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THREE REMARKS ON FUNDAMENTAL GROUPS OF SOME RIEMANNIAN MANIFOLDS

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1. Statement of Results. It is an important object of global Riemannian geometry to study the relation between topological structure and metrical structure of Riemannian manifolds. In this note we shall give some elementary results about the fundamental groups of Sasakian manifolds, Kählerian manifolds, and manifolds of non-positive sectional curvature.

(a) Let M be a regular contact manifold with Sasakian structure (ϕ, ξ, η, g), that is, a regular contact manifold with the metric which is an odd dimensional analgoue of Kählerian metric. In particular, if M is compact, it is a principal circle bundle over a compact Kählerian manifold $'M$, where the contact form η is a connection form of this principal bundle. (S. Sasaki [6]).

Let D denote the distribution defined by $\eta = 0$. Then for a unit tangent vector $X \in D$ we define ϕ-holomorphic sectional curvature $H(X)$ of X by $H(X) = K(X, \phi X)$, where $K(X, \phi X)$ is the sectional curvature of the plane spanned by X and ϕX. It is known that the sectional curvature $K(X, \xi) = 1$ for $X \in D$.

Let $p: M \longrightarrow {}'M$ denote the fibering of M stated above, where $'M$ is a compact Kählerian manifold. If we put $'X = p_*(X)$, then $'X$ is the unit tangent vector of $'M$ and X is the horizontal lift of $'X$ with respect to the connection η. We have

$$(1) \qquad 'H('X) = H(X) + 3$$

where $'H('X)$ denotes the holomorphic sectional curvature of $'M$. (S. Tanno [7]). Now we get

THEOREM A. *Let M^d be a compact regular Sasakian manifold of dimension d with $h \leqq H(X) \leqq H$ for any unit tangent vector $X \in D$ where h, H are constants. If $h > -3$ holds then the fundamental group $\pi_1(M)$ is cyclic and if $(h+3)/(H+3) > (d-3)/2(d-1)$ holds, then $\pi_1(M)$ is finite cyclic.*

This improves a result of M. Harada ([4]) which has been proved by completely different way.

(b) For the proof of theorem A we use a result of Y. Tsukamoto ([9]) which states that each compact Kählerian manifold of positive holomorphic sectional curvature must be simply connected. On the other hand there is a close relation between simply connectedness and a fixed point theorem for some transformations of a Riemannian manifold. (A. Weinstein [10]). Following the method of [10], we can prove

THEOREM B. *Let M be a compact Kählerian manifold of positive holomorphic sectional curvature. Then every automorphism f of its Kählerian structure has a fixed point.*

For the case of positive curvature see T. Frankel. ([3]). The above theorem of Y. Tsukamoto follows immediately from Theorem B.

(c) Finally we shall remark the following result which improves slightly a result of A. Preismann ([5]).

THEOREM C. *Let M be a compact Riemannian manifold with non-positive sectional curvature. If there is a point where Ricci cvrvature is negative definite, then $\pi_1(M)$ has a trivial center, in particular, $\pi_1(M)$ can not be abelian.*

2. Proof of Theorem A. We consider the homotopy exact sequence of the bundle $p: M \longrightarrow 'M$ stated in (a), that is,

$$(2) \qquad \cdots \longrightarrow \pi_2(S^1) \xrightarrow{i_*} \pi_2(M) \xrightarrow{p_*} \pi_2('M) \longrightarrow \cdots$$
$$\xrightarrow{\Delta} \pi_1(S^1) \xrightarrow{i_*} \pi_1(M) \xrightarrow{p_*} \pi_1('M) \longrightarrow \cdots.$$

If $h > -3$ holds, then by (1) and the result of Y. Tsukamoto stated in §1 (b), we have $\pi_1('M) = 0$, and hence $\pi_1(M) = Z/\mathrm{Im}\Delta$ must by cyclic.

Next we assume $(h+3)/(H+3) > (d-3)/2(d-1)$, and show that $\mathrm{Im}\Delta \neq 0$. To the contrary, suppose $\mathrm{Im}\Delta = 0$. Then $\pi_1(M)$ has rank 1. Since $\pi_1(M)$ is abelian, by Hurewicz theorem $H_1(M, Z)$ and $\pi_1(M)$ are isomorphic. On the other hand we know that $H_1(M, Z)$ has rank zero by a theorem of S. Tanno ([8]) if $(h+3)/(H+3) > (d-3)/2(d-1)$ holds. This contradiction completes the proof of Theorem A.

3. proof of Theorem B. Let us suppose that f has no fixed point. Put $l = d(x_0, f(x_0)) = \underset{x \in M}{\mathrm{Min}}\ d(x, f(x)) > 0$. Let $c: [0, l] \longrightarrow M$ be a minimizing geodesic (parametrized by arc length) joining x_0 and $f(x_0)$, and β be a geodesic starting from x_0. Now we consider a variation of c defined by

$$(3)\qquad \psi_\beta(\mathrm{t}, \mathrm{u}) = \exp_{c(t)} u \left\{ \left(1 - \frac{t}{l}\right) \tau_t^0 \dot{\beta}(0) + \frac{t}{l} (\tau^{-1})_t^l f_* \dot{\beta}(0) \right\}$$

where τ_t^0 denotes the parallel translation along c from $x_0 = c(0)$ to $c(t)$, and $(\tau^{-1})_t^l$ denotes the parallel translation along c^{-1} from $f(x_0) = c(l)$ to $c(t)$.

Then the associated vector field of this variation is

$$Y_\beta(t) = \left(1 - \frac{t}{l}\right) \tau_t^0 \dot{\beta}(0) + \frac{t}{l} (\tau^{-1})_t^l f_* \dot{\beta}(0).$$

Firstly we have $\dot{c}(l) = f_* \dot{c}(0)$. In fact calculating the first variation formula for ψ_c, we have

$$\begin{aligned}(4)\qquad 0 &= \left\langle \frac{\partial \psi}{\partial u}(l, 0), \frac{\partial \psi}{\partial t}(l, 0) \right\rangle - \left\langle \frac{\partial \psi}{\partial u}(0, 0), \frac{\partial \psi}{\partial t}(0, 0) \right\rangle \\ &= \langle Y_c(l), \dot{c}(l) \rangle - \langle Y_c(0), \dot{c}(0) \rangle \\ &= \langle f_* \dot{c}(0), \dot{c}(l) \rangle - 1.\end{aligned}$$

Thus $f_* \dot{c}(0) = \dot{c}(l)$ holds good. Let J denote the almost complex structure of M.

Next we consider the case where $\dot{\beta}(0) = J\dot{c}(0)$ holds. Then the curve $u \longrightarrow \psi(l, u) = \exp_{f(x_0)} u(f_* J\dot{c}(0)) = f \exp_{x_0} u J\dot{c}(0)$ is the f-image of the curve $u \longrightarrow \psi(0, u)$. Furtheremore in this case the associated vector field $Y(t)$ satisfies

$$\begin{aligned}(5)\qquad Y(t) &= \left(1 - \frac{t}{l}\right) \tau_t^0 J\dot{c}(0) + \frac{t}{l} (\tau^{-1})_t^l f_* J\dot{c}(0) \\ &= \left(1 - \frac{t}{l}\right) J\dot{c}(t) + \frac{t}{l} J(\tau^{-1})_t^l \dot{c}(l) \\ &= J\dot{c}(t),\end{aligned}$$

since J is a parallel tensor field, f is an automorphism of Kählerian structure, and $\dot{c}(l) = f_* \dot{c}(0)$ holds.

Thus we have by the second variation formula for ψ_β,

$$(6) \qquad 0 \leqq L''(0) = \int_0^l \{<Y', Y'> - K(Y, \dot{c})<Y, Y>\}dt$$
$$= -\int_0^l H(\dot{c}(t))dt < 0.$$

This contradiction completes the proof of Theorem B.

4. proof of Theorem C. A. Preismann's theorem ([5]) assumes that M is of negative sectional curvature and his proof depends on the comparison theorem on the angles of a triangle.

Our proof depends on the Wolf's method ([11]) and the convexity of some function.

We consider elements of $\pi_1(M)$ as deck transformations of the universal covering space $\widetilde{M}$ of M. Then it is well known that $\widetilde{M}$ is diffeomorphic to a euclidean space.

Let the center of $\pi_1(M)$ have a non trivial element γ, then γ is a bounded isometry (J. A. Wolf [11]).

Then every bounded isometry of complete simply connected Riemannian manifold $\widetilde{M}$ of non-positive sectional curvature must be a Clifford translation. Wolf proved this fact by a result of L. W. Green. But we can give a simpler proof. Let x, y be any pair of points in $\widetilde{M}$, and σ be a geodesic line from x to y. Then $f(t) = d^2(\sigma(t), \gamma(\sigma(t)))$ is a convex function. (H. Busemann [2], Bishop-O'Neil [1]). Since γ is a bounded transformation, we have $f'(t) \leqq 0$, that is, $d(x, \gamma(x)) \geqq d(y, \gamma(y))$. By changing the role of x and y, we have $d(x, \gamma(x)) = d(y, \gamma(y))$.

Next, let x_0 be a point at which Ricci curvature is negative definite. Let c be the minimizing geodesic joining x_0 and $\gamma(x_0)$, and $X_1 = \dot{c}(0), X_2, \cdots X_d$ ($d = \dim M$) be an orthonormal frame at x_0. For any i $(2 \leqq i \leqq d)$ we consider the geodesic c_i starting from x_0 with $\dot{c}_i(0) = X_i$ and take the variation of c defined by joining $c_i(t)$ and $\gamma(c_i(t))$ by the unique minimizing geodesic. $X_i(t)$ denotes the associated vector field of this variation. Then clearly $X_i(0) = X_i$ holds.

Now the second variation $L_i''(0)$ of this variation must be zero, since γ is a Clifford translation. Then we have

$$(7) \qquad 0 = \sum_{i=2}^d L_i''(0) = \int_0^{d(x_0, \gamma(x_0))} \left\{ \sum_{i=2}^d <X_i'(t), X_i'(t)> \right.$$
$$\left. - \sum_{i=2}^d K(\dot{c}(t), X_i(t))<X_i(t), X_i(t)> \right\} dt > 0,$$

because at $t=0$, $\sum_{i=2}^{d} K(\dot{c}(t), X_i(t)) < X_i(t), X_i(t) > = \mathrm{Ricci}(\dot{c}(0)) < 0$. This contradiction concludes that γ must be identity.

REMARK. Let M, N be compact differentiable manifolds with abelian $\pi_1(M)$. Then $M \times N$ carries no Riemannian metric with non-positive sectional curvature such that there is one point of $M \times N$ where Ricci curvature is negative definite.

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