpha = \mu, \tau. \quad (73)$$

The transition probabilities in matter can thus be written as

$$P_{\alpha\beta} = \delta_{\alpha\beta} - 4 \sum_{a=1}^3 \sum_{\substack{b=1 \\ a < b}}^3 U_{\alpha a}^M U_{\beta a}^M U_{\alpha b}^M U_{\beta b}^M \sin^2 \tilde{x}_{ab}, \quad \alpha, \beta = e, \mu, \tau. \quad (74)$$

The above formulas for the transition probabilities are thus in effect exactly the same as the formulas in Eq. (57).

When the matter density goes to zero, i.e., $A \rightarrow 0$, then $U_{\alpha a}^M \rightarrow U_{\alpha a}$ ($\theta_i^M \rightarrow \theta_i$) and $\tilde{x}_{ab} \rightarrow x_{ab}$. Also Eqs. (57) and (74) will become identical to Eq. (53) when $A = 0$.

To connect this treatment to the usual treatment in terms of the auxiliary matter mixing angles, we have to find the matrix R . Using Eq. (66), the matrix R , which diagonalizes the matrix T , can be constructed from the eigenvectors of the matrix T as

$$R = \begin{pmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \mathbf{v}_3^T \end{pmatrix}, \quad (75)$$

i.e.,

$$R^{-1} = R^T = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{pmatrix}, \quad (76)$$

where

$$\mathbf{v}_a = \begin{pmatrix} \alpha_a \\ \beta_a \\ 1 \end{pmatrix} z_a, \quad a = 1, 2, 3.$$

Here

$$\alpha_a = \frac{T_{12}(T_{33} - \lambda_a) - T_{13}T_{23}}{T_{23}(T_{11} - \lambda_a) - T_{12}T_{13}}, \quad \beta_a = \frac{T_{12}(T_{33} - \lambda_a) - T_{13}T_{23}}{T_{13}(T_{22} - \lambda_a) - T_{12}T_{23}}, \quad z_a = \frac{1}{\sqrt{\alpha_a^2 + \beta_a^2 + 1}}, \quad a = 1, 2, 3.$$

The orthogonal matrix R is of course diagonalizing both T and \mathcal{H}_m .

Equation (65) can now be written in matrix form as

$$\begin{aligned} \begin{pmatrix} U_{e1}^M & U_{e2}^M & U_{e3}^M \\ U_{\mu1}^M & U_{\mu2}^M & U_{\mu3}^M \\ U_{\tau1}^M & U_{\tau2}^M & U_{\tau3}^M \end{pmatrix} &\equiv \begin{pmatrix} C_2^M C_3^M & S_3^M C_2^M & S_2^M \\ -S_3^M C_1^M - S_1^M S_2^M C_3^M & C_1^M C_3^M - S_1^M S_2^M S_3^M & S_1^M C_2^M \\ S_1^M S_3^M - S_2^M C_1^M C_3^M & -S_1^M C_3^M - S_2^M S_3^M C_1^M & C_1^M C_2^M \end{pmatrix} \\ &= \begin{pmatrix} C_2 C_3 & S_3 C_2 & S_2 \\ -S_3 C_1 - S_1 S_2 C_3 & C_1 C_3 - S_1 S_2 S_3 & S_1 C_2 \\ S_1 S_3 - S_2 C_1 C_3 & -S_1 C_3 - S_2 S_3 C_1 & C_1 C_2 \end{pmatrix} \\ &\times \begin{pmatrix} \alpha_1 z_1 & \alpha_2 z_2 & \alpha_3 z_3 \\ \beta_1 z_1 & \beta_2 z_2 & \beta_3 z_3 \\ z_1 & z_2 & z_3 \end{pmatrix} \\ &\equiv \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu1} & U_{\mu2} & U_{\mu3} \\ U_{\tau1} & U_{\tau2} & U_{\tau3} \end{pmatrix} \begin{pmatrix} (\mathbf{v}_1)_1 & (\mathbf{v}_2)_1 & (\mathbf{v}_3)_1 \\ (\mathbf{v}_1)_2 & (\mathbf{v}_2)_2 & (\mathbf{v}_3)_2 \\ (\mathbf{v}_1)_3 & (\mathbf{v}_2)_3 & (\mathbf{v}_3)_3 \end{pmatrix}, \end{aligned} \quad (77)$$

where $S_i^M \equiv \sin \theta_i^M$ and $C_i^M \equiv \cos \theta_i^M$ for $i = 1, 2, 3$. We can then obtain the matter mixing angles from the above matrix relation as

$$\theta_1^M = \arctan \frac{S_1^M}{C_1^M} = \arctan \frac{(-S_3 C_1 - S_1 S_2 C_3) \alpha_3 z_3 + (C_1 C_3 - S_1 S_2 S_3) \beta_3 z_3 + S_1 C_2 z_3}{(S_1 S_3 - S_2 C_1 C_3) \alpha_3 z_3 + (-S_1 C_3 - S_2 S_3 C_1) \beta_3 z_3 + C_1 C_2 z_3}, \quad (78)$$

$$\theta_2^M = \arcsin S_2^M = \arcsin(C_2 C_3 \alpha_3 z_3 + S_3 C_2 \beta_3 z_3 + S_2 z_3), \quad (79)$$

$$\theta_3^M = \arctan \frac{S_3^M}{C_3^M} = \arctan \frac{C_2 C_3 \alpha_2 z_2 + S_3 C_2 \beta_2 z_2 + S_2 z_2}{C_2 C_3 \alpha_1 z_1 + S_3 C_2 \beta_1 z_1 + S_2 z_1}. \quad (80)$$

The conditions for resonance are given by the extremal points of the probability amplitudes. These will coincide with extremal points for the modulus of the amplitudes, thus for the products $U_{\alpha a}^M U_{\beta a}^M$ or for the expressions in the right-hand side of Eq. (70) as functions of A .

VI. DISCUSSION

The main result of our analysis is given by the time evolution operator for the neutrinos when passing through matter with constant matter density in Eq. (46) expressed as a finite sum of elementary functions in the matrix elements of the Hamiltonian. Also the transition probabilities in Eq. (61) and the neutrino energies in presence of matter $\tilde{E}_a = \lambda_a$, where $a = 1, 2, 3$, given by Eqs. (36)–(38) belong to our main results as do the expressions for the matter mixing angles in Eqs. (71)–(73). In our treatment these auxiliary matter mixing angles play no independent role and are not really needed. It is convenient, though, to express the matter mixing angles in the matrix elements of the Hamiltonian to look for the resonance conditions.

As an illustration of the resonance phenomena, we have plotted the energy differences $|\lambda_a - \lambda_b|$, where $a, b = 1, 2, 3$, $a \neq b$, and the behavior of the matter mixing angles θ_1^M , θ_2^M , θ_3^M as a function of the matter density A for two different cases. First for two mass squared differences that are close and bimaximal mixing [20], i.e., two large mixing angles and one small. The results are presented in Figs. 1 and 2. Then for two widely separated mass squared differences and all the mixing angles small. These results are presented in Figs. 3 and 4. The energy differences $|\lambda_a - \lambda_b|$ are related to the effective matter mass squared differences $\Delta\tilde{m}_{ab}^2$ by the following relations

$$|\Delta\tilde{m}_{ab}^2| = 2E|\lambda_a - \lambda_b|, \quad a, b = 1, 2, 3, \quad a \neq b.$$

As can be seen from Figs. 1–4, the resonances occur (i.e., the mixing is maximal) when the energy levels in presence of matter approach each other as function of the matter density A . As the mass squared differences approach each other in magnitude, the resonances move closer together and the mixing angle θ_1^M , which looks quite flat (constant) at large separations of the mass squared differences, will have a modest increase above the location of the (higher) resonance of the mixing angle θ_2^M . The smaller the vacuum mixing angles are, the shaper are the resonance peaks. For bimaximal mixing the resonance occurs only for the small mixing angle, but even for matter densities above the resonance region the values of the auxiliary mixing angles are changed appreciably from their vacuum values, as is shown in Fig. 2.

Our results for the auxiliary matter mixing angles are in agreement with the previous approximative and numerical calculations in Refs. [8,9,11,12,14].

We have also calculated the transition probabilities $P_{\alpha\beta}$ for neutrino oscillations in matter as functions of the matter density A . The results are illustrated in Figs. 5 and 6. These two figures correspond to the only physically measurable quantities. Our illustration is chosen to correspond to values of the mass squared differences that are close to those obtained from analyses of atmospheric neutrino data. In Fig. 5(c), we can see a sharp drop in the probability P_{ee} at a matter density of about $5 \cdot 10^{-14}$ eV (same parameter values as in Figs. 1 and 2) for a sensitivity of $L/E \approx 1.6 \cdot 10^5$ eV⁻² as the neutrinos pass through the Earth. This corresponds to an enhancement of $P_{e\mu}$ and $P_{e\tau}$ at the same value of A , whereas the probability $P_{\mu\tau}$ wiggles in the same region. Due to unitarity only three of these probabilities are of course independent.

The Cayley–Hamilton formalism could in principle be used to derive neutrino oscillation formulas in matter for arbitrary numbers of neutrino flavors. However, the calculations for

four or even more flavors will be much more tedious than in the case of three neutrino flavors. For four or more flavors, the unitary mixing matrices have more complicated structures than in the three flavor case. Another complication, for more than four flavors, is that one will not, in general, be able to find analytical solutions to the characteristic equation, since only characteristic equations up to degree four are analytically solvable.

In Appendix C, we have given, as a comparison and reference, the derivation of the neutrino oscillation formulas in matter for two neutrino flavors also using the Cayley–Hamilton formalism. We observe that the derivation for two flavors is much easier to make than that for three flavors. The evolution operator $U_m(L)$ can of course be written in a much more compact form for two flavors than for three; compare Eqs. (45) and (C5).

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APPENDIX A: INTRODUCING THE RELATIVISTIC LIMIT

We will here introduce the relativistic limit by reparametrizing the expressions for the energy differences. Any two of E_{21} , E_{32} , and E_{13} can be chosen as independent variables. Thus, we use $E_{ab} = \Delta m_{ab}^2 / (E_a + E_b)$ for $a, b = 1, 2, 3$ and $a \neq b$, where $\Delta m_{ab}^2 \equiv m_a^2 - m_b^2$. Without lack of generality, we will assume that there is a mass ordering with $m_1 < m_2 < m_3$. Apart from the masses, we will take the momentum as a common independent variable and assume that the momentum is the same for all components of the mixed state. However, it is convenient to use instead the energy as an independent quantity. This choice can again be made in several ways. For definiteness, we will here choose as physical variables $E \equiv (E_1 + E_2)/2$, $\Delta m^2 \equiv \Delta m_{21}^2$, and $\Delta M^2 \equiv \Delta m_{32}^2$. This gives

$$E_{21} = \frac{\Delta m^2}{2E}, \quad (\text{A1})$$

$$E_{32} = -\left(E + \frac{\Delta m^2}{2E}\right) + \sqrt{\left(E + \frac{\Delta m^2}{2E}\right)^2 + \Delta M^2}. \quad (\text{A2})$$

Since in the applications $\Delta M/E \ll 1$ ($\lesssim 10^{-6}$), we find the excellent approximation

$$E_{32} \simeq \frac{\Delta M^2}{2E}. \quad (\text{A3})$$

Equations (A1) and (A3) lead to the following expressions for the coefficients c_2 , c_1 , and c_0 :

$$c_2 = 0, \quad (\text{A4})$$

$$c_1 \simeq -\frac{1}{3}A^2 + \frac{A}{6E} [U_{e1}^2 (\Delta M^2 + 2\Delta m^2) + U_{e2}^2 (\Delta M^2 - \Delta m^2) - U_{e3}^2 (2\Delta M^2 + \Delta m^2)] \\ - \frac{1}{12E^2} (\Delta M^4 + \Delta m^4 + \Delta M^2 \Delta m^2), \quad (\text{A5})$$

$$c_0 \simeq -\frac{2}{27}A^3 + \frac{A^2}{18E} [U_{e1}^2 (\Delta M^2 + 2\Delta m^2) + U_{e2}^2 (\Delta M^2 - \Delta m^2) - U_{e3}^2 (2\Delta M^2 + \Delta m^2)] \\ + \frac{A}{36E^2} [U_{e1}^2 (2\Delta M^2 + \Delta m^2) (\Delta M^2 - \Delta m^2) + U_{e2}^2 (2\Delta M^2 + \Delta m^2) (\Delta M^2 + 2\Delta m^2) \\ - U_{e3}^2 (\Delta M^2 + 2\Delta m^2) (\Delta M^2 - \Delta m^2) - (\Delta M^4 + \Delta m^4 + \Delta M^2 \Delta m^2)] \\ - \frac{1}{216E^3} (2\Delta M^2 + \Delta m^2) (\Delta M^2 + 2\Delta m^2) (\Delta M^2 - \Delta m^2). \quad (\text{A6})$$

APPENDIX B: THE ENTRIES OF THE MATRICES \tilde{T} AND \tilde{T}^2

The entries of the 3×3 real symmetric matrix T^2 are

$$(T^2)_{11} = \frac{1}{3} [A^2 (U_{e1}^2 + \frac{1}{3}) + 2A (U_{e1}^2 - \frac{1}{3}) (E_{12} + E_{13}) + \frac{1}{3} (E_{12} + E_{13})^2], \quad (\text{B1})$$

$$(T^2)_{22} = \frac{1}{3} [A^2 (U_{e2}^2 + \frac{1}{3}) + 2A (U_{e2}^2 - \frac{1}{3}) (E_{21} + E_{23}) + \frac{1}{3} (E_{21} + E_{23})^2], \quad (\text{B2})$$

$$(T^2)_{33} = \frac{1}{3} [A^2 (U_{e3}^2 + \frac{1}{3}) + 2A (U_{e3}^2 - \frac{1}{3}) (E_{31} + E_{32}) + \frac{1}{3} (E_{31} + E_{32})^2], \quad (\text{B3})$$

$$(T^2)_{12} = (T^2)_{21} = \frac{1}{3} U_{e1} U_{e2} A (A + E_{13} + E_{23}), \quad (\text{B4})$$

$$(T^2)_{13} = (T^2)_{31} = \frac{1}{3} U_{e1} U_{e3} A (A + E_{12} + E_{32}), \quad (\text{B5})$$

$$(T^2)_{23} = (T^2)_{32} = \frac{1}{3} U_{e2} U_{e3} A (A + E_{21} + E_{31}). \quad (\text{B6})$$

For the matrices \tilde{T} and \tilde{T}^2 we have the following expressions for the entries:

$$\tilde{T}_{\alpha\beta} = \sum_{a=1}^3 \sum_{b=1}^3 U_{\alpha a} U_{\beta b} T_{ab}, \quad \alpha, \beta = e, \mu, \tau, \quad (\text{B7})$$

and

$$(\tilde{T}^2)_{\alpha\beta} = \sum_{a=1}^3 \sum_{b=1}^3 U_{\alpha a} U_{\beta b} (T^2)_{ab}, \quad \alpha, \beta = e, \mu, \tau. \quad (\text{B8})$$

For $\alpha \neq \beta$ Eq. (B7) simplifies to

$$\tilde{T}_{\alpha\beta} = \sum_{a=1}^3 U_{\alpha a} U_{\beta a} E_a. \quad (\text{B9})$$

APPENDIX C: THE TWO FLAVOR NEUTRINO CASE

Considering only a two flavor neutrino oscillation model instead of a three neutrino, the free Hamiltonian in the mass eigenstate basis and the potential matrix are

$$H_m = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \quad \text{and} \quad V_f = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}. \quad (\text{C1})$$

In the mass eigenstate basis, the potential matrix is given by

$$V_m = U^{-1} V_f U, \quad (\text{C2})$$

where

$$U = \begin{pmatrix} U_{e1} & U_{e2} \\ U_{\mu 1} & U_{\mu 2} \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

is the usual 2×2 orthogonal matrix, which describes neutrino mixing with two neutrino flavors. Thus, we have

$$V_m = A \begin{pmatrix} U_{e1}^2 & U_{e1}U_{e2} \\ U_{e1}U_{e2} & U_{e2}^2 \end{pmatrix} \quad (\text{C3})$$

and

$$\mathcal{H}_m = H_m + V_m = \begin{pmatrix} E_1 + AU_{e1}^2 & AU_{e1}U_{e2} \\ AU_{e1}U_{e2} & E_2 + AU_{e2}^2 \end{pmatrix}. \quad (\text{C4})$$

Again, we have to find the evolution operator $U_m(L) = e^{-i\mathcal{H}_m L}$. Similar but simpler formulas than in the three flavor neutrino case now give

$$U_m(L) = e^{-i\mathcal{H}_m L} = \phi e^{-iLT} = \phi \sum_{a=1}^2 e^{-iL\lambda_a} \frac{1}{2\lambda_a} (\lambda_a I + T) = \phi \left(\cos(L\tilde{\lambda}) - i \frac{1}{\tilde{\lambda}} \sin(L\tilde{\lambda}) T \right), \quad (\text{C5})$$

where $\phi \equiv e^{-iL \text{tr } \mathcal{H}_m / 2}$, $T \equiv \mathcal{H}_m - (\text{tr } \mathcal{H}_m)I/2$, $\lambda_1 = -\tilde{\lambda}$, $\lambda_2 = \tilde{\lambda}$, and $\tilde{\lambda} = \sqrt{-\det T}$. Here

$$\tilde{\lambda} \equiv \sqrt{\frac{1}{4}A^2 - \frac{x_{21}}{L} \left(A \cos 2\theta - \frac{x_{21}}{L} \right)}, \quad x_{21} \equiv \frac{E_{21}L}{2} = \frac{\Delta m^2 L}{4E}, \quad \Delta m^2 \equiv m_2^2 - m_1^2.$$

The quantity $\tilde{\lambda}$ comes from the solutions, λ_1 and λ_2 , to the characteristic equation $\lambda^2 + c_1\lambda + c_0 = 0$, where $c_1 = -\text{tr } T = 0$ and $c_0 = \det T = -A^2/4 + AU_{e1}^2 E_{21}/2 + AU_{e2}^2 E_{12}/2 + E_{12}E_{21}/4 = -A^2/4 + x_{21} (A \cos 2\theta - x_{21}/L) / L$.

The evolution operator $U_m(L)$ can now be written in matrix form as

$$U_m(L) = \phi \begin{pmatrix} \alpha & \beta \\ \beta & \alpha^* \end{pmatrix}, \quad (\text{C6})$$

where $\alpha = \cos(L\tilde{\lambda}) - i (A \cos 2\theta - 2x_{21}/L) \sin(L\tilde{\lambda})/(2\tilde{\lambda})$ and $\beta = -iA \sin 2\theta \sin(L\tilde{\lambda})/(2\tilde{\lambda})$. Similarly, the evolution operator $U_f(L)$ is

$$\begin{aligned}
U_f(L) &= UU_m(L)U^{-1} \\
&= \phi \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \beta & \alpha^* \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \\
&= \phi \begin{pmatrix} \alpha \cos^2 \theta + \alpha^* \sin^2 \theta + \beta \sin 2\theta & \frac{\alpha^* - \alpha}{2} \sin 2\theta + \beta \cos 2\theta \\ \frac{\alpha^* - \alpha}{2} \sin 2\theta + \beta \cos 2\theta & \alpha \sin^2 \theta + \alpha^* \cos^2 \theta - \beta \sin 2\theta \end{pmatrix}. \tag{C7}
\end{aligned}$$

The $e\mu$ -element of the $U_f(L)$ matrix is

$$(U_f(L))_{e\mu} = \frac{\alpha^* - \alpha}{2} \sin 2\theta + \beta \cos 2\theta = -i\phi \frac{x_{21}}{L\tilde{\lambda}} \sin 2\theta \sin(L\tilde{\lambda}). \tag{C8}$$

Taking the absolute value and squaring Eq. (C8), we obtain the transition probability $P_{e\mu}$ as

$$\begin{aligned}
P_{e\mu} &= |(U_f(L))_{e\mu}|^2 = \frac{x_{21}^2}{(L\tilde{\lambda})^2} \sin^2 2\theta \sin^2(L\tilde{\lambda}) \\
&= \frac{x_{21}^2 \sin^2 2\theta}{L^2 [A^2/4 - x_{21} (A \cos 2\theta - x_{21}/L) / L]} \sin^2(L\tilde{\lambda}) \\
&= \frac{\sin^2 2\theta}{1 + AL [AL/(4x_{21}) - \cos 2\theta] / x_{21}} \sin^2(L\tilde{\lambda}) \\
&= \frac{\sin^2 2\theta}{\sin^2 2\theta + [\cos 2\theta - AL/(2x_{21})]^2} \sin^2(L\tilde{\lambda}), \tag{C9}
\end{aligned}$$

where $L\tilde{\lambda} = \sqrt{x_{21}^2 - ALx_{21} [\cos 2\theta - AL/(4x_{21})]}$. Introducing the definitions

$$\sin^2 2\theta^M \equiv \frac{\sin^2 2\theta}{\sin^2 2\theta + (\cos 2\theta - A/E_{21})^2} \tag{C10}$$

and

$$\tilde{x}_{21} \equiv \sqrt{x_{21}^2 - ALx_{21} \left(\cos 2\theta - \frac{AL}{4x_{21}} \right)} = x_{21} \sqrt{\sin^2 2\theta + \left(\cos 2\theta - \frac{A}{E_{21}} \right)^2}, \tag{C11}$$

we can write

$$P_{e\mu} = \sin^2 2\theta^M \sin^2 \tilde{x}_{21}. \tag{C12}$$

One could also immediately identify the definitions (C10) and (C11) from Eq. (C8) by observing that the factor $x_{21} \sin 2\theta / (L\tilde{\lambda})$ should correspond to $\sin 2\theta^M$ and that the argument $L\tilde{\lambda}$ in the factor $\sin(L\tilde{\lambda})$ should correspond to \tilde{x}_{21} . This identification corresponds to the one in Eq. (70) above in Sec. V. The factor $-i\phi$ is just a complex phase factor.

Setting

$$\cos 2\theta = \frac{A}{E_{21}} \tag{C13}$$

in Eqs. (C10) and (C12) gives

$$\sin^2 2\theta^M = 1 \quad (\text{maximal mixing}), \quad (\text{C14})$$

$$P_{e\mu} = \sin^2 \sqrt{\left(x_{21} + \frac{AL}{2}\right) \left(x_{21} - \frac{AL}{2}\right)} = \sin^2 \left(x_{21} \sqrt{\left(1 + \frac{A}{E_{21}}\right) \left(1 - \frac{A}{E_{21}}\right)} \right), \quad (\text{C15})$$

where Eq. (C13) is the so-called MSW resonance condition.

Inserting $A = 0$ into Eq. (C12) leads, of course, back to $P_{e\mu} = \sin^2 2\theta \sin^2 x_{21}$, the old transition probability formula for ν_e - ν_μ oscillations in vacuum.

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FIGURES

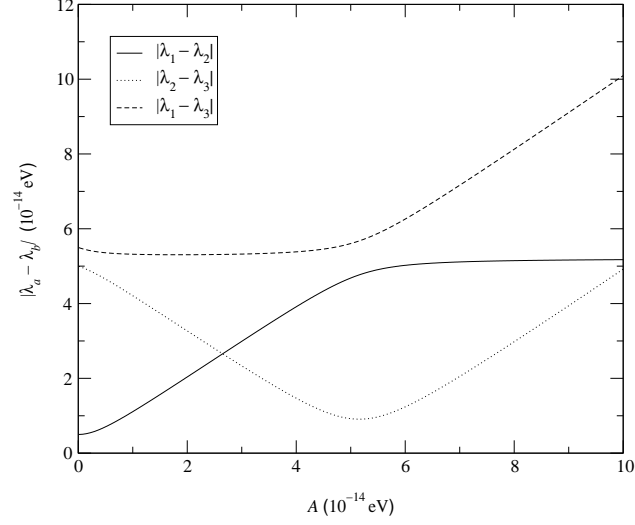


FIG. 1. The differences $|\lambda_a - \lambda_b|$, $a, b = 1, 2, 3$, $a \neq b$, as a function of the matter density A for $\theta_1 = \theta_3 = 45^\circ$ (bimaximal mixing), $\theta_2 = 5^\circ$, $\Delta m^2 = 10^{-4} \text{ eV}^2$, $\Delta M^2 = 10^{-3} \text{ eV}^2$, and $E = 10 \text{ GeV}$.

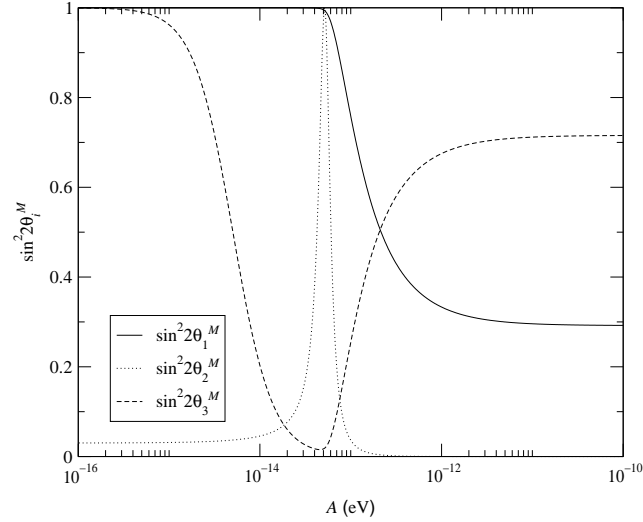


FIG. 2. The quantities $\sin^2 2\theta_i^M$, $i = 1, 2, 3$, as a function of the matter density A for $\theta_1 = \theta_3 = 45^\circ$ (bimaximal mixing), $\theta_2 = 5^\circ$, $\Delta m^2 = 10^{-4} \text{ eV}^2$, $\Delta M^2 = 10^{-3} \text{ eV}^2$, and $E = 10 \text{ GeV}$. Note that $\sin^2 2\theta_i^M = 1$ corresponds to maximal mixing.

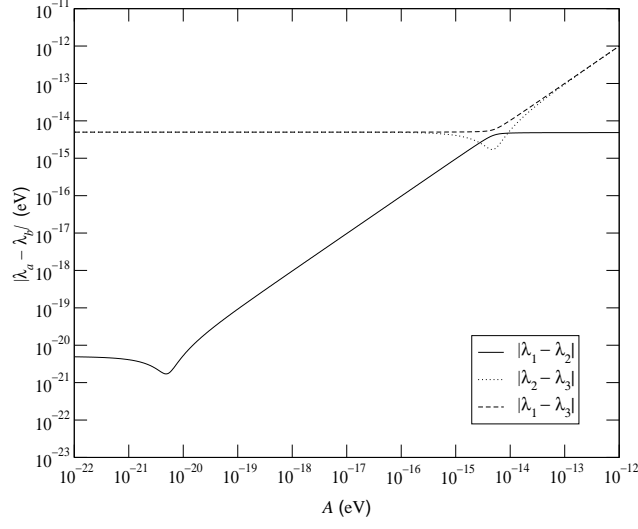


FIG. 3. The differences $|\lambda_a - \lambda_b|$, $a, b = 1, 2, 3$, $a \neq b$, as a function of the matter density A for $\theta_1 = \theta_2 = \theta_3 = 10^\circ$, $\Delta m^2 = 10^{-10} \text{ eV}^2$, $\Delta M^2 = 10^{-4} \text{ eV}^2$, and $E = 10 \text{ GeV}$.

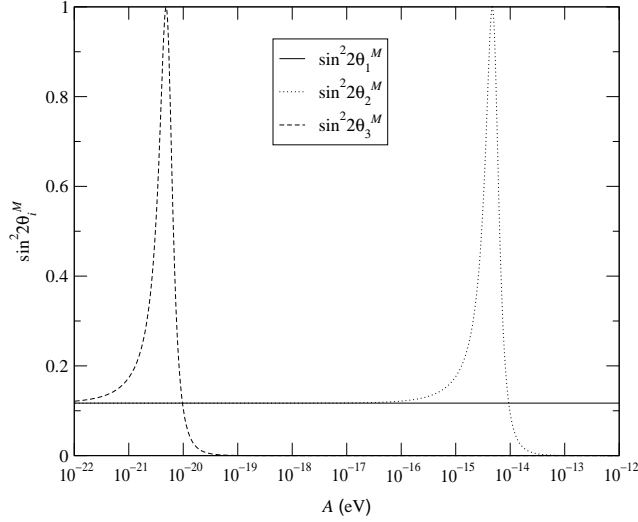


FIG. 4. The quantities $\sin^2 2\theta_i^M$, $i = 1, 2, 3$, as a function of the matter density A for $\theta_1 = \theta_2 = \theta_3 = 10^\circ$, $\Delta m^2 = 10^{-10} \text{ eV}^2$, $\Delta M^2 = 10^{-4} \text{ eV}^2$, and $E = 10 \text{ GeV}$. Note that $\sin^2 2\theta_i^M = 1$ corresponds to maximal mixing.

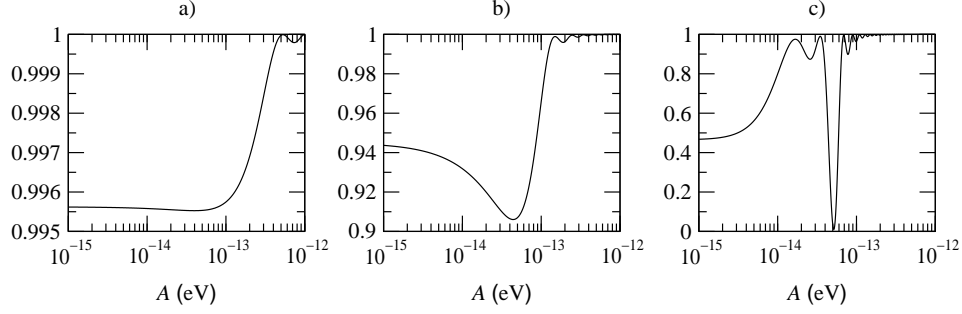


FIG. 5. The transition probability P_{ee} as a function of the matter density A for $\theta_1 = \theta_3 = 45^\circ$ (bimaximal mixing), $\theta_2 = 5^\circ$, $\Delta m^2 = 10^{-4} \text{ eV}^2$, $\Delta M^2 = 10^{-3} \text{ eV}^2$, and $L/E = \eta(2R_\oplus/10 \text{ GeV}) \simeq 3.23 \times 10^4 \text{ eV}^{-2} \eta$, where $R_\oplus \simeq 6378 \text{ km} \simeq 1.62 \times 10^{14} \text{ eV}^{-1}$ is the (equatorial) radius of the Earth. (a) $\eta = 1/5$, (b) $\eta = 1$, and (c) $\eta = 5$.

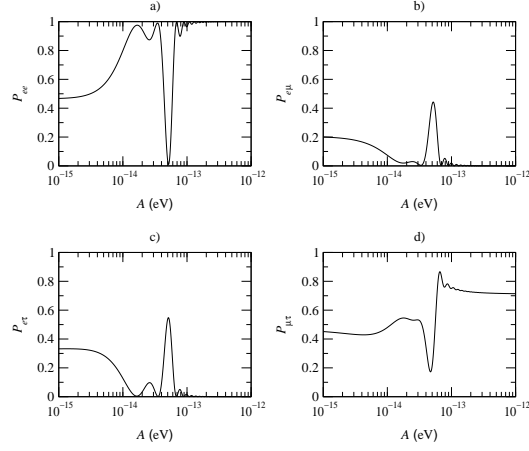


FIG. 6. Transition probabilities as functions of the matter density A , using the same parameter values as for Fig. 5(c). (a) P_{ee} , (b) $P_{e\mu}$, (c) $P_{e\tau}$, and (d) $P_{\mu\tau}$.