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THREE LIMIT CYCLES IN DISCONTINUOUS PIECEWISE LINEAR DIFFERENTIAL SYSTEMS WITH TWO ZONES

JAUME LLIBRE[1] AND ENRIQUE PONCE[2]

Abstract. In this paper we study a planar piecewise linear differential system formed by two regions separated by a straight line so that one system has a real unstable focus and the other a virtual stable focus which coincides with the real one. This system was introduced by S.-M. Huan and X.-S. Yang in [8] who numerically showed that it can exhibit 3 limit cycles surrounding the real focus. This is the first example that a non–smooth piecewise linear differential system with two zones can have 3 limit cycles surrounding a unique equilibrium. We provide a rigorous proof of this numerical result.

1. Introduction and statement of the main result

The analysis of piecewise linear differential systems can be traced back to Andronov and coworkers [1] and still continues to receive attention by researchers. Effectively, in recent years there has been an upsurge of interest from the mathematical community in understanding their dynamical richness, as such systems are widely used to model many real processes and different modern devices, see for instance [4] and references therein. Recently, they have been shown to be also relevant as idealized models of cell activity, see [3, 11, 12].

The case of continuous piecewise linear systems, when they have only two linearity regions separated by a straight line is the simplest possible configuration in piecewise linear systems. We remark that even in this seemingly simple case, only after a thorough analysis it was possible to establish the existence at most of one limit cycle for such systems, see [6]. The reason for that misleading simplicity of piecewise linear systems is twofold. First, even one can easily integrate solutions in any linearity region, the time that each orbit requires to pass from a linearity region to each other is unknown and so the matching of the corresponding solutions is an intricate problem. Second, the number of parameters to consider in order to be sure that one copes with all possible configurations is typically not small, so that the achievement of efficient canonical forms with fewer parameters is crucial.

Discontinuous piecewise linear systems with only two linearity regions separated by a straight line have been studied recently in [7, 8], among other papers. In [7] some results about the existence of two limit cycles appeared, so that the authors conjectured that the maximum number of limit cycles for this class of piecewise

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linear differential systems is exactly two. This conjecture is analogous to Conjecture 1 in the discussion included in [11]. However, by considering a specific family of discontinuous PWL differential systems with two linear zones sharing the equilibrium position, in [8] strong numerical evidence about the existence of three limit cycles was obtained. The example in [8] represents up to the best of our knowledge the first non-smooth piecewise linear differential system with two zones with 3 limit cycles surrounding a unique equilibrium. We will provide in this paper a rigorous proof of the the existence of such 3 limit cycles.

The planar non-smooth piecewise linear differential system with two zones separated by a straight line corresponding to the example 5.1 of Huan and Yang in [8] is

$$\dot{\mathbf{X}} = \begin{cases} A^+\mathbf{X} & \text{if } x \geq 1, \\ A^-\mathbf{X} & \text{if } x < 1, \end{cases} \tag{1}$$

where $\mathbf{X} = (x, y)^T$ with

$$A^+ = \begin{pmatrix} \dfrac{19}{500} & -\dfrac{1}{10} \\ \dfrac{1}{10} & \dfrac{19}{500} \end{pmatrix}, \quad \text{and} \quad A^- = \begin{pmatrix} 1 & -5 \\ \dfrac{377}{1000} & -\dfrac{13}{10} \end{pmatrix}.$$

The dot denotes derivative with respect to the independent variable t, that we call here the *time*. The point $(0, 0)$ is a virtual unstable focus for the system $\dot{\mathbf{X}} = A^+\mathbf{X}$ and a real stable focus for the system $\dot{\mathbf{X}} = A^-\mathbf{X}$. Our main result is the following.

Theorem 1. *The planar non–smooth piecewise linear differential system with two zones* (1) *has* 3 *limit cycles surrounding its unique equilibrium point located at the origin.*

Theorem 1 is proved in section 2. For proving it we shall use ideas from the paper [2]. The three limit cycles cross the line $x = 1$ and have alternating stabilities, see Figure 1.

2. Proof of Theorem 1

The flow of system (1) enters to the half plane $x > 1$ through the half straight line $\{(1, y) : y < 1/5\}$, and exits it through the half straight line $\{(1, y) : y > 19/50\}$. Therefore if the system (1) has crossing periodic orbits (by concatenating orbits of each linearity region) these must surround the segment $\{(1, y) : 1/5 \leq y \leq 19/50\}$. This segment is the so-called sliding set where some dynamics could be defined using the Filippov's method, see [5]. In contrast to the crossing periodic orbits, we note that periodic orbits using some part of the sliding set through the Filippov's dynamics (a situation not possible here) are called sliding periodic orbits.

To alleviate all the expressions in this proof, we start by making a rescaling in time different for each zone, which amounts to multiply each matrix for an adequate constant, namely 10 for the zone $x > 1$ and $4/3$ for the zone $x < 1$. Thus we work with a topologically equivalent system to system (1), where from now on we assume

$$A^+ = \begin{pmatrix} \dfrac{19}{50} & -1 \\ 1 & \dfrac{19}{50} \end{pmatrix}, \quad \text{and} \quad A^- = \begin{pmatrix} \dfrac{4}{3} & -\dfrac{20}{3} \\ \dfrac{377}{750} & -\dfrac{26}{15} \end{pmatrix}.$$

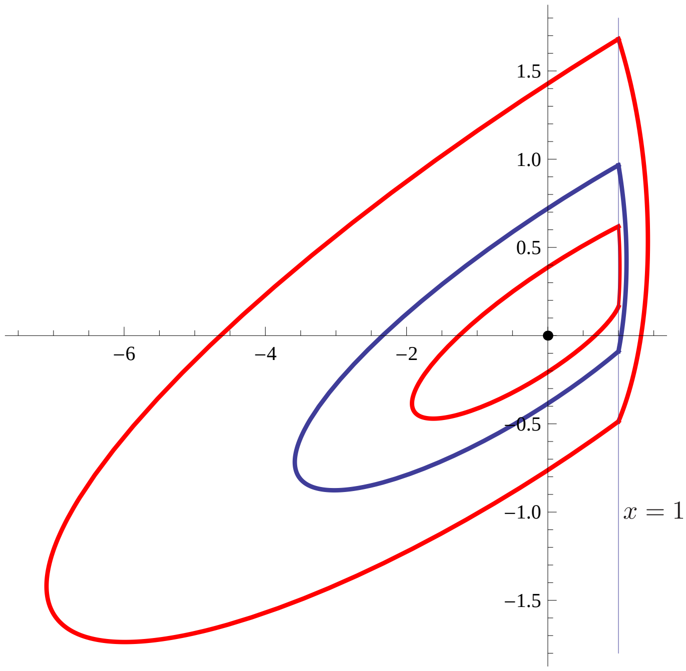

FIGURE 1. The three limit cycles surrounding the origin and having different stabilities (stable in blue, unstable in red).

The solution $(x^+(t), y^+(t))$ of system $\dot{\mathbf{X}} = A^+\mathbf{X}$ which pass through the point $(1, Y)$ when the time $t = 0$ is

$$x^+(t) = e^{19t/50}\left(\cos t - Y \sin t\right),$$
$$y^+(t) = e^{19t/50}\left(Y \cos t + \sin t\right),$$

and the solution $(x^-(t), y^-(t))$ of system $\dot{\mathbf{X}} = A^-\mathbf{X}$ which pass through the point $(1, Y)$ when the time $t = 0$ is

$$x^-(t) = \frac{1}{15}e^{-t/5}\left(15 \cos t - 100Y \sin t + 23 \sin t\right),$$
$$y^-(t) = \frac{1}{750}e^{-t/5}\left(750Y \cos t - 1150Y \sin t + 377 \sin t\right).$$

Assume that for the point $(1, Y)$ with $Y > 19/50$ pass a periodic solution $(x^+(t), y^+(t)) \cup (x^-(t), y^-(t))$. Then if $t^+ > 0$ is the smallest time such that $x^+(-t^+) = 1$, and $t^- > 0$ is the smallest time such that $x^-(t^-) = 1$, we have that $y^+(-t^+) = y^-(t^-) < 1/5$. Hence a periodic solution of system (1) is characterized by a solution (t^+, t^-, Y) of the system

$$\begin{aligned} f_1(t^+, t^-, Y) &= x^+(-t^+) - 1 = 0, \\ f_2(t^+, t^-, Y) &= x^-(t^-) - 1 = 0, \\ f_3(t^+, t^-, Y) &= y^+(-t^+) - y^-(t^-) = 0. \end{aligned} \tag{2}$$

We shall prove that there are three isolated solutions (t_k^+, t_k^-, Y_k) for $k = 1, 2, 3$ of system (2) near

$$\begin{aligned} t_1^+ &= 1.486632803655017270687520165 95..,\\ t_1^- &= 3.451083322960971568017700333 78..,\\ Y_1 &= 1.681194510518939969909463945 80..,\\[4pt] t_2^+ &= 0.856683222921113530964676935 01..,\\ t_2^- &= 3.782341118660762639035233933 33..,\\ Y_2 &= 0.965799855846682255135449274 84..,\\[4pt] t_3^+ &= 0.391783885984432806504662019 97..,\\ t_3^- &= 4.665072695545692237740663055 15..,\\ Y_3 &= 0.618854165182525013760673237 68... \end{aligned}$$

So these three solutions will correspond to isolated periodic orbits of system (1), i.e. to limit cycles of that system. For proving the existence of these three isolated solutions of system (2) we shall use the Newton-Kantorovich Theorem.

Let $B_r(x_0)$ be the points $x \in \mathbb{R}^n$ such that $|x - x_0| < r$, i.e. the open ball of center x_0 and radius r. We denote by $\overline{B_r(x_0)}$ the closure of $B_r(x_0)$.

Theorem 2 (Newton-Kantorovich Theorem)**.** *Given a function $f : C \subset \mathbb{R}^n \to \mathbb{R}^n$ and a convex $C_0 \subset C$, assume that f is $\mathcal{C}^1$ in C_0 and that the following assumptions hold:*

(a) $|Df(z) - Df(z')| \leq \gamma |z - z'|$ *for all* $z, z' \in C_0$,
(b) $|Df(z_0)^{-1} f(z_0)| \leq \alpha$,
(c) $|Df(z_0)^{-1}| \leq \beta$,

for some $z_0 \in C_0$. Consider

$$h = \alpha\beta\gamma, \qquad r_{1,2} = \frac{1 \pm \sqrt{1 - 2h}}{h}\alpha.$$

If $h \leq 1/2$ and $\overline{B_{r_1}(z_0)} \subset C_0$, then the sequence $\{z_k\}$ defined by

$$z_{k+1} = z_k - Df(z_k)^{-1} f(z_k) \quad \text{for} \quad k = 0, 1, ...$$

is contained in $B_{r_1}(z_0)$ and converges to the unique zero of $f(z)$ contained in $C_0 \cap B_{r_2}(z_0)$.

Proof. See [10]. □

We shall apply Theorem 2 to our function $f = (f_1, f_2, f_3)$. So $n = 3$, $C = \mathbb{R}^3$ and C_0 will be

$$\begin{aligned} C_0^1 &= [1.48, 1.49] \times [3.45, 3.46] \times [1.68, 1.69],\\ C_0^2 &= [0.85, 0.86] \times [3.78, 3.79] \times [0.96, 0.97],\\ C_0^3 &= [0.39, 0.40] \times [4.66, 4.67] \times [0.61, 0.62], \end{aligned} \tag{3}$$

for the solutions of system (2) near (t_k^+, t_k^-, Y_k) for $k = 1, 2, 3$ respectively.

The norm that we shall use in the statement of Theorem 2 will be the norm $|\ |_\infty$, i.e.

$$|z|_\infty = \max_i |z_i| \quad \text{if } z = (z_1, z_2, z_3).$$

In what follows sometimes we shall use (z_1, z_2, z_3) instead of (t^+, t^-, Y) for simplifying the notation.

In order to find the constants α, β and γ for which the inequalities (a), (b) and (c) of Theorem 2 hold for every one of our convex sets C_0 we need some preliminary results.

Let $z=(z_1,z_2,z_3)$ and $z'=(z_1',z_2',z_3')$. If $g: C_0\to\mathbb{R}$ is $\mathcal{C}^1$ and $z,z'\in C_0$, then

$$\begin{aligned}|g(z)-g(z')| \leq\ & |g(z_1,z_2,z_3)-g(z_1',z_2,z_3)|+\\ & |g(z_1',z_2,z_3)-g(z_1',z_2',z_3)|+\\ & |g(z_1',z_2',z_3)-g(z_1',z_2',z_3')|\\ \leq\ & \left[\frac{\partial g}{\partial z_1}\right]|z_1-z_1'|+\left[\frac{\partial g}{\partial z_2}\right]|z_2-z_2'|+\left[\frac{\partial g}{\partial z_3}\right]|z_3-z_3'|\\ \leq\ & 3\max\left\{\left[\frac{\partial g}{\partial z_1}\right],\left[\frac{\partial g}{\partial z_2}\right],\left[\frac{\partial g}{\partial z_3}\right]\right\}|z-z'|_\infty,\end{aligned}$$

where $[\partial g/\partial z_k]$ denotes the maximum of $|\partial g/\partial z_k|$ on C_0.

The matrix norm $|\ |_\infty$ of a matrix $A=(a_{ij})$ is given by

$$|A|_\infty=\max_i\left\{\sum_j|a_{ij}|\right\},$$

see for instance [9]. Therefore for our $f=(f_1,f_2,f_3)$ with $z=(z_1,z_2,z_3)$ we obtain

$$\begin{aligned}|Df(z)-Df(z')|_\infty &= \max_{1\leq i\leq 3}\left\{\sum_{j=1}^{3}\left|\frac{\partial f_i}{\partial z_j}(z)-\frac{\partial f_i}{\partial z_j}(z')\right|\right\}\\ &\leq \max_{1\leq i\leq 3}\left\{3\max_{1\leq j\leq 3}\left|\frac{\partial f_i}{\partial z_j}(z)-\frac{\partial f_i}{\partial z_j}(z')\right|\right\}\\ &\leq 3\max_{1\leq i,j\leq 3}\left\{\left|\frac{\partial f_i}{\partial z_j}(z)-\frac{\partial f_i}{\partial z_j}(z')\right|\right\}\\ &\leq 9\max_{1\leq i,j,h\leq 3}\left[\frac{\partial^2 f_i}{\partial z_j\partial z_h}\right]|z-z'|_\infty.\end{aligned}$$

Consequently, for finding an estimation for γ, we must compute the second partial derivative of the functions f_i, and after we must bound them in C_0. Thus we have

$$\begin{aligned}\frac{\partial^2 f_1}{(\partial t^+)^2} &= \frac{1}{2500}e^{-19t^+/50}\left(-(1900Y+2139)\cos t^+ +(1900-2139Y)\sin t^+\right),\\ \frac{\partial^2 f_1}{\partial t^+\partial Y} &= \frac{1}{50}e^{-19t^+/50}\left(50\cos t^+-19\sin t^+\right),\\ \frac{\partial^2 f_2}{(\partial t^-)^2} &= -\frac{1}{375}e^{-t^-/5}\left(10(100Y-59)\cos t^- +6(400Y-67)\sin t^-\right),\\ \frac{\partial^2 f_2}{\partial t^-\partial Y} &= \frac{4}{3}e^{-t^-/5}\left(\sin t^- -5\cos t^-\right),\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 f_3}{(\partial t^+)^2} &= \frac{1}{2500} e^{-19t^+/50} \left((1900 - 2139Y)\cos t^+ + (1900Y + 2139)\sin t^+ \right), \\
\frac{\partial^2 f_3}{\partial t^+ \partial Y} &= -\frac{1}{50} e^{-19t^+/50} \left(19\cos t^+ + 50 \sin t^+ \right), \\
\frac{\partial^2 f_3}{(\partial t^-)^2} &= \frac{13}{9375} e^{-t^-/5} \left(5(50Y + 29)\cos t^- + 6(58 - 225Y)\sin t^- \right), \\
\frac{\partial^2 f_3}{\partial t^- \partial Y} &= \frac{26}{75} e^{-t^-/5} \left(5\cos t^- + 2\sin t^- \right).
\end{aligned}$$

All the second partial derivatives which do not appear explicitly previously are zero.

We note that all the second partial derivatives can be written in the form

$$\frac{\partial^2 f_k}{\partial z_i \partial z_j} = \sum_l c_l g_l(z_k),$$

where the functions g_l are linear combinations of the functions sinus, cosinus and exponential. So we can bound such partial derivatives taking absolute value of the coefficients c_l and bounding the sinus and cosinus by 1. Then the bounds for these derivatives are

$$\begin{aligned}
\left| \frac{\partial^2 f_1}{(\partial t^+)^2} \right| &\le \frac{4039}{2500} e^{-19t^+/50}(Y + 1), \\
\left| \frac{\partial^2 f_1}{\partial t^+ \partial Y} \right| &\le \frac{69}{50} e^{-19t^+/50}, \\
\left| \frac{\partial^2 f_2}{(\partial t^-)^2} \right| &\le \frac{8}{375} e^{-t^-/5}(425Y + 124), \\
\left| \frac{\partial^2 f_2}{\partial t^- \partial Y} \right| &\le 8 e^{-t^-/5}, \\
\left| \frac{\partial^2 f_3}{(\partial t^+)^2} \right| &\le \frac{4039}{2500} e^{-19t^+/50}(Y + 1), \\
\left| \frac{\partial^2 f_3}{\partial t^+ \partial Y} \right| &\le \frac{69}{50} e^{-19t^+/50}, \\
\left| \frac{\partial^2 f_3}{(\partial t^-)^2} \right| &\le \frac{13}{9375} e^{-t^-/5}(1600Y + 493), \\
\left| \frac{\partial^2 f_3}{\partial t^- \partial Y} \right| &\le \frac{182}{75} e^{-t^-/5}.
\end{aligned}$$

Now we shall apply Theorem 2 to C_0^1 for proving that in this convex set the unique solution of system (2) is the one near (t_1^+, t_1^-, Y_1). As the upper bounds functions that we have obtained are decreasing functions in the variables t^+, t^- but increasing in Y, their maxima in C_0^1 take place when $t^+ = 1.48$, $t^- = 3.45$ and

$Y = 1.69$. Therefore we get

$$\left|\frac{\partial^2 f_1}{(\partial t^+)^2}\right| \leq 2.47651, \quad \left|\frac{\partial^2 f_1}{\partial t^+ \partial Y}\right| \leq 0.78638, \quad \left|\frac{\partial^2 f_2}{(\partial t^-)^2}\right| \leq 9.01232,$$
$$\left|\frac{\partial^2 f_2}{\partial t^- \partial Y}\right| \leq 4.01261, \quad \left|\frac{\partial^2 f_3}{(\partial t^+)^2}\right| \leq 2.47651, \quad \left|\frac{\partial^2 f_3}{\partial t^+ \partial Y}\right| \leq 0.78638,$$
$$\left|\frac{\partial^2 f_3}{(\partial t^-)^2}\right| \leq 2.22358, \quad \left|\frac{\partial^2 f_3}{\partial t^- \partial Y}\right| \leq 1.21716.$$

Hence an upper bound for all the second partial derivatives is 9.02. Then, from

$$|Df(z) - Df(z')|_\infty \leq 9 \max_{1\leq i,j,h\leq 3} \left[\frac{\partial^2 f_i}{\partial z_j \partial z_h}\right] |z - z'|_\infty,$$

if we take $\gamma = 82 \geq 9 \cdot 9.02$, we get that this γ satisfies assumption (a) of Theorem 2.

Taking $z_0 = (t_1^+, t_1^-, Y_1)$ we have

$$Df(z_0) = \begin{pmatrix} -0.86606 & 0 & 0.56639 \\ 0 & 4.57374 & 1.01822 \\ -0.81529 & -1.34517 & 0.29123 \end{pmatrix},$$

where the computations have been made with an accuracy of 10^{-20}, but here we present the numbers only with 5 decimals. The inverse of $Df(z_0)$ is

$$Df(z_0)^{-1} \approx B = \begin{pmatrix} -11.86035.. & 3.34468.. & 11.37230.. \\ 3.64432.. & -0.91992.. & -3.87124.. \\ -16.36989.. & 5.11429.. & 17.38917.. \end{pmatrix},$$

again the computations have been made with an accuracy of 10^{-20}, but we only present the result only with 5 decimals. To control the error in the computation of $Df(t_1^+, t_1^-, Y_1)^{-1}$ we will use the following lemma, for a proof see Lemma 4.4.14 of [10].

Lemma 3. *Let A be an $n \times n$ real matrix and B an approximation of A^{-1}. Then*

$$|A^{-1}| \leq \frac{|B|}{1 - |Id - AB|}.$$

The matrix $Df(t_1^+, t_1^-, Y_1)B$ satisfies that $|Id - Df(t_1^+, t_1^-, Y_1)B|_\infty \leq 10^{-18}$ (doing the computations with an accuracy of 10^{-20}), and since $|B|_\infty < 38.874$ then, by lemma 3, $|Df(z_0)^{-1}|_\infty < 38.874$.

Now, since

$$f(z_0) = \begin{pmatrix} 1.65139.. \cdot 10^{-16} \\ -3.87795.. \cdot 10^{-17} \\ 1.45690.. \cdot 10^{-16} \end{pmatrix},$$

we have

$$|Df(z_0)^{-1} f(z_0)|_\infty \leq |Df(z_0)^{-1}|_\infty |f(z_0)|_\infty \leq 6.45 \cdot 10^{-15}.$$

So, taking $\alpha = 6.5 \cdot 10^{-15}$ and $\beta = 39$, the assumptions (b) and (c) of Theorem 2 are satisfied. Since $h = 2.07.. \cdot 10^{-11}$, $r_1 = 0.00062539..$ and $r_2 = 6.50000.. \cdot 10^{-15}$, it follows that $h \leq 1/2$ and $B_{r_1}(z_0) \subset C_0$. Hence, from Theorem 2 the function $f(z)$ has a zero $\overline{z_0}$ in $C_0 \cap B_{r_2}(z_0)$. Consequently, the non–smooth piecewise linear differential system (1) has a limit cycle $\gamma(t) = (x(t), y(t))$ such that $\gamma(0) = \overline{z_0}$.

The proof of the existence of the other two limit cycles passing near the points (t_k^+, t_k^-, Y_k) for $k = 2, 3$ is completely similar. We only provide the values of (α, β, γ), i.e.

$$(8 \cdot 10^{-15}, 57, 49) \quad \text{and} \quad (8 \cdot 10^{-15}, 73, 69)$$

for $k = 2$ and $k = 3$, respectively. This completes the proof of Theorem 1.

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[1] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

E-mail address: jllibre@mat.uab.cat

[2] E.T.S. Ingenieros, Camino de los Descubrimientos, 41092 Sevilla, Spain

E-mail address: eponcem@us.es