Three particles in a finite volume

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K. Polejaeva and A. Rusetsky

Helmholtz–Institut für Strahlen– und Kernphysik
and Bethe Center for Theoretical Physics, Universität Bonn
D–53115 Bonn, Germany

Abstract

Within the non-relativistic potential scattering theory, we derive a generalized version of the Lüscher formula, which includes three-particle inelastic channels. Faddeev equations in a finite volume are discussed in detail. It is proved that, even in the presence of the three-particle intermediate states, the discrete spectrum in a finite box is determined by the infinite-volume elements of the scattering S-matrix up to corrections, exponentially suppressed at large volumes.

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1 Introduction

The nature of the Roper resonance $N(1440)$ has been an open question for decades. It is quite unnatural – at least, from the point of view of the quark models [1, 2] – that its mass turns out to be lower than that of the negative-parity ground state $N(1535)$. Different phenomenological approaches have been employed so far to explain such a level ordering. For example, $N(1440)$ was assumed to be a hybrid state with excited gluon field configurations [3, 4], a breathing mode of the ground state [5], and a five-quark (meson-baryon) state [6]. The properties of the Roper resonance have been studied within the Skyrme model [7] and the bag model [8].

Up to now, numerous calculations of the excited spectrum of the nucleon [9–24] (see *e.g.*, ref. [25] for a recent state-of-art review) have not resolved the issue. In particular, a reverse level ordering (the same as in the constituent quark model) between $N(1440)$ and $N(1535)$ has been reported in some lattice calculations [14], albeit the situation there is far from being clear. It should be also pointed out that a large chiral curvature for the Roper resonance mass is not compatible with the findings of ref. [26], where the dependence of the Roper mass on the pion mass was investigated within the framework of Chiral Perturbation Theory. According to these findings, the level crossing between $N(1440)$ and $N(1535)$ states, emerging as a result of chiral extrapolation to the small quark masses, does not constitute to a plausible scenario.

There is, however, one issue that has not been addressed in all above investigations so far. The $N(1440)$ is a resonance and not a stable state which would correspond to an isolated energy level in lattice simulations. This, in particular, means that the energy levels measured on the lattice will be volume-dependent and the true resonance pole position should be extracted from the volume-dependent spectrum.

The case of the elastic low-lying resonances on the lattice has been investigated in detail. Namely, the Lüscher formula [27] enables one to uniquely relate the discrete energy levels in a finite box to the elastic scattering phase shift in the infinite volume, measured at the same energy. This eventually opens the way for the extraction of the parameters of the elastic resonances – their masses and widths – in the lattice QCD (for illustration see, *e.g.*, refs. [28, 29]).

The case of the inelastic resonances, however, is more complicated. Albeit the Lüscher approach can be straightforwardly generalized to the case of the coupled two-particle channels [30–32], the disentanglement of physical observables becomes a more delicate affair, since there is more than one observable at a single energy. In order to circumvent this problem, in refs. [32, 33] the use of the twisted boundary conditions or asymmetric boxes was advocated for the $\pi\pi - K\bar{K}$ coupled-channel system. Moreover, in ref. [33] it has been argued that, using unitarized ChPT (UChPT) in a finite volume, it is possible first to directly fit the parameters of the chiral potential to the energy spectrum measured on the lattice and eventually to determine the physical observables (the phase shifts, resonance parameters, etc) from the solution of the scattering equations. The feasibility of such a procedure was demonstrated in the example with synthetic data [33]. For the further applications of this method, see refs. [34, 35].

Inelastic resonances, which have a significant decay rate into the three and more particle final states, have received much less attention in the literature so far. The Roper resonance $N(1440)$, which decays into the inelastic three-particle channels with an approximately 40 %

probability, is an example of such a system. *A priori,* one may expect significant finite-volume effects in this decay, which can not be evaluated by using the standard Lüscher approach. For this reason, at the moment it is not clear, whether the reverse level ordering between $N(1440)$ and $N(1535)$ in lattice simulations, mentioned above, can not be at least partially attributed to these finite-volume effects. Putting it differently, one may ask, whether the effect still persists for the true resonance positions in the infinite volume. We therefore conclude that it is highly desirable to construct a framework that will allow one to systematically calculate the finite-volume effects, coming from the tree-body final states.

Formulating a counterpart of the Lüscher approach in a three-body case represents a major challenge. For this reason, at the first step, we want to simplify the problem as much as possible. Namely, we consider a non-relativistic quantum-mechanical model with coupled two-particle and three-particle channels. Multi-particle channels (4 and more particles) are neglected from the beginning. Using these approximations, in particular, the technical complications, related to the necessity of Lorentz-boosting the two-particle sub-systems in the three-particle state to their respective center-of-mass (CM) frames can be avoided. Moreover, using the potential model instead of the effective field theory (EFT) framework, we avoid the discussion of the proper counting rules for the multi-particle intermediate states, which are generated by loops. The core of the problem, which consists in the study of the finite-volume effects coming from the three-particle intermediate states, remains however unaffected by the above approximations. At the next step, we plan to carry out a full-fledged investigation of the problem.

The central issue addressed in our study, can be formulated as follows. In case of the elastic two-body scattering, the Lüscher formula relates the discrete spectrum in a finite box to the scattering S-matrix element in the infinite volume, up to the corrections that are exponentially suppressed in the box size. Putting it differently, this is a relation between the *observables* in a finite and in the infinite volumes: the details of the potential do not matter if the box size is much larger than the typical radius of interaction. It is intuitively clear that the statement must remain valid even if there is a contribution from the three-particle intermediate state – as far as the size of the box remains much larger than the scales characterizing the interactions. In our paper, we prove this statement for the model described in the previous paragraph. It is expected that this proof can be extended beyond this particular model.

The layout of the present paper is as follows. In section 2 we discuss the quantum-mechanical model which describes the scattering in the coupled two- and three- particle channels. Considering the same model in a finite volume, we derive an equation which predicts the volume-dependent energy levels. Further, in section 3 we re-derive the Lüscher formula for the two-body scattering applying a new method, which can be used with minor modifications in the 3-body case as well. This is done in section 4, in which we derive the three-body analog of the Lüscher formula. Section 5 contains our conclusions.

2 Three-body problem in a finite volume

2.1 The model in the infinite volume

In this section, we describe a model which will be later used to study the finite-volume effects in the three-body sector. In order to make the presentation self-contained, below we explicitly display the standard formulas in case of the infinite volume. We consider a non-relativistic quantum-mechanical system of three spinless non-identical particles with the masses m_α, $\alpha = 1,2,3$. The Hamiltonian of the model is given by a sum of a free Hamiltonian $\mathbf{H}_0$, pair interaction Hamiltonian $\mathbf{H}_{2\to 2}$ and the Hamiltonian $\mathbf{H}_{2\to 3}$ that describes the transition from two- to three-particle state $1+2\to 1+2+3$. In order to simplify the following expressions, we do not include the three-particle force into the Hamiltonian. As shown later, the induced three-particle force will anyway emerge from the two-particle interactions, so one could have indeed added it from the beginning without much effort. The explicit expression of the full Hamiltonian in terms of the creation/annihilation operators is given by

$$
\begin{aligned}
\mathbf{H} &= \mathbf{H}_0 + \mathbf{H}_{2\to 2} + \mathbf{H}_{2\to 3} \doteq \mathbf{H}_0 + \mathbf{H}_{\mathsf{I}}\,, \\
\mathbf{H}_0 &= \sum_{\alpha=1}^{3}\sum_{\mathbf{k}}\left(m_\alpha + \frac{\mathbf{k}^2}{2m_\alpha}\right) a_\alpha(\mathbf{k}) a_\alpha^\dagger(\mathbf{k})\,, \\
\mathbf{H}_{2\to 2} &= \sum_{(\alpha\beta)=(12),(23),(13)}^{3}\;\sum_{\mathbf{k}_1\cdots\mathbf{k}_4}(2\pi)^3\delta^3(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\,V_{\alpha\beta}(\mathbf{k}_1,\mathbf{k}_2;\mathbf{k}_3,\mathbf{k}_4) \\
&\times\; a_\alpha(\mathbf{k}_1)a_\beta(\mathbf{k}_2)a_\alpha^\dagger(\mathbf{k}_3)a_\beta^\dagger(\mathbf{k}_4)\,, \\
\mathbf{H}_{2\to 3} &= \sum_{\mathbf{k}_1\cdots\mathbf{k}_5}(2\pi)^3\delta^3(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4-\mathbf{k}_5) \\
&\times \left\{\Gamma(\mathbf{k}_1,\mathbf{k}_2;\mathbf{k}_3,\mathbf{k}_4,\mathbf{k}_5)\,a_1(\mathbf{k}_1)a_2(\mathbf{k}_2)a_1^\dagger(\mathbf{k}_3)a_2^\dagger(\mathbf{k}_4)a_3^\dagger(\mathbf{k}_5) + \mathsf{h.c.}\right\}, \qquad (1)
\end{aligned}
$$

where $a_\alpha(\mathbf{k})$ and $a_\alpha^\dagger(\mathbf{k})$ denote the annihilation and creation operators for the particle α, respectively. Further, $V_{12} \doteq V_3$, $V_{23} \doteq V_1$, $V_{31} = V_2$ denote the pair potentials, so that $\mathbf{H}_{2\to 2} = \mathbf{V}_1+\mathbf{V}_2+\mathbf{V}_3 = \sum_{\alpha=1}^{3}\mathbf{V}_\alpha$. Finally, the vertex Γ describes the transition 1+2 $\to$ 1+2+3.

Next, we introduce the free and full resolvents, which are defined by

$$
\mathbf{G}_0(z) = \frac{1}{\mathbf{H}_0 - z - i0}\,, \qquad \mathbf{G}(z) = \frac{1}{\mathbf{H} - z - i0}\,. \qquad (2)
$$

The scattering matrix, which is expressed through the resolvents as

$$
\mathbf{G}(z) = \mathbf{G}_0(z) + \mathbf{G}_0(z)\mathbf{T}(z)\mathbf{G}_0(z)\,, \qquad (3)
$$

obeys the Lippmann-Schwinger (LS) equation[1]

$$\mathbf{T}(z) = (-\mathbf{H}_\mathsf{I}) + (-\mathbf{H}_\mathsf{I})\mathbf{G}_0(z)\mathbf{T}(z)\,. \tag{4}$$

In this paper, we are primarily aimed at the modeling of the πN scattering in the Roper resonance region – that is, in the presence of an open three-particle channel. To this end, we consider the scattering amplitude for the process $1+2 \to 1+2$. Introducing the projection operators $\mathbf{P}$ and $\mathbf{Q} = \mathbb{I} - \mathbf{P}$ (where $\mathbb{I}$ stands for the unit operator), which project onto the two-particle state $|12\rangle$ and the three-particle state $|123\rangle$, respectively, we obtain the equation for the effective two-body operator $\mathbf{T}_\mathsf{P}(z) = \mathbf{P}\mathbf{T}(z)\mathbf{P}$

$$\mathbf{T}_\mathsf{P}(z) = \mathbf{W}(z) + \mathbf{W}(z)\mathbf{G}_\mathsf{P}(z)\mathbf{T}_\mathsf{P}(z)\,, \qquad \mathbf{W}(z) = (-\mathbf{H}_{2\to 2}) + \mathbf{H}_{2\to 3}\mathbf{G}_3(z)(\mathbf{H}_{2\to 3})^\dagger\,, \tag{5}$$

where the three-particle Green's function obeys the equation

$$\mathbf{G}_3(z) = \mathbf{G}_\mathsf{Q} + \mathbf{G}_\mathsf{Q}(-\mathbf{H}_{2\to 2})\mathbf{G}_3(z)\,, \tag{6}$$

and the following notations are used

$$\mathbf{G}_\mathsf{P} = \mathbf{P}\mathbf{G}_0(z)\mathbf{P}\,, \qquad \mathbf{G}_\mathsf{Q} = \mathbf{Q}\mathbf{G}_0(z)\mathbf{Q}\,. \tag{7}$$

Note that the effective one-channel potential $\mathbf{W}(z)$ becomes non-Hermitian above the three-particle threshold $z > m_1 + m_2 + m_3$.

Next, we define the three-particle scattering amplitude $\mathbf{R}(z)$ through

$$\mathbf{G}_3(z) = \mathbf{G}_\mathsf{Q} + \mathbf{G}_\mathsf{Q}\mathbf{R}(z)\mathbf{G}_\mathsf{Q}\,. \tag{8}$$

The quantity $\mathbf{R}(z)$ can be expressed as

$$\mathbf{R}(z) = \sum_{\alpha,\beta=1}^{3} \mathbf{M}_{\alpha\beta}(z)\,, \tag{9}$$

where $\mathbf{M}_{\alpha\beta}(z)$ obeys Faddeev equations (see, *e.g.*, [36])

$$\mathbf{M}_{\alpha\beta}(z) = \delta_{\alpha\beta}\mathbf{T}_\alpha(z) + \mathbf{T}_\alpha(z)\mathbf{G}_\mathsf{Q}(z)\sum_{\gamma=1}^{3}(1-\delta_{\alpha\gamma})\mathbf{M}_{\gamma\beta}(z)\,, \tag{10}$$

and $\mathbf{T}_\alpha(z)$ denote the scattering amplitudes in the channel α

$$\mathbf{T}_\alpha(z) = (-\mathbf{V}_\alpha) + (-\mathbf{V}_\alpha)\mathbf{G}_\mathsf{Q}(z)\mathbf{T}_\alpha(z)\,. \tag{11}$$

In order to write down these equations explicitly in momentum space, it is useful to work in Jacobi basis

$$\mathbf{P} = \mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3\,, \quad \mathbf{p}_\alpha = \mathbf{k}_\alpha\,, \quad \mathbf{q}_\alpha = \frac{m_\gamma \mathbf{k}_\beta - m_\beta \mathbf{k}_\gamma}{m_\gamma + m_\beta}\,, \qquad (\alpha\beta\gamma) = (123), (231), (312)\,. \tag{12}$$

[1]Note that the sign convention in the LS equation below coincides with the one in the field theory and is opposite to the one usually adopted in the potential scattering theory.

Separating the center-of-mass (CM) motion from the matrix elements

$$
\begin{aligned}
\langle \mathbf{k}_1'\mathbf{k}_2'\mathbf{k}_3'|\mathbf{M}_{\alpha\beta}(z)|\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\rangle &= (2\pi)^3\delta^3(\mathbf{k}_1'+\mathbf{k}_2'+\mathbf{k}_3'-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\langle \mathbf{p}_\alpha'\mathbf{q}_\alpha'|\mathbf{m}_{\alpha\beta}(z)|\mathbf{p}_\beta\mathbf{q}_\beta\rangle\,,\\
\langle \mathbf{k}_1'\mathbf{k}_2'\mathbf{k}_3'|\mathbf{G}_{\mathsf{Q}}(z)|\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\rangle &= (2\pi)^3\delta^3(\mathbf{k}_1'+\mathbf{k}_2'+\mathbf{k}_3'-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\\
&\times\ \frac{(2\pi)^3\delta^3(\mathbf{p}_\alpha'-\mathbf{p}_\alpha)\,(2\pi)^3\delta^3(\mathbf{q}_\alpha'-\mathbf{q}_\alpha)}{M+\dfrac{\mathbf{p}_\alpha^2}{2M_\alpha}+\dfrac{\mathbf{q}_\alpha^2}{2\mu_\alpha}-z-i0}\,,\\
\langle \mathbf{k}_1'\mathbf{k}_2'\mathbf{k}_3'|\mathbf{T}_\alpha(z)|\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\rangle &= (2\pi)^3\delta^3(\mathbf{k}_1'+\mathbf{k}_2'+\mathbf{k}_3'-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)(2\pi)^3\delta^3(\mathbf{p}_\alpha'-\mathbf{p}_\alpha)\\
&\times\ \langle \mathbf{q}_\alpha'|\boldsymbol{\tau}_\alpha\left(z-m_\alpha-\frac{\mathbf{p}_\alpha^2}{2M_\alpha}\right)|\mathbf{q}_\alpha\rangle\,, \qquad (13)
\end{aligned}
$$

where $\boldsymbol{\tau}_\alpha(z)$ is the two-body scattering amplitude, and

$$
M = m_\alpha+m_\beta+m_\gamma\,,\qquad M_\alpha=\frac{m_\alpha(m_\beta+m_\gamma)}{m_\alpha+m_\beta+m_\gamma}\,,\qquad \mu_\alpha=\frac{m_\beta m_\gamma}{m_\beta+m_\gamma}\,, \qquad (14)
$$

we finally obtain

$$
\begin{aligned}
&\langle \mathbf{p}_\alpha'\mathbf{q}_\alpha'|\mathbf{m}_{\alpha\beta}(z)|\mathbf{p}_\beta\mathbf{q}_\beta\rangle = \delta_{\alpha\beta}(2\pi)^3\delta^3(\mathbf{p}_\alpha'-\mathbf{p}_\alpha)\langle \mathbf{q}_\alpha'|\boldsymbol{\tau}_\alpha\left(z-m_\alpha-\frac{\mathbf{p}_\alpha^2}{2M_\alpha}\right)|\mathbf{q}_\alpha\rangle\\
&+\ \sum_{\gamma=1}^{3}(1-\delta_{\alpha\gamma})\int\frac{d^3\mathbf{p}_\alpha''}{(2\pi)^3}\frac{d^3\mathbf{q}_\alpha''}{(2\pi)^3}\frac{d^3\mathbf{p}_\gamma''}{(2\pi)^3}\frac{d^3\mathbf{q}_\gamma''}{(2\pi)^3}(2\pi)^3\delta^3(\mathbf{p}_\alpha'-\mathbf{p}_\alpha'')\langle \mathbf{q}_\alpha'|\boldsymbol{\tau}_\alpha\left(z-m_\alpha-\frac{(\mathbf{p}_\alpha')^2}{2M_\alpha}\right)|\mathbf{q}_\alpha''\rangle\\
&\times\ \frac{(2\pi)^3\delta^3(\mathbf{p}_\alpha''-\mathbf{p}_\alpha''(\mathbf{p}_\gamma'',\mathbf{q}_\gamma''))(2\pi)^3\delta^3(\mathbf{q}_\alpha''-\mathbf{q}_\alpha''(\mathbf{p}_\gamma'',\mathbf{q}_\gamma''))}{M+\dfrac{(\mathbf{p}_\alpha'')^2}{2M_\alpha}+\dfrac{(\mathbf{q}_\alpha'')^2}{2\mu_\alpha}-z-i0}\langle \mathbf{p}_\gamma''\mathbf{q}_\gamma''|\mathbf{m}_{\gamma\beta}(z)|\mathbf{p}_\beta\mathbf{q}_\beta\rangle\,. \qquad (15)
\end{aligned}
$$

The relations between the momenta $\mathbf{p}_\alpha,\mathbf{p}_\beta,\mathbf{p}_\gamma$ and $\mathbf{q}_\alpha,\mathbf{q}_\beta,\mathbf{q}_\gamma$ for different channels are given by

$$
\begin{aligned}
\mathbf{p}_\alpha &= -\frac{m_\alpha}{m_\alpha+m_\beta}\mathbf{p}_\gamma+\mathbf{q}_\gamma\,, \qquad \mathbf{q}_\alpha=-\frac{m_\beta(m_\alpha+m_\beta+m_\gamma)}{(m_\alpha+m_\beta)(m_\beta+m_\gamma)}\mathbf{p}_\gamma-\frac{m_\gamma}{m_\beta+m_\gamma}\mathbf{q}_\gamma\,,\\
\mathbf{p}_\alpha &= -\frac{m_\alpha}{m_\alpha+m_\gamma}\mathbf{p}_\beta-\mathbf{q}_\beta\,, \qquad \mathbf{q}_\alpha=\frac{m_\gamma(m_\alpha+m_\beta+m_\gamma)}{(m_\alpha+m_\gamma)(m_\beta+m_\gamma)}\mathbf{p}_\beta-\frac{m_\beta}{m_\beta+m_\gamma}\mathbf{q}_\beta\,. \qquad (16)
\end{aligned}
$$

Here, $(\alpha\beta\gamma)=(123),\ (231),\ (312)$.

Further, one may express the effective one-channel potential through the solution of Faddeev equations. Removing first the CM motion, one gets

$$
\langle \mathbf{k}_1'\mathbf{k}_2'|\mathbf{W}(z)|\mathbf{k}_1\mathbf{k}_2\rangle = (2\pi)^3\delta^3(\mathbf{k}_1'+\mathbf{k}_2'-\mathbf{k}_1-\mathbf{k}_2)\langle \mathbf{q}_3'|\mathbf{w}(z)|\mathbf{q}_3\rangle\,, \qquad (17)
$$

with

$$\langle \mathbf{q}'|\mathbf{w}(z)|\mathbf{q}\rangle = -\bar{V}_3(\mathbf{q}',\mathbf{q}) + \sum_{\alpha,\beta=1}^{3} \int \frac{d^3\mathbf{p}''_\alpha}{(2\pi)^3}\frac{d^3\mathbf{q}''_\alpha}{(2\pi)^3}\frac{d^3\mathbf{p}'''_\beta}{(2\pi)^3}\frac{d^3\mathbf{q}'''_\beta}{(2\pi)^3}$$
$$\times \; \bar{\Gamma}_\alpha(\mathbf{q}';\mathbf{p}''_\alpha\mathbf{q}''_\alpha)\langle \mathbf{p}''_\alpha\mathbf{q}''_\alpha|\mathbf{g}_{3,\alpha\beta}(z)|\mathbf{p}'''_\beta\mathbf{q}'''_\beta\rangle(\bar{\Gamma}_\beta(\mathbf{q};\mathbf{p}'''_\beta\mathbf{q}'''_\beta))^* \,. \tag{18}$$

In the above expressions, the following notations are used

$$\bar{\Gamma}_\alpha(\mathbf{q};\mathbf{p}_\alpha\mathbf{q}_\alpha) = \Gamma\Bigg(\mathbf{q},-\mathbf{q};\mathbf{k}_\alpha=\mathbf{p}_\alpha,\mathbf{k}_\beta=\mathbf{q}_\alpha-\frac{m_\beta}{m_\beta+m_\gamma}\,\mathbf{p}_\alpha,\mathbf{k}_\gamma=-\mathbf{q}_\alpha-\frac{m_\gamma}{m_\beta+m_\gamma}\,\mathbf{p}_\alpha\Bigg)\,,$$
$$\bar{V}_\alpha(\mathbf{q}',\mathbf{q}) = V_\alpha(\mathbf{q}',-\mathbf{q}';\mathbf{q},-\mathbf{q})\,, \tag{19}$$

and

$$\langle \mathbf{p}''_\alpha\mathbf{q}''_\alpha|\mathbf{g}_{3,\alpha\beta}(z)|\mathbf{p}'''_\beta\mathbf{q}'''_\beta\rangle = \delta_{\alpha3}\delta_{\beta3}\frac{(2\pi)^3\delta^3(\mathbf{p}''_3-\mathbf{p}'''_3)(2\pi)^3\delta^3(\mathbf{q}''_3-\mathbf{q}'''_3)}{M+\dfrac{(\mathbf{p}''_3)^2}{2M_3}+\dfrac{(\mathbf{q}''_3)^2}{2\mu_3}-z-i0}$$
$$+ \; \frac{1}{M+\dfrac{(\mathbf{p}''_\alpha)^2}{2M_\alpha}+\dfrac{(\mathbf{q}''_\alpha)^2}{2\mu_\alpha}-z-i0}\,\langle \mathbf{p}''_\alpha\mathbf{q}''_\alpha|\mathbf{m}_{\alpha\beta}(z)|\mathbf{p}'''_\beta\mathbf{q}'''_\beta\rangle\,\frac{1}{M+\dfrac{(\mathbf{p}'''_\beta)^2}{2M_\beta}+\dfrac{(\mathbf{q}'''_\beta)^2}{2\mu_\beta}-z-i0} \tag{20}$$

Finally, the LS equation for the two-body scattering matrix $\mathbf{T}_\mathsf{P}(z)$ after removing the CM motion

$$\langle \mathbf{k}'_1\mathbf{k}'_2|\mathbf{T}_\mathsf{P}(z)|\mathbf{k}_1\mathbf{k}_2\rangle = (2\pi)^3\delta^3(\mathbf{k}'_1+\mathbf{k}'_2-\mathbf{k}_1-\mathbf{k}_2)\langle \mathbf{q}'_3|\mathbf{t}_\mathsf{P}(z)|\mathbf{q}_3\rangle\,, \tag{21}$$

takes the form

$$\langle \mathbf{q}'|\mathbf{t}_\mathsf{P}(z)|\mathbf{q}\rangle = \langle \mathbf{q}'|\mathbf{w}(z)|\mathbf{q}\rangle + \int \frac{d^3\mathbf{q}''}{(2\pi)^3}\,\frac{\langle \mathbf{q}'|\mathbf{w}(z)|\mathbf{q}''\rangle\,\langle \mathbf{q}''|\mathbf{t}_\mathsf{P}(z)|\mathbf{q}\rangle}{m_1+m_2+\dfrac{(\mathbf{q}'')^2}{2\mu_3}-z-i0}\,. \tag{22}$$

The above formulas simplify considerably, if the pair potentials have separable form. For completeness, in Appendix A we list the pertinent expressions in the separable model for the case of three identical particles.

2.2 The model in a finite volume

Now, let us put the system described by the Hamiltonian in eq. (1), in a finite cubic box of a size L. Assuming periodic boundary conditions, the momenta of all free particles take discrete values $\mathbf{k}_\alpha = 2\pi\mathbf{n}_\alpha/L\,,\ \alpha=1,2,3$ and $\mathbf{n}_\alpha\in\mathbb{Z}^3$. The only difference between the infinite-volume and finite-volume cases consists in replacing the momentum-space integrals in all scattering equations by the sums over the discrete momenta.

The finite-volume counterpart of the LS equation with the effective two-body potential in eq. (22) is given by

$$\langle \mathbf{q}'|\mathbf{t}_{\mathsf{P}}^L(z)|\mathbf{q}\rangle = \langle \mathbf{q}'|\mathbf{w}^L(z)|\mathbf{q}\rangle + \frac{1}{L^3}\sum_{\mathbf{q}''} \frac{\langle \mathbf{q}'|\mathbf{w}^L(z)|\mathbf{q}''\rangle\,\langle \mathbf{q}''|\mathbf{t}_{\mathsf{P}}^L(z)|\mathbf{q}\rangle}{m_1+m_2+\dfrac{(\mathbf{q}'')^2}{2\mu_3}-z}\,. \tag{23}$$

Here and below, we attach the superscript "L" to the quantities defined in a finite volume. Note also that in the CM frame, the relative three momentum in the intermediate state is given by $\mathbf{k}_1''=-\mathbf{k}_2''=\mathbf{q}''$. This means, that the summation momentum $\mathbf{q}''$ takes the discrete values $\mathbf{q}''=2\pi\mathbf{n}/L\,,\ \mathbf{n}\in\mathbb{Z}^3$.

The scattering amplitude $\mathbf{t}_{\mathsf{P}}^L(z)$ has an infinite tower of poles, corresponding to the discrete energy spectrum of a system in a finite box. It can be shown that the locations of these poles are determined by the Lüscher formula. Indeed, performing the partial-wave expansion in eq. (23)

$$\begin{aligned}
\langle \mathbf{q}'|\mathbf{t}_{\mathsf{P}}^L(z)|\mathbf{q}\rangle &= 4\pi\sum_{l'm',lm} Y_{l'm'}(\hat{\mathbf{q}}')t^L_{l'm',lm}(q',q;z)Y^*_{lm}(\hat{\mathbf{q}})\,,\\
\langle \mathbf{q}'|\mathbf{w}^L(z)|\mathbf{q}\rangle &= 4\pi\sum_{l'm',lm} Y_{l'm'}(\hat{\mathbf{q}}')w^L_{l'm',lm}(q',q;z)Y^*_{lm}(\hat{\mathbf{q}})\,,
\end{aligned} \tag{24}$$

where $\hat{\mathbf{q}}$ denotes the unit vector in the direction of $\mathbf{q}$ and $Y_{lm}(\hat{\mathbf{q}})$ stands for the spherical function. The position of the poles is determined by the equation

$$\begin{aligned}
\det\mathcal{D} &= 0\,,\\
\mathcal{D}_{l'm',lm} &= \delta_{l'l}\delta_{m'm} - \sum_{l''m''}\frac{\mu_3 p}{2\pi}K^L_{l'm',l''m''}(p,p;z(p))\mathcal{M}_{l''m'',lm}(\nu)\,,\\
\mathcal{M}_{l'm',lm}(\nu) &= \frac{(-)^{l'}}{\pi^{3/2}}\sum_{j=|l'-l|}^{l'+l}\sum_{s=-j}^{j}\frac{i^j}{\nu^{j+1}}Z_{js}(1;\nu^2)C_{l'm',js,lm}\,.
\end{aligned} \tag{25}$$

Here,

$$\nu=\frac{pL}{2\pi}\,, \tag{26}$$

the quantity $Z_{js}(1;\nu^2)$ denotes the Lüscher zeta-function, and the symbols $C_{l'm',js,lm}$ are given by (see, *e.g.* [27])

$$C_{l'm',js,lm}=(-)^m i^{l'-j+l}\sqrt{(2l'+1)(2j+1)(2l+1)}\begin{pmatrix} l' & j & l\\ m' & s & -m\end{pmatrix}\begin{pmatrix} l' & j & l\\ 0 & 0 & 0\end{pmatrix}. \tag{27}$$

Further, the quantity $K^L(q,q,z(q))$ with $z(q)=m_1+m_2+\dfrac{q^2}{2\mu_3}$ is the on-shell solution of the

LS equation up to the exponentially suppressed contributions in the box size L. It is given by

$$K^L_{l'm',lm}(q',q;z(q)) = w^L_{l'm',lm}(q',q;z(q)) + \frac{\mu_3}{2\pi}\,\mathsf{P.V.}\int_0^\infty \frac{(q'')^2dq''}{(q'')^2-q^2} \times \sum_{l''m''} w^L_{l'm',l''m''}(q',q'';z(q))K^L_{l''m'',lm}(q'',q;z(q))\,. \quad (28)$$

Here, P.V. stands for the principal-value integral.

In the above equations, the mixing of the partial waves occurs, because the rotational symmetry is broken on the cubic lattice. The equation can be partially diagonalized in the basis of the irreducible representations of the cubic group [27, 37]. In case of the fermions, the basis of the irreducible representations of the double cover of the cubic group should be used [38].

The expressions simplify considerably, if, *e.g.*, one assumes that only the S-wave scattering contributes. Then, the Lüscher formula can be rewritten as

$$\phi(\nu) = -\delta^L(p) + \pi n\,, \quad n = 0, 1, \cdots\,, \quad \tan\phi(\nu) = -\frac{\pi^{3/2}\nu}{Z_{00}(1;\nu^2)}\,. \quad (29)$$

The quantity $\delta^L(p)$ stands for the so-called *pseudophase* [30, 32, 33]

$$\tan\delta^L(p) = \frac{\mu_3 p}{2\pi}\,K^L_{00,00}(p,p;z(p))\,. \quad (30)$$

If the energy z is below the three-particle threshold, the volume-dependence in the effective two-body potential is exponentially suppressed in L. Consequently, up to the exponentially suppressed contributions, the pseudophase $\delta^L(p)$ does not depend on L and on the level index n (the latter dependence arises through the dependence of the effective potential on $L = L_n(p)$ at a given value of p). In this case, $\delta^L(p) = \delta(p)$ coincides with the conventional elastic scattering phase.

Above the 3-body threshold, the effective potential is given by (cf. eq. (18))

$$\langle \mathbf{q}'|\mathbf{w}^L(z)|\mathbf{q}\rangle = -\bar V_3(\mathbf{q}',\mathbf{q}) + \frac{1}{L^{12}}\sum_{\alpha,\beta=1}^{3}\sum_{\mathbf{p}''_\alpha\mathbf{q}''_\alpha\mathbf{p}'''_\beta\mathbf{q}'''_\beta}\bar\Gamma_\alpha(\mathbf{q}';\mathbf{p}''_\alpha\mathbf{q}''_\alpha) \times \langle \mathbf{p}''_\alpha\mathbf{q}''_\alpha|\mathbf{g}^L_{3,\alpha\beta}(z)|\mathbf{p}'''_\beta\mathbf{q}'''_\beta\rangle(\bar\Gamma_\beta(\mathbf{q};\mathbf{p}'''_\beta\mathbf{q}'''_\beta))^*\,. \quad (31)$$

Note that in the above equation, we do not attach the superscript L to the quantities $\bar V_3$ and $\bar\Gamma$, which are the same in a finite and in the infinite volumes. Further, the summation in eq. (31) runs over the momenta (cf. eq. (12))

$$\mathbf{p}''_\alpha = \frac{2\pi\mathbf{n}''_\alpha}{L}\,, \qquad \mathbf{p}'''_\beta = \frac{2\pi\mathbf{n}'''_\beta}{L}\,,$$
$$\mathbf{q}''_\alpha = \frac{2\pi}{L}\left(\mathbf{l}''_\alpha + \frac{m_\beta}{m_\beta+m_\gamma}\mathbf{n}''_\alpha\right), \qquad \mathbf{q}'''_\beta = \frac{2\pi}{L}\left(\mathbf{l}'''_\beta + \frac{m_\gamma}{m_\gamma+m_\alpha}\mathbf{n}'''_\beta\right),$$
$$\mathbf{n}''_\alpha, \mathbf{n}'''_\beta, \mathbf{l}''_\alpha, \mathbf{l}'''_\beta \in \mathbb{Z}^3\,. \quad (32)$$

The finite-volume version of eq. (20) reads

$$\langle \mathbf{p}_\alpha''\mathbf{q}_\alpha''|\mathbf{g}^L_{3,\alpha\beta}(z)|\mathbf{p}_\beta'''\mathbf{q}_\beta'''\rangle = \frac{\delta_{\alpha 3}\delta_{\beta 3}L^3\delta_{\mathbf{p}_3''\mathbf{p}_3'''}L^3\delta_{\mathbf{q}_3''\mathbf{q}_3'''}}{M+\dfrac{(\mathbf{p}_3'')^2}{2M_3}+\dfrac{(\mathbf{q}_3'')^2}{2\mu_3}-z}$$

$$+ \quad \frac{1}{M+\dfrac{(\mathbf{p}_\alpha'')^2}{2M_\alpha}+\dfrac{(\mathbf{q}_\alpha'')^2}{2\mu_\alpha}-z}\langle \mathbf{p}_\alpha''\mathbf{q}_\alpha''|\mathbf{m}^L_{\alpha\beta}(z)|\mathbf{p}_\beta'''\mathbf{q}_\beta'''\rangle \frac{1}{M+\dfrac{(\mathbf{p}_\beta''')^2}{2M_\beta}+\dfrac{(\mathbf{q}_\beta''')^2}{2\mu_\beta}-z}\,. \tag{33}$$

Faddeev equations in a finite volume are written as (cf. eq. (15))

$$\begin{aligned}\langle \mathbf{p}_\alpha'\mathbf{q}_\alpha'|\mathbf{m}^L_{\alpha\beta}(z)|\mathbf{p}_\beta\mathbf{q}_\beta\rangle &= \delta_{\alpha\beta}L^3\delta_{\mathbf{p}_\alpha'\mathbf{p}_\alpha}\langle \mathbf{q}_\alpha'|\boldsymbol{\tau}^L_\alpha(z;\mathbf{p}_\alpha)|\mathbf{q}_\alpha\rangle + \frac{1}{L^{12}}\sum_{\gamma=1}^{3}(1-\delta_{\alpha\gamma})\sum_{\mathbf{p}_\alpha'',\mathbf{q}_\alpha'',\mathbf{p}_\gamma'',\mathbf{q}_\gamma''}L^3\delta_{\mathbf{p}_\alpha'\mathbf{p}_\alpha''}\\ &\times\langle \mathbf{q}_\alpha'|\boldsymbol{\tau}^L_\alpha(z;\mathbf{p}_\alpha)|\mathbf{q}_\alpha''\rangle\frac{L^3\delta_{\mathbf{p}_\alpha'',\mathbf{p}_\alpha''(\mathbf{p}_\gamma'',\mathbf{q}_\gamma'')}L^3\delta_{\mathbf{q}_\alpha'',\mathbf{q}_\alpha''(\mathbf{p}_\gamma'',\mathbf{q}_\gamma'')}}{M+\dfrac{(\mathbf{p}_\alpha'')^2}{2M_\alpha}+\dfrac{(\mathbf{q}_\alpha'')^2}{2\mu_\alpha}-z}\langle \mathbf{p}_\gamma''\mathbf{q}_\gamma''|\mathbf{m}^L_{\gamma\beta}(z)|\mathbf{p}_\beta\mathbf{q}_\beta\rangle\,,\end{aligned} \tag{34}$$

where the two-body scattering amplitude in a finite volume is a solution of the equation

$$\langle \mathbf{q}_\alpha'|\boldsymbol{\tau}^L_\alpha(z;\mathbf{p}_\alpha)|\mathbf{q}_\alpha\rangle = (-\bar V_\alpha(\mathbf{q}_\alpha',\mathbf{q}_\alpha)) + \frac{1}{L^3}\sum_{\mathbf{q}_\alpha''}\frac{(-\bar V_\alpha(\mathbf{q}_\alpha',\mathbf{q}_\alpha''))\langle \mathbf{q}_\alpha''|\boldsymbol{\tau}^L_\alpha(z;\mathbf{p}_\alpha)|\mathbf{q}_\alpha\rangle}{M+\dfrac{\mathbf{p}_\alpha^2}{2M_\alpha}+\dfrac{(\mathbf{q}_\alpha'')^2}{2\mu_\alpha}-z}\,. \tag{35}$$

Note that the momentum $\mathbf{p}_\alpha$ enters in the definition of $\boldsymbol{\tau}^L_\alpha$ twice: in the argument $z \to z - m_\alpha - \dfrac{\mathbf{p}_\alpha^2}{2M_\alpha}$, and through the shifting of the summation over the momenta $\mathbf{q}_\alpha''$ which, according to eq. (32), is no longer proportional to an integer number. In the infinite volume, the summation is replaced by an integration over the whole momentum space, and the dependence on $\mathbf{p}_\alpha$ remains only in the argument.

As in the infinite volume, the equations simplify considerably, if we assume pair interactions of the separable form. The pertinent equations are listed in Appendix B.

2.3 Singularities of the effective two-body potential

As mentioned above, below the three-particle threshold the finite-volume contributions to $\mathbf{w}^L(z)$ are exponentially suppressed. Therefore, up to such suppressed terms, the effective potential coincides with the regular function $\mathbf{w}(z)$, which is defined in the infinite volume.

Above the three-particle threshold, the singularities emerge in $\mathbf{w}^L(z)$. In the vicinity of these singularities the pseudophase rapidly changes by π. These singularities may strongly affect the finite volume spectrum in the vicinity and above the inelastic threshold. For this reason, below we shall discuss them in detail.

Potentially, the matrix element $\langle \mathbf{q}'|\mathbf{w}^L(z)|\mathbf{q}\rangle$ may become singular, when

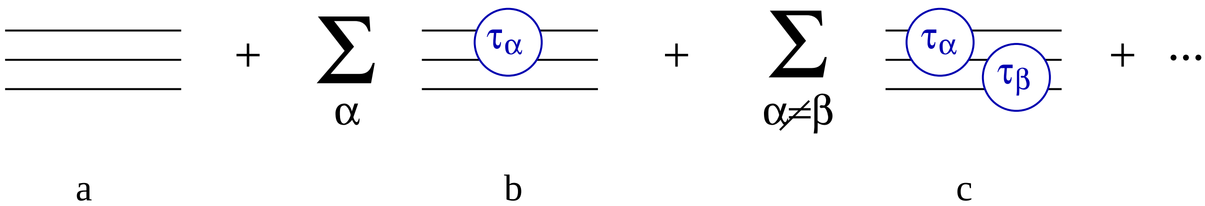

Figure 1: A multiple-scattering series for the three-body Green's function.

i) the three-particle denominators $M+\dfrac{\mathbf{p}_\alpha^2}{2M_\alpha}+\dfrac{\mathbf{q}_\alpha^2}{2\mu_\alpha}-z$ in eq. (33) vanish;

ii) the matrix elements of the operator $\mathbf{m}^L_{\alpha\beta}(z)$, which are solutions of the Faddeev equations in a finite volume, develop a pole at those values of energy, which do not coincide with the poles of the three-particle energy denominators.

We are going to demonstrate below that the singularities of the first type cancel, and only of the second type survive. In order to prove this statement, we mention that, from eqs. (32) and (35) it follows that the two-body scattering matrix $\boldsymbol{\tau}^L_\alpha(z)$ vanishes exactly for those values of z where the three-particle propagators develop a pole. These energies are given by the following equation

$$z=M+\frac{(\mathbf{p}''_\alpha)^2}{2M_\alpha}+\frac{1}{2\mu_\alpha}\left(\mathbf{p}''_\gamma+\frac{m_\gamma}{m_\beta+m\gamma}\,\mathbf{p}''_\alpha\right)^2,\quad(\mathbf{p}''_\alpha,\mathbf{p}''_\gamma)=\frac{2\pi}{L}\,(\mathbf{n}''_\alpha,\mathbf{n}''_\gamma)\,,\quad\mathbf{n}''_\alpha,\mathbf{n}''_\gamma\in\mathbb{Z}^3\,.\tag{36}$$

Let us show now that there are no three-particle singularities of the first type in the matrix elements of the three-particle Green's function. For simplicity, we check this property only for the lowest three-body singularity at $z=M$.

Let us consider the multiple scattering series for the three-body Green's function which is shown in Fig. 1. According to this, the sum of the matrix elements of $\mathbf{g}^L_{3\alpha\beta}(z)$ is written as (cf. eq. (33))

$$\sum_{\alpha,\beta=1}^{3}\langle\mathbf{p}'_\alpha\mathbf{q}'_\alpha|\mathbf{g}^L_{3\alpha\beta}(z)|\mathbf{p}_\beta\mathbf{q}_\beta\rangle=\frac{L^3\delta_{\mathbf{p}'_3\mathbf{p}_3}L^3\delta_{\mathbf{q}'_3\mathbf{q}_3}}{M+\dfrac{\mathbf{p}_3^2}{2M_3}+\dfrac{\mathbf{q}_3^2}{2\mu_3}-z}$$
$$+\;\sum_{\alpha=1}^{3}\frac{L^3\delta_{\mathbf{p}'_\alpha\mathbf{p}_\alpha}\langle\mathbf{q}'_\alpha|\boldsymbol{\tau}^L_\alpha(z;\mathbf{p}_\alpha)|\mathbf{q}_\alpha\rangle}{\left(M+\dfrac{(\mathbf{p}'_3)^2}{2M_3}+\dfrac{(\mathbf{q}'_3)^2}{2\mu_3}-z\right)\left(M+\dfrac{\mathbf{p}_3^2}{2M_3}+\dfrac{\mathbf{q}_3^2}{2\mu_3}-z\right)}+\cdots\,.\tag{37}$$

Here, we have used the property

$$M+\frac{\mathbf{p}_\alpha^2}{2M_\alpha}+\frac{\mathbf{q}_\alpha^2}{2\mu_\alpha}=M+\frac{\mathbf{p}_\beta^2}{2M_\beta}+\frac{\mathbf{q}_\beta^2}{2\mu_\beta}\,,\qquad\alpha,\beta=1,2,3\quad\textit{and}\quad\alpha\neq\beta\,.\tag{38}$$

The first term in eq. (37) has a pole at $z = M$, when $\mathbf{p}'_3 = \mathbf{p}_3 = \mathbf{q}'_3 = \mathbf{q}_3 = \mathbf{0}$. The denominator in the second term has a double zero here, but the matrix element of the operator $\boldsymbol{\tau}^L_\alpha(z;\mathbf{p}_\alpha)$ also has a zero, so that only a single pole remains. This property can be easily checked in higher orders. Introducing the notation

$$y_\alpha = \frac{1}{L^3}\,\frac{\langle \mathbf{0}|\boldsymbol{\tau}^L_\alpha(z;\mathbf{0})|\mathbf{0}\rangle}{M-z}\,, \tag{39}$$

we obtain (see eqs. (33) and (34))

$$\sum_{\alpha,\beta=1}^{3} \langle \mathbf{00}|\mathbf{g}^L_{3\alpha\beta}(z)|\mathbf{00}\rangle = \frac{L^6}{M-z}\Big\{1 + (y_1+y_2+y_3) + (y_1(y_2+y_3) + y_2(y_1+y_3) + y_3(y_1+y_2)) + \cdots\Big\} + \textsf{non-sing}\,, \tag{40}$$

where "non-sing" stands for the terms that are non-singular at $z = M$.

Summing up the series (see Appendix C), we find

$$\sum_{\alpha,\beta=1}^{3} \langle \mathbf{00}|\mathbf{g}^L_{3\alpha\beta}(z)|\mathbf{00}\rangle = \frac{L^6}{M-z}\,\frac{x_1x_2x_3}{x_1x_2+x_2x_3+x_3x_1-2x_1x_2x_3} + \textsf{non-sing}\,, \tag{41}$$

where $x_\alpha = 1+y_\alpha$. Let us now show that, in fact, the first term in eq. (41) is also non-singular. To this end, we consider the equation for the operator $\boldsymbol{\tau}^L_\alpha$

$$\langle \mathbf{q}'_\alpha|\boldsymbol{\tau}^L_\alpha(z;\mathbf{0})|\mathbf{q}_\alpha\rangle = (-\bar V_\alpha(\mathbf{q}'_\alpha,\mathbf{q}_\alpha)) + \frac{1}{L^3}\sum_{\mathbf{q}''_\alpha} \frac{(-\bar V_\alpha(\mathbf{q}'_\alpha,\mathbf{q}''_\alpha))\langle \mathbf{q}''_\alpha|\boldsymbol{\tau}^L_\alpha(z;\mathbf{0})|\mathbf{q}_\alpha\rangle}{M + \dfrac{(\mathbf{q}''_\alpha)^2}{2\mu_\alpha} - z}\,. \tag{42}$$

In order to separate the singularity at $z = M$, we single out the term $\mathbf{q}''_\alpha = 0$ in the above sum. It is then straightforward to see that

$$y_\alpha = \frac{y_{\mathsf{R}\alpha}}{1-y_{\mathsf{R}\alpha}}\,,\qquad y_{\mathsf{R}\alpha} = \frac{1}{L^3}\,\frac{\langle \mathbf{0}|\boldsymbol{\tau}^L_{\mathsf{R}\alpha}(z;\mathbf{0})|\mathbf{0}\rangle}{M-z}\,, \tag{43}$$

where

$$\langle \mathbf{q}'_\alpha|\boldsymbol{\tau}^L_{\mathsf{R}\alpha}(z;\mathbf{0})|\mathbf{q}_\alpha\rangle = (-\bar V_\alpha(\mathbf{q}'_\alpha,\mathbf{q}_\alpha)) + \frac{1}{L^3}\sum_{\mathbf{q}''_\alpha\neq\mathbf{0}} \frac{(-\bar V_\alpha(\mathbf{q}'_\alpha,\mathbf{q}''_\alpha))\langle \mathbf{q}''_\alpha|\boldsymbol{\tau}^L_{\mathsf{R}\alpha}(z;\mathbf{0})|\mathbf{q}_\alpha\rangle}{M + \dfrac{(\mathbf{q}''_\alpha)^2}{2\mu_\alpha} - z}\,. \tag{44}$$

Since the term with $\mathbf{q}''_\alpha = \mathbf{0}$ is absent in eq. (44), the quantity $\langle \mathbf{q}'_\alpha|\boldsymbol{\tau}^L_{\mathsf{R}\alpha}(z;\mathbf{0})|\mathbf{q}_\alpha\rangle$ is non-singular at $z \to M$. Now, the validity of our statement after eq. (41) follows from the fact that $x_\alpha = 1 + y_\alpha = O(z-M)$ as $z \to M$ and, consequently, the three-particle singularities cancel in the sum of the matrix elements of $\mathbf{g}^L_{3\alpha\beta}(z)$.

In conclusion of this section we summarize the main points:

i) The singularities of the effective two-body potential $\mathbf{w}^L(z)$ are caused by the singularities of the matrix elements of the operator $\mathbf{m}_{\alpha\beta}(z)$ which emerge as a result of solution of Faddeev equations in a finite volume and thus have a non-perturbative origin.

ii) In other words, the singularities do not emerge, when the energy coincides with the eigenvalues of the *free* Hamiltonian in the box. The singularity structure is determined by the eigenvalues of the *full* Hamiltonian.

iii) Individual terms in the finite-volume multiple-scattering series, however, contain the singularities determined by the *free* Hamiltonian. In order to make them disappear, multiple-scattering series should be summed up to all orders. The Lüscher formula ensures such a summation in the two-particle case. In this section we have explicitly demonstrated that the summation can cure the problem in the three-particle case as well.

iv) The results of this section may serve as a warning against an approximate truncation of the multiple-scattering series in a finite volume. While in the infinite volume this may affect only the numerical precision, the singularity structure of the effective potential can be modified drastically as a result of such a truncation in a finite volume. Consequently, if the approximations are made in the three-body case, the singularity structure of the effective potential should be examined very carefully.

3 An alternative derivation of the Lüscher formula

In order to find the energy spectrum in a finite volume, the equations, which are considered in section 2.2, can be solved numerically, applying, *e.g.*, the method of ref. [39]. However, these equations contain potentials as well as off-shell two-body scattering matrices, which are model-dependent. The central question is, whether the predicted energy levels are also model-dependent. In other words, if two different potential models lead to the same S-matrix in the infinite volume, can the finite-volume spectra in these models be different?

In case of the two-particle elastic scattering, the answer is given by the Lüscher formula, which relates the finite-volume spectrum to the (on-shell) S-matrix element. Our aim is to rewrite the three-particle equations in a finite volume in a similar fashion, in terms of the on-shell S-matrix elements only. In order to do this, in this section we consider a novel derivation of the Lüscher formula. The method used here can be generalized for the case of three particles, as shown in section 4.

Let us consider the sum

$$S_2 = \frac{1}{L^3}\sum_{\mathbf{p}} \frac{\Phi(\mathbf{p})}{\mathbf{p}^2 - q_0^2}\,, \qquad \mathbf{p} = \frac{2\pi\mathbf{n}}{L}\,, \quad \mathbf{n} \in \mathbb{Z}^3\,, \tag{45}$$

where $\Phi(\mathbf{p})$ denotes a regular function of $\mathbf{p}$. Next, we perform a partial-wave expansion

$$\Phi(\mathbf{p}) = \sum_{lm} \frac{\Phi_l(p)}{p^l}\,\mathcal{Y}_{lm}(\mathbf{p})\,, \qquad p = |\mathbf{p}|, \tag{46}$$

where $\mathcal{Y}_{lm}(\mathbf{p}) = p^l Y_{lm}(\hat{\mathbf{p}})$. The sum in eq. (45) can be rewritten in the following form

$$
\begin{aligned}
S_2 &= \frac{1}{L^3}\sum_{\mathbf{p}}^{|\mathbf{p}|<\Lambda}\sum_{lm}\frac{\mathcal{Y}_{lm}(\mathbf{p})}{\mathbf{p}^2-q_0^2}\left(\frac{\Phi_l(p)}{p^l}-\frac{\Phi_l(q_0)}{q_0^l}f(q_0^2/\mu^2)\right) \\
&+ \frac{\Phi_l(q_0)}{q_0^l}f(q_0^2/\mu^2)\,\frac{1}{L^3}\sum_{\mathbf{p}}^{|\mathbf{p}|<\Lambda}\sum_{lm}\frac{\mathcal{Y}_{lm}(\mathbf{p})}{\mathbf{p}^2-q_0^2}\,. \qquad (47)
\end{aligned}
$$

Note that we have introduced the momentum cutoff Λ, in order to regularize intermediate expressions. The cutoff disappears from the final expressions[2], if the function $\Phi(\mathbf{p})$ falls off sufficiently fast with $\mathbf{p}$ Further, we have introduced a regulator $f(x)$ with the following properties:

1. The function $f(x)$ is bounded and smooth (has any number of derivatives) on the whole interval $x \in]-\infty,\infty[$.
2. $f(x) = 1$ if $x \geq 0$.
3. $\lim_{x\to 0^-} f(x) = 1$ and $\lim_{x\to 0^-} f^{(n)}(x) = 0$ for all $n \neq 0$.
4. The function $f(x)$ vanishes exponentially when $x \to -\infty$.

Otherwise, the function $f(x)$ is arbitrary. An example of such a function is

$$
f(x) = \begin{cases} 1 & , \quad \text{if} \quad x \geq 0 \\ \exp\left(\dfrac{1}{1-\exp(x^{-2})}\right) & , \quad \text{if} \quad x < 0\,. \end{cases} \qquad (48)
$$

The choice of the scale μ in the regulator f is also arbitrary. For example, one could choose μ to coincide with the mass of the lightest particle.

Taking now into account the fact that the expression $\Phi_l(p)/p^l$ is a regular function of p^2 in the vicinity of $p^2 = 0$, the regular summation theorem [40] can be applied. Up to the exponentially suppressed terms in L, one may replace the first sum in eq. (47) by the integral

$$
\begin{aligned}
S_2 &= \int^{|\mathbf{p}|<\Lambda}\frac{d^3\mathbf{p}}{(2\pi)^3}\sum_{lm}\frac{\mathcal{Y}_{lm}(\mathbf{p})}{\mathbf{p}^2-q_0^2-i0}\left(\frac{\Phi_l(p)}{p^l}-\frac{\Phi_l(q_0)}{q_0^l}f(q_0^2/\mu^2)\right) \\
&+ \frac{\Phi_l(q_0)}{q_0^l}f(q_0^2/\mu^2)\,\frac{1}{L^3}\sum_{\mathbf{p}}^{|\mathbf{p}|<\Lambda}\sum_{lm}\frac{\mathcal{Y}_{lm}(\mathbf{p})}{\mathbf{p}^2-q_0^2}\,, \qquad (49)
\end{aligned}
$$

where $q_0^2 \to q_0^2 + i0$ prescription has been chosen arbitrarily (the numerator of the integrand vanishes at $p^2 = q_0^2$, so the prescription does not matter). Simplifying the above expression, we

[2] At finite values of Λ, rapidly oscillating terms at $L \to \infty$ may occur for the sharp cutoff, so a mathematically rigorous procedure is to use a smooth cutoff at a momentum scale Λ. In order to ease the notations, we however proceed further with s sharp cutoff. The oscillating terms are briefly considered in Appendix D. It is shown there that these – as expected – are harmless.

arrive at

$$S_2 = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{\Phi(\mathbf{p})}{\mathbf{p}^2-q_0^2-i0} + \frac{\sqrt{-q_0^2-i0}}{(4\pi)^{3/2}}\,\Phi_0(q_0)f(q_0^2/\mu^2)$$
$$+ \frac{1}{4\pi^2L}f(q_0^2/\mu^2)\sum_{lm}\frac{\Phi_l(q_0)}{\nu^l}\,Z_{lm}(1;\nu^2)\,, \qquad (50)$$

where $\nu=q_0L/(2\pi)$, and $Z_{lm}(1;\nu^2)$ stands for the Lüscher zeta-function

$$Z_{lm}(1;\nu^2)=\lim_{\lambda\to\infty}\left\{\sum_{\mathbf{n}\in\mathbb{Z}^3}\theta(\lambda^2-\mathbf{n}^2)\,\frac{\mathcal{Y}_{lm}(\mathbf{n})}{\mathbf{n}^2-\nu^2}-\delta_{l0}\delta_{m0}\sqrt{4\pi}\lambda\right\}, \qquad \lambda=\frac{\Lambda L}{2\pi}\,. \qquad (51)$$

Formally, the above equation can be rewritten in the following manner

$$\frac{1}{L^3}\sum_{\mathbf{k}}\frac{(2\pi)^3\delta^3(\mathbf{p}-\mathbf{k})}{\mathbf{k}^2-q_0^2} = \mathsf{P.V.}\,\frac{1}{\mathbf{p}^2-q_0^2}+\sum_{lm}\frac{2}{\nu^{l+1}}\,Y^*_{lm}(\hat{\mathbf{p}})Z_{lm}(1;\nu^2)\Delta(\mathbf{p}^2,q_0^2)\,,$$
$$\mathsf{P.V.}\,\frac{1}{\mathbf{p}^2-q_0^2} = \frac{1}{\mathbf{p}^2-q_0^2-i0}-i\pi\Delta(\mathbf{p}^2,q_0^2)\,, \qquad (52)$$

where the quantity $\Delta(\mathbf{p}^2,q_0^2)$ coincides with the conventional Dirac δ-function $\delta(\mathbf{p}^2-q_0^2)$ for $q_0^2\geq 0$ and is defined through the action on the smooth test functions

$$-i\pi\int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\Delta(\mathbf{p}^2,q_0^2)\Phi(\mathbf{p})=\frac{\sqrt{-q_0^2-i0}}{(4\pi)^{3/2}}\,\Phi_0(q_0)f(q_0^2/\mu^2) \qquad (53)$$

(recall that $f(q_0^2/\mu^2)=1$ for $q_0^2\geq 0$). For $q_0^2<0$, the action of the distribution Δ on a test function is defined through the analytic continuation of $\Phi_0(q_0)$ in eq. (53). The regulator f serves the purpose to effectively cut the contributions with $-q_0^2>\mu^2$.

To summarize, the momentum-space two-body Green's function in a finite volume, which is given by

$$G_0^L(\mathbf{k};z)=\frac{2\mu}{L^3}\sum_{\mathbf{p}}\frac{(2\pi)^3\delta^3(\mathbf{p}-\mathbf{k})}{\mathbf{p}^2-q_0^2}\,, \qquad (54)$$

where μ is the reduced mass and $z=m_1+m_2+\dfrac{q_0^2}{2\mu}$, can be decomposed into the following parts

$$G_0^L(\mathbf{k};z) = 2\mu\left\{\frac{1}{\mathbf{k}^2-q_0^2-i0}-i\pi\Delta(\mathbf{k}^2,q_0^2)+\Delta(\mathbf{k}^2,q_0^2)\sum_{lm}Y^*_{lm}(\hat{\mathbf{k}})\frac{2}{\nu^{l+1}}\,Z_{lm}(1;\nu^2)\right\}$$
$$\doteq G_0(\mathbf{k};z)+G_{\mathsf{U}}(\mathbf{k};z)+G_{\mathsf{F}}(\mathbf{k};z)\doteq G_{\mathsf{K}}(\mathbf{k};z)+G_{\mathsf{F}}(\mathbf{k};z)\,. \qquad (55)$$

The equation (55) gives the splitting of the finite-volume two-body Green's function into the infinite-volume part $G_{\mathsf{K}}=G_0+G_{\mathsf{U}}$ (corresponding to the principal-value prescription at the singularity) and the correction G_{F}. The latter is proportional to the on-shell factor $\Delta(\mathbf{k}^2,q_0^2)$, which projects onto the energy shell. This fact plays a crucial role in the proof of the statement that the finite-volume spectrum is determined only by the infinite-volume S-matrix elements which are defined on shell.

The Lüscher formula can be derived straightforwardly, using eq. (55). To this end, we take $q_0^2 \geq 0$. Then, $\Delta(\mathbf{k}^2,q_0^2)$ is replaced by the conventional δ-function. First, note that, substituting the free Green's function by the quantity defined in eq. (54), the two-body LS equation in a finite volume can be written formally in the same form as in the infinite volume

$$\langle\mathbf{p}|\mathbf{T}^L(z)|\mathbf{q}\rangle = \langle\mathbf{p}|(-\mathbf{V})|\mathbf{q}\rangle + \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\langle\mathbf{p}|(-\mathbf{V})|\mathbf{k}\rangle G_0^L(\mathbf{k};z)\langle\mathbf{k}|\mathbf{T}^L(z)|\mathbf{q}\rangle\,. \tag{56}$$

Next, we define

$$\begin{aligned}\mathbf{T} &= (-\mathbf{V}) + (-\mathbf{V})\mathbf{G}_0\mathbf{T}\,,\\ \mathbf{K} &= (-\mathbf{V}) + (-\mathbf{V})(\mathbf{G}_0+\mathbf{G}_{\mathsf{U}})\mathbf{K} = \mathbf{T}+\mathbf{T}\mathbf{G}_{\mathsf{U}}\mathbf{K}\,,\\ \mathbf{T}^L &= (-\mathbf{V}) + (-\mathbf{V})(\mathbf{G}_0+\mathbf{G}_{\mathsf{U}}+\mathbf{G}_{\mathsf{F}})\mathbf{T}^L = \mathbf{K}+\mathbf{K}\mathbf{G}_{\mathsf{F}}\mathbf{T}^L\,.\end{aligned} \tag{57}$$

It is immediately seen that $\mathbf{T}$ and $\mathbf{K}$ are the two-body T- and K-matrices in the infinite volume, respectively. The relation between the K-matrix in the infinite volume and the T-matrix in a finite volume $\mathbf{T}^L$ is given by the last equation in eq. (57). After carrying out the partial-wave expansion

$$\begin{aligned}\langle\mathbf{p}|\mathbf{T}^L(z)|\mathbf{q}\rangle &= 4\pi\sum_{l'm'}\sum_{lm}Y_{l'm'}(\hat{\mathbf{p}})T^L_{l'm',lm}(p,q;z)Y^*_{lm}(\hat{\mathbf{q}})\,,\\ \langle\mathbf{p}|\mathbf{K}(z)|\mathbf{q}\rangle &= 4\pi\sum_{lm}Y_{lm}(\hat{\mathbf{p}})K_l(p,q;z)Y^*_{lm}(\hat{\mathbf{q}})\,,\end{aligned} \tag{58}$$

and integrating over the angles, this equation can be rewritten in the algebraic form

$$T^L_{l'm',lm}(p,q;z) = \delta_{ll'}\delta_{mm'}K_l(p,q;z) + \frac{\mu q_0}{2\pi}\,K_{l'}(p,q_0;z)\sum_{l''m''}\mathcal{M}_{l'm',l''m''}(\nu)T^L_{l''m'',lm}(q_0,q,q_0^2)\,, \tag{59}$$

where the matrix $\mathcal{M}_{l'm',lm}(\nu)$ is displayed in eq. (25). Going to the mass shell $p=q=q_0$ and requiring that the above system of linear equations is singular (T^L becomes infinity, that means that the determinant of the above system of linear equations vanishes), we finally arrive at the Lüscher formula, see eq. (25). Note that, since in case of an elastic two-body scattering considered here, the K-matrix in the Lüscher formula stands for the infinite-volume K-matrix, $K^L_{l'm',lm}$ in eq. (25) should be substituted by $\delta_{ll'}\delta_{mm'}K_l$. Note also that, within the normalization used, the on-shell K-matrix is related to the scattering phase shift, according to

$$\tan\delta_l(q_0) = \frac{\mu p}{2\pi}\,K_l(q_0,q_0;z)\,. \tag{60}$$

In conclusion, several remarks are in order:

i) The decomposition of the two-particle Green's function in a finite volume, given in eq. (55), is valid, if both sides of the same equation are integrated with the same regular test function. In general, it does not hold, if integrated with singular functions. In the Born series of the LS equation, the two-particle Green's function is always integrated with the potential, whose singularities are assumed to lie far away from the integration contour.

ii) If $\nu^2 < 0$, then $Z_{lm}(1;\nu^2) = -\delta_{l0}\delta_{m_0}\pi^{3/2}\sqrt{-\nu^2 - i0}$, up to the exponentially suppressed contributions. This implies that the sum of all the terms, proportional to $\Phi_0(q_0)$ for $\nu^2 < 0$ in eq. (50), is in fact exponentially suppressed. Hence, the quantity S_2 does not depend on the values of the function $\Phi(\mathbf{p})$ outside the original range $|\mathbf{p}| > 0$, as it should.

iii) The reason why we still retain these terms and do not replace the quantity $\Delta(\mathbf{p}^2, q_0^2)$ by $\delta(\mathbf{p}^2 - q_0^2)$ everywhere, is the fact that the principal-value integral, which was defined above, is the regular function of q_0^2 (no unitary cusp present). This property plays no role in the two-particle scattering (even in the multi-channel case) but, as we shall see below, becomes critical in case of three-particles. Note also that in ref. [32], which deals with the multi-channel scattering for the coupled $\pi\pi$-$K\bar{K}$ channels, we have effectively used the same splitting as in eq. (55) but with $f(x) = 1$.

4 Three particles in a finite volume

4.1 Splitting of the three-particle Green's function

In analogy with eq. (54) we define the finite-volume three-body Green's function in a channel $\alpha = 1, 2, 3$ as (in order to ease the notations, we suppress the channel index α in all momenta)

$$
\begin{aligned}
G_{0\alpha}^L(\mathbf{k}, \mathbf{l}; z) &= \frac{1}{L^6} \sum_{\mathbf{pq}} \frac{(2\pi)^3\delta^3(\mathbf{p} - \mathbf{k})(2\pi)^3\delta^3(\mathbf{q} - \mathbf{l})}{M + \dfrac{\mathbf{p}^2}{2M_\alpha} + \dfrac{\mathbf{q}^2}{2\mu_\alpha} - z}, \\
\mathbf{q} &= \tilde{\mathbf{p}} + \frac{m_\beta}{m_\beta + m_\gamma}\,\mathbf{p}, \qquad (\mathbf{p}, \tilde{\mathbf{p}}) = \frac{2\pi}{L}\,(\mathbf{n}, \tilde{\mathbf{n}}), \quad \mathbf{n}, \tilde{\mathbf{n}} \in \mathbb{Z}^3,
\end{aligned} \tag{61}
$$

so that, in the above equation, the summation is carried out over the integers $\mathbf{n}, \tilde{\mathbf{n}}$. In order to find the desired form of the splitting, we separate the 2-particle Green's function in the above expression

$$
\begin{aligned}
G_{0\alpha}^L(\mathbf{k}, \mathbf{l}; z) &= \frac{1}{L^3} \sum_{\mathbf{p}} (2\pi)^3\delta^3(\mathbf{p} - \mathbf{k}) \frac{2\mu_\alpha}{L^3} \sum_{\mathbf{q}} \frac{(2\pi)^3\delta^3(\mathbf{q} - \mathbf{l})}{\mathbf{q}^2 - q_{0\alpha}^2}, \\
q_{0\alpha}^2 &= 2\mu_\alpha\left(z - M - \frac{\mathbf{p}^2}{2M_\alpha}\right).
\end{aligned} \tag{62}
$$

Acting in the similar way as in the two-body case (see section 3), we obtain

$$
\begin{aligned}
G_{0\alpha}^L(\mathbf{k},\mathbf{l};z) &= \frac{1}{L^3}\sum_{\mathbf{p}}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{k})\Bigg\{\frac{1}{M+\dfrac{\mathbf{p}^2}{2M_\alpha}+\dfrac{\mathbf{l}^2}{2\mu_\alpha}-z-i0} \\
&- i\pi\Delta\Bigg(\mathbf{l}^2,2\mu_\alpha\Bigg(M+\frac{\mathbf{p}^2}{2M_\alpha}-z\Bigg)\Bigg) \\
&+ \Delta\Bigg(\mathbf{l}^2,2\mu_\alpha\Bigg(M+\frac{\mathbf{p}^2}{2M_\alpha}-z\Bigg)\Bigg)\sum_{lm}Y_{lm}^*(\hat{\mathbf{l}})\frac{2}{\nu_\alpha^l}Z_{lm}^{\mathbf{a}}(1;\nu_\alpha^2)\Bigg\}, \qquad (63)
\end{aligned}
$$

where $\nu_\alpha = q_{0\alpha}L/(2\pi)$, $\mathbf{a}=\big[m_\beta/(m_\beta+m_\gamma)\big]\,\mathbf{p}L/(2\pi)$, and the Lüscher zeta-function in the moving frame is defined as (cf. the pertinent expression in the CM frame, eq. (51))

$$
Z_{lm}^{\mathbf{a}}(1;\nu_\alpha^2)=\lim_{\lambda\to\infty}\Bigg\{\sum_{\mathbf{n}\in\mathbb{Z}^3}\theta(\lambda^2-(\mathbf{n}+\mathbf{a})^2)\,\frac{\mathcal{Y}_{lm}(\mathbf{n}+\mathbf{a})}{(\mathbf{n}+\mathbf{a})^2-\nu_\alpha^2}-\delta_{l0}\delta_{m0}\sqrt{4\pi}\lambda\Bigg\}. \qquad (64)
$$

Since the principal-value integral over the variable $\mathbf{l}$, containing a regular function of the arguments $\mathbf{p},\mathbf{l}$, is a regular function of the remaining variable $\mathbf{p}$, one may use the regular summation theorem in this variable and rewrite eq. (63) in the following form (cf. eq. (55))

$$
\begin{aligned}
G_{0\alpha}^L(\mathbf{k},\mathbf{l};z) &= \frac{1}{M+\dfrac{\mathbf{k}^2}{2M_\alpha}+\dfrac{\mathbf{l}^2}{2\mu_\alpha}-z-i0} - i\pi\Delta\Bigg(\mathbf{l}^2,2\mu_\alpha\Bigg(M+\frac{\mathbf{k}^2}{2M_\alpha}-z\Bigg)\Bigg) \\
&+ \frac{1}{L^3}\sum_{\mathbf{p}}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{k})\,\Delta\Bigg(\mathbf{l}^2,2\mu_\alpha\Bigg(M+\frac{\mathbf{p}^2}{2M_\alpha}-z\Bigg)\Bigg)\sum_{lm}Y_{lm}^*(\hat{\mathbf{l}})\frac{2}{\nu_\alpha^l}Z_{lm}^{\mathbf{a}}(1;\nu_\alpha^2) \\
&\doteq G_{0\alpha}(\mathbf{k},\mathbf{l};z)+G_{\mathsf{U}\alpha}(\mathbf{k},\mathbf{l};z)+G_{\mathsf{F}\alpha}(\mathbf{k},\mathbf{l};z) \\
&\doteq G_{\mathsf{K}\alpha}(\mathbf{k},\mathbf{l};z)+\frac{1}{L^3}\sum_{\mathbf{p}}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{k})\tilde{G}_{\mathsf{F}\alpha}(\mathbf{p},\mathbf{l};z)\,. \qquad (65)
\end{aligned}
$$

Needless to say that the above splitting is valid, if both sides of eq. (65) are integrated with a regular function of the momentum variables $\mathbf{k}$ and $\mathbf{l}$. Further, since the variables $\mathbf{k}$ and $\mathbf{p}$ are not restricted from above, the argument of the distribution Δ can become positive and arbitrarily large, independent of the choice of the variables $\mathbf{l}^2$ and z. This means that one has to necessarily deal with the analytic continuation of the pertinent amplitudes below threshold. Note however that this sub-threshold contribution is effectively cut by the regulator f for the momenta $-q_{0\alpha}^2>\mu^2$. Assuming the range of the potentials much smaller than the inverse of the lightest mass in the system, one does not expect to encounter any singularities in the analytic continuation for $-q_{0\alpha}^2<\mu^2$. The above argumentation serves to justify the introduction of the regulator f.

Using the representation of the three-particle Green's function, given in eq. (65), one may try to repeat the same steps as in eq. (57), also for the three-particle T-matrix. In the three-particle case, however, a new complication arises, related to the presence of the disconnected contributions (the diagrams where a spectator particle propagates freely, when the other two particles interact). These diagrams contain the factor $L^3\delta_{\mathbf{p}_\alpha\mathbf{q}_\alpha}$ ($(2\pi)^3\delta^3(\mathbf{p}_\alpha-\mathbf{q}_\alpha)$) in a finite (infinite) volume. This factor is not a regular function and is L-dependent (in a finite volume). Consequently, the method, which was used in the two-particle case, can be applied here first after these disconnected contributions are removed. Below we shall demonstrate, how this can be achieved.

4.2 Effective two-body K-matrix in the presence of the three-particle intermediate states

As was shown in section 3, in order to obtain the Lüscher formula from eq. (23), one may substitute here the splitting of the two-particle Green's function, given in eq. (55). The result is given in eq. (25). The effective two-body K-matrix obeys the equation

$$\langle\mathbf{q}'|\mathbf{K}^L(z)|\mathbf{q}\rangle=\langle\mathbf{q}'|\mathbf{w}^L(z)|\mathbf{q}\rangle+\mathsf{P.V.}\int\frac{d^3\mathbf{q}''}{(2\pi)^3}\,\frac{\langle\mathbf{q}'|\mathbf{w}^L(z)|\mathbf{q}''\rangle\,\langle\mathbf{q}''|\mathbf{K}^L(z)|\mathbf{q}\rangle}{m_1+m_2+\dfrac{(\mathbf{q}'')^2}{2\mu_3}-z}\,. \tag{66}$$

Performing the partial-wave expansion of the above equation, we arrive at eq. (28). Note that, in difference to eq. (59), which refers to the elastic case, the effective K-matrix in the above equation still depends on L above the three-particle threshold.

Taking into account the expression for the potential $\mathbf{w}^L(z)$, given in eq. (31), we get

$$\begin{aligned}\langle\mathbf{q}'|\mathbf{K}^L(z)|\mathbf{q}\rangle &= \langle\mathbf{q}'|\mathbf{K}_3(z)|\mathbf{q}\rangle+\frac{1}{L^{12}}\sum_{\mathbf{p}''_\alpha\mathbf{q}''_\alpha\mathbf{p}'''_\beta\mathbf{q}'''_\beta}\tilde\Gamma_\alpha(\mathbf{q}';\mathbf{p}''_\alpha\mathbf{q}''_\alpha)\\ &\times \langle\mathbf{p}''_\alpha\mathbf{q}''_\alpha|\tilde{\mathbf{g}}_3^L(z)|\mathbf{p}'''_\beta\mathbf{q}'''_\beta\rangle(\tilde\Gamma_\beta(\mathbf{q};\mathbf{p}'''_\beta\mathbf{q}'''_\beta))^*\,,\end{aligned} \tag{67}$$

where K_3 is defined in the infinite volume

$$\langle\mathbf{q}'|\mathbf{K}_3(z)|\mathbf{q}\rangle=(-\bar V_3(\mathbf{q}',\mathbf{q}))+\mathsf{P.V.}\int\frac{d^3\mathbf{q}''}{(2\pi)^3}\,\frac{(-\bar V_3(\mathbf{q}',\mathbf{q}''))\,\langle\mathbf{q}''|\mathbf{K}_3(z)|\mathbf{q}\rangle}{m_1+m_2+\dfrac{(\mathbf{q}'')^2}{2\mu_3}-z}\,. \tag{68}$$

Further, the quantity $\tilde\Gamma_\alpha$ is defined through $\bar\Gamma_\alpha$ in the following manner

$$\tilde\Gamma_\alpha(\mathbf{q};\mathbf{p}''_\alpha\mathbf{q}''_\alpha)=\bar\Gamma_\alpha(\mathbf{q};\mathbf{p}''_\alpha\mathbf{q}''_\alpha)+\int\frac{d^3\mathbf{q}''}{(2\pi)^3}\,\frac{\langle\mathbf{q}'|\mathbf{K}_3(z)|\mathbf{q}''\rangle\bar\Gamma_\alpha(\mathbf{q}'';\mathbf{p}''_\alpha\mathbf{q}''_\alpha)}{m_1+m_2+\dfrac{(\mathbf{q}'')^2}{2\mu_3}-z}\,, \tag{69}$$

and the Green's function $\tilde{\mathbf{g}}_3^L(z)$ obeys the following equation

$$\begin{aligned}\langle\mathbf{p}'_\alpha\mathbf{q}'_\alpha|\tilde{\mathbf{g}}_3^L(z)|\mathbf{p}_\beta\mathbf{q}_\beta\rangle &= \langle\mathbf{p}'_\alpha\mathbf{q}'_\alpha|\mathbf{g}_3^L(z)|\mathbf{p}_\beta\mathbf{q}_\beta\rangle+\frac{1}{L^{12}}\sum_{\mathbf{p}''_\gamma\mathbf{q}''_\gamma\mathbf{p}'''_\sigma\mathbf{q}'''_\sigma}\langle\mathbf{p}'_\alpha\mathbf{q}'_\alpha|\mathbf{g}_3^L(z)|\mathbf{p}''_\gamma\mathbf{q}''_\gamma\rangle\\ &\times\langle\mathbf{p}''_\gamma\mathbf{q}''_\gamma|(-\mathbf{V}_4(z))|\mathbf{p}'''_\sigma\mathbf{q}'''_\sigma\rangle\langle\mathbf{p}'''_\sigma\mathbf{q}'''_\sigma|\tilde{\mathbf{g}}_3^L(z)|\mathbf{p}_\beta\mathbf{q}_\beta\rangle\,.\end{aligned} \tag{70}$$

Here, $\mathbf{g}_3^L(z)=\sum_{\alpha,\beta=1}^3\mathbf{g}_{3\alpha\beta}^L(z)$ and γ,σ can be chosen arbitrarily. Further, $\mathbf{V}_4(z)$ denotes the effective three-particle potential, which has emerged after "projecting out" the two-particle intermediate state. This potential is given by the following expression

$$\langle\mathbf{p}'_\alpha\mathbf{q}'_\alpha|(-\mathbf{V}_4(z))|\mathbf{p}_\beta\mathbf{q}_\beta\rangle = \mathsf{P.V.}\int\frac{d^3\mathbf{q}'}{(2\pi)^3}\frac{d^3\mathbf{q}}{(2\pi)^3}(\bar\Gamma_\alpha(\mathbf{q}';\mathbf{p}'_\alpha\mathbf{q}'_\alpha))^*\left\{\frac{(2\pi)^3\delta^3(\mathbf{q}'-\mathbf{q})}{m_1+m_2+\dfrac{\mathbf{q}^2}{2\mu_3}-z}\right.$$
$$\left.+\frac{\langle\mathbf{q}'|\mathbf{K}_3(z)|\mathbf{q}\rangle}{\left(m_1+m_2+\dfrac{(\mathbf{q}')^2}{2\mu_3}-z\right)\left(m_1+m_2+\dfrac{\mathbf{q}^2}{2\mu_3}-z\right)}\right\}\bar\Gamma_\beta(\mathbf{q};\mathbf{p}_\beta\mathbf{q}_\beta)\,. \tag{71}$$

We would like to stress that the effective potential $\mathbf{V}_4(z)$ is defined in the infinite volume and is a smooth function of its arguments.

To summarize, after "projecting out" the two-particle intermediate state, an effective three-particle force $\mathbf{V}_4(z)$ appears in Faddeev equations, even though we initially assumed that no such force is present. The effective two-body K-matrix in a finite volume is given explicitly by eq. (67). In order to evaluate the finite-volume effects in the three-body Green's function, which enters this expression, one should consider Faddeev equations in a finite volume.

4.3 The three-particle counterpart of the Lüscher formula

In order to derive the three-particle analog of the Lüscher formula, we have to deal with the three-body equations in the presence of the three-particle force. To this end, one may use, *e.g.*, the formalism described in the papers [41] – with the potentials replaced by the K-matrix elements and the free Green's function replaced by $\mathbf{G}_\mathsf{F}$. However, as discussed above, there exists a problem related to the presence of the disconnected parts. Namely, in the presence of the Kronecker-δ, contained in the disconnected parts, the use of the splitting procedure for the three-particle propagator according to eq. (65) can not be justified mathematically. In order to circumvent this problem, we act by using a trial and error method. Namely, we first apply the splitting in the three-body LS equations, as if there were no disconnected parts, and further use the Faddeev trick. In the resulting equations, the disconnected terms, containing the δ-functions, emerge in a finite volume. At the next stage, we discard these singular terms by hand, thus making a *conjecture* about the correct form of the equations. At the final step, we check this conjecture explicitly, by considering the multiple-scattering series that emerge from the resulting equations, and showing that this series coincides with the original multiple-scattering series in a finite volume.

Symbolically, the result for the effective two-body K-matrix can be written as follows

$$\mathbf{K}^L=\mathbf{K}_{2\to2}+\mathbf{K}_{2\to3}(\mathbf{G}_\mathsf{F}+\mathbf{G}_\mathsf{F}\mathbf{R}_\mathsf{F}\mathbf{G}_\mathsf{F})\mathbf{K}_{3\to2}\,, \tag{72}$$

where $\mathbf{K}_{i\to j}$ denote the pertinent K-matrix elements in the infinite volume[3], $\mathbf{G}_\mathsf{F}$ stands for the finite-volume part of the three-particle Green's function (see eq. (65)), and the quantity $\mathbf{R}_\mathsf{F}$ is

[3] Above threshold, the three-body K-matrix coincides with the one defined, *e.g.*, in refs. [42, 43].

given by

$$
\begin{aligned}
\mathbf{R}_{\mathsf{F}} &= \sum_{\mu,\nu=1}^{4} \mathbf{R}_{\mu\nu} + \sum_{\alpha=1}^{3} \boldsymbol{\theta}_{\alpha}\,, \\
\mathbf{R}_{4\beta} &= \boldsymbol{\theta}_4 \mathbf{G}_{\mathsf{F}} \Big(\boldsymbol{\theta}_{\beta} + \sum_{\gamma=1}^{3} \mathbf{R}_{\gamma\beta} \Big)\,, \\
\mathbf{R}_{\alpha\beta} &= \boldsymbol{\theta}_{\alpha} \mathbf{G}_{\mathsf{F}} \Big(\sum_{\gamma=1}^{3} (1-\delta_{\alpha\gamma}) \mathbf{R}_{\gamma\beta} + \mathbf{R}_{4\beta} \Big)\,, \\
\mathbf{R}_{\alpha 4} &= \boldsymbol{\theta}_{\alpha} \mathbf{G}_{\mathsf{F}} \sum_{\gamma=1}^{3} (1-\delta_{\alpha\gamma}) \mathbf{R}_{\gamma 4} + \boldsymbol{\theta}_{\alpha} \mathbf{G}_{\mathsf{F}} \mathbf{R}_{44}\,, \\
\mathbf{R}_{44} &= \boldsymbol{\theta}_4 + \boldsymbol{\theta}_4 \mathbf{G}_{\mathsf{F}} \sum_{\gamma=1}^{3} \mathbf{R}_{\gamma 4}\,.
\end{aligned} \tag{73}
$$

Here, $\mu,\nu = 1,\cdots,4$, whereas $\alpha,\beta,\gamma = 1,\cdots,3$ and

$$
\boldsymbol{\theta}_{\mu} = \mathbf{K}_{\mu} + \mathbf{K}_{\mu} \mathbf{G}_{\mathsf{F}} \boldsymbol{\theta}_{\mu}\,, \qquad \mathbf{K}_{\mu} = (-\mathbf{V}_{\mu}) + (-\mathbf{V}_{\mu})(\mathbf{G}_{\mathsf{0}} + \mathbf{G}_{\mathsf{U}}) \mathbf{K}_{\mu}\,, \qquad \mathbf{K}_{3\to 3} = \sum_{\mu=1}^{4} \mathbf{K}_{\mu}\,. \tag{74}
$$

Note that in eq. (73) (third line), we have omitted the terms of the type $\boldsymbol{\theta}_{\alpha}\mathbf{G}_{\mathsf{F}}\boldsymbol{\theta}_{\beta}$ with $\alpha \neq \beta$. Physically, such terms correspond to the finite-volume corrections in the disconnected diagrams and emerge, if one faithfully applies the splitting procedure even to the disconnected piece. This omission will be justified below by showing that eq. (73) produces – diagram by diagram – the correct splitting of the infinite- and finite-volume parts in the multiple-scattering series[4]. Here we also would like to mention that this system of equations is formally identical to the equations that relate the T-matrix and K-matrix elements, with a replacement $\mathbf{R} \to \mathbf{K}$, $\mathbf{K} \to \mathbf{T}$ and $\mathbf{G}_{\mathsf{F}} \to \mathbf{G}_{\mathsf{U}}$.

The above equations can be written down in the explicit form

$$
\langle \mathbf{q}' | \mathbf{K}^{L} | \mathbf{q} \rangle = \langle \mathbf{q}' | \mathbf{K}_{2\to 2} | \mathbf{q} \rangle + \langle \mathbf{q}' | \mathbf{K}_{2\to 3} \mathbf{G}_{\mathsf{F}} \mathbf{K}_{3\to 2} | \mathbf{q} \rangle + \langle \mathbf{q}' | \mathbf{K}_{2\to 3} \mathbf{G}_{\mathsf{F}} \mathbf{R}_{\mathsf{F}} \mathbf{G}_{\mathsf{F}} \mathbf{K}_{3\to 2} | \mathbf{q} \rangle\,, \tag{75}
$$

where[5]

$$
\langle \mathbf{q}' | \mathbf{K}_{2\to 3} \mathbf{G}_{\mathsf{F}} \mathbf{K}_{3\to 2} | \mathbf{q} \rangle = \frac{1}{L^3} \sum_{\mathbf{p}_{\alpha}} \int \frac{d^3 \mathbf{q}_{\alpha}}{(2\pi)^3} \langle \mathbf{q}' | \mathbf{K}^{(\alpha)}_{2\to 3} | \mathbf{p}_{\alpha} \mathbf{q}_{\alpha} \rangle \tilde{G}_{\mathsf{F}\alpha}(\mathbf{p}_{\alpha}, \mathbf{q}_{\alpha}; z) \langle \mathbf{p}_{\alpha} \mathbf{q}_{\alpha} | \mathbf{K}^{(\alpha)}_{3\to 2} | \mathbf{q} \rangle\,, \tag{76}
$$

[4]The physical meaning of this prescription is very transparent. The finite-volume corrections emerge only in the loop diagrams. However, due to the presence of the Kronecker-delta in the disconnected diagrams, a first iteration of the disconnected diagrams in the Faddeev equations gives a connected diagram without a loop. Its explicit expression is identical in a finite and the infinite volumes. The loops (and, consequently, the finite-volume corrections) emerge first in the second iteration, see fig. 2

[5]Note the superscript (α) in the K-matrix elements. It emerges because $\mathbf{G}_{\mathsf{U}\alpha}$ explicitly depends on the channel α through the regulator $f(q^2_{0\alpha}/\mu^2)$, see eq. (65).

Figure 2: a) One iteration of the disconnected diagrams produces a connected diagram with no loops; b) Loops emerge first after two iterations.

The last term in eq. (75) is given by

$$\langle \mathbf{q}'|\mathbf{K}_{2\to 3}\mathbf{G}_{\mathsf{F}}\mathbf{R}_{\mathsf{F}}\mathbf{G}_{\mathsf{F}}\mathbf{K}_{3\to 2}|\mathbf{q}\rangle = \langle \mathbf{q}'|\mathbf{K}_{2\to 3}\mathbf{G}_{\mathsf{F}}\left(\sum_{\alpha=1}^{3}\boldsymbol{\theta}_\alpha + \sum_{\mu,\nu=1}^{4}\mathbf{R}_{\mu\nu}\right)\mathbf{G}_{\mathsf{F}}\mathbf{K}_{3\to 2}|\mathbf{q}\rangle\,, \tag{77}$$

where

$$\langle \mathbf{q}'|\mathbf{K}_{2\to 3}\mathbf{G}_{\mathsf{F}}\boldsymbol{\theta}_\alpha\mathbf{G}_{\mathsf{F}}\mathbf{K}_{3\to 2}|\mathbf{q}\rangle = \frac{1}{L^3}\sum_{\mathbf{p}_\alpha}\int\frac{d^3\mathbf{q}'_\alpha}{(2\pi)^3}\frac{d^3\mathbf{q}_\alpha}{(2\pi)^3}\,\langle \mathbf{q}'|\mathbf{K}^{(\alpha)}_{2\to 3}|\mathbf{p}_\alpha\mathbf{q}'_\alpha\rangle\tilde G_{\mathsf{F}\alpha}(\mathbf{p}_\alpha,\mathbf{q}'_\alpha;z)$$
$$\times\ \langle \mathbf{q}'_\alpha|\boldsymbol{\theta}_\alpha(z;\mathbf{p}_\alpha)|\mathbf{q}_\alpha\rangle\tilde G_{\mathsf{F}\alpha}(\mathbf{p}_\alpha,\mathbf{q}_\alpha;z)\langle \mathbf{p}_\alpha\mathbf{q}_\alpha|\mathbf{K}^{(\alpha)}_{3\to 2}|\mathbf{q}\rangle \tag{78}$$

and

$$\langle \mathbf{q}'|\mathbf{K}_{2\to 3}\mathbf{G}_{\mathsf{F}}\mathbf{R}_{\mu\nu}\mathbf{G}_{\mathsf{F}}\mathbf{K}_{3\to 2}|\mathbf{q}\rangle = \frac{1}{L^6}\sum_{\mathbf{p}'_\alpha\mathbf{p}_\beta}\int\frac{d^3\mathbf{q}'_\alpha}{(2\pi)^3}\frac{d^3\mathbf{q}_\beta}{(2\pi)^3}\,\langle \mathbf{q}'|\mathbf{K}^{(\alpha)}_{2\to 3}|\mathbf{p}'_\alpha\mathbf{q}'_\alpha\rangle\tilde G_{\mathsf{F}\alpha}(\mathbf{p}'_\alpha,\mathbf{q}'_\alpha;z)$$
$$\times\ \langle \mathbf{p}'_\alpha\mathbf{q}'_\alpha|\mathbf{R}_{\mu\nu}|\mathbf{p}_\beta\mathbf{q}_\beta\rangle\tilde G_{\mathsf{F}\beta}(\mathbf{p}_\beta,\mathbf{q}_\beta;z)\langle \mathbf{p}_\beta\mathbf{q}_\beta|\mathbf{K}^{(\beta)}_{3\to 2}|\mathbf{q}\rangle\,. \tag{79}$$

Note that if $\mu=4$ and/or $\nu=4$ above, the pertinent indexes α,β can be chosen arbitrarily.

Next, $\boldsymbol{\theta}_\alpha$ are determined through

$$\langle \mathbf{q}'_\alpha|\boldsymbol{\theta}_\alpha(z;\mathbf{p}_\alpha)|\mathbf{q}_\alpha\rangle = \langle \mathbf{q}'_\alpha|\mathbf{K}_\alpha\left(z-m_\alpha-\frac{\mathbf{p}_\alpha^2}{2M_\alpha}\right)|\mathbf{q}_\alpha\rangle$$
$$+\int\frac{d^3\mathbf{l}_\alpha}{(2\pi)^3}\,\langle \mathbf{q}'_\alpha|\mathbf{K}_\alpha\left(z-m_\alpha-\frac{\mathbf{p}_\alpha^2}{2M_\alpha}\right)|\mathbf{l}_\alpha\rangle\tilde G_{\mathsf{F}\alpha}(\mathbf{p}_\alpha,\mathbf{l}_\alpha;z)\langle \mathbf{l}_\alpha|\boldsymbol{\theta}_\alpha(z;\mathbf{p}_\alpha)|\mathbf{q}_\alpha\rangle\,. \tag{80}$$

In this expression, $\mathbf{K}_\alpha$ stands for the *two-particle* K-matrix. Performing partial-wave expansion in this equation and integrating over the absolute value $|\mathbf{l}_\alpha|$, we arrive at a set of *algebraic* equations that relate $\mathbf{K}_\alpha$ and $\boldsymbol{\theta}_\alpha$. These equations mix all partial waves. Hence, in order to solve them, a partial-wave truncation is necessary. It is easy to recognize that eq. (80) is nothing but the Lüscher formula for a two-particle sub-system in the moving frame (the third particle plays the role of a spectator). The result was of course expected from the beginning.

The equation for $\boldsymbol{\theta}_4$ is given by

$$\langle \mathbf{p}'_\alpha \mathbf{q}'_\alpha | \boldsymbol{\theta}_4 | \mathbf{p}_\beta \mathbf{q}_\beta \rangle = \langle \mathbf{p}'_\alpha \mathbf{q}'_\alpha | \mathbf{K}_4 | \mathbf{p}_\beta \mathbf{q}_\beta \rangle + \frac{1}{L^3} \sum_{\mathbf{k}_\alpha} \int \frac{d^3 \mathbf{l}_\alpha}{(2\pi)^3} \langle \mathbf{p}'_\alpha \mathbf{q}'_\alpha | \mathbf{K}_4 | \mathbf{k}_\alpha \mathbf{l}_\alpha \rangle \tilde{G}_{\mathsf{F}\alpha}(\mathbf{k}_\alpha, \mathbf{l}_\alpha; z) \langle \mathbf{k}_\alpha \mathbf{l}_\alpha | \boldsymbol{\theta}_4 | \mathbf{p}_\beta \mathbf{q}_\beta \rangle \, . \quad (81)$$

It is easy to see that, after performing the integration over the absolute value of the relative momentum $\mathbf{l}_\alpha$, eq. (81) transforms into the matrix equation that can be solved straightforwardly by using numerical methods. The matrix indexes here include the discrete values of the spectator momenta $\mathbf{p}_\alpha, \mathbf{q}_\beta, \mathbf{k}_\alpha$. In order to solve this equation, a method analogous to the one described in ref. [39] can be used.

Finally, we write down the equations (73) in the explicit form

$$\langle \mathbf{p}'_\alpha \mathbf{q}'_\alpha | \mathbf{R}_{4\beta} | \mathbf{p}_\beta \mathbf{q}_\beta \rangle = \int \frac{d^3 \mathbf{q}''_\beta}{(2\pi)^3} \langle \mathbf{p}'_\alpha \mathbf{q}'_\alpha | \boldsymbol{\theta}_4 | \mathbf{p}_\beta \mathbf{q}''_\beta \rangle \tilde{G}_{\mathsf{F}\beta}(\mathbf{p}_\beta, \mathbf{q}''_\beta; z) \langle \mathbf{q}''_\beta | \boldsymbol{\theta}_\beta(z; \mathbf{p}_\beta) | \mathbf{q}_\beta \rangle$$

$$+ \sum_{\gamma=1}^{3} \frac{1}{L^3} \sum_{\mathbf{p}''_\gamma} \int \frac{d^3 \mathbf{q}''_\gamma}{(2\pi)^3} \langle \mathbf{p}'_\alpha \mathbf{q}'_\alpha | \boldsymbol{\theta}_4 | \mathbf{p}''_\gamma \mathbf{q}''_\gamma \rangle \tilde{G}_{\mathsf{F}\gamma}(\mathbf{p}''_\gamma, \mathbf{q}''_\gamma; z) \langle \mathbf{p}''_\gamma \mathbf{q}''_\gamma | \mathbf{R}_{\gamma\beta} | \mathbf{p}_\beta \mathbf{q}_\beta \rangle \, , \quad (82)$$

$$\langle \mathbf{p}'_\alpha \mathbf{q}'_\alpha | \mathbf{R}_{\alpha\beta} | \mathbf{p}_\beta \mathbf{q}_\beta \rangle = \sum_{\gamma=1}^{3} (1 - \delta_{\alpha\gamma}) \int \frac{d^3 \mathbf{q}''_\alpha}{(2\pi)^3} \langle \mathbf{q}'_\alpha | \boldsymbol{\theta}(\mathbf{p}'_\alpha; z) | \mathbf{q}''_\alpha \rangle \tilde{G}_{\mathsf{F}\alpha}(\mathbf{p}'_\alpha, \mathbf{q}''_\alpha; z) \langle \bar{\mathbf{p}}''_\gamma \bar{\mathbf{q}}''_\gamma | \mathbf{R}_{\gamma\beta} | \mathbf{p}_\beta \mathbf{q}_\beta \rangle$$

$$+ \int \frac{d^3 \mathbf{q}''_\alpha}{(2\pi)^3} \langle \mathbf{q}'_\alpha | \boldsymbol{\theta}(\mathbf{p}'_\alpha; z) | \mathbf{q}''_\alpha \rangle \tilde{G}_{\mathsf{F}\alpha}(\mathbf{p}'_\alpha, \mathbf{q}''_\alpha; z) \langle \mathbf{p}'_\alpha \mathbf{q}''_\alpha | \mathbf{R}_{4\beta} | \mathbf{p}_\beta \mathbf{q}_\beta \rangle \, , \quad (83)$$

$$\langle \mathbf{p}'_\alpha \mathbf{q}'_\alpha | \mathbf{R}_{\alpha 4} | \mathbf{p}_\beta \mathbf{q}_\beta \rangle = \sum_{\gamma=1}^{3} (1 - \delta_{\alpha\gamma}) \int \frac{d^3 \mathbf{q}''_\alpha}{(2\pi)^3} \langle \mathbf{q}'_\alpha | \boldsymbol{\theta}(\mathbf{p}'_\alpha; z) | \mathbf{q}''_\alpha \rangle \tilde{G}_{\mathsf{F}\alpha}(\mathbf{p}'_\alpha, \mathbf{q}''_\alpha; z) \langle \bar{\mathbf{p}}''_\gamma \bar{\mathbf{q}}''_\gamma | \mathbf{R}_{\gamma 4} | \mathbf{p}_\beta \mathbf{q}_\beta \rangle$$

$$+ \int \frac{d^3 \mathbf{q}''_\alpha}{(2\pi)^3} \langle \mathbf{q}'_\alpha | \boldsymbol{\theta}(\mathbf{p}'_\alpha; z) | \mathbf{q}''_\alpha \rangle \tilde{G}_{\mathsf{F}\alpha}(\mathbf{p}'_\alpha, \mathbf{q}''_\alpha; z) \langle \mathbf{p}'_\alpha \mathbf{q}''_\alpha | \mathbf{R}_{44} | \mathbf{p}_\beta \mathbf{q}_\beta \rangle \, , \quad (84)$$

$$\langle \mathbf{p}'_\alpha \mathbf{q}'_\alpha | \mathbf{R}_{44} | \mathbf{p}_\beta \mathbf{q}_\beta \rangle = \langle \mathbf{p}'_\alpha \mathbf{q}'_\alpha | \boldsymbol{\theta}_{44} | \mathbf{p}_\beta \mathbf{q}_\beta \rangle$$

$$+ \sum_{\gamma=1}^{3} \frac{1}{L^3} \sum_{\mathbf{p}''_\gamma} \int \frac{d^3 \mathbf{q}''_\gamma}{(2\pi)^3} \langle \mathbf{p}'_\alpha \mathbf{q}'_\alpha | \boldsymbol{\theta}_{44} | \mathbf{p}''_\gamma \mathbf{q}''_\gamma \rangle \tilde{G}_{\mathsf{F}\gamma}(\mathbf{p}''_\gamma, \mathbf{q}''_\gamma; z) \langle \mathbf{p}''_\gamma \mathbf{q}''_\gamma | \mathbf{R}_{\gamma 4} | \mathbf{p}_\beta \mathbf{q}_\beta \rangle \, , \quad (85)$$

where

$$\bar{\mathbf{p}}''_\gamma = -\frac{m_\gamma}{m_\beta + m_\gamma} \mathbf{p}'_\alpha - \mathbf{q}''_\alpha \, , \qquad \bar{\mathbf{q}}''_\gamma = \frac{m_\beta (m_\alpha + m_\beta + m_\gamma)}{(m_\alpha + m_\beta)(m_\beta + m_\gamma)} \mathbf{p}'_\alpha - \frac{m_\alpha}{m_\alpha + m_\beta} \mathbf{q}''_\alpha \, . \quad (86)$$

4.4 The term without a potential insertion

Below we shall start examining the multiple scattering series which are obtained from eq. (67) diagram by diagram, and shall demonstrate – for a few illuminating cases – that our equations indeed produce the desired splitting of the infinite- and finite-volume contributions.

We start from the trivial case of a diagram without rescattering, shown in fig. 1a. This diagram should be folded with the vertex functions $\tilde{\Gamma}, \tilde{\Gamma}^*$. As a result, we arrive at the following expression

$$I_0 = \frac{1}{L^6} \sum_{\mathbf{p}_3\mathbf{q}_3} \frac{\tilde{\Gamma}_3(\mathbf{q}';\mathbf{p}_3\mathbf{q}_3)(\tilde{\Gamma}_3(\mathbf{q};\mathbf{p}_3\mathbf{q}_3))^*}{M + \dfrac{\mathbf{p}_3^2}{2M_3} + \dfrac{\mathbf{q}_3^2}{2\mu_3} - z} \,. \tag{87}$$

This expression is of the type already considered in section 4.1, since the product of two vertices is a regular function of the momenta. Consequently, up to exponentially suppressed terms,

$$\begin{aligned} I_0 &= \mathsf{P.V.} \int \frac{d^3\mathbf{p}_3}{(2\pi)^3} \frac{d^3\mathbf{q}_3}{(2\pi)^3} \frac{\tilde{\Gamma}_3(\mathbf{q}';\mathbf{p}_3\mathbf{q}_3)(\tilde{\Gamma}_3(\mathbf{q};\mathbf{p}_3\mathbf{q}_3))^*}{M + \dfrac{\mathbf{p}_3^2}{2M_3} + \dfrac{\mathbf{q}_3^2}{2\mu_3} - z} \\ &+ \frac{1}{L^3} \sum_{\mathbf{p}_3} \int \frac{d^3\mathbf{q}_3}{(2\pi)^3} \tilde{\Gamma}_3(\mathbf{q}';\mathbf{p}_3\mathbf{q}_3) \tilde{G}_{\mathsf{F}3}(\mathbf{p}_3,\mathbf{q}_3;z)(\tilde{\Gamma}_3(\mathbf{q};\mathbf{p}_3\mathbf{q}_3))^* \,. \end{aligned} \tag{88}$$

4.5 Disconnected contributions

The corresponding diagram is shown in fig. 1b. Expanding $\boldsymbol{\tau}_\alpha$ in Born series, one obtains the diagrams with one, two, ... insertions of the potential. The explicit expression of a diagram with one insertion is given by

$$I_1^d = \sum_{\alpha=1}^{3} \frac{1}{L^9} \sum_{\mathbf{p}_\alpha\mathbf{q}'_\alpha\mathbf{q}_\alpha} \frac{\tilde{\Gamma}_\alpha(\mathbf{q}';\mathbf{p}_\alpha\mathbf{q}'_\alpha)(-\bar{V}_\alpha(\mathbf{q}'_\alpha,\mathbf{q}_\alpha))(\tilde{\Gamma}_\alpha(\mathbf{q};\mathbf{p}_\alpha\mathbf{q}_\alpha))^*}{\left(M + \dfrac{\mathbf{p}_\alpha^2}{2M_\alpha} + \dfrac{(\mathbf{q}'_\alpha)^2}{2\mu_\alpha} - z\right)\left(M + \dfrac{\mathbf{p}_\alpha^2}{2M_\alpha} + \dfrac{\mathbf{q}_\alpha^2}{2\mu\alpha} - z\right)} \,. \tag{89}$$

Carrying out the splitting of the infinite- and finite-volume contributions explicitly, it is easy to see that, up to the exponentially suppressed terms, the above equation can be rewritten as

$$\begin{aligned} I_1^d &= \sum_{\alpha=1}^{3} \int \frac{d^3\mathbf{p}_\alpha}{(2\pi)^3} \frac{d^3\mathbf{q}'_\alpha}{(2\pi)^3} \frac{d^3\mathbf{q}_\alpha}{(2\pi)^3} \tilde{\Gamma}_\alpha(\mathbf{q}';\mathbf{p}_\alpha\mathbf{q}'_\alpha) G_{\mathsf{K}\alpha}(\mathbf{p}_\alpha,\mathbf{q}'_\alpha;z)(-\bar{V}_\alpha(\mathbf{q}'_\alpha,\mathbf{q}_\alpha)) \\ &\qquad \times G_{\mathsf{K}\alpha}(\mathbf{p}_\alpha,\mathbf{q}_\alpha;z)(\tilde{\Gamma}_\alpha(\mathbf{q};\mathbf{p}_\alpha\mathbf{q}_\alpha))^* \\ &+ \sum_{\alpha=1}^{3} \frac{1}{L^3} \sum_{\mathbf{p}_\alpha} \int \frac{d^3\mathbf{q}'_\alpha}{(2\pi)^3} \frac{d^3\mathbf{q}_\alpha}{(2\pi)^3} \tilde{\Gamma}_\alpha(\mathbf{q}';\mathbf{p}_\alpha\mathbf{q}_\alpha) U_{1\alpha}^d(\mathbf{q}'_\alpha,\mathbf{q}_\alpha;\mathbf{p}_\alpha,z)(\tilde{\Gamma}_\alpha(\mathbf{q};\mathbf{p}_\alpha\mathbf{q}_\alpha))^* \,, \end{aligned} \tag{90}$$

where

$$U^d_{1\alpha}(\mathbf{q}'_\alpha,\mathbf{q}_\alpha;\mathbf{p}_\alpha,z) = G_{\mathsf{K}\alpha}(\mathbf{p}_\alpha,\mathbf{q}'_\alpha;z)(-\bar V_\alpha(\mathbf{q}'_\alpha,\mathbf{q}_\alpha))\tilde G_{\mathsf{F}\alpha}(\mathbf{p}_\alpha,\mathbf{q}_\alpha;z) + \tilde G_{\mathsf{F}\alpha}(\mathbf{p}_\alpha,\mathbf{q}'_\alpha;z)(-\bar V_\alpha(\mathbf{q}'_\alpha,\mathbf{q}_\alpha))G_{\mathsf{K}\alpha}(\mathbf{p}_\alpha,\mathbf{q}_\alpha;z) + \tilde G_{\mathsf{F}\alpha}(\mathbf{p}_\alpha,\mathbf{q}'_\alpha;z)(-\bar V_\alpha(\mathbf{q}'_\alpha,\mathbf{q}_\alpha))\tilde G_{\mathsf{F}\alpha}(\mathbf{p}_\alpha,\mathbf{q}_\alpha;z)\,. \tag{91}$$

The generalization for the diagrams with many potential insertions between the same two particles is straightforward. Further, recalling the definition of $\mathbf{K}_{2\to 3}$ and $\mathbf{K}_{3\to 2}$, it is easy to ensure that the diagrams considered in sections 4.4 and 4.5, can be unambiguously identified with the pertinent diagrams emerging in the perturbative expansion of the quantity $\mathbf{K}_{2\to 2}+\mathbf{K}_{2\to 3}(\mathbf{G}_\mathsf{F}+\mathbf{G}_\mathsf{F}\sum_{\alpha=1}^3\boldsymbol{\theta}_\alpha\mathbf{G}_\mathsf{F})\mathbf{K}_{3\to 2}$.

4.6 Connected contributions

The connected contributions emerge as a result of the expansion of the diagrams shown in fig. 1c in powers of the potentials $\bar V_\alpha$. The lowest-order term has $O(\bar V_\alpha^2)$ and is given by

$$I_2^c = \sum_{\alpha\neq\beta}\frac{1}{L^{12}}\sum_{\mathbf{p}'_\alpha\mathbf{q}'_\alpha\mathbf{p}_\beta\mathbf{q}_\beta}\frac{\tilde\Gamma_\alpha(\mathbf{q}';\mathbf{p}'_\alpha\mathbf{q}'_\alpha)(-\bar V_\alpha(\mathbf{q}'_\alpha,\mathbf{p}_\beta+\dfrac{m_\beta}{M-m_\alpha}\mathbf{p}'_\alpha))}{\left(M+\dfrac{(\mathbf{p}'_\alpha)^2}{2M_\alpha}+\dfrac{(\mathbf{q}'_\alpha)^2}{2\mu_\alpha}-z\right)\left(M+\dfrac{(\mathbf{p}'_\alpha)^2}{2M_\alpha}+\dfrac{(\mathbf{p}_\beta+\dfrac{m_\beta}{M-m_\alpha}\mathbf{p}'_\alpha)^2}{2\mu_\alpha}-z\right)}$$
$$\times\;\frac{(-\bar V_\beta(-\mathbf{p}'_\alpha-\dfrac{m_\alpha}{M-m_\beta}\mathbf{p}_\beta,\mathbf{q}_\beta))(\tilde\Gamma_\beta(\mathbf{q};\mathbf{p}_\beta\mathbf{q}_\beta))^*}{\left(M+\dfrac{\mathbf{p}_\beta^2}{2M_\beta}+\dfrac{\mathbf{q}_\beta^2}{2\mu_\beta}-z\right)}\,. \tag{92}$$

Up to the exponentially suppressed terms, this expression can be rewritten as

$$I_2^c = \sum_{\alpha\neq\beta}\int\frac{d^3\mathbf{p}'_\alpha}{(2\pi)^3}\frac{d^3\mathbf{q}'_\alpha}{(2\pi)^3}\frac{d^3\mathbf{p}_\beta}{(2\pi)^3}\frac{d^3\mathbf{q}_\beta}{(2\pi)^3}\,\tilde\Gamma_\alpha(\mathbf{q}';\mathbf{p}'_\alpha\mathbf{q}'_\alpha)U^{c,0}_{2\alpha\beta}(\mathbf{p}'_\alpha\mathbf{q}'_\alpha;\mathbf{p}_\beta\mathbf{q}_\beta)(\tilde\Gamma_\beta(\mathbf{q};\mathbf{p}_\beta\mathbf{q}_\beta))^*$$
$$+\sum_{\alpha\neq\beta}\frac{1}{L^3}\sum_{\mathbf{p}'_\alpha}\int\frac{d^3\mathbf{q}'_\alpha}{(2\pi)^3}\frac{d^3\mathbf{p}_\beta}{(2\pi)^3}\frac{d^3\mathbf{q}_\beta}{(2\pi)^3}\,\tilde\Gamma_\alpha(\mathbf{q}';\mathbf{p}'_\alpha\mathbf{q}'_\alpha)U^{c,1}_{2\alpha\beta}(\mathbf{p}'_\alpha\mathbf{q}'_\alpha;\mathbf{p}_\beta\mathbf{q}_\beta)(\tilde\Gamma_\beta(\mathbf{q};\mathbf{p}_\beta\mathbf{q}_\beta))^*$$
$$+\sum_{\alpha\neq\beta}\frac{1}{L^3}\sum_{\mathbf{p}_\beta}\int\frac{d^3\mathbf{p}'_\alpha}{(2\pi)^3}\frac{d^3\mathbf{q}'_\alpha}{(2\pi)^3}\frac{d^3\mathbf{q}_\beta}{(2\pi)^3}\,\tilde\Gamma_\alpha(\mathbf{q}';\mathbf{p}'_\alpha\mathbf{q}'_\alpha)U^{c,2}_{2\alpha\beta}(\mathbf{p}'_\alpha\mathbf{q}'_\alpha;\mathbf{p}_\beta\mathbf{q}_\beta)(\tilde\Gamma_\beta(\mathbf{q};\mathbf{p}_\beta\mathbf{q}_\beta))^*$$
$$+\sum_{\alpha\neq\beta}\frac{1}{L^6}\sum_{\mathbf{p}'_\alpha\mathbf{p}_\beta}\int\frac{d^3\mathbf{q}'_\alpha}{(2\pi)^3}\frac{d^3\mathbf{q}_\beta}{(2\pi)^3}\,\tilde\Gamma_\alpha(\mathbf{q}';\mathbf{p}'_\alpha\mathbf{q}'_\alpha)U^{c,3}_{2\alpha\beta}(\mathbf{p}'_\alpha\mathbf{q}'_\alpha;\mathbf{p}_\beta\mathbf{q}_\beta)(\tilde\Gamma_\beta(\mathbf{q};\mathbf{p}_\beta\mathbf{q}_\beta))^*\,, \tag{93}$$

where

$$U^{c,0}_{2\alpha\beta}(\mathbf{p}'_\alpha\mathbf{q}'_\alpha;\mathbf{p}_\beta\mathbf{q}_\beta) = G_{\mathsf{K}\alpha}(\mathbf{p}'_\alpha,\mathbf{q}'_\alpha;z)(-\bar V_\alpha(\mathbf{q}'_\alpha,\bar{\mathbf{q}}''_\alpha))G_{\mathsf{K}\alpha}(\mathbf{p}'_\alpha,\bar{\mathbf{q}}''_\alpha;z)(-\bar V_\beta(\bar{\mathbf{q}}'''_\beta,\mathbf{q}_\beta))G_{\mathsf{K}\beta}(\mathbf{p}_\beta,\mathbf{q}_\beta;z), \tag{94}$$

$$
\begin{aligned}
U^{c,1}_{2\alpha\beta}(\mathbf{p}'_\alpha\mathbf{q}'_\alpha;\mathbf{p}_\beta\mathbf{q}_\beta) \;=\; & G_{\mathsf{K}\alpha}(\mathbf{p}'_\alpha,\mathbf{q}'_\alpha;z)(-\bar{V}_\alpha(\mathbf{q}'_\alpha,\bar{\mathbf{q}}''_\alpha))\tilde{G}_{\mathsf{F}\alpha}(\mathbf{p}'_\alpha,\bar{\mathbf{q}}''_\alpha;z)(-\bar{V}_\beta(\bar{\mathbf{q}}'''_\beta,\mathbf{q}_\beta))G_{\mathsf{K}\beta}(\mathbf{p}_\beta,\mathbf{q}_\beta;z) \\
+\; & \tilde{G}_{\mathsf{F}\alpha}(\mathbf{p}'_\alpha,\mathbf{q}'_\alpha;z)(-\bar{V}_\alpha(\mathbf{q}'_\alpha,\bar{\mathbf{q}}''_\alpha))G_{\mathsf{K}\alpha}(\mathbf{p}'_\alpha,\bar{\mathbf{q}}''_\alpha;z)(-\bar{V}_\beta(\bar{\mathbf{q}}'''_\beta,\mathbf{q}_\beta))G_{\mathsf{K}\beta}(\mathbf{p}_\beta,\mathbf{q}_\beta;z) \\
+\; & \tilde{G}_{\mathsf{F}\alpha}(\mathbf{p}'_\alpha,\mathbf{q}'_\alpha;z)(-\bar{V}_\alpha(\mathbf{q}'_\alpha,\bar{\mathbf{q}}''_\alpha))\tilde{G}_{\mathsf{F}\alpha}(\mathbf{p}'_\alpha,\bar{\mathbf{q}}''_\alpha;z)(-\bar{V}_\beta(\bar{\mathbf{q}}'''_\beta,\mathbf{q}_\beta))G_{\mathsf{K}\beta}(\mathbf{p}_\beta,\mathbf{q}_\beta;z)\,,
\end{aligned}
\tag{95}
$$

$$
\begin{aligned}
U^{c,2}_{2\alpha\beta}(\mathbf{p}'_\alpha\mathbf{q}'_\alpha;\mathbf{p}_\beta\mathbf{q}_\beta) \;=\; & G_{\mathsf{K}\alpha}(\mathbf{p}'_\alpha,\mathbf{q}'_\alpha;z)(-\bar{V}_\alpha(\mathbf{q}'_\alpha,\bar{\mathbf{q}}''_\alpha))G_{\mathsf{K}\beta}(\mathbf{p}_\beta,\bar{\mathbf{q}}'''_\beta;z)(-\bar{V}_\beta(\bar{\mathbf{q}}'''_\beta,\mathbf{q}_\beta))\tilde{G}_{\mathsf{F}\beta}(\mathbf{p}_\beta,\mathbf{q}_\beta;z) \\
+\; & G_{\mathsf{K}\alpha}(\mathbf{p}'_\alpha,\mathbf{q}'_\alpha;z)(-\bar{V}_\alpha(\mathbf{q}'_\alpha,\bar{\mathbf{q}}''_\alpha))\tilde{G}_{\mathsf{F}\beta}(\mathbf{p}_\beta,\bar{\mathbf{q}}'''_\beta;z)(-\bar{V}_\beta(\bar{\mathbf{q}}'''_\beta,\mathbf{q}_\beta))\tilde{G}_{\mathsf{F}\beta}(\mathbf{p}_\beta,\mathbf{q}_\beta;z),
\end{aligned}
\tag{96}
$$

$$
U^{c,3}_{2\alpha\beta}(\mathbf{p}'_\alpha\mathbf{q}'_\alpha;\mathbf{p}_\beta\mathbf{q}_\beta) \;=\; \tilde{G}_{\mathsf{F}\alpha}(\mathbf{p}'_\alpha,\mathbf{q}'_\alpha;z)\frac{(-\bar{V}_\alpha(\mathbf{q}'_\alpha,\bar{\mathbf{q}}''_\alpha))(-\bar{V}_\beta(\bar{\mathbf{q}}'''_\beta,\mathbf{q}_\beta))}{\left(M+\dfrac{\mathbf{p}^2_\beta}{2M_\beta}+\dfrac{(\bar{\mathbf{q}}'''_\beta)^2}{2\mu_\beta}-z\right)}\tilde{G}_{\mathsf{F}\beta}(\mathbf{p}_\beta,\mathbf{q}_\beta;z)\,,
\tag{97}
$$

and

$$
\bar{\mathbf{q}}''_\alpha = \mathbf{p}_\beta + \frac{m_\beta}{M-m_\alpha}\,\mathbf{p}'_\alpha\,, \qquad \bar{\mathbf{q}}'''_\beta = -\mathbf{p}'_\alpha - \frac{m_\alpha}{M-m_\beta}\,\mathbf{p}_\beta\,. \tag{98}
$$

It is seen that individual terms in eqs. (94)-(97) can be unambiguously identified with the pertinent terms in the multiple-scattering series that emerges from the system of equations displayed in section 4.3. Moreover, as already mentioned before, it is seen that, in order to make these two series coincide, one should omit the terms of the type $\boldsymbol{\theta}_\alpha\mathbf{G}_{\mathsf{F}}\boldsymbol{\theta}_\beta$ *with* $\alpha\neq\beta$ in the equation for $\mathbf{R}_{\alpha\beta}$ (third line in eq. (73)).

The triple scattering is relegated to Appendix E. It can be checked again that to this order the multiple-scattering series which are produced by the system of equations displayed in section 4.3 can be unambiguously identified. The generalization to the higher orders is straightforward.

4.7 Induced three-body force

The inclusion of the induced three-body force, which is described by the operator $\mathbf{K}_4$, does not lead to any complication, since $\mathbf{K}_4$ is a smooth function of all momenta. Moreover, at this stage it is seen that a genuine three-body force could be included without any further ado.

4.8 Analytic continuation and the cusp effect

In the three-body counterpart of the Lüscher formula, the *on-shell* T-matrix elements *below threshold* are necessarily present, due to the fact that the on-shell projector Δ defined in eq. (53) does not vanish for $q_0^2 < 0$. Instead, a smooth regulator $f(q_0^2/\mu^2)$ is introduced in eq. (53), which effectively suppresses the contributions with $-q_0^2 > \mu^2$. However, since in the original sum the variable $\mathbf{p}^2$ was positively defined, a question naturally arises, whether such an analytic continuation is needed at all. Indeed, as it is known, one can avoid this procedure in the two-particle case, even if multiple scattering channels are considered.

The reason why the analytic continuation in the three-particle case can not be avoided, lies in the following: if, instead of the smooth function f, a sharp cutoff $\theta(q_0^2)$ is introduced, the principal-value integral will have *a unitary cusp* at threshold, proportional to the factor $\sqrt{-q_0^2}$. In the two-particle case, the quantity q_0^2 depends on z only and the cusp does not cause a problem. In the three-particle case, q_0^2 depends in addition on the spectator momentum $\mathbf{p}_\alpha$, see eq. (62). Due to the presence of the cusp, one can not apply the regular summation theorem: the finite volume corrections are suppressed by a power of L, not by exponentials. In order to see this, let us consider a simple example with the cusp present

$$I = \frac{1}{L^3}\sum_{\mathbf{p}} |\mathbf{p}|\,\theta(\Lambda^2-\mathbf{p}^2) = \int^\Lambda d^3\mathbf{k}\,|\mathbf{k}|\,\frac{1}{L^3}\sum_{\mathbf{p}}\delta^3(\mathbf{p}-\mathbf{k})\,. \tag{99}$$

Using Poisson's summation formula, the above expression can be transformed into

$$I = \int^\Lambda \frac{d^3\mathbf{k}}{(2\pi)^3}\,|\mathbf{k}| + \sum_{\mathbf{n}\in\mathbb{Z}^3\backslash\mathbf{0}} \int^\Lambda \frac{d^3\mathbf{k}}{(2\pi)^3}\,|\mathbf{k}|\,e^{i\mathbf{n}\mathbf{k}L} = I_1 + I_2\,. \tag{100}$$

Carrying out the integration in the second term, it is easy to see that it vanishes as L^{-2} and not as an exponential, see Appendix D.

To summarize, the price to pay for the exponential fall-off of the finite-volume corrections in the regular terms[6] that contain principal-value integrals is that these principal-value integrals should be defined through the analytic continuation below threshold. If one wishes to use the information from the physical region only, the finite-volume corrections in the regular terms will be power-suppressed, not exponentially suppressed.

5 Conclusions

i) In this paper we have derived the three-body counterpart of the Lüscher formula. The pertinent expressions are displayed in section 4.3.

ii) The fundamental property of the finite-volume spectrum, which follows from this formula, is that the spectrum is completely determined by the S-matrix elements for the transitions

[6] More precisely, the finite-volume corrections in the regular terms fall off faster than any inverse power of L [40].

$2 \to 2$,$2 \to 3$ and $3 \to 3$ in the infinite volume. Consequently, two different potential models with the same S-matrix elements lead to the same spectra up to the exponentially suppressed corrections.

iii) The equations given in section 4.3 have a complicated structure. However, due to the presence of the on-shell factor $\Delta(\mathbf{k}^2, q_0^2)$, the dimensionality of equations is reduced as compared to the original Faddeev equations. Namely, in the Faddeev equations we have two momenta describing the three-particle intermediate state. The integration over one of these momenta is removed by the on-shell factor. This is similar to the conventional Lüscher formula, which becomes an algebraic equation (after the truncation of the partial-wave expansion), whereas the original Lippmann-Schwinger equation was an integral equation in one momentum variable. Despite this simplification, a direct numerical solution of the three-body equations in a finite volume, as described in section 2.2, may still prove to be less challenging.

iv) In this paper, we restrict ourselves to the non-relativistic potential model. Considering the processes within the field theory will introduce several novel aspects, *e.g.*, relativistic effects, particle creation and annihilation, etc. However, we do not expect that these effects will upset the proof given in the present paper. Further investigations are planned in this direction in the future, and the results will be reported elsewhere.

v) It is legitimate to ask, whether it is possible to use equations from section 4.3 in order to extract the S-matrix elements from the measured spectrum. Due to the complex nature of these equations, a straightforward extraction can be very complicated. Adopting a strategy similar to that of ref. [33] – introducing a phenomenologically reasonable parameterization of the potential and fitting the parameters of the potential to the lattice data – could be more promising. The physical observables in the infinite volume (S-matrix elements, resonance pole positions) can be then obtained from the solution of the scattering equations.

vi) From the above discussion it becomes clear that it would be very interesting to carry out calculations of the finite-volume spectrum in different realistic models (see, *e.g.* [44, 45]), which predict both the Roper resonance and $N(1535)$ in the infinite volume. These calculations may shed light on the puzzle related to the "wrong" level ordering for such systems in lattice QCD. The findings of the present paper guarantee that the spectrum will stay stable with respect to the choice of a model and the variation of its parameters, provided the S-matrix elements in the infinite volume remain the same.

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A Three identical particles with a separable potential

In this Appendix, we list the expressions, which appeared first in section 2, in a special case of the pair potentials having separable form. Further, we restrict ourselves to the case of three identical particles with the masses $m_1 = m_2 = m_3 = m$.

The free-particle states of n identical particles are normalized, according to

$$|\mathbf{k}_1 \cdots \mathbf{k}_n\rangle = \frac{1}{\sqrt{n!}}\, a^\dagger(\mathbf{k}_1)\cdots a^\dagger(\mathbf{k}_n)|0\rangle\,. \tag{A.1}$$

In case of the identical particles, there is only one term in the two-body Hamiltonian $\mathbf{H}_{2\to 2}$, given by eq. (1) – the sum over α, β drops out. Introducing the appropriate symmetry factors, the two-particle potential in the separable model can be written in the form

$$\bar{V}_\alpha(\mathbf{q}',\mathbf{q}) = \frac{1}{2!}\, v(\mathbf{q}')v(\mathbf{q})\,. \tag{A.2}$$

Assuming that the vertex $\bar{\Gamma}_\alpha$ is also separable

$$\bar{\Gamma}_\alpha(\mathbf{q}';\mathbf{p},\mathbf{q}) = \frac{1}{\sqrt{2!3!}}\, \lambda(\mathbf{q}')f(\mathbf{p},\mathbf{q})\,. \tag{A.3}$$

The effective two-particle potential then can be written as

$$\langle \mathbf{q}'|\mathbf{w}(z)|\mathbf{q}\rangle = -v(\mathbf{q}')v(\mathbf{q}) + \lambda(\mathbf{q}')d(z)\lambda(\mathbf{q})\,, \tag{A.4}$$

where $d(z)$ is given by

$$d(z) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d^3\mathbf{q}}{(2\pi)^3}\,\frac{f^2(\mathbf{p},\mathbf{q})}{3m + \dfrac{3\mathbf{p}^2}{4m} + \dfrac{\mathbf{q}^2}{m} - z - i0}$$
$$+ \int \frac{d^3\mathbf{p}'}{(2\pi)^3}\frac{d^3\mathbf{q}'}{(2\pi)^3}\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d^3\mathbf{q}}{(2\pi)^3}\,\frac{f(\mathbf{p}',\mathbf{q}')\langle \mathbf{p}'\mathbf{q}'|\hat{\mathbf{m}}(z)|\mathbf{p}\mathbf{q}\rangle f(\mathbf{p},\mathbf{q})}{\left(3m + \dfrac{3(\mathbf{p}')^2}{4m} + \dfrac{(\mathbf{q}')^2}{m} - z - i0\right)\left(3m + \dfrac{3\mathbf{p}^2}{4m} + \dfrac{\mathbf{q}^2}{m} - z - i0\right)}\,, \tag{A.5}$$

where

$$\langle \mathbf{p}'\mathbf{q}'|\hat{\mathbf{m}}(z)|\mathbf{p}\mathbf{q}\rangle = \frac{1}{3}\sum_{\alpha,\beta=1}^{3} v(\mathbf{q}'_\alpha)\Big\{(2\pi)^3\delta^3(\mathbf{p}'_\alpha - \mathbf{p}_\beta)\tau\Big(z - m - \frac{3\mathbf{p}_\alpha^2}{4m}\Big)$$
$$+ \tau\Big(z - m - \frac{3(\mathbf{p}'_\alpha)^2}{4m}\Big)\langle \mathbf{p}'_\alpha|\mathbf{Y}(z)|\mathbf{p}_\beta\rangle\tau\Big(z - m - \frac{3\mathbf{p}_\beta^2}{4m}\Big)\Big\}v(\mathbf{q}_\beta)\,, \tag{A.6}$$

with

$$\mathbf{p}_1 = -\frac{1}{2}\mathbf{p} + \mathbf{q}\,,\quad \mathbf{p}_2 = -\frac{1}{2}\mathbf{p} - \mathbf{q}\,,\quad \mathbf{p}_3 = \mathbf{p}\,,$$
$$\mathbf{q}_1 = -\frac{3}{4}\mathbf{p} - \frac{1}{2}\mathbf{q}\,,\quad \mathbf{q}_2 = \frac{3}{4}\mathbf{p} - \frac{1}{2}\mathbf{q}\,,\quad \mathbf{q}_3 = \mathbf{q}\,, \tag{A.7}$$

and similarly for $\mathbf{p}'_\alpha, \mathbf{q}'_\alpha$.

Further, the quantity $\tau(z)$ is defined by

$$\tau(z) = -\left(1 + \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{v^2(\mathbf{q})}{2m + \dfrac{\mathbf{q}^2}{m} - z - i0}\right)^{-1} . \tag{A.8}$$

Finally, the quantity $\langle \mathbf{p}'|\mathbf{Y}(z)|\mathbf{p}\rangle$ obeys the equation

$$\langle \mathbf{p}'|\mathbf{Y}(z)|\mathbf{p}\rangle = 2Z(\mathbf{p}',\mathbf{p}) + \int \frac{d^3\mathbf{p}''}{(2\pi)^3} 2Z(\mathbf{p}',\mathbf{p}'')\tau\left(z - m - \frac{3(\mathbf{p}'')^2}{4m}\right)\langle \mathbf{p}''|\mathbf{Y}(z)|\mathbf{p}\rangle , \tag{A.9}$$

with the kernel

$$Z(\mathbf{p}',\mathbf{p}) = \frac{v(\mathbf{p}'/2 + \mathbf{p})v(\mathbf{p}' + \mathbf{p}/2)}{3m + \dfrac{(\mathbf{p}')^2 + \mathbf{p}^2 + \mathbf{p}'\mathbf{p}}{m} - z - i0} . \tag{A.10}$$

B Separable potential in a finite volume

Below, as in Appendix A, we restrict ourselves to the case of three identical particles interacting with the separable pair potentials. The finite-volume versions of eqs. (A.5)-(A.9) are obtained by merely replacing the integrals by the sums. We do not display these equations explicitly, with one exception: the finite-volume version of eq. (A.8) is given by

$$\tau^L(z) = -\left(1 + \frac{1}{L^3}\sum_{\mathbf{q}} \frac{v^2(\mathbf{q})}{2m + \dfrac{\mathbf{q}^2}{m} - z}\right)^{-1} , \tag{B.1}$$

where, as already mentioned above, the sum over the momentum $\mathbf{q}$ runs over (cf. eq. (32))

$$\mathbf{q} = \frac{2\pi}{L}\left(\mathbf{l} + \frac{1}{2}\mathbf{n}\right) , \qquad \mathbf{l},\mathbf{n} = \mathbb{Z}^3 . \tag{B.2}$$

Here, the shift of the discrete variable $\mathbf{l}$ is related to the CM motion of a two-particle pair in the rest frame of three particles.

Using the regular summation theorem [40], we obtain that, up to the terms exponentially suppressed at large values of L,

$$\tau^L(z(p)) = \left(\frac{mv^2(p)}{4\pi}\right)^{-1}\left(p\cot\delta(p) - \frac{2}{\sqrt{\pi}L}Z^\theta_{00}(1;\nu^2)\right)^{-1} , \quad z(p) = 2m + \frac{p^2}{m} ,$$

$$p\cot\delta(p) = -\frac{4\pi}{mv^2(p)}\left(1 + \frac{m}{2\pi^2}\,\text{P.V.}\int_0^\infty \frac{dq\,q^2v^2(q)}{q^2 - p^2}\right) ,$$

$$Z^\theta_{00}(1;\nu^2) = \frac{1}{\sqrt{4\pi}}\sum_{\mathbf{l}\in\mathbb{Z}^3} \frac{1}{(\mathbf{l} + \boldsymbol{\theta}/2\pi)^2 - \nu^2} , \quad \nu = \frac{pL}{2\pi} , \quad \theta_i = 2\pi\left(\frac{n_i}{2} - \left[\frac{n_i}{2}\right]\right) . \tag{B.3}$$

In other words, the components θ_i of the vector $\boldsymbol{\theta}$ are either 0 or π, for the even/odd components of the vector $\mathbf{n}$.

It is interesting to compare the above equations to the three-body equations in a finite volume, which were considered in refs. [46–48]. A straightforward comparison leads to the conclusion that the sole difference between these equations consists in the replacement $Z_{00}^{\theta}(1;\nu^2)$ by $Z_{00}(1;\nu^2)$, *i.e.*, in neglecting the CM motion of the two-particle sub-systems in the rest frame of three particles. Note also that the parameter $\boldsymbol{\theta}$ is related to the topological phase considered in ref. [49].

C Summation in eq. (41)

The multiple-scattering series in eq. (40) can be reproduced through the following system of linear equations

$$\begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 & y_2 & y_3 \\ y_1 & 0 & y_3 \\ y_1 & y_2 & 0 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} . \tag{C.1}$$

It is straightforward to ensure that

$$\frac{1}{2}(T_1+T_2+T_3-1) = 1+(y_1+y_2+y_3)+(y_1(y_2+y_3)+y_2(y_1+y_3)+y_3(y_1+y_2))+\cdots\,, \tag{C.2}$$

which has exactly the same structure as the series in eq. (40).

On the other hand, eq. (C.1) can be solved directly. The solution of this equation is given in eq. (41).

D The asymptotic behavior of the last term in eq. (100)

Carrying out the integration in $\mathbf{k}$, the last term in eq. (100) can be written as

$$I_2 = \frac{1}{2\pi^2 L^4} \sum_{\mathbf{n}\in\mathbb{Z}\backslash\mathbf{0}} \left\{ -\frac{2}{|\mathbf{n}|^4} + 2\,\mathrm{Re}\,J_4(x) + 2x\;\,\mathrm{Im}\,J_3(x) - x^2\,\mathrm{Re}\,J_2(x) \right\}, \qquad x = L\Lambda\,, \tag{D.1}$$

where

$$J_i(z) = \sum_{\mathbf{n}\in\mathbb{Z}\backslash\mathbf{0}} \frac{z^{|\mathbf{n}|}}{|\mathbf{n}|^i}\,, \qquad z = e^{ix}\,. \tag{D.2}$$

Using the relation from ref. [39]

$$\sum_{\mathbf{n}\in\mathbb{Z}} z^{|\mathbf{n}|} = g_P(z) \qquad g_P(z) = (\theta_3(0,z))^3\,, \tag{D.3}$$

where $\theta_3(0,z)$ is the elliptic θ-function

$$\theta_3(0,z) = \sum_{k=-\infty}^{\infty} z^{k^2}\,, \tag{D.4}$$

which obeys the following integral representation,

$$\theta_3(0,z) = -i\int_{i-\infty}^{i+\infty} du\, z^{u^2}\cot(\pi u)\,, \tag{D.5}$$

a set of differential equations, which relate $J_i(z)$ with $g_P(z)$, can be derived. For example, for $i=1$, the pertinent equation has the form

$$\frac{dJ_1(z)}{dz} = \frac{g_P(z)-1}{z}\,. \tag{D.6}$$

Equations for $i = 2,3,4,\cdots$ can be obtained in the similar fashion. Integrating these differential equations, the following expression for $J_i(z)$ can be straightforwardly obtained

$$J_i(z) = \frac{(-1)^{i-1}}{(i-1)!}\int_0^1 dy\,(\ln y)^{i-1}\frac{g_P(zy)-1}{y}\,, \qquad i = 2,3,4\,. \tag{D.7}$$

Further, from eq. (D.4) it follows that

$$|\theta_3(0, ye^{ix})| \leq \theta_3(0,y)\,, \qquad \theta_3(0,y) \geq 1\,, \qquad \textit{for } y \geq 0\,. \tag{D.8}$$

From the above equation, we readily obtain

$$|(\theta_3(0,ye^{ix}))^3 - 1| \leq (\theta_3(0,y))^3 - 1\,. \tag{D.9}$$

Consequently,

$$|J_i(z)| \leq \frac{1}{(i-1)!}\int_0^1 dy|\ln y|^{i-1}\frac{|g_P(y)-1|}{y}\,. \tag{D.10}$$

The above integral converges at $y=0$. Further, since the series in eq. (D.4) converges for $|y|<1$, the divergence in the integral may occur only on the upper limit $y=1$. Indeed, using the integral representation, one finds that $\theta_3(0,y) \sim (1-y)^{-1/2}$ as $y \to 1^-$. However, due to the fact that $\ln y$ vanishes as $y \to 1$, the singularity in the integrand is of the integrable type. Consequently, $|J_i(z)|$are uniformly bound from above and therefore, the quantity $L^2 I_2$ is also bound at $L \to \infty$.

Last but not least, we wish to address here the issue of using a sharp cutoff at a momentum Λ. As seen, *e.g.*, from eqs. (D.1) and (D.2), the expression for the quantity I_2 contains a sum of rapidly oscillating terms proportional to $\sin(|\mathbf{n}|L\Lambda)$ and $\cos(|\mathbf{n}|L\Lambda)$, as $L \to \infty$. These terms are the artifacts of using a sharp cutoff and should disappear when the cutoff is removed, since no observable effect in the infrared should emerge from the ultraviolet cutoff. However, the expressions diverge at $\Lambda \to \infty$. In order to tackle this problem, the simplest way is to introduce an additional smooth cutoff in the expressions

$$I_2 = \sum_{\mathbf{n}\in\mathbb{Z}\setminus\mathbf{0}} J_n(\varepsilon)\,, \qquad J_n(\varepsilon) = \int^{\Lambda} \frac{d^3\mathbf{p}}{(2\pi)^3}\,\exp(-\varepsilon\mathbf{p}^2)\,|\mathbf{p}|e^{i\mathbf{n}\mathbf{p}L}\,, \tag{D.11}$$

and then consider the limits $\Lambda \to \infty$ and $\varepsilon \to 0$ (in this order). It is easy to show that

$$\lim_{\varepsilon\to 0}\lim_{\Lambda\to\infty} J_n = \frac{1}{2\pi^2 L^4}\,\frac{1}{\mathbf{n}^4}\,\lim_{\varepsilon'\to 0}\int_0^\infty dp\,p^2 e^{-\varepsilon' p^2}\sin p\,, \qquad \varepsilon' = \frac{\varepsilon}{\mathbf{n}^2 L^2}\,. \tag{D.12}$$

Now using the equality [50]

$$\lim_{\varepsilon'\to 0}\int_0^\infty dp\,p^2 e^{-\varepsilon' p^2}\sin p = \lim_{\varepsilon'\to 0}\left(-\frac{d}{d\varepsilon'}\left\{\frac{1}{2\varepsilon'}\,{}_1F_1\left(1;\frac{3}{2};-\frac{1}{4\varepsilon'}\right)\right\}\right) = -2\,, \tag{D.13}$$

where ${}_1F_1(a;b;z)$ denotes the confluent hypergeometric function, we finally get

$$I_2 = -\frac{1}{\pi^2 L^4}\sum_{\mathbf{n}\in\mathbb{Z}\setminus\mathbf{0}}\frac{1}{|\mathbf{n}|^4} = O(L^{-4})\,. \tag{D.14}$$

Comparing with eq. (D.1), one sees that all the oscillating terms disappear, as expected. However, the final expression is still only power-suppressed in L, not exponentially suppressed.

E Triple scattering diagram

Below we shall consider the triple scattering diagram, which is given by

$$\begin{aligned}
I_3^c = {} & \sum_{\alpha\neq\beta,\beta\neq\gamma}\frac{1}{L^{15}}\sum_{\mathbf{p}''_\alpha\mathbf{p}'_\beta\mathbf{p}_\gamma\mathbf{q}''_\alpha\mathbf{q}_\gamma}\frac{\tilde\Gamma_\alpha(\mathbf{q}';\mathbf{p}''_\alpha\mathbf{q}''_\alpha)(-\bar V_\alpha(\mathbf{q}''_\alpha,\mathbf{p}'_\beta+\dfrac{m_\beta}{M-m_\alpha}\,\mathbf{p}''_\alpha))}{\left(M+\dfrac{(\mathbf{p}''_\alpha)^2}{2M_\alpha}+\dfrac{(\mathbf{q}''_\alpha)^2}{2\mu_\alpha}-z\right)}\\
\times\; & \frac{(-\bar V_\beta(-\mathbf{p}''_\alpha-\dfrac{m_\alpha}{M-m_\beta}\,\mathbf{p}'_\beta,-\mathbf{p}_\gamma-\dfrac{m_\gamma}{M-m_\beta}\,\mathbf{p}'_\beta))(-\bar V_\gamma(\mathbf{p}'_\beta+\dfrac{m_\beta}{M-m_\gamma}\,\mathbf{p}_\gamma,\mathbf{q}_\gamma))}{\left(M+\dfrac{(\mathbf{p}'_\beta)^2}{2M_\beta}+\dfrac{(-\mathbf{p}''_\alpha-\dfrac{m_\alpha}{M-m_\beta}\,\mathbf{p}'_\beta)^2}{2\mu_\beta}-z\right)}\\
\times\; & \frac{(\tilde\Gamma_\gamma(\mathbf{q};\mathbf{p}_\gamma\mathbf{q}_\gamma))^*}{\left(M+\dfrac{\mathbf{p}_\gamma^2}{2M_\gamma}+\dfrac{\mathbf{q}_\gamma^2}{2\mu_\gamma}-z\right)\left(M+\dfrac{(\mathbf{p}'_\beta)^2}{2M_\beta}+\dfrac{(-\mathbf{p}_\gamma-\dfrac{m_\gamma}{M-m_\beta}\,\mathbf{p}'_\beta)^2}{2\mu_\beta}-z\right)}\,.
\end{aligned} \tag{E.1}$$

In this expression, first the summations over the momenta $\mathbf{q}''_\alpha, \mathbf{q}_\gamma$ are carried out, similarly as in section 4.6. Further, with the use of the formal relation (see Appendix F)

$$\begin{aligned}
&\frac{1}{L^3}\sum_{\mathbf{k}'_\beta} \frac{(2\pi)^3\delta^3(\mathbf{p}'_\beta-\mathbf{k}'_\beta)}{\left(M+\frac{(\mathbf{p}''_\alpha)^2}{2M_\alpha}+\frac{(\mathbf{k}'_\beta+\frac{m_\beta}{M-m_\alpha}\,\mathbf{p}''_\alpha)^2}{2\mu_\alpha}-z\right)\left(M+\frac{(\mathbf{p}_\gamma)^2}{2M_\gamma}+\frac{(\mathbf{k}'_\beta+\frac{m_\beta}{M-m_\gamma}\,\mathbf{p}_\gamma)^2}{2\mu_\gamma}-z\right)} \\
&\qquad = G_{\mathsf{K}\alpha}\left(\mathbf{p}''_\alpha, \mathbf{p}'_\beta+\frac{m_\beta}{M-m_\alpha}\,\mathbf{p}''_\alpha; z\right) G_{\mathsf{K}\gamma}\left(\mathbf{p}_\gamma, \mathbf{p}'_\beta+\frac{m_\beta}{M-m_\gamma}\,\mathbf{p}_\gamma; z\right) \\
&\qquad + \frac{\tilde{G}_{\mathsf{F}\alpha}\left(\mathbf{p}''_\alpha, \mathbf{p}'_\beta+\frac{m_\beta}{M-m_\alpha}\,\mathbf{p}''_\alpha; z\right)}{\left(M+\frac{(\mathbf{p}_\gamma)^2}{2M_\gamma}+\frac{(\mathbf{p}'_\beta+\frac{m_\beta}{M-m_\gamma}\,\mathbf{p}_\gamma)^2}{2\mu_\gamma}-z\right)} \\
&\qquad + \frac{\tilde{G}_{\mathsf{F}\gamma}\left(\mathbf{p}_\gamma, \mathbf{p}'_\beta+\frac{m_\beta}{M-m_\gamma}\,\mathbf{p}_\gamma; z\right)}{\left(M+\frac{(\mathbf{p}''_\alpha)^2}{2M_\alpha}+\frac{(\mathbf{p}'_\beta+\frac{m_\beta}{M-m_\alpha}\,\mathbf{p}''_\alpha)^2}{2\mu_\alpha}-z\right)}\,,
\end{aligned} \tag{E.2}$$

as well as the symmetry properties of the denominators

$$\begin{aligned}
\frac{(\mathbf{p}'_\beta)^2}{2M_\beta}+\frac{(\mathbf{p}''_\alpha+\frac{m_\alpha}{M-m_\beta}\,\mathbf{p}'_\beta)^2}{2\mu_\beta} &= \frac{(\mathbf{p}''_\alpha)^2}{2M_\alpha}+\frac{(\mathbf{p}'_\beta+\frac{m_\beta}{M-m_\alpha}\,\mathbf{p}''_\alpha)^2}{2\mu_\alpha} \\
\frac{(\mathbf{p}'_\beta)^2}{2M_\beta}+\frac{(\mathbf{p}_\gamma+\frac{m_\gamma}{M-m_\beta}\,\mathbf{p}'_\beta)^2}{2\mu_\beta} &= \frac{(\mathbf{p}_\gamma)^2}{2M_\gamma}+\frac{(\mathbf{p}'_\beta+\frac{m_\beta}{M-m_\gamma}\,\mathbf{p}_\gamma)^2}{2\mu_\gamma}\,,
\end{aligned} \tag{E.3}$$

the above expression can be rewritten in the following form

$$
\begin{aligned}
I_2^c &= \sum_{\alpha\neq\beta,\beta\neq\gamma} \int \frac{d^3\mathbf{q}''_\alpha}{(2\pi)^3}\frac{d^3\mathbf{q}_\gamma}{(2\pi)^3} \Bigg\{ \\
&\times \int \frac{d^3\mathbf{p}''_\alpha}{(2\pi)^3}\frac{d^3\mathbf{p}'_\beta}{(2\pi)^3}\frac{d^3\mathbf{p}_\gamma}{(2\pi)^3} \tilde{\Gamma}_\alpha(\mathbf{q}';\mathbf{p}''_\alpha\mathbf{q}''_\alpha) U^{c,0}_{3\alpha\beta\gamma}(\mathbf{p}''_\alpha\mathbf{q}''_\alpha;\mathbf{p}'_\beta;\mathbf{p}_\gamma\mathbf{q}_\gamma)(\tilde{\Gamma}_\gamma(\mathbf{q};\mathbf{p}_\gamma\mathbf{q}_\gamma))^* \\
&+ \frac{1}{L^3}\sum_{\mathbf{p}''_\alpha} \int \frac{d^3\mathbf{p}'_\beta}{(2\pi)^3}\frac{d^3\mathbf{p}_\gamma}{(2\pi)^3} \tilde{\Gamma}_\alpha(\mathbf{q}';\mathbf{p}''_\alpha\mathbf{q}''_\alpha) U^{c,1}_{3\alpha\beta\gamma}(\mathbf{p}''_\alpha\mathbf{q}''_\alpha;\mathbf{p}'_\beta;\mathbf{p}_\gamma\mathbf{q}_\gamma)(\tilde{\Gamma}_\gamma(\mathbf{q};\mathbf{p}_\gamma\mathbf{q}_\gamma))^* \\
&+ \frac{1}{L^3}\sum_{\mathbf{p}'_\beta} \int \frac{d^3\mathbf{p}''_\alpha}{(2\pi)^3}\frac{d^3\mathbf{p}_\gamma}{(2\pi)^3} \tilde{\Gamma}_\alpha(\mathbf{q}';\mathbf{p}''_\alpha\mathbf{q}''_\alpha) U^{c,2}_{3\alpha\beta\gamma}(\mathbf{p}''_\alpha\mathbf{q}''_\alpha;\mathbf{p}'_\beta;\mathbf{p}_\gamma\mathbf{q}_\gamma)(\tilde{\Gamma}_\gamma(\mathbf{q};\mathbf{p}_\gamma\mathbf{q}_\gamma))^* \\
&+ \frac{1}{L^3}\sum_{\mathbf{p}_\gamma} \int \frac{d^3\mathbf{p}''_\alpha}{(2\pi)^3}\frac{d^3\mathbf{p}'_\beta}{(2\pi)^3} \tilde{\Gamma}_\alpha(\mathbf{q}';\mathbf{p}''_\alpha\mathbf{q}''_\alpha) U^{c,3}_{3\alpha\beta\gamma}(\mathbf{p}''_\alpha\mathbf{q}''_\alpha;\mathbf{p}'_\beta;\mathbf{p}_\gamma\mathbf{q}_\gamma)(\tilde{\Gamma}_\gamma(\mathbf{q};\mathbf{p}_\gamma\mathbf{q}_\gamma))^* \\
&+ \frac{1}{L^6}\sum_{\mathbf{p}'_\beta\mathbf{p}_\gamma} \int \frac{d^3\mathbf{p}''_\alpha}{(2\pi)^3} \tilde{\Gamma}_\alpha(\mathbf{q}';\mathbf{p}''_\alpha\mathbf{q}''_\alpha) U^{c,4}_{3\alpha\beta\gamma}(\mathbf{p}''_\alpha\mathbf{q}''_\alpha;\mathbf{p}'_\beta;\mathbf{p}_\gamma\mathbf{q}_\gamma)(\tilde{\Gamma}_\gamma(\mathbf{q};\mathbf{p}_\gamma\mathbf{q}_\gamma))^* \\
&+ \frac{1}{L^6}\sum_{\mathbf{p}''_\alpha\mathbf{p}'_\beta} \int \frac{d^3\mathbf{p}_\gamma}{(2\pi)^3} \tilde{\Gamma}_\alpha(\mathbf{q}';\mathbf{p}''_\alpha\mathbf{q}''_\alpha) U^{c,5}_{3\alpha\beta\gamma}(\mathbf{p}''_\alpha\mathbf{q}''_\alpha;\mathbf{p}'_\beta;\mathbf{p}_\gamma\mathbf{q}_\gamma)(\tilde{\Gamma}_\gamma(\mathbf{q};\mathbf{p}_\gamma\mathbf{q}_\gamma))^* \\
&+ \frac{1}{L^6}\sum_{\mathbf{p}''_\alpha\mathbf{p}_\gamma} \int \frac{d^3\mathbf{p}'_\beta}{(2\pi)^3} \tilde{\Gamma}_\alpha(\mathbf{q}';\mathbf{p}''_\alpha\mathbf{q}''_\alpha) U^{c,6}_{3\alpha\beta\gamma}(\mathbf{p}''_\alpha\mathbf{q}''_\alpha;\mathbf{p}'_\beta;\mathbf{p}_\gamma\mathbf{q}_\gamma)(\tilde{\Gamma}_\gamma(\mathbf{q};\mathbf{p}_\gamma\mathbf{q}_\gamma))^* \Bigg\}, \qquad \text{(E.4)}
\end{aligned}
$$

where

$$
\begin{aligned}
U^{c,0}_{3\alpha\beta\gamma}(\mathbf{p}''_\alpha\mathbf{q}''_\alpha;\mathbf{p}'_\beta;\mathbf{p}_\gamma\mathbf{q}_\gamma) &= G_{\mathsf{K}\alpha}(\mathbf{p}''_\alpha,\mathbf{q}''_\alpha;z)(-\bar{V}_\alpha(\mathbf{q}''_\alpha,\bar{\mathbf{q}}''_\alpha))G_{\mathsf{K}\alpha}(\mathbf{p}''_\alpha,\bar{\mathbf{q}}''_\alpha;z)(-\bar{V}_\beta(\bar{\mathbf{q}}''_\beta,\bar{\mathbf{q}}'''_\beta)) \\
&\times G_{\mathsf{K}\beta}(\mathbf{p}'_\beta,\bar{\mathbf{q}}'''_\beta;z)(-\bar{V}_\gamma(\bar{\mathbf{q}}''_\gamma,\mathbf{q}_\gamma))G_{\mathsf{K}\gamma}(\mathbf{p}_\gamma,\mathbf{q}_\gamma;z) \qquad \text{(E.5)}
\end{aligned}
$$

$$
\begin{aligned}
U^{c,1}_{3\alpha\beta\gamma}(\mathbf{p}''_\alpha\mathbf{q}''_\alpha;\mathbf{p}'_\beta;\mathbf{p}_\gamma\mathbf{q}_\gamma) &= \tilde{G}_{\mathsf{F}\alpha}(\mathbf{p}''_\alpha,\mathbf{q}''_\alpha;z)(-\bar{V}_\alpha(\mathbf{q}''_\alpha,\bar{\mathbf{q}}''_\alpha))G_{\mathsf{K}\alpha}(\mathbf{p}''_\alpha,\bar{\mathbf{q}}''_\alpha;z)(-\bar{V}_\beta(\bar{\mathbf{q}}''_\beta,\bar{\mathbf{q}}'''_\beta)) \\
&\times G_{\mathsf{K}\beta}(\mathbf{p}'_\beta,\bar{\mathbf{q}}'''_\beta;z)(-\bar{V}_\gamma(\bar{\mathbf{q}}''_\gamma,\mathbf{q}_\gamma))G_{\mathsf{K}\gamma}(\mathbf{p}_\gamma,\mathbf{q}_\gamma;z) \\
&+ \tilde{G}_{\mathsf{F}\alpha}(\mathbf{p}''_\alpha,\mathbf{q}''_\alpha;z)(-\bar{V}_\alpha(\mathbf{q}''_\alpha,\bar{\mathbf{q}}''_\alpha))\tilde{G}_{\mathsf{F}\alpha}(\mathbf{p}''_\alpha,\bar{\mathbf{q}}''_\alpha;z)(-\bar{V}_\beta(\bar{\mathbf{q}}''_\beta,\bar{\mathbf{q}}'''_\beta)) \\
&\times G_{\mathsf{K}\beta}(\mathbf{p}'_\beta,\bar{\mathbf{q}}'''_\beta;z)(-\bar{V}_\gamma(\bar{\mathbf{q}}''_\gamma,\mathbf{q}_\gamma))G_{\mathsf{K}\gamma}(\mathbf{p}_\gamma,\mathbf{q}_\gamma;z) \qquad \text{(E.6)}
\end{aligned}
$$

$$
\begin{aligned}
U^{c,2}_{3\alpha\beta\gamma}(\mathbf{p}''_\alpha\mathbf{q}''_\alpha;\mathbf{p}'_\beta;\mathbf{p}_\gamma\mathbf{q}_\gamma) &= G_{\mathsf{K}\alpha}(\mathbf{p}''_\alpha,\mathbf{q}''_\alpha;z)(-\bar{V}_\alpha(\mathbf{q}''_\alpha,\bar{\mathbf{q}}''_\alpha))\tilde{G}_{\mathsf{F}\alpha}(\mathbf{p}''_\alpha,\bar{\mathbf{q}}''_\alpha;z)(-\bar{V}_\beta(\bar{\mathbf{q}}''_\beta,\bar{\mathbf{q}}'''_\beta))\\
&\times\ G_{\mathsf{K}\beta}(\mathbf{p}'_\beta,\bar{\mathbf{q}}'''_\beta;z)(-\bar{V}_\gamma(\bar{\mathbf{q}}''_\gamma,\mathbf{q}_\gamma))G_{\mathsf{K}\gamma}(\mathbf{p}_\gamma,\mathbf{q}_\gamma;z)\\
&+\ G_{\mathsf{K}\alpha}(\mathbf{p}''_\alpha,\mathbf{q}''_\alpha;z)(-\bar{V}_\alpha(\mathbf{q}''_\alpha,\bar{\mathbf{q}}''_\alpha))G_{\mathsf{K}\beta}(\mathbf{p}'_\beta,\bar{\mathbf{q}}''_\beta;z)(-\bar{V}_\beta(\bar{\mathbf{q}}''_\beta,\bar{\mathbf{q}}'''_\beta))\\
&\times\ \tilde{G}_{\mathsf{F}\beta}(\mathbf{p}'_\beta,\bar{\mathbf{q}}'''_\beta;z)(-\bar{V}_\gamma(\bar{\mathbf{q}}''_\gamma,\mathbf{q}_\gamma))G_{\mathsf{K}\gamma}(\mathbf{p}_\gamma,\mathbf{q}_\gamma;z)\\
&+\ G_{\mathsf{K}\alpha}(\mathbf{p}''_\alpha,\mathbf{q}''_\alpha;z)(-\bar{V}_\alpha(\mathbf{q}''_\alpha,\bar{\mathbf{q}}''_\alpha))\tilde{G}_{\mathsf{F}\beta}(\mathbf{p}'_\beta,\bar{\mathbf{q}}''_\beta;z)(-\bar{V}_\beta(\bar{\mathbf{q}}''_\beta,\bar{\mathbf{q}}'''_\beta))\\
&\times\ \tilde{G}_{\mathsf{F}\beta}(\mathbf{p}'_\beta,\bar{\mathbf{q}}'''_\beta;z)(-\bar{V}_\gamma(\bar{\mathbf{q}}''_\gamma,\mathbf{q}_\gamma))G_{\mathsf{K}\gamma}(\mathbf{p}_\gamma,\mathbf{q}_\gamma;z) \qquad \text{(E.7)}
\end{aligned}
$$

$$
\begin{aligned}
U^{c,3}_{3\alpha\beta\gamma}(\mathbf{p}''_\alpha\mathbf{q}''_\alpha;\mathbf{p}'_\beta;\mathbf{p}_\gamma\mathbf{q}_\gamma) &= G_{\mathsf{K}\alpha}(\mathbf{p}''_\alpha,\mathbf{q}''_\alpha;z)(-\bar{V}_\alpha(\mathbf{q}''_\alpha,\bar{\mathbf{q}}''_\alpha))G_{\mathsf{K}\beta}(\mathbf{p}'_\beta,\bar{\mathbf{q}}''_\beta;z)(-\bar{V}_\beta(\bar{\mathbf{q}}''_\beta,\bar{\mathbf{q}}'''_\beta))\\
&\times\ G_{\mathsf{K}\gamma}(\mathbf{p}_\gamma,\bar{\mathbf{q}}''_\gamma;z)(-\bar{V}_\gamma(\bar{\mathbf{q}}''_\gamma,\mathbf{q}_\gamma))\tilde{G}_{\mathsf{F}\gamma}(\mathbf{p}_\gamma,\mathbf{q}_\gamma;z)\\
&+\ G_{\mathsf{K}\alpha}(\mathbf{p}''_\alpha,\mathbf{q}''_\alpha;z)(-\bar{V}_\alpha(\mathbf{q}''_\alpha,\bar{\mathbf{q}}''_\alpha))G_{\mathsf{K}\beta}(\mathbf{p}'_\beta,\bar{\mathbf{q}}''_\beta;z)(-\bar{V}_\beta(\bar{\mathbf{q}}''_\beta,\bar{\mathbf{q}}'''_\beta))\\
&\times\ \tilde{G}_{\mathsf{F}\gamma}(\mathbf{p}_\gamma,\bar{\mathbf{q}}''_\gamma;z)(-\bar{V}_\gamma(\bar{\mathbf{q}}''_\gamma,\mathbf{q}_\gamma))\tilde{G}_{\mathsf{F}\gamma}(\mathbf{p}_\gamma,\mathbf{q}_\gamma;z) \qquad \text{(E.8)}
\end{aligned}
$$

$$
\begin{aligned}
U^{c,4}_{3\alpha\beta\gamma}(\mathbf{p}''_\alpha\mathbf{q}''_\alpha;\mathbf{p}'_\beta;\mathbf{p}_\gamma\mathbf{q}_\gamma) &= G_{\mathsf{K}\alpha}(\mathbf{p}''_\alpha,\mathbf{q}''_\alpha;z)(-\bar{V}_\alpha(\mathbf{q}''_\alpha,\bar{\mathbf{q}}''_\alpha))\tilde{G}_{\mathsf{F}\beta}(\mathbf{p}'_\beta,\bar{\mathbf{q}}''_\beta;z)\\
&\times\ \frac{(-\bar{V}_\beta(\bar{\mathbf{q}}''_\beta,\bar{\mathbf{q}}'''_\beta))(-\bar{V}_\gamma(\bar{\mathbf{q}}''_\gamma,\mathbf{q}_\gamma))}{\left(M+\dfrac{(\mathbf{p}_\gamma)^2}{2M_\gamma}+\dfrac{(\bar{\mathbf{q}}''_\gamma)^2}{2\mu_\gamma}-z\right)}\tilde{G}_{\mathsf{F}\gamma}(\mathbf{p}_\gamma,\mathbf{q}_\gamma;z) \qquad \text{(E.9)}
\end{aligned}
$$

$$
\begin{aligned}
U^{c,5}_{3\alpha\beta\gamma}(\mathbf{p}''_\alpha\mathbf{q}''_\alpha;\mathbf{p}'_\beta;\mathbf{p}_\gamma\mathbf{q}_\gamma) &= \tilde{G}_{\mathsf{F}\alpha}(\mathbf{p}''_\alpha,\mathbf{q}''_\alpha;z)\frac{(-\bar{V}_\alpha(\mathbf{q}''_\alpha,\bar{\mathbf{q}}''_\alpha))(-\bar{V}_\beta(\bar{\mathbf{q}}''_\beta,\bar{\mathbf{q}}'''_\beta))}{\left(M+\dfrac{(\mathbf{p}'_\beta)^2}{2M_\beta}+\dfrac{(\bar{\mathbf{q}}''_\beta)^2}{2\mu_\beta}-z\right)}\\
&\times\ \tilde{G}_{\mathsf{F}\beta}(\mathbf{p}'_\beta,\bar{\mathbf{q}}''_\beta;z)(-\bar{V}_\gamma(\bar{\mathbf{q}}''_\gamma,\mathbf{q}_\gamma))G_{\mathsf{K}\gamma}(\mathbf{p}_\gamma,\mathbf{q}_\gamma;z) \qquad \text{(E.10)}
\end{aligned}
$$

$$
\begin{aligned}
U^{c,6}_{3\alpha\beta\gamma}(\mathbf{p}''_\alpha\mathbf{q}''_\alpha;\mathbf{p}'_\beta;\mathbf{p}_\gamma\mathbf{q}_\gamma) &= \tilde{G}_{\mathsf{F}\alpha}(\mathbf{p}''_\alpha,\mathbf{q}''_\alpha;z)(-\bar{V}_\alpha(\mathbf{q}''_\alpha,\bar{\mathbf{q}}''_\alpha))G_{\mathsf{K}\alpha}(\mathbf{p}''_\alpha,\bar{\mathbf{q}}''_\alpha;z)(-\bar{V}_\beta(\bar{\mathbf{q}}''_\beta,\bar{\mathbf{q}}'''_\beta)) \\
&\times G_{\mathsf{K}\gamma}(\mathbf{p}_\gamma,\bar{\mathbf{q}}''_\gamma;z)(-\bar{V}_\gamma(\bar{\mathbf{q}}''_\gamma,\mathbf{q}_\gamma))\tilde{G}_{\mathsf{F}\gamma}(\mathbf{p}_\gamma,\mathbf{q}_\gamma;z) \\
+ & \quad \tilde{G}_{\mathsf{F}\alpha}(\mathbf{p}''_\alpha,\mathbf{q}''_\alpha;z)\frac{(-\bar{V}_\alpha(\mathbf{q}''_\alpha,\bar{\mathbf{q}}''_\alpha))(-\bar{V}_\beta(\bar{\mathbf{q}}''_\beta,\bar{\mathbf{q}}'''_\beta))}{\left(M+\dfrac{(\mathbf{p}'_\beta)^2}{2M_\beta}+\dfrac{(\bar{\mathbf{q}}''_\beta)^2}{2\mu_\beta}-z\right)}\tilde{G}_{\mathsf{F}\gamma}(\mathbf{p}_\gamma,\bar{\mathbf{q}}''_\gamma;z)(-\bar{V}_\gamma(\bar{\mathbf{q}}''_\gamma,\mathbf{q}_\gamma))\tilde{G}_{\mathsf{F}\gamma}(\mathbf{p}_\gamma,\mathbf{q}_\gamma;z) \\
+ & \quad \tilde{G}_{\mathsf{F}\alpha}(\mathbf{p}''_\alpha,\mathbf{q}''_\alpha;z)(-\bar{V}_\alpha(\mathbf{q}''_\alpha,\bar{\mathbf{q}}''_\alpha))\tilde{G}_{\mathsf{F}\alpha}(\mathbf{p}''_\alpha,\bar{\mathbf{q}}''_\alpha;z)\frac{(-\bar{V}_\beta(\bar{\mathbf{q}}''_\beta,\bar{\mathbf{q}}'''_\beta))(-\bar{V}_\gamma(\bar{\mathbf{q}}''_\gamma,\mathbf{q}_\gamma))}{\left(M+\dfrac{(\mathbf{p}_\gamma)^2}{2M_\gamma}+\dfrac{(\bar{\mathbf{q}}''_\gamma)^2}{2\mu_\gamma}-z\right)}\tilde{G}_{\mathsf{F}\gamma}(\mathbf{p}_\gamma,\mathbf{q}_\gamma;z)
\end{aligned}
\tag{E.11}
$$

where

$$
\begin{aligned}
\bar{\mathbf{q}}''_\alpha &= \mathbf{p}'_\beta+\frac{m_\beta}{M-m_\alpha}\,\mathbf{p}''_\alpha\,, \\
\bar{\mathbf{q}}''_\beta &= -\mathbf{p}''_\alpha-\frac{m_\alpha}{M-m_\beta}\,\mathbf{p}'_\beta\,, \\
\bar{\mathbf{q}}'''_\beta &= -\mathbf{p}_\gamma-\frac{m_\gamma}{M-m_\beta}\,\mathbf{p}'_\beta\,, \\
\bar{\mathbf{q}}''_\gamma &= \mathbf{p}'_\beta+\frac{m_\beta}{M-m_\gamma}\,\mathbf{p}_\gamma\,.
\end{aligned}
\tag{E.12}
$$

It is straightforward to observe that the above terms reproduce the multiple-scattering series of the Faddeev equations in a finite volume (see section 4.3) up to $O(\bar{V}_\alpha^3)$.

F Product of two energy denominators

Consider the expression

$$
J=\frac{1}{L^3}\sum_{\mathbf{p}}\frac{\phi(\mathbf{p})}{(a^2-(\mathbf{p}+\mathbf{c}_1)^2)(b^2-(\mathbf{p}+\mathbf{c}_2)^2)}\doteq\frac{1}{L^3}\sum_{\mathbf{p}}\phi(\mathbf{p})d_1d_2\,, \tag{F.1}
$$

where $\phi(\mathbf{p})$ denotes a regular function.

If $\mathbf{c}_1=\mathbf{c}_2=\mathbf{0}$ and $a^2\neq b^2$, the proof of the desired relation given in eq. (E.2) immediately follows from the identity

$$
\frac{1}{(a^2-\mathbf{p}^2)(b^2-\mathbf{p}^2)}=\frac{1}{b^2-a^2}\left(\frac{1}{a^2-\mathbf{p}^2}-\frac{1}{b^2-\mathbf{p}^2}\right) \tag{F.2}
$$

by using the splitting from eq. (52) in the individual terms on the r.h.s. of this equation.

In the generic case, let us consider the partial-wave expansion

$$\begin{aligned}\phi(\mathbf{p})d_2 &= \frac{\phi(\mathbf{p}+\mathbf{c}_1-\mathbf{c}_1)}{b^2-(\mathbf{p}+\mathbf{c}_1-(\mathbf{c}_1-\mathbf{c}_2))^2} \doteq \frac{\phi_1(\mathbf{p}+\mathbf{c}_1)}{b^2-(\mathbf{p}+\mathbf{c}_1-(\mathbf{c}_1-\mathbf{c}_2))^2}\\ &= \sum_{lm}\mathcal{Y}_{lm}(\mathbf{p}+\mathbf{c}_1)\mathcal{Y}^*_{lm}(\mathbf{c}_1-\mathbf{c}_2)K_{1l}((\mathbf{p}+\mathbf{c}_1)^2)\,, \end{aligned}\tag{F.3}$$

and, analogously,

$$\phi(\mathbf{p})d_1 = \sum_{lm}\mathcal{Y}_{lm}(\mathbf{p}+\mathbf{c}_2)\mathcal{Y}^*_{lm}(\mathbf{c}_1-\mathbf{c}_2)K_{2l}((\mathbf{p}+\mathbf{c}_2)^2)\,. \tag{F.4}$$

Further, define the quantities

$$\begin{aligned}\overline{\phi_1 d_2} &= \sum_{lm}\mathcal{Y}_{lm}(\mathbf{p}+\mathbf{c}_1)\mathcal{Y}^*_{lm}(\mathbf{c}_1-\mathbf{c}_2)K_{1l}(a^2)\,,\\ \overline{\phi_2 d_1} &= \sum_{lm}\mathcal{Y}_{lm}(\mathbf{p}+\mathbf{c}_2)\mathcal{Y}^*_{lm}(\mathbf{c}_1-\mathbf{c}_2)K_{2l}(b^2)\end{aligned}\tag{F.5}$$

Then, the quantity J from eq. (F.1) can be rewritten as

$$\begin{aligned}J &= \frac{1}{L^3}\sum_{\mathbf{p}}\left\{\left[\phi d_1 d_2 - \theta_1 d_1\overline{\phi_1 d_2} - \theta_2 d_2\overline{\phi_2 d_1}\right] + \theta_1 d_1\overline{\phi_1 d_2} + \theta_2 d_2\overline{\phi_2 d_1}\right\},\\ \theta_1 &= f(a^2/\mu^2)\theta(\Lambda^2-(\mathbf{p}+\mathbf{c}_1)^2)\,, \qquad \theta_2 = f(b^2/\mu^2)\theta(\Lambda^2-(\mathbf{p}+\mathbf{c}_2)^2)\,.\end{aligned}\tag{F.6}$$

If

$$\begin{aligned}&a^2>0\,,\quad b^2>0\,,\\ &|(a^2-b^2)-(\mathbf{c}_1-\mathbf{c}_2)^2| > 2|a|\,|\mathbf{c}_1-\mathbf{c}_2|, \qquad |(a^2-b^2)+(\mathbf{c}_1-\mathbf{c}_2)^2| > 2|b|\,|\mathbf{c}_1-\mathbf{c}_2|\,,\end{aligned}\tag{F.7}$$

then the quantities $\overline{\phi_1 d_2}$, $\overline{\phi_2 d_1}$ are non-singular and one may use the regular summation theorem. Namely, the expression in the square brackets in eq. (F.6) is non-singular, so the summation can be replaced by integration there. Using the same technique as in section 3, one straightforwardly arrives at eq. (E.2). Finally, the relation for the generic values of the parameters $a, b, \mathbf{c}_1, \mathbf{c}_2$ can be obtained by analytic continuation of both sides of eq. (E.2) in these parameters.

References

[1] N. Isgur and G. Karl, Phys. Lett. B **72** (1977) 109.

[2] N. Isgur and G. Karl, Phys. Rev. D **19** (1979) 2653 [Erratum-ibid. D **23** (1981) 817].

[3] Z. P. Li, V. Burkert and Z. J. Li, Phys. Rev. D **46** (1992) 70.

[4] C. E. Carlson and N. C. Mukhopadhyay, Phys. Rev. Lett. **67** (1991) 3745.

[5] P. A. M. Guichon, Phys. Lett. B **164** (1985) 361.

[6] O. Krehl, C. Hanhart, S. Krewald and J. Speth, Phys. Rev. C **62** (2000) 025207 [arXiv:nucl-th/9911080].

[7] I. Zahed, U.-G. Meißner and U. B. Kaulfuss, Nucl. Phys. A **426** (1984) 525.

[8] U.-G. Meißner and J. W. Durso, Nucl. Phys. A **430** (1984) 670.

[9] M. Gockeler, R. Horsley, D. Pleiter, P. E. L. Rakow, G. Schierholz, C. M. Maynard and D. G. Richards [QCDSF Collaboration and UKQCD Collaboration and LHPC Collaboration], Phys. Lett. B **532** (2002) 63 [arXiv:hep-lat/0106022].

[10] W. Melnitchouk *et al.*, Phys. Rev. D **67** (2003) 114506 [arXiv:hep-lat/0202022].

[11] F. X. Lee and D. B. Leinweber, Nucl. Phys. Proc. Suppl. **73** (1999) 258 [arXiv:hep-lat/9809095].

[12] F. X. Lee, S. J. Dong, T. Draper, I. Horvath, K. F. Liu, N. Mathur and J. B. Zhang, Nucl. Phys. Proc. Suppl. **119** (2003) 296 [arXiv:hep-lat/0208070].

[13] R. G. Edwards, U. M. Heller and D. G. Richards [LHP Collaboration], Nucl. Phys. Proc. Suppl. **119** (2003) 305 [arXiv:hep-lat/0303004].

[14] N. Mathur *et al.*, Phys. Lett. B **605** (2005) 137 [arXiv:hep-ph/0306199].

[15] S. Sasaki, T. Blum and S. Ohta, Phys. Rev. D **65** (2002) 074503 [hep-lat/0102010].

[16] S. Sasaki, Prog. Theor. Phys. Suppl. **151** (2003) 143 [arXiv:nucl-th/0305014].

[17] K. Sasaki, S. Sasaki and T. Hatsuda, Phys. Lett. B **623** (2005) 208 [hep-lat/0504020].

[18] S. Basak *et al.*, Phys. Rev. D **76** (2007) 074504 [arXiv:0709.0008 [hep-lat]].

[19] S. Cohen *et al.*, PoS **LAT2009** (2009) 112 [arXiv:0911.3373 [hep-lat]].

[20] J. Bulava *et al.*, Phys. Rev. D **79** (2009) 034505 [arXiv:0901.0027 [hep-lat]].

[21] J. Bulava *et al.*, Phys. Rev. D **82** (2010) 014507 [arXiv:1004.5072 [hep-lat]].

[22] G. P. Engel, C. B. Lang, M. Limmer, D. Möhler and A. Schäfer [BGR [Bern- Graz-Regensburg] Collaboration], Phys. Rev. D **82** (2010) 034505 [arXiv:1005.1748 [hep-lat]].

[23] M. S. Mahbub, W. Kamleh, D. B. Leinweber, P. J. Moran and A. G. Williams [CSSM Lattice collaboration], arXiv:1011.5724 [hep-lat].

[24] H. W. Lin and S. D. Cohen, arXiv:1108.2528 [hep-lat].

[25] H. W. Lin, Chin. J. Phys. **49** (2011) 827 [arXiv:1106.1608 [hep-lat]].

[26] B. Borasoy, P. C. Bruns, U.-G. Meißner and R. Lewis, Phys. Lett. B **641** (2006) 294 [arXiv:hep-lat/0608001].

[27] M. Lüscher, Nucl. Phys. B **354** (1991) 531.

[28] M. Gockeler, R. Horsley, Y. Nakamura, D. Pleiter, P. E. L. Rakow, G. Schierholz and J. Zanotti [QCDSF Collaboration], PoS **LATTICE2008** (2008) 136 [arXiv:0810.5337 [hep-lat]].

[29] S. Aoki *et al.* [CS Collaboration], Phys. Rev. D **84** (2011) 094505 [arXiv:1106.5365 [hep-lat]].

[30] M. Lage, U.-G. Meißner and A. Rusetsky, Phys. Lett. B **681** (2009) 439 [arXiv:0905.0069 [hep-lat]].

[31] C. Liu, X. Feng and S. He, Int. J. Mod. Phys. A **21** (2006) 847 [arXiv:hep-lat/0508022].

[32] V. Bernard, M. Lage, U.-G. Meißner and A. Rusetsky, JHEP **1101** (2011) 019 [arXiv:1010.6018 [hep-lat]].

[33] M. Döring, U.-G. Meißner, E. Oset and A. Rusetsky, Eur. Phys. J. A **47** (2011) 139 [arXiv:1107.3988 [hep-lat]].

[34] A. M. Torres, L. R. Dai, C. Koren, D. Jido and E. Oset, arXiv:1109.0396 [hep-lat].

[35] M. Döring and U.-G. Meißner, arXiv:1111.0616 [hep-lat].

[36] V. B. Belyaev, "Lectures On The Theory Of Few Body Systems," *Berlin, Germany: Springer (1990) 134 p. (Springer series in nuclear and particle physics)*

[37] T. Luu and M. J. Savage, Phys. Rev. D **83** (2011) 114508 [arXiv:1101.3347 [hep-lat]].

[38] V. Bernard, M. Lage, U.-G. Meißner and A. Rusetsky, JHEP **0808** (2008) 024 [arXiv:0806.4495 [hep-lat]].

[39] M. Döring, J. Haidenbauer, U.-G. Meißner and A. Rusetsky, Eur. Phys. J. A **47** (2011) 163 [arXiv:1108.0676 [hep-lat]].

[40] M. Lüscher, Commun. Math. Phys. **105** (1986) 153 (1986).

[41] K. L. Kowalski, Phys. Rev. D **7** (1973) 1806; Nucl. Phys. A **264** (1976) 173.

[42] I. Manning, Phys. Rev. D **5** (1972) 1472.

[43] K. L. Kowalski, Phys. Rev. D **5** (1972) 395.

[44] M. Döring, C. Hanhart, F. Huang, S. Krewald and U.-G. Meißner, Nucl. Phys. A **829** (2009) 170 [arXiv:0903.4337 [nucl-th]].

[45] M. Döring, C. Hanhart, F. Huang, S. Krewald and U.-G. Meißner, Phys. Lett. B **681** (2009) 26 [arXiv:0903.1781 [nucl-th]].

[46] S. Kreuzer and H. W. Hammer, Phys. Lett. B **673** (2009) 260 [arXiv:0811.0159 [nucl-th]].

[47] S. Kreuzer and H. W. Hammer, Eur. Phys. J. A **43** (2010) 229 [arXiv:0910.2191 [nucl-th]].

[48] S. Kreuzer and H. W. Hammer, Phys. Lett. B **694** (2011) 424 [arXiv:1008.4499 [hep-lat]].

[49] S. Bour, S. König, D. Lee, H. W. Hammer and U.-G. Meißner, Phys. Rev. D **84** (2011) 091503 [arXiv:1107.1272 [nucl-th]].

[50] I.S. Gradshteyn and I.M. Ryzhik, Table of Integrals, Series and Products, Academic Press (2007).