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*Research Article*

# Three-Point Boundary Value Problems for Conformable Fractional Differential Equations

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We study a fractional differential equation using a recent novel concept of fractional derivative with initial and three-point boundary conditions. We first obtain Green's function for the linear problem and then we study the nonlinear differential equation.

## 1. Introduction

In this paper, we study a class of differential equations supplemented with three-point boundary conditions. Precisely, we consider the following problem:

$$\mathrm{D}^{\alpha}(\mathrm{D}+\lambda)\,x(t)=f(t,x(t)),\quad t\in[0,1], \tag{1}$$

$$x(0)=0,\qquad x'(0)=0,\qquad x(1)=\beta x(\eta), \tag{2}$$

where $\mathrm{D}^{\alpha}$ is the conformable fractional derivative of order $\alpha\in(1,2]$, D is the ordinary derivative, $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a known continuous function, $\lambda$ and $\beta$ are real numbers, $\lambda>0$, and $\eta\in(0,1)$.

Fractional calculus and fractional differential equations are relevant areas of research. There are several concepts of fractional derivatives, some classical, such as Riemann-Liouville or Caputo definitions, and some novel, such as conformable fractional derivative [1], $\beta$-derivative [2], or a new definition [3, 4]. The relation between these definitions and their potential applications needs further study.

The conformable fractional derivative aims at extending the usual derivative satisfying some natural properties (see [1]) and gives a new solution for some fractional differential equations. In this paper we present a boundary value problem involving this fractional derivative.

Sequential fractional differential equations have been considered for other types of fractional derivatives, see, for example, [5, 6].

The paper is organized as follows. In Section 2 we recall some concepts relative to the conformable fractional calculus. In Section 3 we solve the corresponding linear problem and obtain Green's function. In Section 4 we study the nonlinear problem and finally, in Section 5, we present an example to illustrate the applicability of our results.

## 2. Preliminaries

We recall some definitions and results concerning conformable fractional derivative.

*Definition 1* (see [1]). Given a function $x:[0,+\infty)\to\mathbb{R}$, the conformable fractional derivative of order $\alpha\in(0,1]$ of $x$ at $t$ is defined by

$$\mathrm{D}^{\alpha}x(t)=\lim_{\varepsilon\to 0}\frac{x\left(t+\varepsilon t^{1-\alpha}\right)-x(t)}{\varepsilon}, \tag{3}$$

for all $t>0$. If $f$ is $\alpha$-differentiable in some interval $(0,a)$ with $a>0$, then we define

$$\mathrm{D}^{\alpha}x(0)=\lim_{t\to 0^{+}}\mathrm{D}^{\alpha}x(t), \tag{4}$$

whenever the limit of the right hand side exists.

We remark that if $x$ is differentiable, then

$$D^{\alpha} x(t) = t^{1-\alpha} x'(t). \tag{5}$$

Reciprocally, if $D^{\alpha}x(t)$ exists, then for $t \neq 0$ we have

$$\begin{aligned} x'(t) &= \lim_{\delta \to 0} \frac{x(t+\delta) - x(t)}{\delta} \\ &= \lim_{\varepsilon \to 0} \frac{x\left(t + \varepsilon t^{1-\alpha} - x(t)\right)}{\varepsilon t^{1-\alpha}} = t^{\alpha-1} D^{\alpha} x(t). \end{aligned} \tag{6}$$

Hence, $D^{\alpha}x(t) = t^{1-\alpha}x'(t)$. Of course, for $t = 0$ this is not valid and it would be useful to deal with equations and solutions with singularities.

*Definition 2.* Let $\alpha \in (n, n+1]$ and let $x$ be an $n$-differentiable function at $t > 0$; the fractional conformable derivative of order $\alpha$ at $t > 0$ is given by

$$D^{\alpha} x(t) = \lim_{\varepsilon \to 0} \frac{x^{(\lceil\alpha\rceil-1)}\left(t + \varepsilon t^{\lceil\alpha\rceil-\alpha}\right) - x^{(\lceil\alpha\rceil-1)}(t)}{\varepsilon}, \tag{7}$$

where $\lceil\alpha\rceil$ is the smallest integer greater than or equal to $\alpha$. For $t = 0$ we proceed in a similar way as in Definition 1.

*Definition 3.* Given $\alpha \in (0, 1]$, the fractional integral of order $\alpha$ at $t \geq 0$ is given by

$$D^{-\alpha} x(t) \equiv I^{\alpha} x(t) = I^{1}\left(t^{\alpha-1} x\right)(t) = \int_0^t \frac{x(s)}{s^{1-\alpha}} ds. \tag{8}$$

*Definition 4.* Given $\alpha \in (n-1, n]$, the fractional integral of order $\alpha$ is given by

$$D^{-\alpha} x(t) \equiv I^{\alpha} x(t) = I^{n}\left(t^{\alpha-\lceil\alpha\rceil} x\right)(t), \tag{9}$$

where $I^n$ denotes the operator $I^1$ (usual integration) of order $n$.

*Remark 5.* Some authors (see [7, 8]) have argued that conformable fractional derivative is not a truly fractional operator. This question seems today to still be open and perhaps it is a philosophical issue. However, in any case, the study of boundary value problems involving this new derivative has, in our opinion, a point of interest and deserves to be researched in more detail.

In [1, Theorem 3.1], authors have proved that for $\alpha \in (0, 1]$ and $x$ a given continuous function, $D^{\alpha}I^{\alpha}x(t) = x(t)$ for $t \geq 0$. In this paper, we consider $\alpha \in (1, 2]$, so we need the following results.

**Lemma 6.** *Given $\alpha \in (1, 2]$ and $x$ a continuous function defined in the domain of $I^{\alpha}$, one has that $D^{\alpha}I^{\alpha}x(t) = x(t)$ for $t \geq 0$.*

*Proof.* Since $x$ is continuous, then $I^{\alpha}x(t)$ is twice differentiable. In view of [1, Remark 2.1] we have

$$\begin{aligned} D^{\alpha}\left(I^{\alpha} x\right)(t) &= t^{2-\alpha} \frac{d^2}{dt^2} \int_0^t \int_0^{t_1} x(s) s^{\alpha-2} ds\, dt_1 \\ &= t^{2-\alpha} \frac{d}{dt} \int_0^t x(s) s^{\alpha-2} ds \\ &= t^{2-\alpha} x(t) t^{\alpha-2} = x(t). \end{aligned} \tag{10}$$

Thus, statement of Lemma 6 has been proved. □

**Lemma 7.** *Given $\alpha \in (1, 2]$ and $x : [0, +\infty) \to \mathbb{R}$ an $\alpha$-differentiable function, one has that $D^{\alpha}x(t) = 0$ if and only if $x(t) = c_1 t + c_2$, where $c_1, c_2 \in \mathbb{R}$.*

*Proof.* This fact follows easily in view of the mean value theorem for conformable fractional differentiable functions (see [1, Theorem 2.4]). □

## 3. Linear Boundary Problem

In order to study boundary value problem (1)-(2), we consider now the linear equation

$$D^{\alpha}(D + \lambda) x(t) = \sigma(t), \quad t \in [0, 1], \tag{11}$$

where $1 < \alpha \leq 2$ and $\sigma \in \mathscr{C}[0, 1]$.

**Lemma 8.** *Consider*

$$\beta \neq \frac{\lambda + e^{-\lambda} - 1}{\lambda\eta + e^{-\lambda\eta} - 1}. \tag{12}$$

*Then, the unique solution of (11) subject to the boundary conditions (2) is given by*

$$\begin{aligned} x(t) = &\int_0^t e^{-\lambda(t-s)} \left(\int_0^s \sigma(u) u^{\alpha-2} (s-u) du\right) ds \\ &+ A(t) \left[\beta \int_0^{\eta} e^{-\lambda(\eta-s)} \left(\int_0^s \sigma(u) u^{\alpha-2} (s-u) du\right) ds \right. \\ &\left. - \int_0^1 e^{-\lambda(1-s)} \left(\int_0^s \sigma(u) u^{\alpha-2} (s-u) du\right) ds\right], \end{aligned} \tag{13}$$

*where*

$$\begin{aligned} A(t) &= \frac{1}{\Delta}\left(\lambda t + e^{-\lambda t} - 1\right), \\ \Delta &= \lambda + e^{-\lambda} - 1 - \beta\left(\lambda\eta + e^{-\lambda\eta} - 1\right) \neq 0. \end{aligned} \tag{14}$$

*Proof.* Integrating (11) we obtain

$$(D + \lambda) x(t) = I^{\alpha}\sigma(t) + c_1 t + c_2. \tag{15}$$

Thus, in view of Lemmas 6 and 7, every solution of (15) is a solution for (11).

Let $y(t) = e^{\lambda t}x(t)$. Equation (15) can be rewritten as

$$\mathrm{D}y(t) = \left(\mathrm{I}^{\alpha}\sigma(t) + c_1 t + c_2\right)e^{\lambda t}. \tag{16}$$

Integrating from 0 to $t$, we obtain

$$x(t) = \frac{c_1}{\lambda^2}\left(\lambda t - 1 + e^{-\lambda t}\right) + \frac{c_2}{\lambda}\left(1 - e^{-\lambda t}\right) + c_3 + \int_0^t e^{-\lambda(t-s)}\left(\int_0^s \sigma(u)u^{\alpha-2}(s-u)\,du\right)ds. \tag{17}$$

Imposing boundary conditions (2), we conclude that $c_3 = 0$, $c_2 = 0$, and

$$c_1 = \frac{\lambda^2}{\Delta}\left[\beta\int_0^{\eta} e^{-\lambda(\eta-s)}\left(\int_0^s \sigma(u)u^{\alpha-2}(s-u)\,du\right)ds - \int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \sigma(u)u^{\alpha-2}(s-u)\,du\right)ds\right]. \tag{18}$$

Substituting these values of $c_1$, $c_2$, and $c_3$ in (17), we finally obtain (13) and that expression gives the unique solution. □

We now obtain Green's function corresponding to the fractional differential equations (11) of order $\alpha + 1$ with $1 < \alpha \leq 2$ subject to boundary conditions (2).

By changing the order of integration, we note that

$$\int_0^t e^{-\lambda(t-s)}\left(\int_0^s \sigma(u)u^{\alpha-2}(s-u)\,du\right)ds = \int_0^t\left(\int_u^t e^{\lambda(s-t)}(s-u)\,ds\right)\sigma(u)u^{\alpha-2}du. \tag{19}$$

Hence, solution (17) with $c_2 = c_3 = 0$ takes the form

$$x(t) = \frac{c_1}{\lambda^2}\left(\lambda t - 1 + e^{-\lambda t}\right) + \int_0^t k(t,s)\sigma(s)s^{\alpha-2}ds, \tag{20}$$

where

$$k(t,s) = \int_s^t e^{\lambda(u-t)}(u-s)\,du = \frac{e^{\lambda(s-t)} - \lambda s - 1 + \lambda t}{\lambda^2}. \tag{21}$$

Now, using the boundary condition $x(1) = \beta x(\eta)$, we get

$$c_1 = \frac{\lambda^2}{\Delta}\left(\beta\int_0^{\eta} k(\eta,s)\sigma(s)s^{\alpha-s}ds - \int_0^1 k(1,s)\sigma(s)s^{\alpha-2}ds\right). \tag{22}$$

Therefore, we finally conclude the following:

$$x(t) = \frac{A(t)}{\Delta}\left(\beta\int_0^{\eta} k(\eta,s)\sigma(s)s^{\alpha-s}ds - \int_0^1 k(1,s)\sigma(s)s^{\alpha-2}ds\right) + \int_0^t k(t,s)\sigma(s)s^{\alpha-2}ds; \tag{23}$$

so, we deduce the following result.

**Theorem 9.** *The unique solution of (11) subject to boundary conditions (2) is given by*

$$x(t) = \int_0^1 G(t,s)\sigma(s)s^{\alpha-2}ds, \tag{24}$$

*where*

$$G(t,s) = \begin{cases} -k(1,s)\psi(t), & \text{if } 0 \leq \max\{\eta,t\} < s \leq 1; \\ -k(1,s)\psi(t) + k(t,s), & \text{if } 0 \leq \eta < s < t \leq 1; \\ (\beta k(\eta,s) - k(1,s))\psi(t), & \text{if } 0 \leq t < s \leq \eta \leq 1; \\ (\beta k(\eta,s) - k(1,s))\psi(t) + k(t,s), & \text{if } 0 \leq s < \min\{\eta,t\} \leq 1, \end{cases} \tag{25}$$

*with*

$$\psi(t) = \frac{A(t)}{\Delta}, \quad t \in [0,1]. \tag{26}$$

*Remark 10.* In other words, corresponding Green's function for the homogeneous problem (11) satisfying the boundary conditions (2) is given by (25).

*Remark 11.* Note that $G(t,s)$ is independent of $\alpha$, but the solution depends, of course, on $\alpha$.

## 4. Nonlinear Problem

Let $\mathscr{C} = \mathscr{C}[0,1]$ be the Banach space of all continuous functions defined in $[0,1]$ endowed with the usual supremum norm defined by $\|x\| = \sup\{|x(t)|,\ t \in [0,1]\}$.

For the sake of convenience, we set

$$B = \frac{1 + A_1\left[|\beta|\eta^{\alpha}\left(1 - e^{-\lambda\eta}\right) + 1 - e^{-\lambda}\right]}{\lambda\alpha(\alpha-1)} > 0, \tag{27}$$

with

$$A_1 = \sup_{t\in[0,1]}|A(t)|, \tag{28}$$

where $A(t)$ is given by (14).

In view of Lemma 8, we transform boundary value problem (1)-(2) into

$$x = \mathscr{T}x, \quad x \in \mathscr{C}, \tag{29}$$

where $\mathscr{T} : \mathscr{C} \to \mathscr{C}$ is defined by

$$(\mathscr{T}x)(t) = \int_0^t e^{-\lambda(t-s)}\left(\int_0^s f(u,x(u))u^{\alpha-2}(s-u)\,du\right)ds$$

$$+ A(t)\left[\beta \int_0^\eta e^{-\lambda(\eta-s)}\left(\int_0^s f(u,x(u))\,u^{\alpha-2}(s-u)\,du\right)ds - \int_0^1 e^{-\lambda(1-s)}\left(\int_0^s f(u,x(u))\,u^{\alpha-2}(s-u)\,du\right)ds\right]. \quad (30)$$

Observe that problem (1)-(2) has solutions if the operator (30) has fixed points.

**Theorem 12.** *Let $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ be a jointly continuous function satisfying the condition*

$$|f(t,v) - f(t,w)| \le L|v-w| \quad \forall t \in [0,1],\ v,w\in\mathbb{R}, \quad (31)$$

*where $L > 0$ is the Lipschitz constant. Then, boundary value problem (1)-(2) has a unique solution if $B < 1/L$, where $B$ is given by (27).*

*Proof.* First, for $\mathcal{T}$ defined by (30), we show that $\mathcal{T}(\mathcal{B}_r) \subset \mathcal{B}_r$, where $\mathcal{B}_r$ is the closed ball of radius $r > 0$ in $\mathcal{C}$; that is, $\mathcal{B}_r = \{x \in \mathcal{C} : \|x\| \le r\}$. Now we consider $M > \sup\{|f(t,0)| : t \in [0,1]\}$ and we choose

$$r > \frac{MB}{1-LB}. \quad (32)$$

For $x \in \mathcal{B}_r$, we have

$$\begin{aligned}
&\|\mathcal{T}x\| \\
&= \sup_{t\in[0,1]}\Bigg|\int_0^t e^{-\lambda(t-s)}\left(\int_0^s f(u,x(u))\,u^{\alpha-2}(s-u)\,du\right)ds \\
&\quad + A(t)\Bigg[\beta\int_0^\eta e^{-\lambda(\eta-s)} \cdot\left(\int_0^s f(u,x(u))\,u^{\alpha-2}(s-u)\,du\right)ds \\
&\quad - \int_0^1 e^{-\lambda(1-s)} \cdot\left(\int_0^s f(u,x(u))\,u^{\alpha-2}(s-u)\,du\right)ds\Bigg]\Bigg| \\
&\le \sup_{t\in[0,1]}\int_0^t e^{-\lambda(t-s)} \cdot\left(\int_0^s \left(|f(u,x(u)) - f(u,0)| + |f(u,0)|\right) \cdot u^{\alpha-2}(s-u)\,du\right)ds \\
&\quad + \sup_{t\in[0,1]}|A(t)|\Bigg[|\beta|\int_0^\eta e^{-\lambda(\eta-s)} \cdot\left(\int_0^s \left(|f(u,x(u)) - f(u,0)| + |f(u,0)|\right)u^{\alpha-2}(s-u)\,du\right)ds \\
&\quad + \int_0^1 e^{-\lambda(1-s)} \cdot\left(\int_0^s \left(|f(u,x(u)) - f(u,0)| + |f(u,0)|\right)u^{\alpha-2}(s-u)\,du\right)ds\Bigg] \\
&\le \sup_{t\in[0,1]}\int_0^t e^{-\lambda(t-s)} \cdot\left(\int_0^s \left(L|x(u)| + |f(u,0)|\right)u^{\alpha-2}(s-u)\,du\right)ds \\
&\quad + A_1\Bigg[|\beta|\int_0^\eta e^{-\lambda(\eta-s)} \cdot\left(\int_0^s \left(L|x(u)| + |f(u,0)|\right)u^{\alpha-2}(s-u)\,du\right)ds \\
&\quad + \int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \left(L|x(u)| + |f(u,0)|\right) \cdot u^{\alpha-2}(s-u)\,du\right)ds\Bigg] \\
&\le (Lr+M)\Bigg\{\sup_{t\in[0,1]}\int_0^t e^{-\lambda(t-s)}\left(\int_0^s u^{\alpha-2}(s-u)\,du\right)ds \\
&\quad + A_1\Bigg[|\beta|\int_0^\eta e^{-\lambda(\eta-s)}\left(\int_0^s u^{\alpha-2}(s-u)\,du\right)ds + \int_0^1 e^{-\lambda(1-s)}\left(\int_0^s u^{\alpha-2}(s-u)\,du\right)ds\Bigg]\Bigg\} \\
&\le \frac{Lr+M}{\alpha(\alpha-1)}\Bigg\{\sup_{t\in[0,1]}\int_0^t e^{-\lambda(t-s)}s^\alpha ds + A_1\left(|\beta|\int_0^\eta e^{-\lambda(\eta-s)}s^\alpha ds + \int_0^1 e^{-\lambda(1-s)}s^\alpha ds\right)\Bigg\} \\
&\le (Lr+M)\frac{1 + A_1\left(|\beta|\eta^\alpha\left(1-e^{-\lambda\eta}\right) + 1 - e^{-\lambda}\right)}{\lambda\alpha(\alpha-1)} \\
&= (Lr+M)B \le r.
\end{aligned} \quad (33)$$

Now, for $x, y \in \mathcal{C}$ and for each $t \in [0,1]$, we obtain

$$\begin{aligned}
&|(\mathcal{T}x)(t) - (\mathcal{T}y)(t)| \\
&= \sup_{t\in[0,1]}|(\mathcal{T}x)(t) - (\mathcal{T}y)(t)| \\
&\le \sup_{t\in[0,1]}\Bigg\{\int_0^t e^{-\lambda(t-s)}\left(\int_0^s |f(u,x(u)) - f(u,y(u))| \cdot u^{\alpha-2}(s-u)\,du\right)ds
\end{aligned}$$

$$+ A(t)\left[\beta\int_0^\eta e^{-\lambda(\eta-s)} \cdot\left(\int_0^s |f(u,x(u)) - f(u,y(u))| u^{\alpha-2} \cdot (s-u)\,du\right)ds + \int_0^1 e^{-\lambda(1-s)} \cdot\left(\int_0^s |f(u,x(u)) - f(u,y(u))| \cdot u^{\alpha-2}(s-u)\,du\right)ds\right]\Bigg\}$$
$$\leq L\|x-y\| \cdot\left\{\sup_{t\in[0,1]}\int_0^t e^{-\lambda(t-s)}\left(\int_0^s u^{\alpha-2}(s-u)\,du\right)ds + \sup_{t\in[0,1]} A(t)\left[\beta\int_0^\eta e^{-\lambda(\eta-s)}\left(\int_0^s u^{\alpha-2}(s-u)\,du\right)ds + \int_0^1 e^{-\lambda(1-s)}\left(\int_0^s u^{\alpha-2}(s-u)\,du\right)ds\right]\right\}$$
$$\leq L\frac{1 + A_1\left(|\beta|\eta^\alpha\left(1-e^{-\lambda\eta}\right) + 1 - e^{-\lambda}\right)}{\lambda\alpha(\alpha-1)}\|x-y\| = BL\|x-y\|. \tag{34}$$

As $B < 1/L$, we conclude that $\mathcal{T}$ is a contraction. Thus, the statement of the theorem follows by the classical Banach fixed point theorem. This concludes the proof. □

Now we recall a known result due to Krasnoselskii (see [9, Theorem 4.4.1]) which we will use to prove existence of at least one solution to (1)-(2).

**Theorem 13.** *Let $N$ be a closed, convex, and nonempty subset of a Banach space $X$. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be operators such that*

(i) *$\mathcal{T}_1x + \mathcal{T}_2y \in N$ whenever $x, y \in N$,*

(ii) *$\mathcal{T}_1$ is compact and continuous,*

(iii) *$\mathcal{T}_2$ is a contraction mapping.*

*Then there exists $z \in N$ such that $z = \mathcal{T}_1z + \mathcal{T}_2z$.*

**Theorem 14.** *Let $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ be a jointly continuous function satisfying the following conditions:*

(H1) *$|f(t,v) - f(t,w)| \leq L|v-w|$ for all $t \in [0,1]$, $v, w \in \mathbb{R}$,*

(H2) *$|f(t,v)| \leq \mu(t)$ for all $(t,v) \in [0,1]\times\mathbb{R}$ with $\mu \in \mathcal{C}$.*

*Then, boundary value problem (1)-(2) has at least one solution on $\mathcal{C}$ if*

$$A_1\frac{|\beta|\eta^\alpha\left(1-e^{-\lambda\eta}\right) + \left(1+e^{-\lambda}\right)}{\lambda\alpha(\alpha-1)} < 1. \tag{35}$$

*Proof.* Letting $\|\mu\| = \sup_{t\in[0,1]}|\mu(t)|$, we fix

$$r \geq \frac{1 + A_1\left[|\beta|\eta^\alpha\left(1-e^{-\lambda\eta}\right) + \left(1-e^{-\lambda}\right)\right]}{\lambda\alpha(\alpha-1)}\|\mu\|, \tag{36}$$

and we consider $\mathcal{B}_r$ as in Theorem 12.

Define the operators $\mathcal{T}_1$ and $\mathcal{T}_2$ as

$$(\mathcal{T}_1x)(t) = \int_0^t e^{-\lambda(t-s)}\left(\int_0^s f(u,x(u))u^{\alpha-2}(s-u)\,du\right)ds,$$
$$(\mathcal{T}_2y)(t) = A(t)\left[\beta\int_0^\eta e^{-\lambda(\eta-s)}\left(\int_0^s f(u,y(u))u^{\alpha-2}(s-u)\,du\right)ds + \int_0^1 e^{-\lambda(1-s)}\left(\int_0^s f(u,y(u))u^{\alpha-2}(s-u)\,du\right)ds\right]. \tag{37}$$

For $x, y \in \mathcal{B}_r$, it follows by (36) that

$$\|\mathcal{T}_1x + \mathcal{T}_2y\| \leq \frac{1 + A_1\left[|\beta|\eta^\alpha\left(1-e^{-\lambda\eta}\right) + \left(1-e^{-\lambda}\right)\right]}{\lambda\alpha(\alpha-1)}\|\mu\| \leq r. \tag{38}$$

Thus, $\mathcal{T}_1x + \mathcal{T}_2y \in \mathcal{B}_r$. In view of condition (35), we have that $\mathcal{T}_2$ is a contraction mapping.

Now we show that $\mathcal{T}_1$ is compact and continuous. The continuity of $f$ implies that the operator $\mathcal{T}_1$ is continuous. In addition, $\mathcal{T}_1$ is uniformly bounded on $\mathcal{B}_r$ as

$$\|\mathcal{T}_1x\| = \sup_{t\in[0,1]}\left|\int_0^t e^{-\lambda(t-s)}\left(\int_0^s f(u,x(u))u^{\alpha-2}(s-u)\,du\right)ds\right| \leq \|\mu\|\sup_{t\in[0,1]}\int_0^t e^{-\lambda(t-s)}\left(\int_0^s u^{\alpha-2}(s-u)\,du\right)ds = \frac{\|\mu\|}{\alpha(\alpha-1)}\int_0^t e^{-\lambda(t-s)}s^\alpha ds \leq \|\mu\|\frac{1-e^{-\lambda}}{\lambda\alpha(\alpha-1)}. \tag{39}$$

Setting $\Omega = [0,1] \times \mathscr{B}_r$, we define $M_r = \sup_{(t,x)\in\Omega}|f(t,x)|$, and consequently we have that, for $0 \le t_2 < t_1 \le 1$,

$$\begin{aligned} &\left|(\mathscr{T}_1 x)(t_1) - (\mathscr{T}_1 x)(t_2)\right| \\ &= \left| \int_0^{t_1} e^{-\lambda(t_1-s)} \left( \int_0^s f(u,x(u))\, u^{\alpha-2} (s-u)\, du \right) ds \right. \\ &\quad \left. - \int_0^{t_2} e^{-\lambda(t_2-s)} \left( \int_0^s f(u,x(u))\, u^{\alpha-2} (s-u)\, du \right) ds \right| \\ &\le \frac{M_r}{\alpha(\alpha-1)} \left(e^{\lambda t_1} - e^{\lambda t_2}\right) \left(e^{-\lambda t_1} t_1{}^{\alpha} - e^{-\lambda t_2} t_2{}^{\alpha}\right), \end{aligned} \tag{40}$$

which is independent of $x$ and tends to zero as $t_2 \to t_1$. This shows that $\mathscr{T}_1$ is relatively compact on $\mathscr{B}_r$. Hence, by the Ascoli-Arzelá Theorem, $\mathscr{T}_1$ is compact on $\mathscr{B}_r$. Thus, all the hypotheses of Theorem 13 are satisfied and the conclusion of Theorem 13 implies that the boundary value problem (1)-(2) has at least one solution on $\mathscr{B}_r \subset \mathscr{C}$, with $r$ satisfying (36). This completes the proof. □

## 5. Example

Consider the boundary value problem over the interval [0, 1], given by:

$$\begin{aligned} &\mathrm{D}^{3/2} (\mathrm{D}+4)\, x(t) = L\left(t^2 + \cos t + \arctan x(t)\right), \\ &x(0) = 0, \qquad x'(0) = 0, \qquad x(1) = x\left(\frac{1}{2}\right). \end{aligned} \tag{41}$$

Here, $f(t,v) = L(t^2 + \cos t + \arctan v)$, $\lambda = 4$, $\beta = 1$, $\eta = 1/2$. Clearly,

$$\begin{aligned} &\left|f(t,v) - f(t,w)\right| \le L\left|\arctan v - \arctan w\right| \le L\left|v-w\right|, \\ &A_1 = \frac{4 + e^{-4} - 1}{4 + e^{-4} - 1 - \left(2 + e^{-2} - 1\right)} \approx 1.6029, \\ &B = \frac{1 + A_1\left(1/2^{3/2}\left(1 - e^{-2}\right) + 1 - e^{-4}\right)}{3} \approx 1.0212. \end{aligned} \tag{42}$$

For $L < 0.9792$, it follows, by Theorem 12, that boundary value problem (41) has a unique solution.

*Remark 15.* Authors of [5] obtained a similar result considering fractional Caputo derivative instead of fractional conformable derivative in (41).

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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