Three-letter-pattern-avoiding permutations and functional equations

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Submitted: Oct 31, 2005; Accepted: May 18, 2006; Published: May 29, 2006
Mathematics Subject Classification: 05A15, 05A05

Abstract

We present an algorithm for finding a system of recurrence relations for the number of permutations of length n that satisfy a certain set of conditions. A rewriting of these relations automatically gives a system of functional equations satisfied by the multivariate generating function that counts permutations by their length and the indices of the corresponding recurrence relations. We propose an approach to describing such equations. In several interesting cases the algorithm recovers and refines, in a unified way, results on τ-avoiding permutations and permutations containing τ exactly once, where τ is any classical, generalized, and distanced pattern of length three.

1 Introduction

Let $\pi = \pi_1\pi_2\dots\pi_n$ be a permutation of length n. Let $\tau = \tau_1\tau_2\dots\tau_k$ be a permutation of length k. An *occurrence* of τ in π is a subsequence $1 \le i_1 < i_2 < \dots < i_k \le n$ such that $(\pi(i_1),\dots,\pi(i_k))$ is order-isomorphic to τ; in this context, τ is usually called a *pattern* (or *classical pattern*). We say that π *contains* τ if there exists an occurrence of τ in π, otherwise, we say that π *avoids* τ (or is τ*-avoiding)*. Herb Wilf [23] raised the question: For a pattern τ, what can you say about $f_\tau(n)$, the number of permutations in S_n that avoid the pattern τ? More generally, what can you say about

- $\widehat{f}_\tau(n)$, the number of permutations in S_n that contain the pattern τ exactly once?
- $f_{\{\tau^1,\dots,\tau^\ell\};(r_1,\dots,r_\ell)}(n)$, the number of permutations in S_n that contain the pattern τ^j exactly r_j times, for each $j = 1, 2, \dots, \ell$?

It follows from the Robinson-Schensted algorithm and the hook-length formula [11] that for any k, the number of permutations with no increasing subsequence of length k is a multisum with binomial coefficients, from which it follows immediately [20] that it is P-recursive (i.e., it satisfies a linear recurrence with polynomial coefficients in n). Noonan and Zeilberger [15] conjectured that for any given finite set of patterns, T, the sequence $f_{\{\tau^1,\dots,\tau^\ell\};(r_1,\dots,r_\ell)}(n)$ is P-recursive.

Pattern-avoidance problems have been extensively studied over the last decade, and one motivation has been to decide the above conjecture, see for instance [1, 2, 5], [12–21]. In these papers, the authors have employed various methods such as generating trees with one or two labels [7], block decompositions [12] and basic algorithms for counting patterns [15] to derive explicit formulas for the number of permutations of length n that satisfy a certain set of conditions. Other authors used generating trees [18, 19] and enumeration schemes [17, 21] to obtain a finite set of recurrence relations for the sequence of permutations of length n that satisfy a certain set of conditions.

In this paper we suggest another approach, a *scanning-elements algorithm*, to study the number of permutations of length n that satisfy a certain set of conditions. Our algorithm is different from the above approaches by, for instance, the following critical points: (1) The generating tree approach (see [7, 18, 19]) based on finding the refining the permutations to obtain the labels of the tree. But our algorithm suggests refining the permutations according to the value(s) of the leftmost element(s) of the permutations. (2) The basic algorithm for counting patterns [15] used techniques involving generating functions with infinitely many variables. But our algorithm deal with generating functions with finite number of variables. (3) The enumeration schemes approach [17, 21] gives describe the permutations as a generating tree with only finitely many labels. But our algorithm describes the number of permutations as a set of recurrence relations.

The major aim of the scanning-elements algorithm is to obtain a finite set of recurrence relations for the sequence of permutations of length n that satisfy a certain set of conditions (for example, avoiding a pattern or containing a pattern exactly once), and to solve these recurrence relations by using the kernel method technique. Unfortunately, as in the above approaches, this is not always easy, and for many enumeration sequences, e.g. the number of τ-avoiding permutations or the number of permutations containing a pattern τ exactly once, $\tau \in S_k$, it may well be impossible for patterns of length $k \geq 4$. We illustrate the use of the scanning-elements algorithm for the number of permutations of length n that either avoid a pattern of length three, or contain a pattern of length three exactly once.

Organization of the paper. The paper is organized as follows. In Section 2 we formulate the scanning-elements algorithm. In Section 3 we solve a linear system of functional equations. Applying the general methods in Sections 2 and 3 we get several applications on pattern-avoiding permutations, see Section 4. As a consequence, we recover and refine, in a unified way, all the results on τ-avoiding permutations, and permutations containing τ exactly once, where τ is any *classical* (Section 4.1), *generalized* (Section 4.2), or *distanced* (Section 4.3) pattern of length three.

2 Scanning-elements algorithm

Let $P(n)$ be any infinite family of finite subsets of S_n, parameterized by n. Denote the cardinality of the set $P(n)$ by $p(n)$, that is, $p(n) = \#P(n)$ for all $n \geq 0$. The main goal of this section is to describe how to derive a recursive structure for the family $P(n)$, for which we need the following definition. Given $b_1, b_2, \ldots, b_k \in \mathbb{N}$, we define

$$P(n; b_1, b_2, \ldots, b_\ell) = \{\pi_1\pi_2 \ldots \pi_n \in P(n) \mid \pi_1\pi_2 \cdots \pi_\ell = b_1b_2 \cdots b_\ell\}$$

and $p(n; b_1, b_2, \ldots, b_\ell) = \#P(n; b_1, b_2, \ldots, b_\ell)$. If no confusion can arise, we will write $P(n)$ instead of $P(n; \emptyset)$ and $p(n)$ instead of $p(n; \emptyset)$. As a direct consequence of the above definition, we have

$$p(n; b_1, \ldots, b_\ell) = \sum\nolimits_{j=1}^{n} p(n; b_1, \ldots, b_\ell, j). \tag{1}$$

Equation (1) suggests writing a recurrence relation for the sequence $\{p(n)\}_{n\geq 0}$ in terms of $p(n; b_1)$, the sequence $\{p(n; b_1)\}_{n,b_1}$ in terms of the sequences $p(n; b_1, b_2)$, and so on. To simplify this process we make the following definition.

Definition 1 *Let $1 \leq s \leq \ell$ and $a_1, \ldots, a_{\ell-s} \in \mathbb{N}$. If there exists a bijection between the sets $P(n; b_1, \ldots, b_\ell)$ and $P(n - s; a_1, \ldots, a_{\ell-s})$, then the set $P(n - s; a_1, \ldots, a_{\ell-s})$ is said to be the* reduction *of the set $P(n; b_1, \ldots, b_\ell)$. If a reduction exists we say that the set $P(n; b_1, \ldots, b_\ell)$ is* reducible *(otherwise it is* irreducible*). An element b is said to be an* $(\ell + 1)$-active element *(resp.* $(\ell + 1)$-inactive element*) of the set $P(n; b_1, \ldots, b_\ell)$ if the set $P(n; b_1, \ldots, b_\ell, b)$ is irreducible (resp. reducible).*

For instance, if $P(n) = S_n(1234)$ then $P(n-1; j_1, j_3)$ is a reduction of $P(n; j_1, j_2, j_3)$ where $1 \leq j_1 < j_3 < j_2 \leq n - 3$. We now describe, in the following three steps, how to obtain the recurrence relation for the sequence $\{p(n)\}_{n\geq 0}$.

First step. Decide what are the 1-inactive (active) elements of $P(n)$, i.e. the sets

$$\overline{I}(n) = \{j \in [n] \mid j \text{ is an 1-inactive element of } P(n)\},$$
$$I(n) = \{j \in [n] \mid j \text{ is an 1-active element of } P(n)\}.$$

As a direct consequence of the above definitions and (1) we have

$$p(n) = p(n; 1) + \cdots + p(n; n) = |\overline{I}(n)| \cdot p(n - 1) + \sum\nolimits_{j\in I(n)} p(n; j) \tag{2}$$

for all $n \geq 1$. In other words, Equation (2) leads to a recurrence relation for the sequence $\{p(n)\}_{n\geq 0}$ in terms of $p(n; j)$.

Second step. In this step we discuss how to obtain a recurrence relation for the sequence $p(n; b_1, \ldots, b_\ell)$ (in particular, for the sequence $p(n; b_1)$).

Given a set $P(n; b_1, \ldots, b_\ell)$, decide if it is reducible or irreducible. If it is irreducible then decide what are the $(\ell+1)$- inactive (active) elements of $P(n; b_1, \ldots, b_\ell)$, i.e. the sets

$$\overline{I}(n; b_1 \ldots, b_\ell) := \{a \in [n]\backslash\{b_1, \ldots, b_\ell\} \mid a \text{ is an } (\ell + 1)\text{-inactive element of } P(n; b_1, \ldots, b_\ell)\},$$
$$I(n; b_1, \ldots, b_\ell) := \{a \in [n]\backslash\{b_1, \ldots, b_\ell\} \mid a \text{ is an } (\ell + 1)\text{-active element of } P(n; b_1, \ldots, b_\ell)\}.$$

As a direct consequence of the above definitions we have for all $\ell \geq 0$,

$$
\begin{aligned}
p(n;b_1,\ldots,b_\ell) &= \sum_{j\in[n]\setminus\{b_1,\ldots,b_\ell\}} p(n;b_1,\ldots,b_\ell,j) \\
&= \sum_{j\in I(n;b_1,b_2,\ldots,b_\ell)} p(n;b_1,\ldots,b_\ell,j) + \sum_{j\in \overline{I}(n;b_1,b_2,\ldots,b_\ell)} p(n;b_1,\ldots,b_\ell,j) \\
&= \sum_{j\in I(n;b_1,b_2,\ldots,b_\ell)} p(n;b_1,\ldots,b_\ell,j) + \sum_{(d,c_1,\ldots,c_{\ell-d})\in U(n;b_1,\ldots,b_\ell)} p(n-d;c_1,\ldots,c_{\ell-d}),
\end{aligned}
\tag{3}
$$

where $U(n;b_1,\ldots,b_\ell)$ is the following multiset

$$
\left\{ (d,c_1,\ldots,c_{\ell-d}) \;\middle|\; \begin{array}{c} P(n-d;c_1,\ldots,c_{\ell-d}) \text{ is a reduction of } P(n;b_1,\ldots,b_\ell,j), \\ 1\leq d\leq \ell,\ j\in \overline{I}(n;b_1,\ldots,b_\ell) \end{array} \right\}.
$$

Sometimes, Equation (3) suggests an algorithm for writing finite or infinite systems of recurrence relations in terms of $p(n-a;b_1,\ldots,b_\ell)$ with $\ell, a\geq 0$. Thus,

$$
\begin{cases}
0: & p(n) = a_0\cdot p(n-1) + \sum_{j_1\in I(n)} p(n;j_1), \\
1: & p(n;j_1) = \sum_{(d,j_1')\in U(n;j_1)} p(n-d;j_1') + \sum_{j_2\in I(n;j_1)} p(n;j_1,j_2), \\
\vdots &
\end{cases}
\tag{4}
$$

where $a_0 = |\overline{I}(n)|$ is the number of 1-inactive elements of $P(n)$. Let us call the i-th row of Equation (4) the *i-th level of* $p(n)$. The equality $I(n;j_1,\ldots,j_k)=\emptyset$ is equivalent to there not existing any $(k+1)$-active element of the set $P(n;j_1,\ldots,j_p)$. Thus, if $\cup_{j_1,\ldots,j_k} I(n;j_1,\ldots,j_k)=\emptyset$ then Equation (4) contains only $k+1$ levels. Hence, we can make the following definition.

Definition 2 *The minimal $k\in\mathbb{N}$ such that the set $I(n;j_1,\ldots,j_k)$ is empty for any $1\leq j_1,\ldots,j_k\leq n$ is called the* depth of the sequence $\{p(n)\}_{n\geq 0}$. *If the depth of a sequence $\{p(n)\}_{n\geq 0}$ does not depend on n then it is said to have* finite depth.

By the definitions, the depth of $p(n)$ equals the number of levels of $p(n)$.

Definition 3 *The sequence $p(n)$ is said to be k-*linear *if its depth is a finite number $k+1$ and $U(n;j_1,\ldots,j_\ell)$ is a* simple multiset*, that is if the number of occurrences of $(d,j_1',\ldots,j_s')$ in $U(n;j_1,\ldots,j_\ell)$ is a constant that does not depend on the parameters $n,j_1,j_2,\ldots,j_\ell$, for any d,s.*

Proposition 4 *Let $\{p(n)\}_{n\geq 0}$ be any k-linear sequence. Then the sequence $\{p(n)\}_{n\geq 0}$ satisfies the following recurrence relation*

$$
\begin{cases}
0: & p(n) = a_0\cdot p(n-1) + \sum_{j_1\in I(n)} p(n;j_1) \\
1: & p(n;j_1) = \sum_{(d,j_1')\in U(n;j_1)} p(n-d;j_1') + \sum_{j_2\in I(n;j_1)} p(n;j_1,j_2) \\
\vdots & \\
k-1: & p(n;j_1,\ldots,j_{k-1}) \\
& \quad = \sum_{s=1}^{k-1} \sum_{(d,j_1',\ldots,j_s')\in U(j_1,\ldots,j_{k-1})} p(n-d;j_1',\ldots,j_s') + \sum_{j_k\in I(n;j_1,\ldots,j_{k-1})} p(n;j_1,\ldots,j_k) \\
k: & p(n;j_1,\ldots,j_k) = \sum_{s=1}^{k} \sum_{(d,j_1',\ldots,j_s')\in U(n;j_1,\ldots,j_k)} p(n-d;j_1',\ldots,j_s'),
\end{cases}
\tag{5}
$$

where $a_0 = |\overline{I}(n)|$ is the number of 1-inactive elements of $p(n)$, and the set $U(n; j_1, \dots, j_\ell)$ is a simple multiset for all $\ell = 1, 2, \dots, k$.

Now, if we assume that the sequence $\{p(n)\}_{n\geq 0}$ satisfies Equation (5), then we can ask if there is an exact formula for the sequence $\{p(n)\}_{n\geq 0}$. To answer this we need the following notation:

$$Q(t; v_1, \dots, v_k) = \sum_{n\geq 0} t^n Q(n; v_1, \dots, v_k) = \sum_{n\geq 0} t^n \left(\sum_{1\leq j_1,\dots,j_k\leq n} p(n; j_1, \dots, j_k) \prod_{i=1}^{k} v_i^{j_i - 1} \right). \quad (6)$$

Now, we can describe our method by the following naive algorithm, whose input is a k-linear sequence and whose output is an exact formula for the sequence $\{p(n)\}_{n\geq 0}$.

Scanning elements algorithm:

(0) Given a sequence $\{p(n)\}_{n\neq 0}$ where $p(n) = \#P(n)$ and $P(n) \subseteq S_n$.

(1) Find a recurrence relation for the sequence $\{p(n)\}_{n\geq 0}$ as described in (4).

(2) Decide if the sequence $\{p(n)\}_{n\geq 0}$ is k-linear. If yes, continue, otherwise stop.

(3) Rewrite (5) in terms of generating functions with k indeterminates $v_1, v_2, \dots, v_k$. This is done by multiplying (5) by $\prod_{i=1}^{k} v_i^{j_i-1}$ and summing over all the possibilities $j_1 \in I(n)$, $j_2 \in I(n; j_1)$,..., $j_k \in I(n; j_1, \dots, j_{k-1})$.

(4) Extract from step (3) a system of functional equations in $k+1$ variables t, v_1, v_2, ..., v_k.

(5) Solve this system to get a formula for $Q(t; 1, 1, \dots, 1)$, which is a formula for the ordinary generating function $\sum_{n\geq 0} p(n)t^n$, as desired.

The above algorithm suggests refining the permutations according to the value(s) of the leftmost element(s), and then apply algebraic techniques. As we see, Step (2) is the crucial one, and often, the sequence $\{p(n)\}_{n\geq 0}$ is not k-linear. However, in the next section we will illustrate the above algorithm for several interesting cases.

3 Linear system of functional equations

As mentioned before, our method yields a multivariate system of functional equations in several variables, which is hard to solve in general (several cases of functional equations with three variables were studied in [7]). Thus, in this section we focus only on the case of linear systems of functional equations with two variables.

Let $\mathbf{P}(x; v) = (p_{ij}(x; v))_{1\leq i,j\leq \ell}$ and $\mathbf{Q}(x; v) = (q_{ij}(x; v))_{1\leq i,j\leq \ell}$ be any two $\ell \times \ell$ matrices of rational functions in x and v, and $\mathbf{b}(x; v) = (b_1(x; v), \dots, b_\ell(x; v))^T$ be any vector of rational functions in x and v. Let $\mathbf{A}(x;v) = (A_1(x;v), \ldots, A_\ell(x;v))^T$. In this section we wish to solve a linear system of functional equations of the following form:

$$\mathbf{P}(x;v)\mathbf{A}(x;v) = \mathbf{b}(x;v) + \mathbf{Q}(x;v)\mathbf{A}(x;1). \tag{7}$$

That is, we want to find ℓ formal power series $A_1(x;v), \ldots, A_\ell(x;v)$ satisfying (7). For ease reference, we will denote (7) by $(\mathbf{P}, \mathbf{b}, \mathbf{Q})$. Using elementary linear algebra (Gaussian elimination) we can assume that the matrix $\mathbf{P} = \mathbf{D}$, where $\mathbf{D}$ is a diagonal matrix. To find a solution for the system $(\mathbf{D}, \mathbf{b}, \mathbf{Q})$ with diagonal matrix $\mathbf{D}$, we state the following theorem.

Theorem 5 *Let $(\mathbf{D}, \mathbf{b}, \mathbf{Q})$ be any linear system of functional equations with ℓ variables $A_1(x;v), \ldots, A_\ell(x;v)$ of power series in x and v such that $\mathbf{D} = diag(d_1(x;v), \ldots, d_\ell(x;v))$ is a diagonal matrix, where $d_i(x;v) \not\equiv 0$ is a rational function for all $i = 1, 2, \ldots, \ell$. Suppose there exists a formal power series $u_i(x)$ such that $d_i(x; u_i(x)) = 0$, $i = 1, 2, \ldots, \ell$, and such that $q_{ij}(x; u_i(x))$ is a formal power series for all i, j, where $\mathbf{Q}(x;v) = (q_{ij}(x;v))_{1 \le i,j \le \ell}$. Then the system $(\mathbf{D}, \mathbf{b}, \mathbf{Q})$ has a unique solution of algebraic functions if and only if $\det(\mathbf{T}(x)) = \det((q_{ij}(x; u_i(x)))_{1 \le i,j \le \ell}) \neq 0$.*

Proof The system $(\mathbf{D}, \mathbf{b}, \mathbf{Q})$ has a unique solution if and only if the system

$$\mathbf{T}(x)\mathbf{A}(x;v) = -\mathbf{s}(x),$$

$\mathbf{s}(x) = (b_1(x; u_1(x)), \ldots, b_\ell(x; u_\ell(x))^T$, has a unique solution, which is equivalent to

$$\det(\mathbf{T}(x)) \neq 0.$$

Moreover, since $d_i(x;v)$ is a rational function, we have that $u_i(x)$ is an algebraic function for all $i = 1, 2, \ldots, \ell$. This implies that $(A_1(x;1), \ldots, A_\ell(x;1))^T$ is a vector of algebraic functions satisfying our system $(\mathbf{D}, \mathbf{b}, \mathbf{Q})$. □

4 Three letter patterns

In this section we deal with several interesting cases of families of permutations by using the scanning-elements algorithm as described in Section 2. In particular, we deal with classical patterns (Section 4.1), *generalized patterns* (Section 4.2), and *distanced patterns* (Section 4.3) of three letters.

4.1 Classical patterns

In this subsection we recover and refine several interesting enumerations on the set of permutations that either avoid or contain a (classical) pattern of length three.

4.1.1 Refining 123-avoiding permutations

Define $P(n) = S_n(123)$. Let $\pi = \pi_1\pi_2\dots\pi_n$ be any permutation of length n. If $\pi_1 > \pi_2$ then π avoids 123 if and only if $\pi_2\pi_3\dots\pi_n$ is a permutation of length $n-1$ that avoids 123, and if $\pi_1 < \pi_2$ then π avoids 123 if and only if $\pi_2 = n$ which is equivalent to saying that $\pi_1\pi_3\pi_4\dots\pi_n$ is a permutation of length $n-1$ that avoids 123. Hence, $P(n-1)$ is a reduction of $P(n;n)$ and $P(n;n-1)$, and $P(n;i,j)$ is a reduction of $P(n-1;j)$ for all $i > j$. Thus, Equation (1) gives

$$p(n) = 2p(n-1) + \sum_{j=1}^{n-2} p(n;j), \qquad p(n;j) = \sum_{i=1}^{j} p(n-1;i), \tag{8}$$

for all $n-2 \geq j \geq 1$. Therefore, the above recurrence relation is a particular case of (5), i.e. $a_0 = 2$, $U(n;j) = \{(1,i) \mid i = 1,2,\dots,j\}$, $I(n;j) = \emptyset$, and can be written in terms of $Q(n;v)$ as $Q(n;v) = v^{n-1}Q(n-1;1) + \frac{1}{1-v}\left(Q(n-1;v) - v^{n-1}Q(n-1;1)\right)$, for all $n \geq 2$. From the definitions we have that $Q(0;v) = Q(1;v) = 1$. Multiplying the above recurrence relation by t^n and summing over all possible $n \geq 2$ we arrive at

$$Q(t;v) = 1 + \frac{t}{1-v}\cdot\left(Q(t;v) - Q(tv;1)\right) + tQ(tv;1). \tag{9}$$

Therefore, by using the kernel method as described in [4], we recover the well known enumeration of 123-avoiding permutations of length n by the n-th Catalan number (see [11]). Moreover, the number of 123-avoiding permutations of length n starting with m ($m = 1,2,\dots,n$) is given by $\binom{m+n-2}{m-1} - \binom{m+n-2}{m-2}$ (where $\binom{a}{b}$ by 0 whenever $a < b$ or $b < 0$).

4.1.2 Refining permutations containing 123 exactly once

Now let us find an explicit formula for $\widehat{f}_{123}(n)$, the number of permutations of length n that contain 123 exactly once. Using similar arguments as in Section 4.1.1 we obtain that for all $n \geq 3$ and $1 \leq i \leq n-2$,

$$\begin{cases} \widehat{f}_{123}(n;i) = g(n;i) + \widehat{f}_{123}(n-1;i) + f_{123}(n-1;i), \\ \widehat{f}_{123}(n;n) = \widehat{f}_{123}(n;n-1) = \widehat{f}_{123}(n-1), \end{cases}$$

where $g(n,i) = \sum_{j=1}^{i-1}\widehat{f}_{123}(n;j)$ satisfies $g(n;i) = g(n-1;1) + g(n-1;2) + \cdots + g(n-1;i)$. Define $G(n;v) = \sum_{i=1}^{n} G(n;i)v^{i-1}$ and $G(t;v) = \sum_{n\geq 0} G(n;v)t^n$. Rewriting the above recurrence relation in terms of the generating functions we obtain that

$$\begin{cases} \left(1 - \frac{t}{v(1-v)}\right)\left(1 - \frac{t}{v}\right)\widehat{F}_{123}(t/v;v) = -\frac{t(v-t)}{v(1-v)}\widehat{F}_{123}(t;1) + \frac{t^2(v^2+t)}{v^3}F_{123}(t,1), \\ \left(1 - \frac{t}{v(1-v)}\right)F_{123}(t/v;v) = 1 - \frac{tv}{1-v}F_{123}(t;1). \end{cases}$$

Therefore, Theorem 5 with $u_1(t) = u_2(t) = \frac{1+\sqrt{1-4t}}{2}$ gives that $\widehat{f}_{123}(n) = \frac{6}{n+3}\binom{2n-1}{n-3}$ (see [14]). Moreover, the number of permutations $\pi \in S_n$ that contain the pattern 123 exactly once and have $\pi_1 = m$ is given by $\binom{n+m-3}{m} - \binom{n+m-3}{m-5} - \frac{3}{m+1}\binom{2m-2}{m}$ if $m = 1,2,\dots,n-2$, and $\frac{6}{n+2}\binom{2n-3}{n-4}$ otherwise.

4.1.3 Refining 132-avoiding permutations

Define $P(n) = S_n(132)$. Let $\pi = \pi_1\pi_2\dots\pi_n \in S_n$ be any permutation of length n. Now, if $\pi_1 > \pi_2$, then π avoids 132 if and only if $\pi_2\pi_3\dots\pi_n$ is a permutation of length $n-1$ that avoids 132, and if $\pi_1 < \pi_2$ then π avoids 132 if and only if $\pi_2 = \pi_1 + 1$ which is equivalent to saying that $\pi_1\pi_3\pi_4\dots\pi_n$ is a permutation of length $n-1$ that avoids 132. Hence, $P(n-1)$ is a reduction of $P(n;n)$ and $P(n;n-1)$, $P(n-1;j)$ is a reduction of $P(n;i,j)$ for all $i > j$, and $P(n-1;i)$ is a reduction of $P(n-1;i,i+1)$. Thus, Equation (5) gives

$$p(n) = 2p(n-1) + \textstyle\sum_{j=1}^{n-2} p(n;j), \quad p(n;j) = \sum_{i=1}^{j} p(n-1;i), \tag{10}$$

for all $n-2 \geq j \geq 1$. Thus, Equations (8), (9) and (10) give that $P(t;v) = F_{132}(t;v) = F_{123}(t;v)$.

4.1.4 Refining permutations containing 132 exactly once

Now let us find an explicit formula for $\widehat{f}_{132}(n)$, the number of permutations of length n that contain 132 exactly once. Using similar arguments as in the proof of Equation (10) we obtain that for all $1 \leq j \leq n-2$,

$$\begin{cases} \widehat{f}_{132}(n;j) = g(n;j) + h(n;j) + f_{132}(n-1;j), \\ \widehat{f}_{132}(n;n-1) = \widehat{f}_{132}(n;n) = \widehat{f}_{132}(n-1), \end{cases} \tag{11}$$

where $g(n;j) = \sum_{i=1}^{j-1} \widehat{f}_{132}(n;j,i)$ and $h(n;j) = \widehat{f}_{132}(n;j,j+1)$ satisfy the following recurrences:

$$\begin{cases} g(n;j) = \sum_{i=1}^{j-1} g(n-1;i) + \sum_{i=1}^{j-1} h(n-1;i) + \sum_{i=1}^{j-2} f_{132}(n-2;i), \\ g(n;n) = \widehat{f}_{132}(n-1), \end{cases}$$

$h(n;j) = g(n-1;j) + h(n-1;j)$, and $h(n;n-1) = h(n;n-2) = \widehat{f}_{132}(n-2)$. Define

$$G(t;v) = \sum_{n\geq 0} G(n;v)t^n = \sum_{n\geq 0} t^n \sum_{j=1}^{n} G(n;j)v^{j-1},$$

$$H(t;v) = \sum_{n\geq 0} H(n;v)t^n = \sum_{n\geq 0} t^n \sum_{j=1}^{n} H(n;j)v^{j-1}.$$

By rewriting the above recurrence relations in terms of generating functions $G(t;v)$ and $H(t;v)$ and using the fact that $F_{132}(t/v;v)$ satisfies $F_{132}(t/v;v) = 1 + \frac{t}{v(1-v)}(F_{132}(t/v;v) - F_{132}(t;1)) + \frac{t}{v}F_{132}(t;1)$ (see Section 4.1.3), we obtain that

$$\begin{cases} \left(1 - \frac{t}{v(1-v)}\right)^2 \widehat{F}_{132}(t/v;v) \\ \qquad = -\frac{t^2(t-v+tv)}{v^3(1-v)} - \frac{t}{1-v}\left(1 - \frac{t}{v(1-v)}\right)\widehat{F}_{132}(t;1) + \frac{t^2(t-v+tv)(v-t)}{v^4(1-v)}F_{132}(t;1), \\ \left(1 - \frac{t}{v(1-v)}\right) F_{132}(t/v;v) \\ \qquad = 1 - \frac{t}{1-v}F_{132}(t;1). \end{cases}$$

To find an explicit formula for $\widehat{F}_{132}(x;1)$ and $F_{132}(x;1)$ let us consider the following system

$$\begin{cases} \frac{\partial}{\partial v}\left[\left(1-\frac{t}{v(1-v)}\right)^2 \widehat{F}_{132}(t/v;v)\right] \\ \qquad\qquad = -\frac{\partial}{\partial v}\frac{t^2(t-v+tv)}{v^3(1-v)} - \frac{\partial}{\partial v}\left(\frac{t}{1-v}-\frac{t^2}{v(1-v)^2}\right)\widehat{F}_{132}(t;1) + \frac{\partial}{\partial v}\frac{t^2(t-v+tv)(v-t)}{v^4(1-v)}F_{132}(t;1), \\ \left(1-\frac{t}{v(1-v)}\right)F_{132}(t/v;v) = 1 - \frac{t}{1-v}F_{132}(t;1). \end{cases}$$

Theorem 5 with $u_1(t) = u_2(t) = \frac{1+\sqrt{1-4t}}{2}$ gives $\widehat{f}_{132}(n) = \binom{2n-3}{n-3}$ (see [6]). Moreover, the number of permutations π of length n containing 132 exactly once and starting with m is given by $\widehat{f}_{132}(n-1)$ if $m = n, n-1$, and $\frac{n^2-4n+6-m}{n-1}\binom{n+m-5}{m-3} + \frac{n-m}{n-1}\binom{n+m-3}{m-1} - \sum_{j=0}^{n-3}\frac{j}{n-2-j}\binom{2n-6-2j}{n-3-j}\binom{2j+m-n}{j+1} - \sum_{j=0}^{n-4}\binom{m+n-4}{n-4-2j}$ otherwise.

4.1.5 Refining 231-avoiding permutations

Up to now, all our results are given by k-linear sequences. Here we present a sequence, 231-avoiding permutations of length n, which does not have the k-linear property. Define $P(n) = S_n(231)$ and let $A_m(n) = \sum_{n-1\geq j_1>\cdots>j_s\geq 2} p(n;j_1,\dots,j_s)$ for all $m \geq 1$ with $A_0(n) = p(n)$. Then, using our scanning-elements algorithm we obtain that $p(n) = p(n;1) + p(n;n) + \sum_{j_1=2}^{n-1} p(n;j_1) = 2p(n-1) + \sum_{j_1=2}^{n-1} p(n;j_1)$, that is, $A_1(n) = p(n) - 2p(n-1)$. Also, for all $m \geq 1$,

$$\begin{aligned} A_m(n) &= A_{m+1}(n) + \sum_{n-1\geq j_1>\cdots>j_m\geq 2} p(n;j_1,\dots,j_m,1) \\ &= A_{m+1}(n) + \sum_{n-2\geq j_1>\cdots>j_m\geq 1} p(n-1;j_1,\dots,j_m) \\ &= A_{m+1}(n) + A_m(n-1) + \sum_{n-3\geq j_1>\cdots>j_{m-1}\geq 1} p(n-1;j_1,\dots,j_{m-1}) \\ &= \cdots = A_{m+1}(n) + A_m(n-1) + \cdots + A_0(n-1-m). \end{aligned}$$

Hence, $p(n)$ is a sequence with depth $n-1$ (depends on n), that is, this sequence is not k-linear. But, the sequence $A_m(n)$ is a 1-linear sequence. Thus, if we define $A_m(x) = \sum_{n\geq 0} A_m(n)x^n$ and $A(x;v) = \sum_{m\geq 0} A_m(x)v^m$, then $A_m(x) = A_{m+1}(x) + xA_m(x) + \cdots + x^{m+1}A_0(x) - x^{m+1}$ with $A_0(x) = \sum_{n\geq 0} p(n)x^n$ and $A_1(x) = (1-2x)A_0(x) - 1 + x$. As a consequence, we arrive at

$$\left(1 - \frac{v(1-x-xv)}{1-xv}\right)A(x;v) = (1-xv)A(x;0) - \frac{v(1-x-xv)}{1-xv}.$$

Therefore, by using the kernel method as described in [4], we recover the well known enumeration of 231-avoiding permutations of length n as c_n, the n-th Catalan number (see [11]). Moreover, the number of 231-avoiding permutations of length n and starting with m is given by $c_{m-1}c_{n-m}$, where c_n is the n-th Catalan number. We remark that we have presented the case of 231-avoiding permutations as it will be a good reference for the enumeration in next sections, where this result can be shown from the fact that the number of 231-avoiding permutations π of length n having $\pi_1 = m$ equals the number of 132-avoiding permutations π of length n having $\pi_m = n$.

4.1.6 Refining permutations containing 231 exactly once

One can try to obtain results similar to Section 4.1.5, but the proof for the case of permutations containing 231 exactly once is similar to the proof of the case of 231-avoiding permutations and extremely cumbersome. One can obtain that the number of permutations of length n containing 231 exactly once and starting with m is given by $c_{m-2}c_{n-m} + \binom{2m-5}{m-4}c_{n-m} + \binom{2n-2m-3}{n-m-3}c_{m-1}$, where c_n is the n-th Catalan number.

4.2 Three letter generalized pattern of type $(2,1)$

In [3] Babson and Steingrímsson introduced *generalized patterns* that allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation. In this subsection, we study the generating functions $F_\tau(x;v)$ and $\widehat{F}_\tau(x;v)$, where $\tau = ab\text{-}c$ and $abc \in S_3$.

4.2.1 Refining 12-3-avoiding permutations

By using the arguments in Section 4.1.1 we obtain that for all $n \geq 2$ and $1 \leq i \leq n-1$, $f_{12\text{-}3}(n;i) = f_{12\text{-}3}(n-2) + \sum_{j=1}^{i-1} f_{12\text{-}3}(n-1;j)$ and $f_{12\text{-}3}(n;n) = f_{12\text{-}3}(n-1)$. Rewriting the above recurrence relations in terms of generating functions we get that

$$\begin{aligned}&F_{12\text{-}3}(t/v;v)\\ &\quad = 1 + \tfrac{t}{v} + \tfrac{t}{1-v}(F_{12\text{-}3}(t/v;v) - F_{12\text{-}3}(t;1)) + \tfrac{t^2}{v^2(1-v)}(F_{12\text{-}3}(t/v;1) - vF_{12\text{-}3}(t;1)).\end{aligned}$$

Therefore, by using the kernel method with $v = 1-t$ we arrive at $F_{12\text{-}3}(t;1) = 1 + \frac{t}{1-t}F_{12\text{-}3}(t/(1-t);1)$. Using the above functional equation repeatedly, we recover the well known enumeration of 12-3-avoiding permutations of length n by B_n, the n-th Bell number (see [8]). Moreover, the number of permutations of length n, $n \geq 2$, that avoid 12-3 and start with m is given by $\sum_{i=0}^{m-1}\binom{m-1}{i}B_{n-2-i}$ if $m = 1, 2, \ldots, n-1$, and B_{n-1} otherwise.

4.2.2 Refining permutations containing 12-3 exactly once

Now we find an explicit formula for the number of permutations of length n that contain 12-3 exactly once. Using similar arguments as in Section 4.1.1 we obtain that for all $n \geq 3$ and $1 \leq i \leq n-2$,

$$\begin{cases} \widehat{f}_{12\text{-}3}(n;i) \\ \qquad = \widehat{f}_{12\text{-}3}(n-1;1) + \cdots + \widehat{f}_{12\text{-}3}(n-1;i-1) + \widehat{f}_{12\text{-}3}(n-2) + f_{12\text{-}3}(n-1), \\ \widehat{f}_{12\text{-}3}(n;n) \\ \qquad = \widehat{f}_{12\text{-}3}(n;n-1) = \widehat{f}_{12\text{-}3}(n-1). \end{cases} \tag{12}$$

Rewriting the above recurrence relations in terms of generating functions we get that

$$\begin{aligned}\widehat{F}_{12\text{-}3}(t/v;v) &= \tfrac{t}{1-v}(\widehat{F}_{12\text{-}3}(t/v;v) - \widehat{F}_{12\text{-}3}(t;1))\\ &\qquad + \tfrac{t^2}{v^2(1-v)}(\widehat{F}_{12\text{-}3}(t/v;1) - v\widehat{F}_{12\text{-}3}(t;1) + F_{12\text{-}3}(t/v;1) - F_{12\text{-}3}(t;1)).\end{aligned}$$

Therefore, by using the kernel method as described in [4] with $v = 1 - t$ we arrive at

$$\widehat{F}_{12\text{-}3}(t;1) = \frac{t}{1-t}(\widehat{F}_{12\text{-}3}(t/(1-t);1) + F_{12\text{-}3}(t/(1-t);1) - F_{12\text{-}3}(t;1)).$$

If we simply apply the above equation repeatedly and in each step perform some rather tedious algebraic manipulations, then we get that the ordinary generating function for the number of permutations in S_n containing 12-3 exactly once is

$$\widehat{F}_{12\text{-}3}(t;1) = \sum_{n\geq 1} \frac{t}{1-nt} \sum_{k\geq 0} \frac{kt^{k+n}}{(1-t)(1-2x)\cdots(1-(k+n)t)},$$

(see [9]).

4.2.3 Refining 21-3-avoiding permutations

Using similar techniques as in Section 4.2.1, we obtain that the generating function $F_{21\text{-}3}(t;v)$ satisfies

$$F_{21\text{-}3}(t/v;v) = 1 - \frac{t}{v} + \frac{t}{v(1-v)}(F_{21\text{-}3}(t/v;1) - vF_{21\text{-}3}(t/v;v)) + \frac{t}{v}F_{21\text{-}3}(t;1).$$

Therefore, the kernel method as described in [4] with $v = 1 + t$ gives

$$F_{21\text{-}3}(t;1) = 1 + \frac{t}{1-t}F_{21\text{-}3}(\frac{t}{1-t};1).$$

Hence, by applying the above functional equation repeatedly, we recover the well known enumeration of 21-3-avoiding permutations of length n by the n-th Bell number (see [8]). Moreover, the number of permutations of length n, $n \geq 2$, that avoid 21-3 and start with m is given by $\sum_{i=0}^{m-1}(-1)^i\binom{m-1}{i}B_{n-1-i}$ if $m = 1, 2, \dots, n-1$, and B_{n-1} otherwise.

4.2.4 Refining permutations containing 21-3 exactly once

Again, using similar techniques as in Section 4.2.1 we obtain that the generating function $F_{21\text{-}3}(t;v)$ satisfies

$$\begin{aligned}&\widehat{F}_{21\text{-}3}(t/v;v)\\ &\quad= \tfrac{t}{v(1-v)}(\widehat{F}_{21\text{-}3}(t/v;1) - v\widehat{F}_{21\text{-}3}(t/v;v)) + \tfrac{t}{v}\widehat{F}_{21\text{-}3}(t;1) + \tfrac{t(1-t)}{v^2}F_{21\text{-}3}(t;1) - \tfrac{t}{v^2}.\end{aligned}$$

Therefore, the kernel method as described in [4] with $v = 1 + t$ gives

$$\widehat{F}_{21\text{-}3}(t;1) = \frac{t}{1-t}\widehat{F}_{21\text{-}3}(\frac{t}{1-t};1) + \frac{t(1-2t)}{1-t}(F_{21\text{-}3}(\frac{t}{1-t};1) - 1).$$

Hence, by applying the above functional equation repeatedly we get that the ordinary generating function for the number of permutations of length n containing 21-3 exactly once is $\sum_{n\geq 1}\frac{t}{1-(n-1)t}\sum_{k\geq 0}\frac{kt^{k+n}}{(1-t)(1-2t)\cdots(1-(k+n)t)}$ (see [9]).

4.2.5 Refining 13-2-avoiding permutations and permutations containing 13-2 exactly once

In [8], Claesson showed that $S_n(13\text{-}2) = S_n(1\text{-}3\text{-}2)$ for all $n \geq 0$. Thus, Section 4.1.3 gives that $F_{13\text{-}2}(t;v) = F_{1\text{-}3\text{-}2}(t;v)$. To exhibit an explicit formula for $\widehat{F}_{13\text{-}2}(t;1)$, we use similar arguments as in the previous subsection. Then one can state that

$$\left(1 - \frac{t}{v(1-v)}\right) \widehat{F}_{13\text{-}2}(t/v;v) = \frac{t}{v^2} F_{13\text{-}2}(t/v;v) - \frac{t}{1-v}\widehat{F}_{13\text{-}2}(t;1) - \frac{t}{v(v-t)}.$$

Using the fact that $\lim_{v \to \frac{1+\sqrt{1-4x}}{2}} F_{13\text{-}2}(t/v;v) = \frac{1+\sqrt{1-4x}}{2\sqrt{1-4x}}$ (Section 4.1.3), we get that $\widehat{f}_{13\text{-}2}(n) = \binom{2n}{n-3}$ (see [9]). Moreover, the number of permutations of length n containing 13-2 exactly once and starting with m, $m = 1, 2, \ldots, n-1$, is given by

$$(m-1)\binom{m+n-2}{n-3} - \sum_{j=0}^{m-1} \frac{j}{m-j}\binom{2(m-1-j)}{m-1-j}\binom{2j+n-m-3}{j+1} - \sum_{j=0}^{m-5}\binom{2j+6}{j}\binom{n+m-9}{n-4-j}.$$

4.3 Distanced patterns

We say a permutation $\pi \in S_n$ avoids the distanced pattern $\tau_1\tau_2\Box\tau_3$ if there is no subsequence $1 \leq a < b < c \leq n$ such that $c > b+1$ and $\pi_a\pi_b\pi_c$ is order-isomorphic to τ (see [10]).

4.3.1 The pattern 12□3

Using similar techniques as in Section 4.2.1 we obtain that the generating function $F_{12\Box3}(t;v)$ satisfies $\left(1 - \frac{t}{v(1-v)} - \frac{t^2}{v^2}\right) F_{12\Box3}(t/v;v) = 1 - \frac{t^2}{v^2} + \frac{t}{1-v} F_{12\Box3}(t;1)$. Therefore, by using the kernel method as described in [4] with $v = v_0(t)$ which is the root of the polynomial $1 - v + t(1-t)v^2 + t^2v^3$ we arrive at $F_{12\Box3}(t;1) = v_0(t)$. By using the Lagrange inversion formula (see [22]) we get that the number of 12□3-avoiding permutations in S_n is given by $\sum_{k\geq0} \frac{1}{n-k}\binom{2n-2k}{n-1-2k}\binom{n-k}{k}$.

4.3.2 The pattern 13□2

Again, using similar techniques as in Section 4.2.1 we obtain that the generating function $F_{13\Box2}(t;v)$ satisfies $F_{13\Box2}(t;v) = F_{12\Box3}(t;v)$. In particular, the number of 13□2-avoiding permutations in S_n is given by $\sum_{k\geq0} \frac{1}{n-k}\binom{2n-2k}{n-1-2k}\binom{n-k}{k}$.

4.3.3 The pattern 23□1

Define

$$A_m(n) = \sum_{n-1\geq j_1 > \cdots > j_m \geq 2} f_{23\Box1}(n; j_1, \ldots, j_m), \;\; B_m(n) = \sum_{n-1\geq j_1 > \cdots > j_m \geq 1} f_{23\Box1}(n; j_1, \ldots, j_m),$$

and $A_0(n) = B_0(n) = f_{23\square 1}(n)$. Using similar arguments as in th proofs of Section 4.1.5 we obtain that $A_1(n) = A_0(n) - 2A_0(n-1)$, $B_1(n) = A_0(n) - A_0(n-1)$, for all $n > m+1 \geq 1$,

$$A_m(n) = A_{m+1}(n) + B_m(n-1) + B_m(n-2) + B_{m-1}(n-3) + \cdots + B_0(n-2-m),$$

and $B_m(n) = A_m(n) + A_{m-1}(n-1) + \cdots + A_0(n-m)$.

Now, by defining $A(x;v) = \sum_{m\geq 0} A_m(x)v^m$ and $B(x;v) = \sum_{m\geq 0} B_m(x)v^m$, we get that

$$\begin{cases} B(x;v) \\ = \frac{1}{1-xv}(A(x;v)-1), \\ A(x;v) \\ = (1-xv+x^2v)A(x;0) + vA(x;v) - \left(x(1+x)v + \frac{x^3v^2}{1-xv}\right)(B(x;v)+1) - (1-x)v. \end{cases}$$

Thus,

$$\left(1 - v + \frac{x(1-x+x/v)}{(1-x)^2}\right) A(x/v;v) = (1-x+x/v)A(x/v;0) + \frac{v^2 - v(2v+1)x + v(v+1)x^2 - x^3}{v(1-x)^2}.$$

Therefore, by using the kernel method as described in [4], we recover the well known enumeration of $23\square 1$-avoiding permutations of length n by $\sum_{k\geq 0} \frac{1}{n-k}\binom{2n-2k}{n-1-2k}\binom{n-k}{k}$ (see [10]).

Acknowledgement. We would like to thank S. Heubach and E. Steingrímsson for their comments that improved the presentation of the paper. We would like to thanks M. Bousquet-Mélou for interesting remarks.

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