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THREE-POINT BOUNDARY VALUE PROBLEMS FOR SINGULAR SYSTEM OF FIRST ORDER DIFFERENTIAL EQUATIONS

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Abstract

In this paper we prove existence and uniqueness results to three point boundary value problems associated with singular system of first order differential equations by the use of generalized inverses of matrices.

1 Introduction

In many problems involving singular behavior of some kind, the ideas of a generalized inverse are implicit. One area of current research on the applications of generalized inverses deals with singular system of differential equations of the form

$$Py' + Qy = f, \quad a \le t \le c \tag{1}$$

where $P(t) \in C^2[a,c]$ and $Q(t) \in C^1[a,c]$ are square matrices of order n and $f(t)$ is a column n vector. Such equations arise in singular perturbations, cheap control problems, and descriptor systems.

This paper presents criteria for the existence and uniqueness of solutions to three point boundary value problems associated with the system (1) in the case of a singular square matrix P, satisfying the boundary conditions

$$My(a) + Ny(b) + Ry(c) = \alpha, \quad (a < b < c) \tag{2}$$

where M, N, and R are square matrices of order n and α is a column n vector. The results are established under the usual assumptions that the related homogeneous boundary value problem

$$Py' + Qy = 0 \tag{3}$$

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$$My(a) + Ny(b) + Ry(c) = 0 \tag{4}$$

has only the trivial solution. Generalized inverse of a matrix is used as a tool to study the existence and uniqueness criteria of (1) and (2).

Several authors [2], [5] have studied existence and uniqueness of solutions to two and three point boundary value problems associated with the system (1), when P is non-singular. Recently Murty and Rao [3] obtained existence and uniqueness criteria to two point boundary value problems associated with the system (1) with the help of generalized inverses. This paper extends the results of Murty and Rao [3], developed for two point boundary value problems to three point boundary value problems.

In section 2 we develop some basic results relating to generalized inverses and also obtain the general solution of the system (1), when P is singular.

Section 3 deals with obtaining existence and uniqueness criteria to three point boundary value problems associated with (1) satisfying (2). Some properties of the related Green's matrices are also studied. The results of this section are highlighted with a suitable example.

2 Preliminaries

In this section we state some useful definitions, results and theorems which are useful for obtaining our main theorem regarding existence and uniqueness of three point boundary value problems.

Result 2.1. [1] *If A is a $m \times n$ matrix of rank r, then there exist non-singular matrices K and S such that*

$$KAS = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

Definition 2.1. [3] A generalized inverse of $m \times n$ matrix A is any $n \times m$ matrix A^- such that $AA^-A = A$.

Result 2.2. [4] *The equation*

$$AX = B \tag{5}$$

is consistent if and only if $(I_m - AA^-)B = 0$, where $A, X,$ and B are matrices of order $m \times n, n \times m,$ and $m \times m$ respectively.

Result 2.3. [4] *If the equation (5) is consistent, then its general solution is of the form*

$$X = A^-B + (I_n - A^-A)U,$$

where U is any arbitrary matrix of order $n \times m$.

The equation (3), when P is non-singular is equivalent to $y' = A(t)y$, where $A = -P^{-1}Q$.

Definition 2.2. Any set of n linearly independent solutions of $y' = A(t)y$ is a fundamental set of solutions. The matrix with these elements as columns is a fundamental matrix for the given equation.

Definition 2.3. The dimension of the solution space of a homogeneous boundary value problem is called the *index of compatibility* of the problem. If the index of compatibility is zero, then we say that the boundary value problem is *incompatible.*

Result 2.4. [5] *Any solution of the system (1), when P is non-singular is of the form*

$$y(t) = Y(t)C + Y(t)\int_a^t Y^{-1}(s)P^{-1}(s)f(s)ds,$$

where $Y(t)$ is a fundamental matrix of the homogeneous equation (3), and C is a constant n-vector.

Now we develop the method of finding solution of equation (1), when P is singular and of rank $r < n$.

From Result 2.1 there exist non-singular matrices K and S such that

$$KPS = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

Then

$$KPSS^{-1}y' + KQy = Kf. \tag{6}$$

If we put $y = Sz$, then $y' = Sz' + S'z$. Therefore (6) becomes

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} S^{-1}[Sz' + S'z] + KQy = Kf. \tag{7}$$

Let us suppose $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$, where z_1 has r elements, then we have

$$z_1' + \phi_1 z_1 + \phi_2 z_2 = g_1 \tag{8}$$

$$\phi_3 z_1 + \phi_4 z_2 = g_2, \tag{9}$$

where

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} S^{-1}S' + KQS = \Phi = \begin{bmatrix} \phi_1 & \phi_2 \\ \phi_3 & \phi_4 \end{bmatrix},$$

ϕ_1 being a square matrix of order r and $Kf = g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$. Hence solving the problem (1) is equivalent to solving the equations (8) and (9). If ϕ_4 is invertible, then $z_2 = \phi_4^{-1}(g_2 - \phi_2 z_1)$ and equation (8) is solved classically.

The above procedure is illustrated with the following example.

Example 2.1. Consider the system (1) with $a = 0$, $c = 1$, $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$,

$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, and $f = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix}$.

Since P is a 3×3 matrix of rank 2 so we can write

$$KPS = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$, $S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$. Therefore

$$\Phi = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} S^{-1}S' + KQS = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 \\ \phi_3 & \phi_4 \end{bmatrix},$$

where $\phi_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\phi_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\phi_3 = [1 \quad -2]$ and $\phi_4 = 1$.

Here $Kf = \begin{bmatrix} 0 \\ t \\ -2t \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$, where $g_1 = \begin{bmatrix} 0 \\ t \end{bmatrix}$, $g_2 = -2t$.

From (9), $z_2 = \phi_4^{-1}(g_2 - \phi_3 z_1)$ and substituting z_2 in (8), we have

$$z'_{11} + z_{11} = 0, \;\; z'_{12} + z_{12} = t.$$

By solving these equations we get $z_{11} = e^{-t}$, $z_{12} = t - 1$, and $z_2 = -(e^{-t} + 2)$.

Therefore $y = Sz = Iz = z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} e^{-t} \\ t-1 \\ -(e^{-t}+2) \end{bmatrix}$.

If ϕ_4 is not invertible, we require the following lemma.

Lemma 2.1. *A necessary and sufficient condition for the existence of a solution z_2 to the equation (9) is $[I_{n-r} - (B\phi_3)(B\phi_3)^-](Bg_2) = 0$ and then*

$$z_2 = \phi_4^-(g_2 - \phi_3 z_1) + (I_{n-r} - \phi_4^-\phi_4)U,$$

where $B = (I_{n-r} - \phi_4\phi_4^-)$, and U is any arbitrary matrix of order $(n-r) \times 1$.

Proof. Solution z_2 for (9) exists if and only if

$$(I_{n-r} - \phi_4\phi_4^-)(g_2 - \phi_3 z_1) = 0$$

$$\text{i.e.} \quad Bg_2 = B\phi_3 z_1. \tag{10}$$

The equation (10) in z_1 is consistent if and only if

$$\left[I_{n-r} - (B\phi_3)(B\phi_3)^-\right](Bg_2) = 0$$

and hence

$$z_1 = (B\phi_3)^-(Bg_2) + \left[I_r - (B\phi_3)^-(B\phi_3)\right]V, \tag{11}$$

where V is any arbitrary matrix of order $r \times 1$. The general solution for z_2 is given by

$$z_2 = \phi_4^-(g_2 - \phi_3 z_1) + (I_{n-r} - \phi_4^-\phi_4)U,$$

where U is any arbitrary matrix of order $(n-r) \times 1$. Hence the proof. □

Substituting the value of z_2 in the equation (8), we have

$$z_1' + (\phi_1 - \phi_2\phi_4^- \phi_3)z_1 + \phi_2\phi_4^- g_2 + \phi_2(I_{n-r} - \phi_4^- \phi_4)U = g_1,$$

$$i.e. \quad z_1' + dz_1 = h, \tag{12}$$

where $d = \phi_1 - \phi_2\phi_4^- \phi_3$, $h = g_1 - \phi_2\phi_4^- g_2 - \phi_2(I_{n-r} - \phi_4^- \phi_4)U$.

If $Z(t)$ is a fundamental matrix for the homogeneous system

$$z_1' + dz_1 = 0, \tag{13}$$

which consists of r linearly independent solutions of (13) as columns, then any solution of (12) is of the form

$$z_1(t) = Z(t)\mathbf{C} + Z(t)\int_a^t Z^{-1}(s)h(s)ds, \tag{14}$$

where $\mathbf{C}$ is a constant column vector of order r.

Since $y = Sz = S_1z_1 + S_2z_2$, the general solution of (1) is of the form

$$\begin{aligned} y(t) &= S_1z_1 + S_2\left[\phi_4^-(g_2 - \phi_3 z_1) + (I_{n-r} - \phi_4^- \phi_4)U\right] \\ &= \left(S_1 - S_2\phi_4^- \phi_3\right) z_1 + S_2\left[\phi_4^- g_2 + (I_{n-r} - \phi_4^- \phi_4)U\right] \\ &= \left(S_1 - S_2\phi_4^- \phi_3\right)\left[Z(t)\mathbf{C} + Z(t)\int_a^t Z^{-1}(s)h(s)ds\right] \\ &\qquad + S_2\left[\phi_4^- g_2 + (I_{n-r} - \phi_4^- \phi_4)U\right] \\ &= T(t)Z(t)\mathbf{C} + T(t)Z(t)\int_a^t Z^{-1}(s)h(s)ds + F(t), \end{aligned} \tag{15}$$

where

$$T(t) = S_1 - S_2\phi_4^- \phi_3 \tag{16}$$

and

$$F(t) = S_2\left[\phi_4^- g_2 + (I_{n-r} - \phi_4^- \phi_4)U\right]. \tag{17}$$

Note 2.1. The condition for (10) and (14) to have a common solution in z_1 is that

$$KV = E, \tag{18}$$

where

$$K = I_r - (B\phi_3)^-(B\phi_3) \tag{19}$$

and

$$E = (B\phi_3)^-(Bg_2) - Z(t)\mathbf{C} - Z(t)\int_a^t Z^{-1}(s)h(s)ds. \tag{20}$$

The equation (18) in V is consistent if and only if

$$(I_r - KK^-)E = 0, \tag{21}$$

then

$$V = K^-E + (I_r - KK^-)W, \tag{22}$$

where W is any arbitrary matrix of order $r \times 1$. Thus we have the following theorem.

Theorem 2.1. *Any solution of the matrix differential system (1), when P is singular, is of the form*

$$y(t) = T(t)Z(t)\mathbf{C} + T(t)Z(t)\int_a^t Z^{-1}(s)h(s)ds + F(t),$$

where T, h, d, and F are defined as before and $Z(t)$ is a fundamental matrix of order $r \times r$ for the homogeneous system (13).

Corrollary 2.1. *If $(I_{n-r} - \phi_4^- \phi_4)U = 0$, then any solution of the homogeneous matrix differential system (3), when P is singular is of the form $T(t)Z(t)\mathbf{C}$.*

3 Main Theorem

In this section we obtain existence and uniqueness criteria for three point boundary value problems associated with the system (1) and (2). The properties of the Green's matrix are also studied. Finally, we illustrate the main theorem of this section with a suitable example.

Theorem 3.1. *Suppose the homogeneous boundary value problem (3), (4) is incompatible. Then there exists a unique solution to the boundary value problem (1), (2) and is of the form*

$$y(t) = \int_a^c G(t,s)h(s)ds + \psi(t),$$

where $G(t,s)$ is the Green's matrix for the homogeneous boundary value problem and $\psi(t)$ is given by

$$\psi(t) = T(t)Z(t)\,[\,D^-\alpha - D^-(MF(a) + NF(b) + RF(c)) + (I_r - D^-D)X\,] + F(t). \tag{23}$$

Proof. Substituting from Theorem 2.1, the general form of y in the boundary condition (2), we have

$$M\,[T(a)Z(a)\mathbf{C} + 0 + F(a)] + N\left[T(b)Z(b)\mathbf{C} + T(b)Z(b)\int_a^b Z^{-1}(s)h(s)ds + F(b)\right]$$

$$+R\left[T(c)Z(c)\mathbf{C} + T(c)Z(c)\int_a^c Z^{-1}(s)h(s)ds + F(c)\right] = \alpha$$

$$\Rightarrow [MT(a)Z(a) + NT(b)Z(b) + RT(c)Z(c)]\,\mathbf{C} + [MF(a) + NF(b) + RF(c)]$$

$$+NT(b)Z(b)\int_a^b Z^{-1}(s)h(s)ds + RT(c)Z(c)\int_a^c Z^{-1}(s)h(s)ds = \alpha$$

$$D\mathbf{C} = \beta, \tag{24}$$

where $D = MT(a)Z(a) + NT(b)Z(b) + RT(c)Z(c)$ is the characteristic matrix for the boundary value problem and

$$\beta = \alpha - NT(b)Z(b)\int_a^b Z^{-1}(s)h(s)ds - RT(c)Z(c)\int_a^c Z^{-1}(s)h(s)ds$$
$$-\left[MF(a) + NF(b) + RF(c)\right].$$

Suppose the equation (24) in $\mathbf{C}$ is consistent.

Then $(I_r - DD^-)\beta = 0$ and $\mathbf{C} = D^-\beta + (I_r - D^-D)X$, where X is any arbitrary matrix of order $r \times 1$.

Substituting this value of $\mathbf{C}$ in $y(t)$, we have

$$y(t) = T(t)Z(t)\left[D^-\beta + (I_r - DD^-)X\right] + T(t)Z(t)\int_a^t Z^{-1}(s)h(s)ds + F(t)$$

$$= T(t)Z(t)D^-\left\{\alpha - [MF(a) + NF(b) + RF(c)] - NT(b)Z(b)\int_a^b Z^{-1}(s)h(s)ds\right.$$
$$\left.-RT(c)Z(c)\int_a^c Z^{-1}(s)h(s)ds\right\} + T(t)Z(t)(I_r - D^-D)X$$
$$+T(t)Z(t)\int_a^t Z^{-1}(s)h(s)ds + F(t)$$

$$= \psi(t) - T(t)Z(t)D^-NT(b)Z(b)\int_a^b Z^{-1}(s)h(s)ds$$
$$-T(t)Z(t)D^-RT(c)Z(c)\int_a^c Z^{-1}(s)h(s)ds + T(t)Z(t)\int_a^t Z^{-1}(s)h(s)ds.$$

Hence

$$y(t) = \int_a^c G(t,s)h(s)ds + \psi(t),$$

where

$$\underset{t\in[a,b]}{G(t,s)} = \begin{cases} T(t)Z(t)\left[I_r - D^-NT(b)Z(b) - D^-RT(c)Z(c)\right]Z^{-1}(s), \\ \qquad\qquad a < s < t \le b < c \\ -T(t)Z(t)D^-\left[NT(b)Z(b) + RT(c)Z(c)\right]Z^{-1}(s), \\ \qquad\qquad a \le t < s < b < c \\ -T(t)Z(t)D^-RT(c)Z(c)Z^{-1}(s), \quad a \underset{(\le)}{<} t < b < s < c \end{cases} \tag{25}$$

$$\underset{t\in[b,c]}{G(t,s)} = \begin{cases} T(t)Z(t)\left[I_r - D^-RT(c)Z(c)\right]Z^{-1}(s), & a < b < s < t \le c \\ -T(t)Z(t)D^-RT(c)Z(c)Z^{-1}(s), & a < b \le t < s < c \\ T(t)Z(t)\left[I_r - D^-NT(b)Z(b) - D^-RT(c)Z(c)\right]Z^{-1}(s), \\ \qquad\qquad a < s < b < t \underset{(\le)}{<} c, \end{cases} \tag{26}$$

and $\psi(t)$ is defined as in (23). $\square$

Note 3.1. In the above theorem the function $\psi(t)$ satisfies the condition

$$DD^{-}\left[M\psi(a)+N\psi(b)+R\psi(c)\right]=DD^{-}\alpha.$$

Proof. Consider

$$\begin{aligned}
&DD^{-}\left[M\psi(a)+N\psi(b)+R\psi(c)\right]\\
&\quad=DD^{-}\left[MT(a)Z(a)+NT(b)Z(b)+RT(c)Z(c)\right]\big[D^{-}\alpha-D^{-}(MF(a)+NF(b)\\
&\qquad\qquad+RF(c))+(I_r-D^{-}D)X\big]+DD^{-}\left[MF(a)+NF(b)+RF(c)\right]\\
&\quad=DD^{-}\alpha-DD^{-}\left(MF(a)+NF(b)+RF(c)\right)+(D-DD^{-}D)X\\
&\qquad\qquad+DD^{-}\left(MF(a)+NF(b)+RF(c)\right)\\
&\quad=DD^{-}\alpha.
\end{aligned}$$

□

Note 3.2. Since U is any arbitrary matrix, U may be chosen in such a way that

$$(I_{n-r}-\phi_4^{-}\phi_4)U=0.$$

Theorem 3.2. *The Green's matrix $G(t,s)$ has the following properties ;*

(i) $G(t,s)$ as a function of t for fixed s has continuous derivatives everywhere except at $t=s$. At $t=s$, G has a jump discontinuity and its jump is given by

$$G(s^{+},s)-G(s^{-},s)=T(s).$$

(ii) G is a formal solution of the homogeneous boundary value problem

$$\begin{gathered}
Py'+Qy=0,\\
DD^{-}\left[MG(a,s)+NG(b,s)+RG(c,s)\right]=0.
\end{gathered}\tag{27}$$

(iii) $G(t,s)$ satisfying the properties (i) and (ii) is unique.

Proof. (i) For $t\in[a,b]$, consider

$$\begin{aligned}
G(s^{+},s)-G(s^{-},s)&=T(s)Z(s)\left[I_r-D^{-}NT(b)Z(b)-D^{-}RT(c)Z(c)\right]Z^{-1}(s)\\
&\qquad+T(s)Z(s)D^{-}\left[NT(b)Z(b)+RT(c)Z(c)\right]Z^{-1}(s)\\
&=T(s)Z(s)Z^{-1}(s)=T(s).
\end{aligned}$$

Similarly we can prove for $t\in[b,c]$.

(ii) The representation of $G(t,s)$ by (25) and (26) shows that $G(t,s)$ is a formal solution of $Py'+Qy=0$, it fails to be a true solution because of the discontinuity at $t=s$. $G(t,s)$ satisfies the boundary conditions.

For fixed $s \in [a, b]$,

$$DD^{-}[MG(a,s)+NG(b,s)+RG(c,s)]$$

$$\begin{aligned}&= DD^{-}\left\{-MT(a)Z(a)D^{-}[NT(b)Z(b)+RT(c)Z(c)]Z^{-1}(s)\right.\\&\qquad +NT(b)Z(b)[I_r - D^{-}NT(b)Z(b) - D^{-}RT(c)Z(c)]Z^{-1}(s)\\&\qquad \left.+RT(c)Z(c)[I_r - D^{-}NT(b)Z(b) - D^{-}RT(c)Z(c)]Z^{-1}(s)\right\}\\&= -DD^{-}D[D^{-}RT(c)Z(c)]Z^{-1}(s) - DD^{-}D[D^{-}NT(b)Z(b)]Z^{-1}(s)\\&\qquad +DD^{-}NT(b)Z(b)Z^{-1}(s) + DD^{-}RT(c)Z(c)Z^{-1}(s) = 0.\end{aligned}$$

Similarly, when $s \in [b, c]$, $DD^{-}[MG(a,s) + NG(b,s) + RG(c,s)] = 0$.

(iii) The property (iii) follows by noting that if $G_1(t,s)$ and $G_2(t,s)$ are Green's matrices, then $G_1(t,s) - G_2(t,s)$ is continuously differentiable and satisfies the homogeneous boundary value problem (27). Since the homogeneous boundary value problem is incompatible, it follows that $G_1(t,s) = G_2(t,s)$. $\square$

The following example illustrates the application of the main theorem.

Example 3.1. Consider the system

$$Py' + Qy = f, \quad 0 \le t \le 1$$

satisfying

$$My(0) + Ny(\tfrac{1}{2}) + Ry(1) = \alpha,$$

where

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, f = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}$$

and

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \alpha = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Since P is a 3×3 matrix of rank 1 so we can write

$$KPS = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. If we take $y = Sz$, where

$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$, $S = [S_1 \;\; S_2]$, $S_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $S_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Let $\phi = KPSS^{-1}S' + KQS = \begin{bmatrix} \phi_1 & \phi_2 \\ \phi_3 & \phi_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$.

Clearly $\phi_1 = 1$, $\phi_2 = [0 \;\; 1]$, $\phi_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and $\phi_4 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$. Then

$\phi_4^- = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$, $Kf = g = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$, where $g_1 = t$, $g_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Let Z be the fundamental solution of $z_1' + dz_1 = 0$, where $d = \phi_1 - \phi_2\phi_4^-\phi_3 = 1 - 0 = 1$. therefore $z_1' + z_1 = 0$ and hence $Z(t) = e^{-t}$. Thus

$$T(t) = S_1 - S_2\phi_4^-\phi_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$D = MT(0)Z(0) + NT(\tfrac{1}{2})Z(\tfrac{1}{2}) + RT(1)Z(1) = \begin{bmatrix} e^{-\frac{1}{2}} \\ 0 \\ 1 + e^{-1} \end{bmatrix},$$

$$D^- = \begin{bmatrix} e^{\frac{1}{2}} & 0 & 0 \end{bmatrix},$$

$$h(t) = g_1 - \phi_2\phi_4^- g_2 - \phi_2\left(I_2 - \phi_4^-\phi_4\right)U = t,$$

$$F(t) = S_2\left[\phi_4^- g_2 + \left(I_2 - \phi_4^-\phi_4\right)U\right] = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

$$\begin{aligned} \psi(t) &= T(t)Z(t)\,[\,D^-\alpha - D^-(MF(0) + NF(\tfrac{1}{2}) + RF(1)) + (I_1 - D^-D)X\,] + F(t) \\ &= \begin{bmatrix} -e^{-t+\frac{1}{2}} & 1 & 0 \end{bmatrix}^T. \end{aligned}$$

$$\underset{t\in[0,\frac{1}{2}]}{G(t,s)} = \begin{cases} 0, & 0 < s < t < \frac{1}{2} < 1 \\ \begin{bmatrix} -e^{-t+\frac{1}{2}} & 0 & 0 \end{bmatrix}^T, & 0 \le t < s < \frac{1}{2} < 1 \\ 0, & 0 \underset{(\le)}{<} t < \frac{1}{2} < s < 1 \end{cases}$$

$$\underset{t\in[\frac{1}{2},1]}{G(t,s)} = \begin{cases} \begin{bmatrix} -e^{-t+\frac{1}{2}} & 0 & 0 \end{bmatrix}^T, & 0 < \frac{1}{2} < s < t < 1 \\ 0, & 0 < \frac{1}{2} < t < s < 1 \\ 0, & 0 < s < \frac{1}{2} < t \underset{(\le)}{<} 1. \end{cases}$$

Hence, the unique solution of the boundary value problem is

$$y(t) = \int_0^1 G(t,s)h(s)ds + \psi(t) = \begin{bmatrix} t - 1 - \frac{1}{2}\,e^{-t+\frac{1}{2}} \\ 1 \\ 0 \end{bmatrix}.$$

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