Thresholds for epidemic spreading in networks

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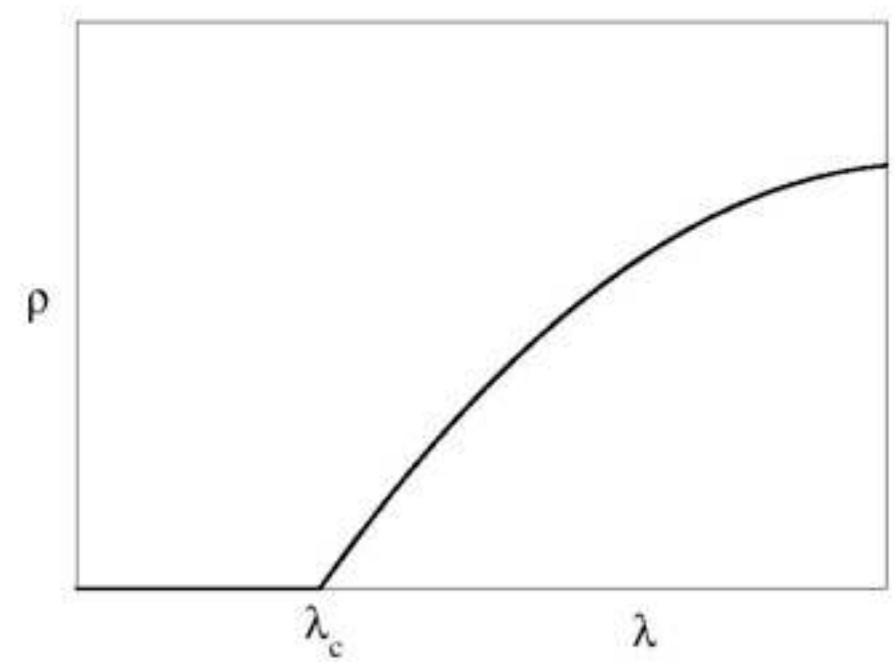
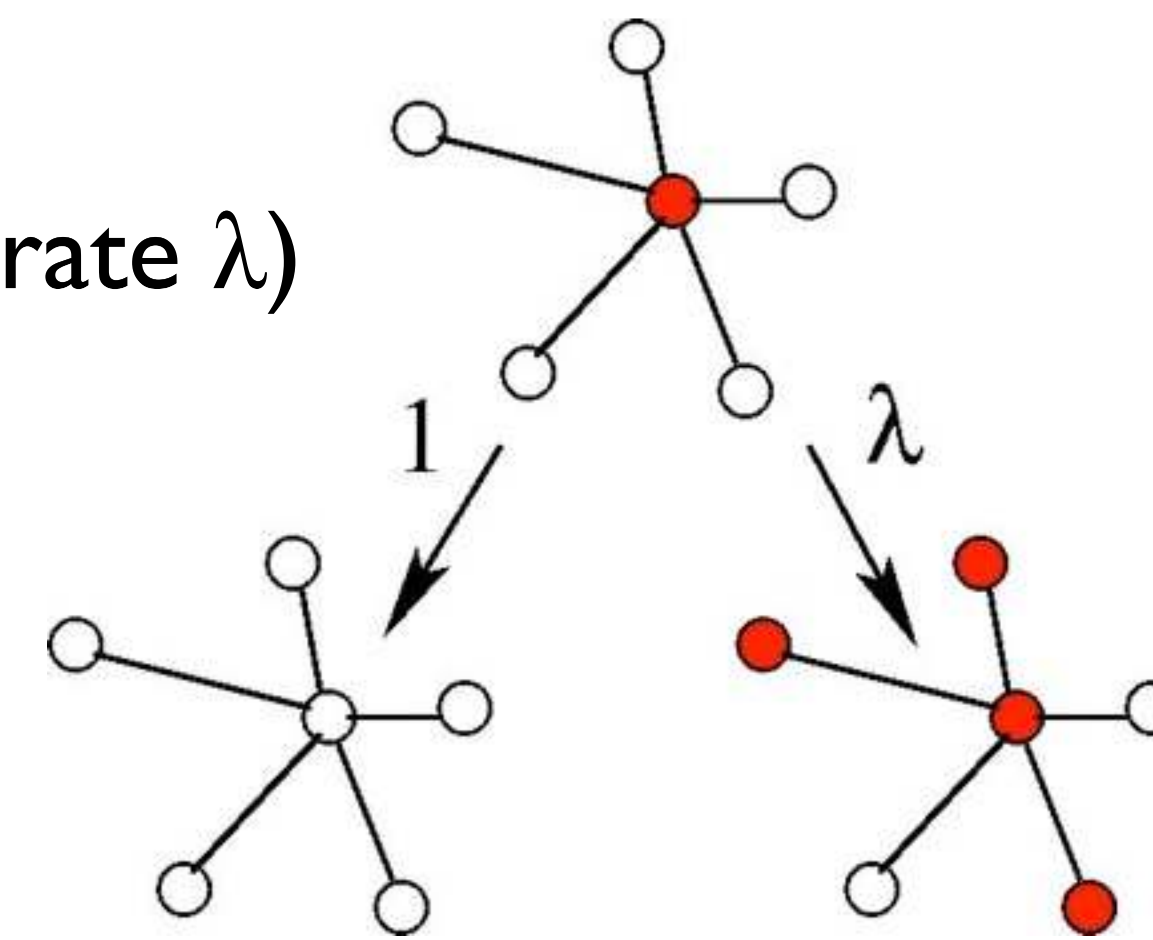
Susceptible-Infected-Susceptible (SIS) model

- Two possible states: susceptible and infected
- Two possible events for infected nodes:
  - Recovery (rate 1)
  - Infection to neighbors (rate λ)





Heterogeneous Mean-Field theory for SIS

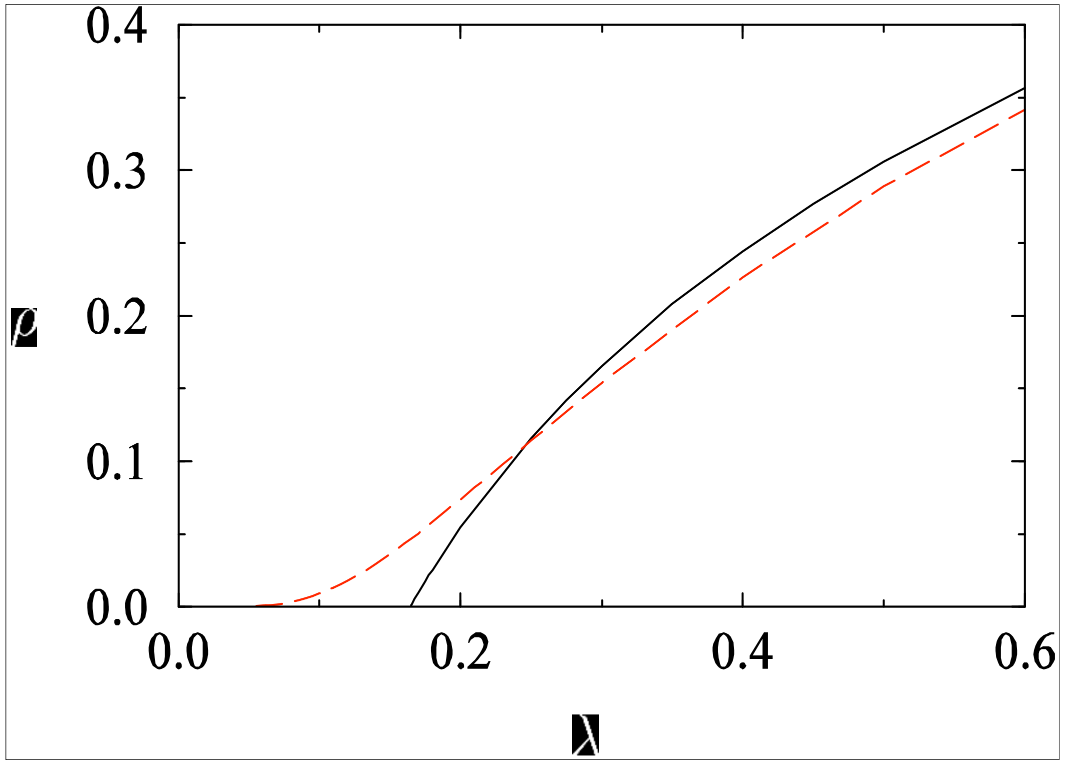
Pastor-Satorras and Vespignani (2001)

- Degree distribution $P(k) \sim k^{-\gamma}$
- ρ_k = density of infected nodes of degree k

$$\dot{\rho_k} = -\rho_k + \lambda k[1-\rho_k]\sum_{k'} P(k'|k)\rho_{k'}$$

- Threshold

$$\lambda_c = \frac{\langle k \rangle}{\langle k^2 \rangle}$$

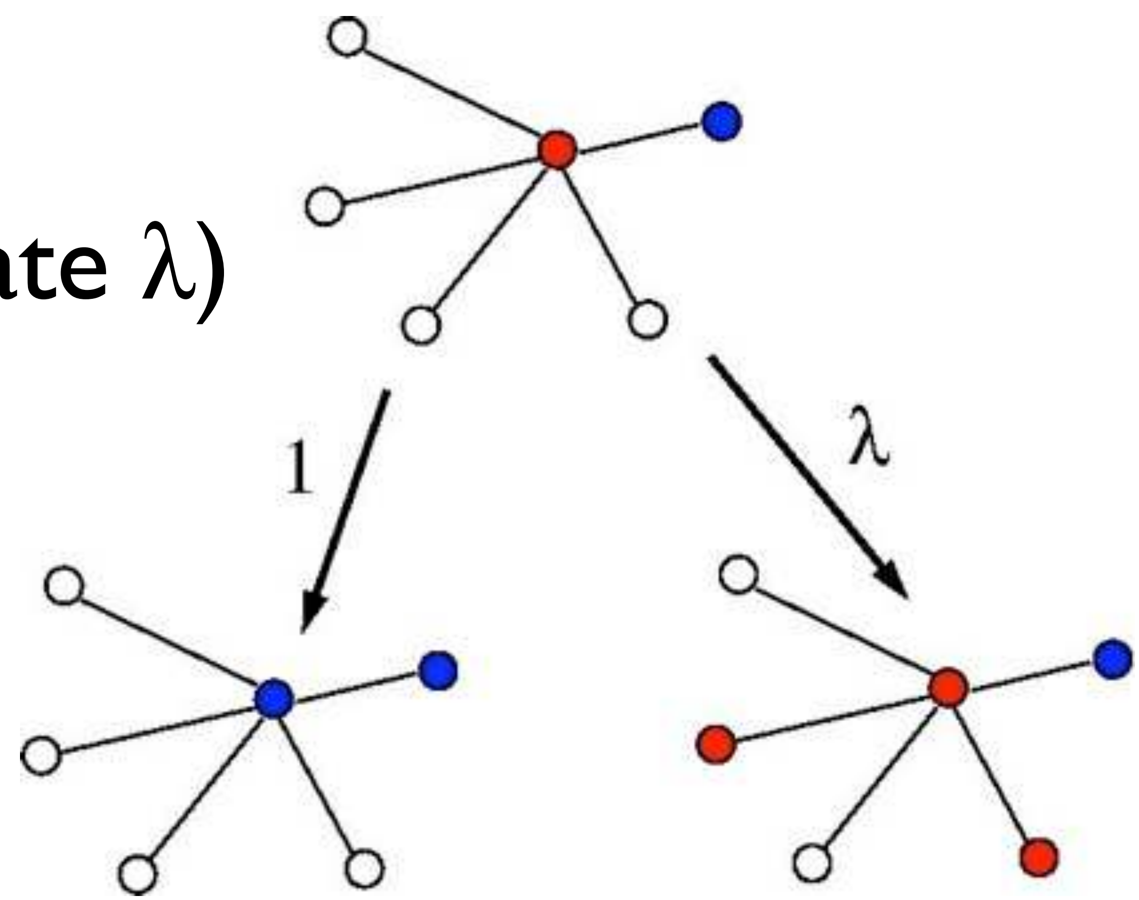


- Scale-free networks: $\gamma < 3$

zero epidemic threshold

Susceptible-Infected-Removed (SIR) model

- Three possible states: susceptible, infected and removed.
- Two possible events for infected nodes:
  - Death/recovery (rate 1)
  - Infection to neighbors (rate λ)
  - Transition between healthy and infected



HMF for SIR

- HMF theory

$$\lambda_c = \frac{\langle k \rangle}{\langle k^2 \rangle - \langle k \rangle}$$

- Zero epidemic threshold for scale-free networks
- Finite epidemic threshold for scale-rich networks

Got the Flu (or Mumps)? Check the Eigenvalue!

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Abstract

For a given, arbitrary graph, what is the epidemic threshold? That is, under what conditions will a virus result in an epidemic? We provide the *super-model* theorem, which generalizes older results in two important, orthogonal dimensions. The theorem shows that (a) for a wide range of virus propagation models (VPM) that include *all* virus propagation models in standard literature (say, [8][5]), and (b) for *any* contact graph, the answer always depends on the first eigenvalue of the connectivity matrix. We give the proof of the theorem, arithmetic examples for popular VPMs, like flu (SIS), mumps (SIR), SIRS and more. We also show the implications of our discovery: easy (although sometimes *counter-intuitive*) answers to 'what-if' questions; easier design and evaluation of immunization policies, and significantly faster agent-based simulations.

Beyond HMF for SIS

- Wang et al., 2003

$$\lambda_c = \frac{1}{\Lambda_N}$$

$$\Lambda_N = \text{Largest eigenvalue of the adjacency matrix}$$

- Chung et al. 2005

$$\Lambda_N = \begin{cases} c_1\sqrt{k_c} & \sqrt{k_c} > \frac{\langle k^2\rangle}{\langle k\rangle}\ln^2(N) \\ c_2\frac{\langle k^2\rangle}{\langle k\rangle} & \frac{\langle k^2\rangle}{\langle k\rangle} > \sqrt{k_c}\ln(N) \end{cases}$$

$$k_c = \text{Largest degree in the network}$$

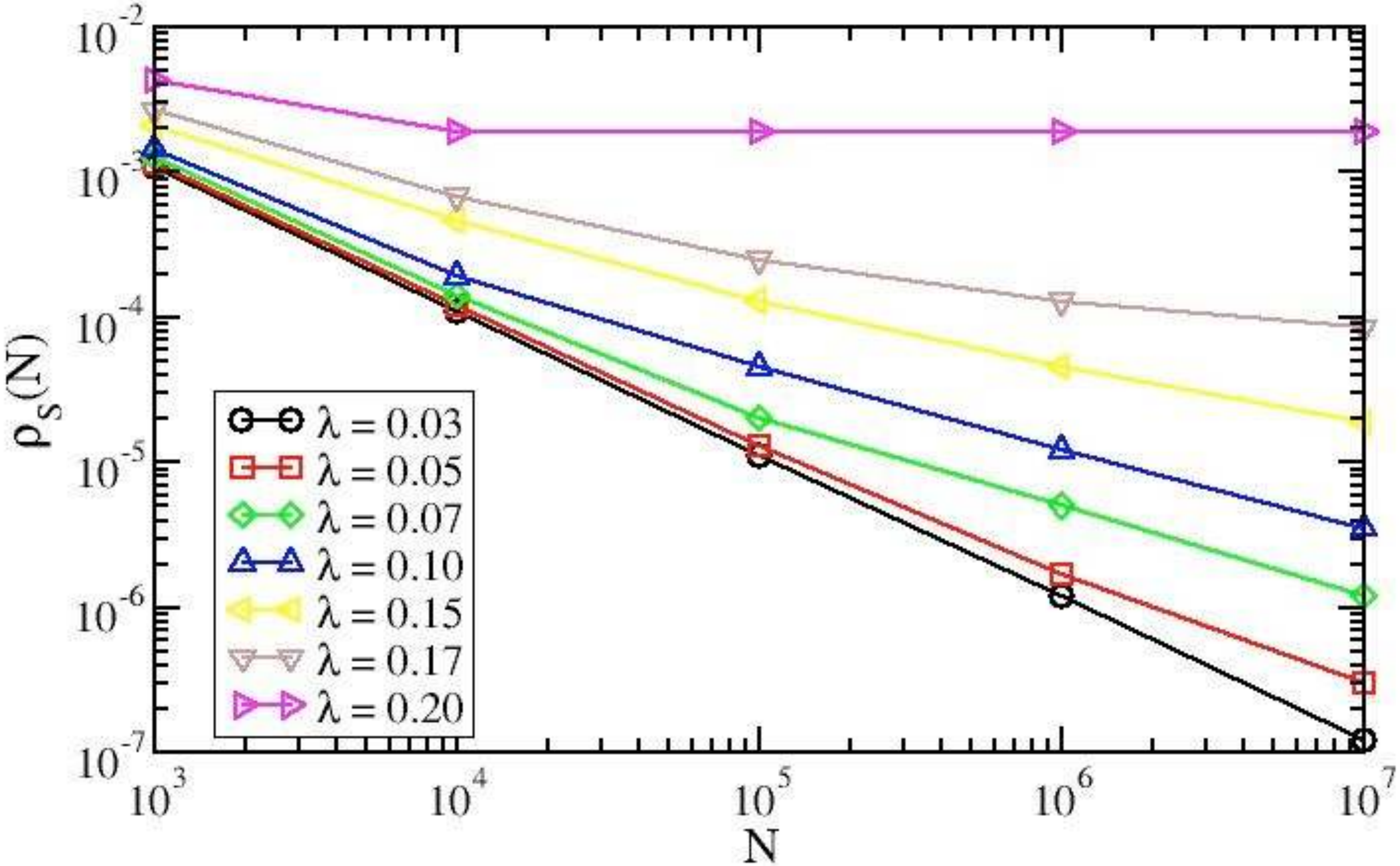
Beyond HMF for SIS

- Summing up

$$\lambda_c \simeq \begin{cases} 1/\sqrt{k_c} & \gamma > 5/2 \\ \frac{\langle k \rangle}{\langle k^2 \rangle} & 2 < \gamma < 5/2 \end{cases}$$

- In any uncorrelated quenched random network with power-law distributed connectivities, the epidemic threshold goes to zero as the system size goes to infinity.
- This has nothing to do with the scale-free nature of the degree distribution.

Finite Size Scaling
SIS $\gamma = 4.5$



Mathematical origin of HMF failure for SIS

- HMF is equivalent to using <u>annealed</u> networks with adjacency matrix

$$a_{ij} \to \bar{a}(k_i, k_j) = \frac{k_i k_j}{\langle k \rangle N}$$

- This matrix has a unique nonzero eigenvalue

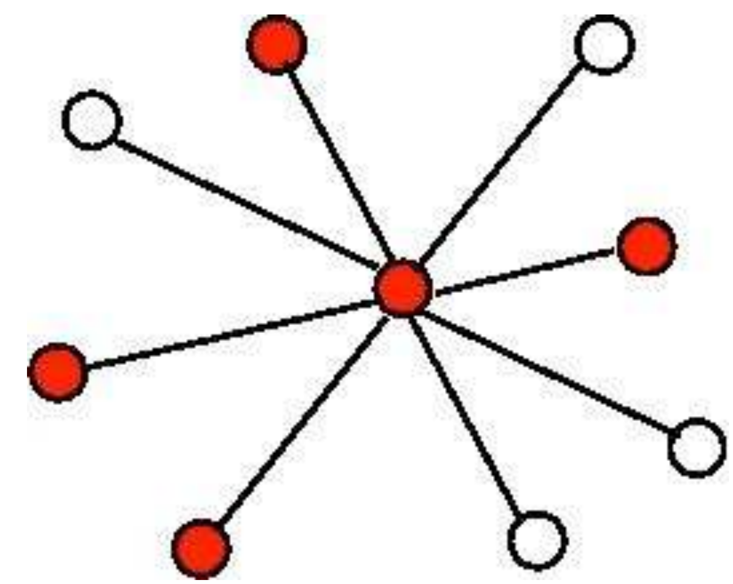
$$\Lambda_N = \frac{\langle k^2 \rangle}{\langle k \rangle}$$

Physical origin of HMF failure for SIS

- Star graph with k_{max}+1 nodes

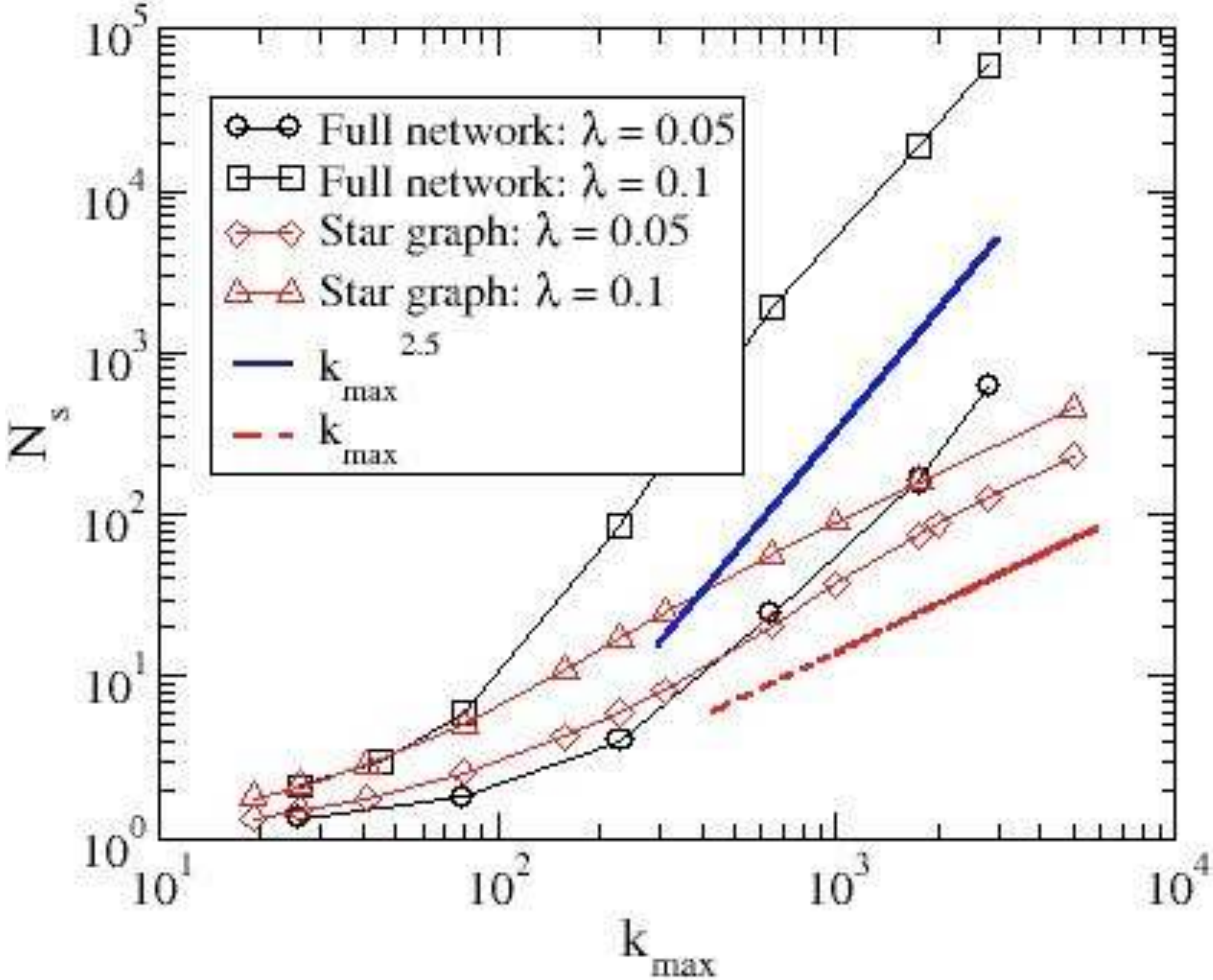
$$\rho_{max} \propto (\lambda^2 k_{max} - 1)$$

$$\rho_1 \propto (\lambda^2 k_{max} - 1)$$



- For $\lambda > 1/\sqrt{k_{max}}$ the hub and its neighbors are a self-sustained core of infected nodes, which spreads the activity to the rest of the system.

A truly endemic state



$\gamma = 3.5$

Extensions

- For Erdos-Renyi graphs

$$\Lambda_N \to \max\{\sqrt{k_{max}}, \langle k \rangle\}$$

Krivelevich et al., 2003

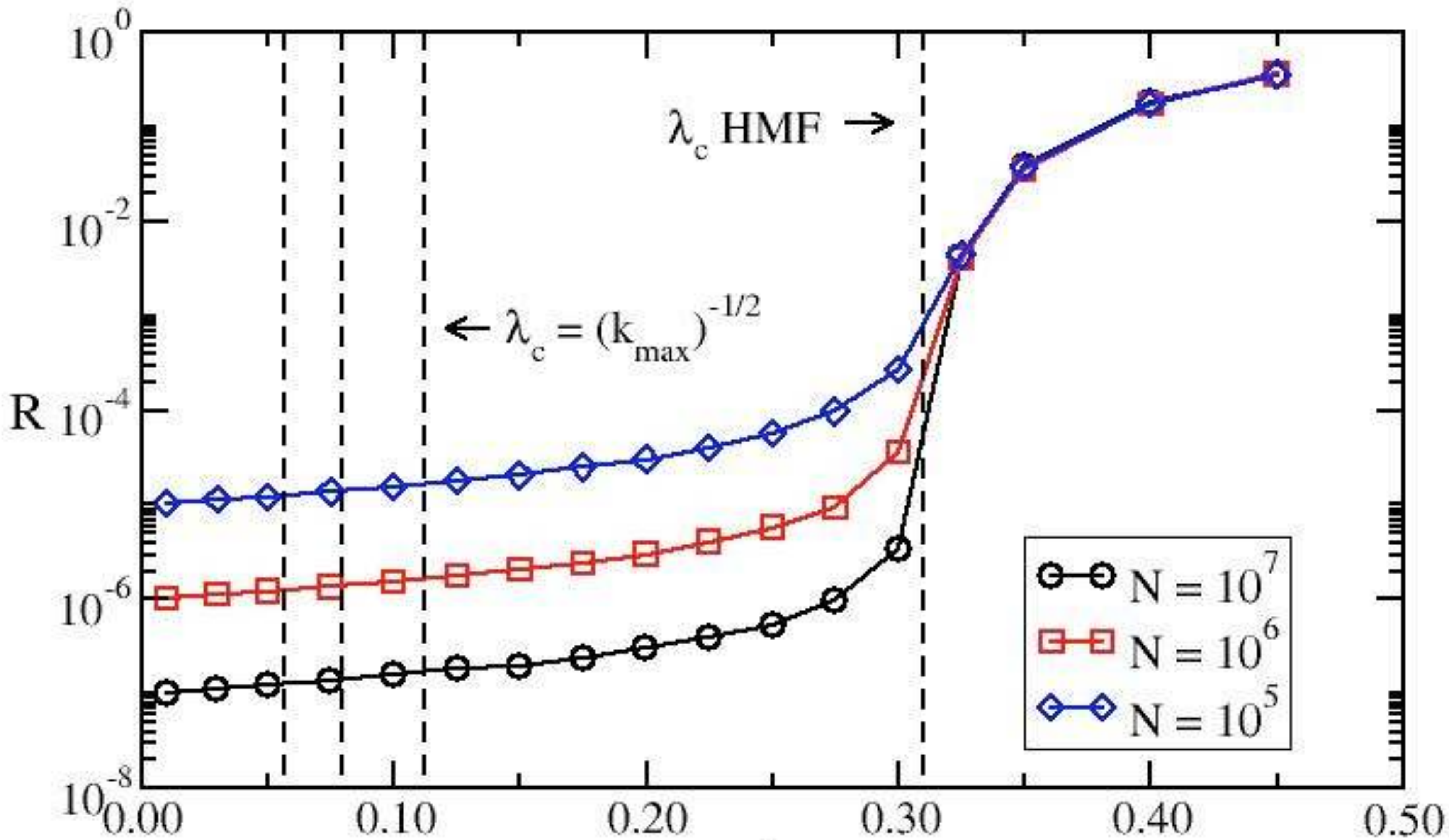
- For correlated graphs

$$\Lambda_N \geq \sqrt{k_{max}}$$

Restrepo et al, 2007

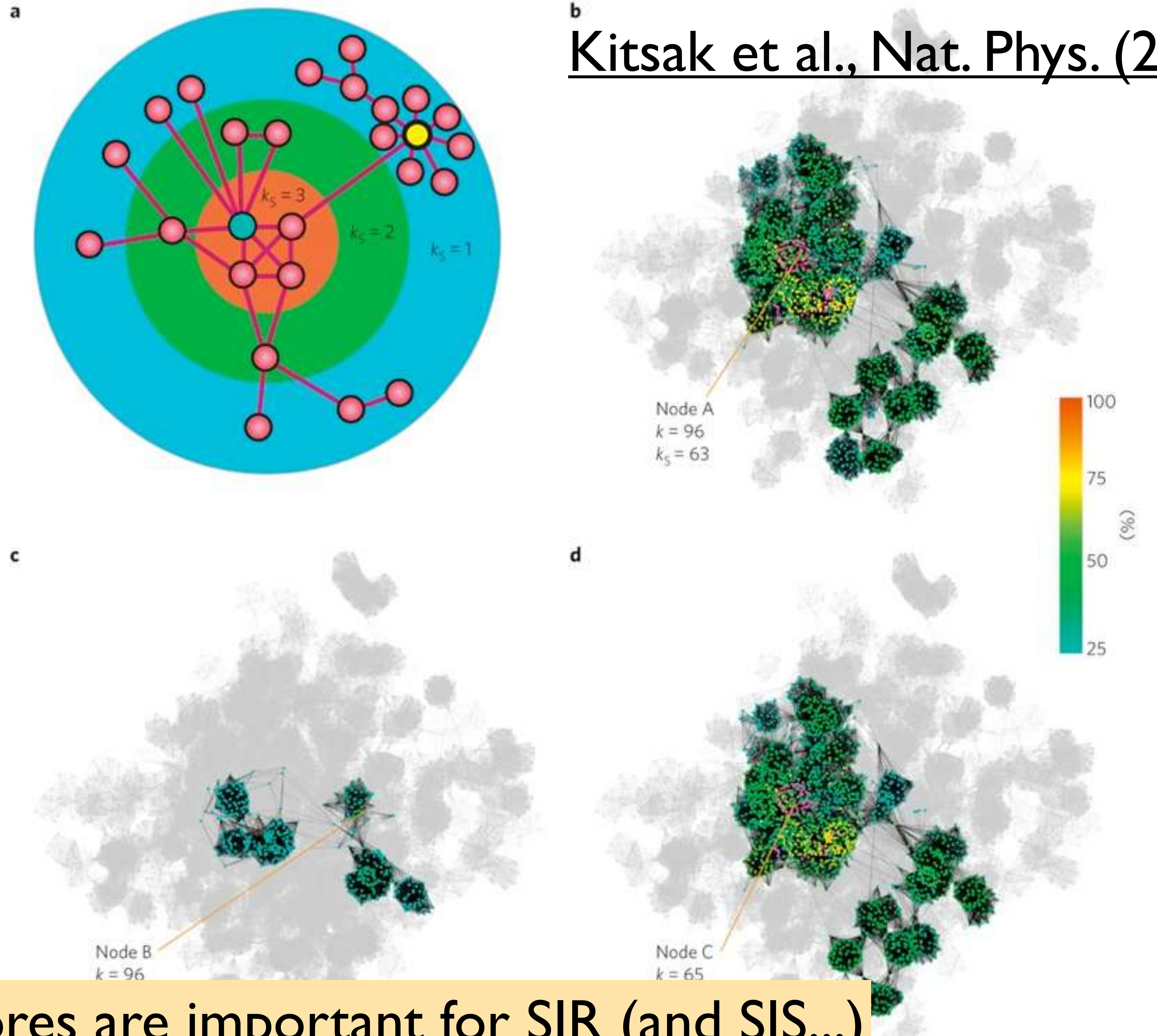
The threshold vanishes for any network

SIR $\gamma = 4.5$



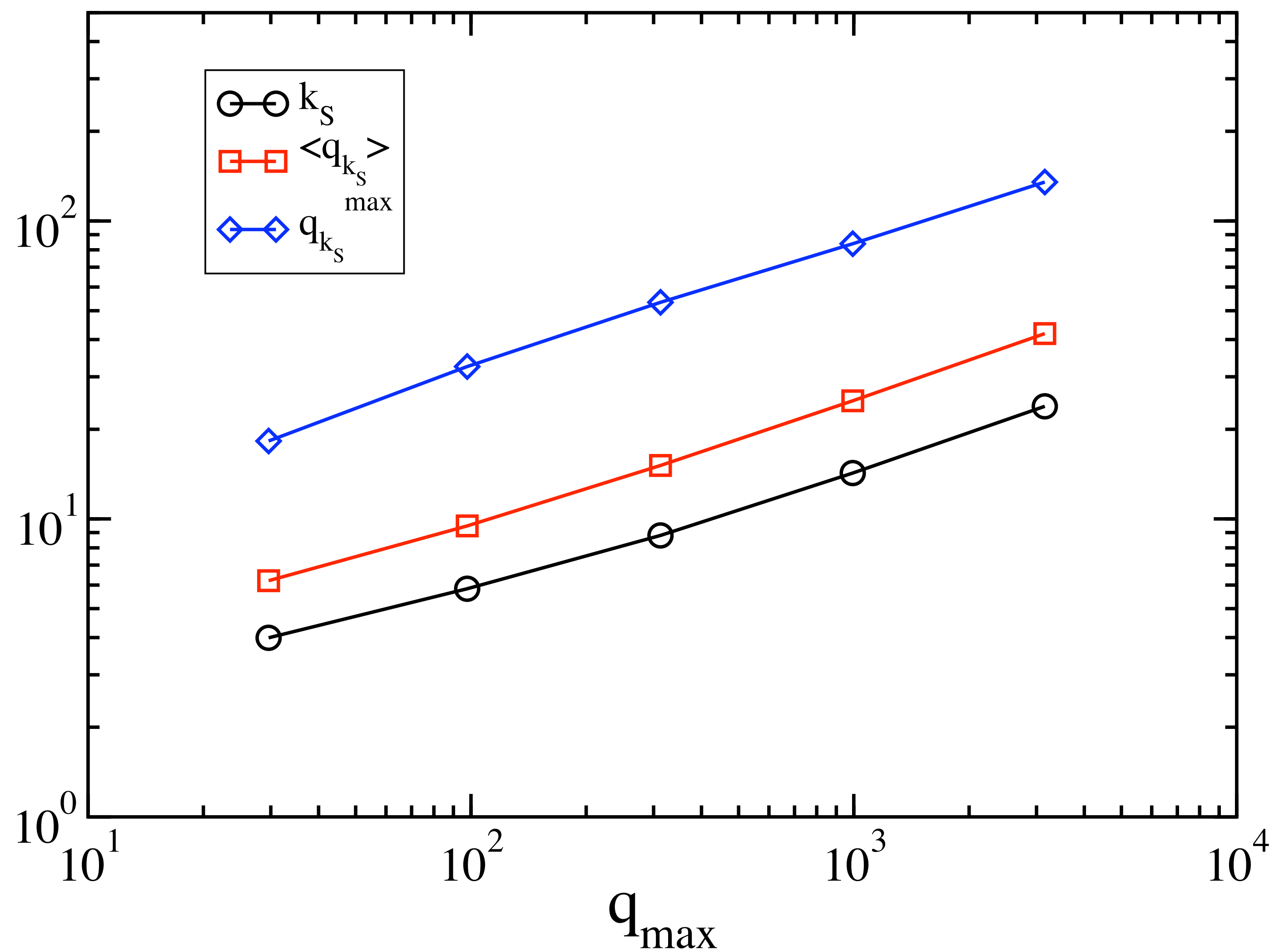
Conjecture: on scale-rich networks, the epidemic threshold is vanishing or finite depending on the presence or absence of a steady-state.

Kitsak et al., Nat. Phys. (2010)



k-cores are important for SIR (and SIS...)

SIS dynamics on maximum k-core



$\gamma = 2.5$

- The maximum k-core has a narrow degree distribution

$$\lambda_c \approx \frac{1}{\langle k \rangle} \sim \frac{1}{k_S}$$

SIS dynamics on maximum k-core

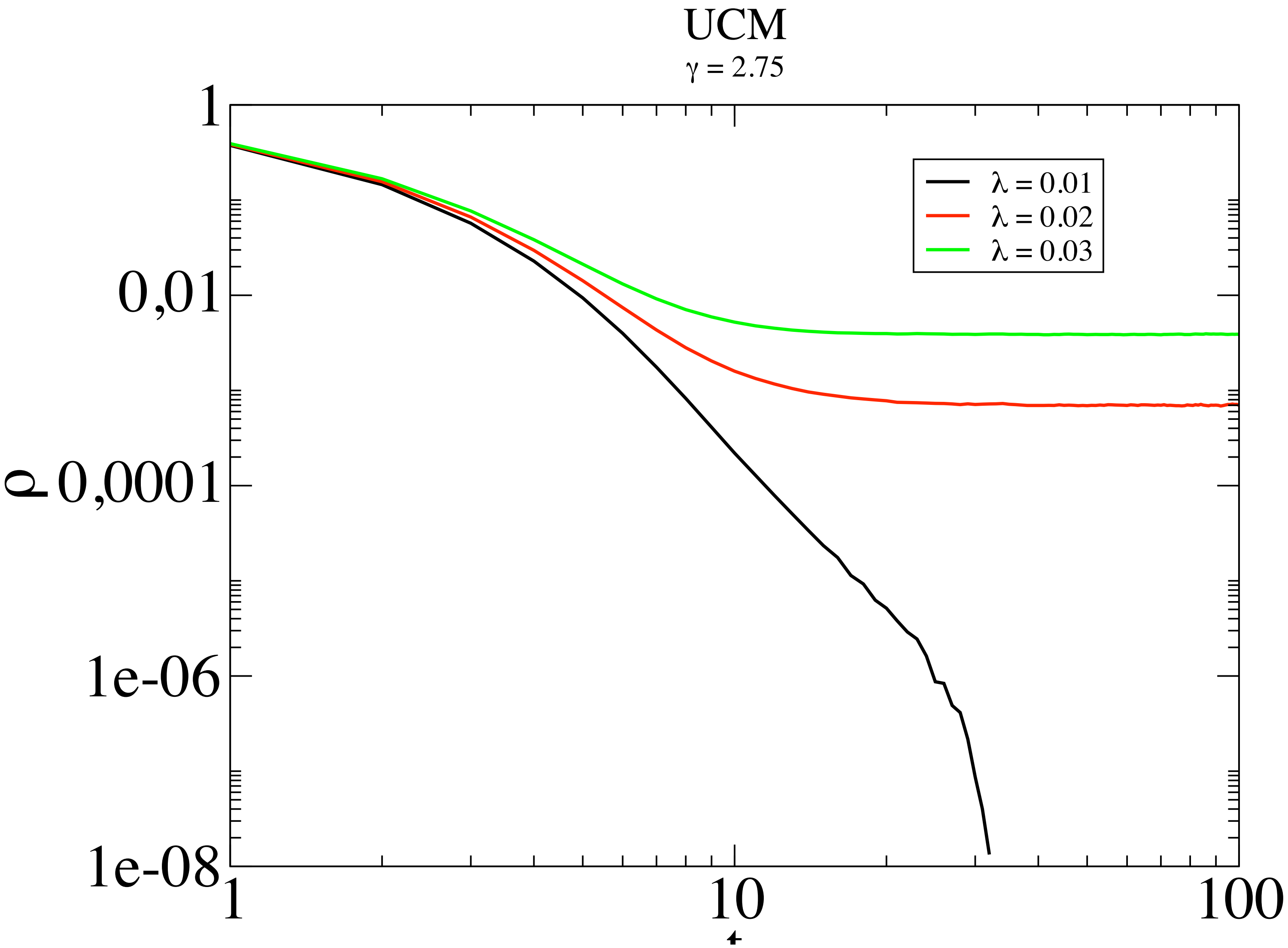
- Maximum k-core index (Dorogovtsev et al, 2007)

$$k_S \approx (\gamma-2)(3-\gamma)^{(3-\gamma)/(\gamma-2)} k_{max} \left(\frac{k_{min}}{k_{max}}\right)^{(\gamma-2)} \sim k_{max}^{3-\gamma}$$

- Summing up

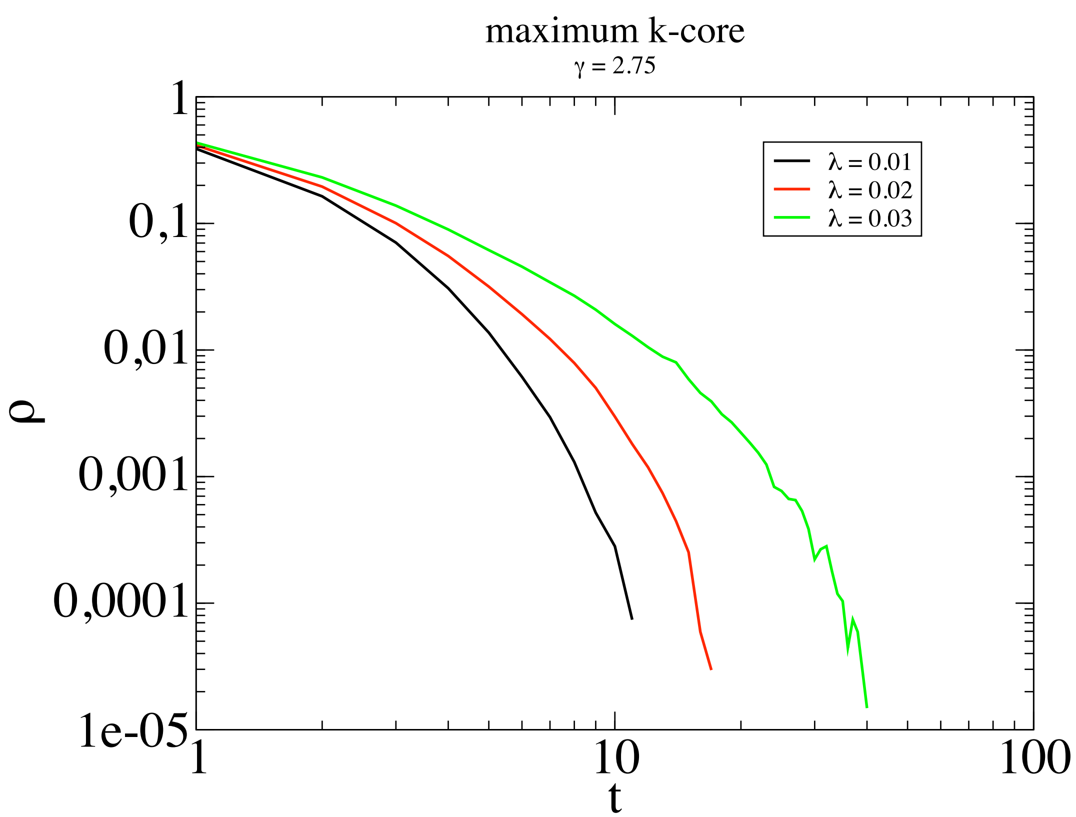
$$\lambda_c \sim 1/k_S \sim k_{max}^{\gamma-3} \sim \langle k \rangle / \langle k^2 \rangle$$

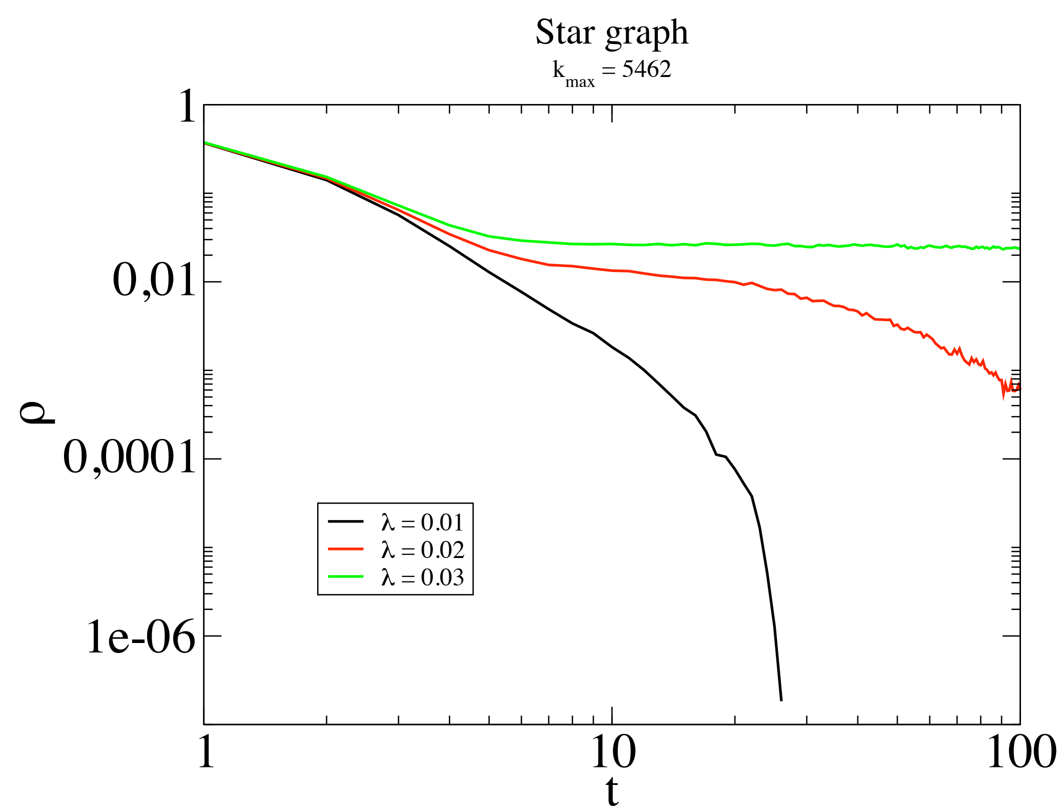
The maximum k-core induces a threshold scaling as the HMF threshold

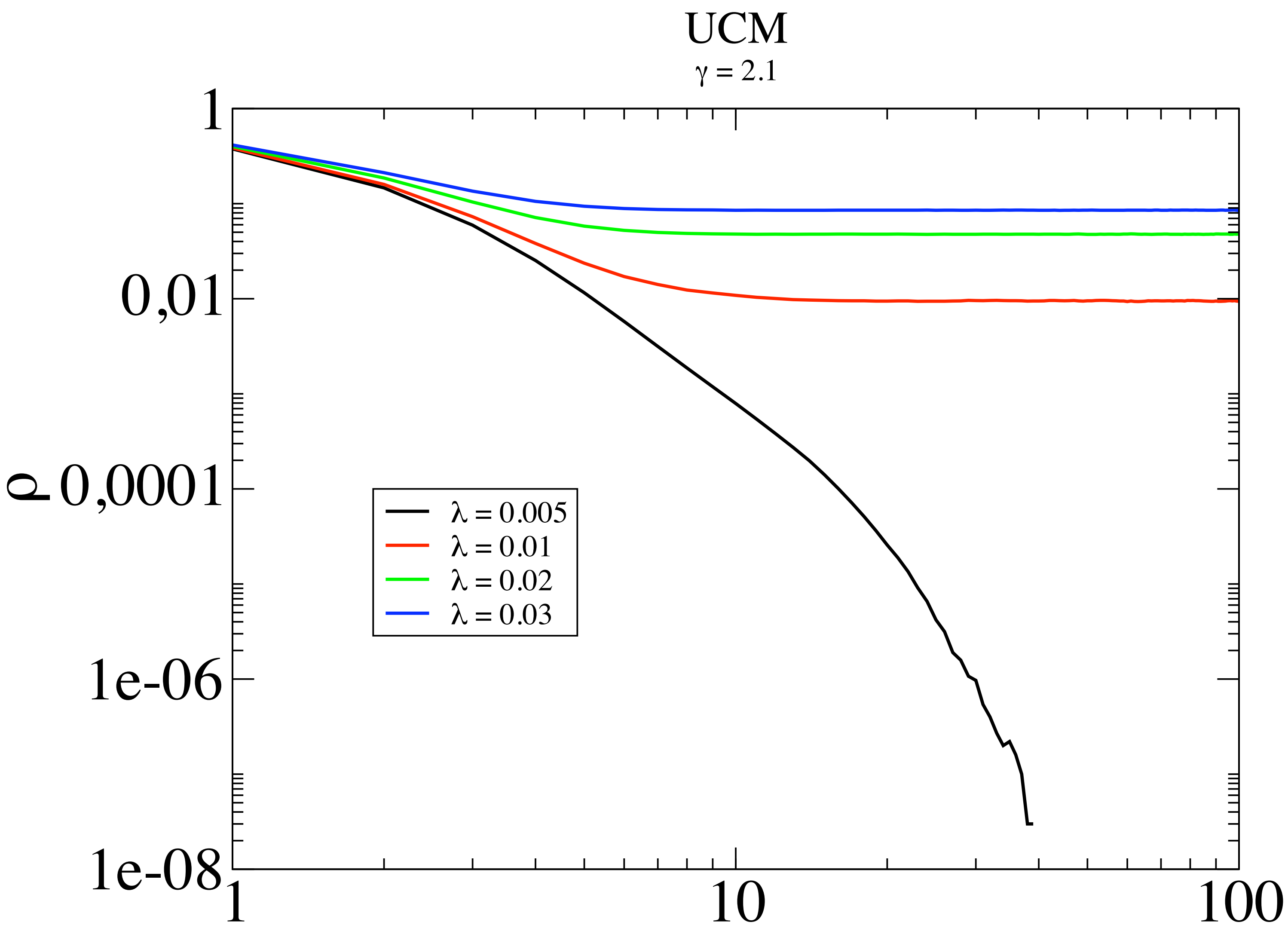


$$1/\Lambda_N = 0.0131$$
$$1/\sqrt{k_{max}} = 0.0135$$
$$\langle k \rangle / \langle k^2 \rangle = 0.0231$$

Transition governed by the hub





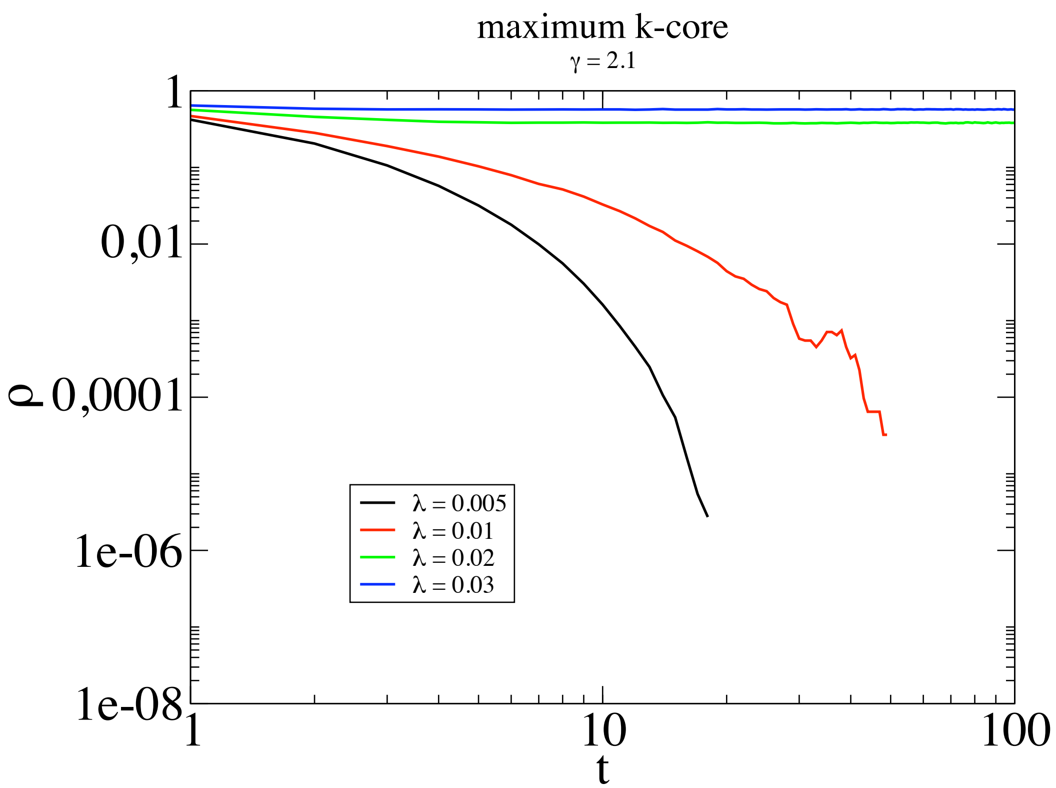


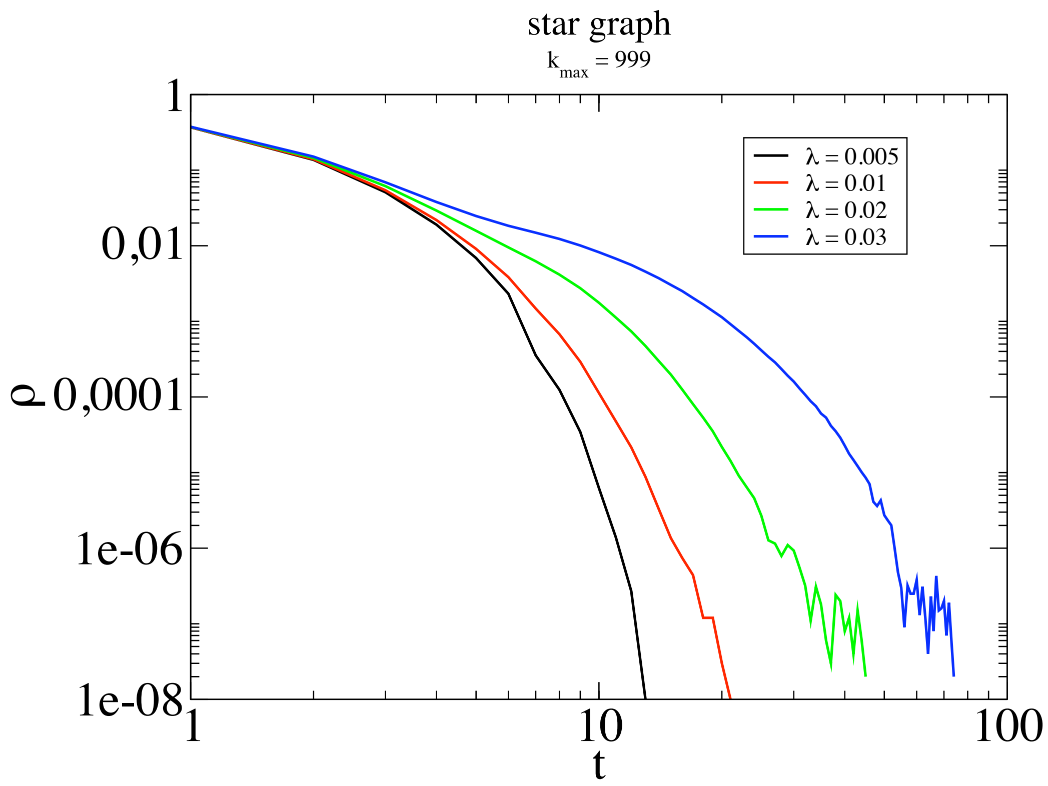
$$1/\Lambda_N = 0.00735$$

$$1/\sqrt{k_{max}} = 0.03163$$

$$\langle k \rangle / \langle k^2 \rangle = 0.00745$$

Transition governed by the maximum k-core





SIR dynamics

- On the maximum k-core

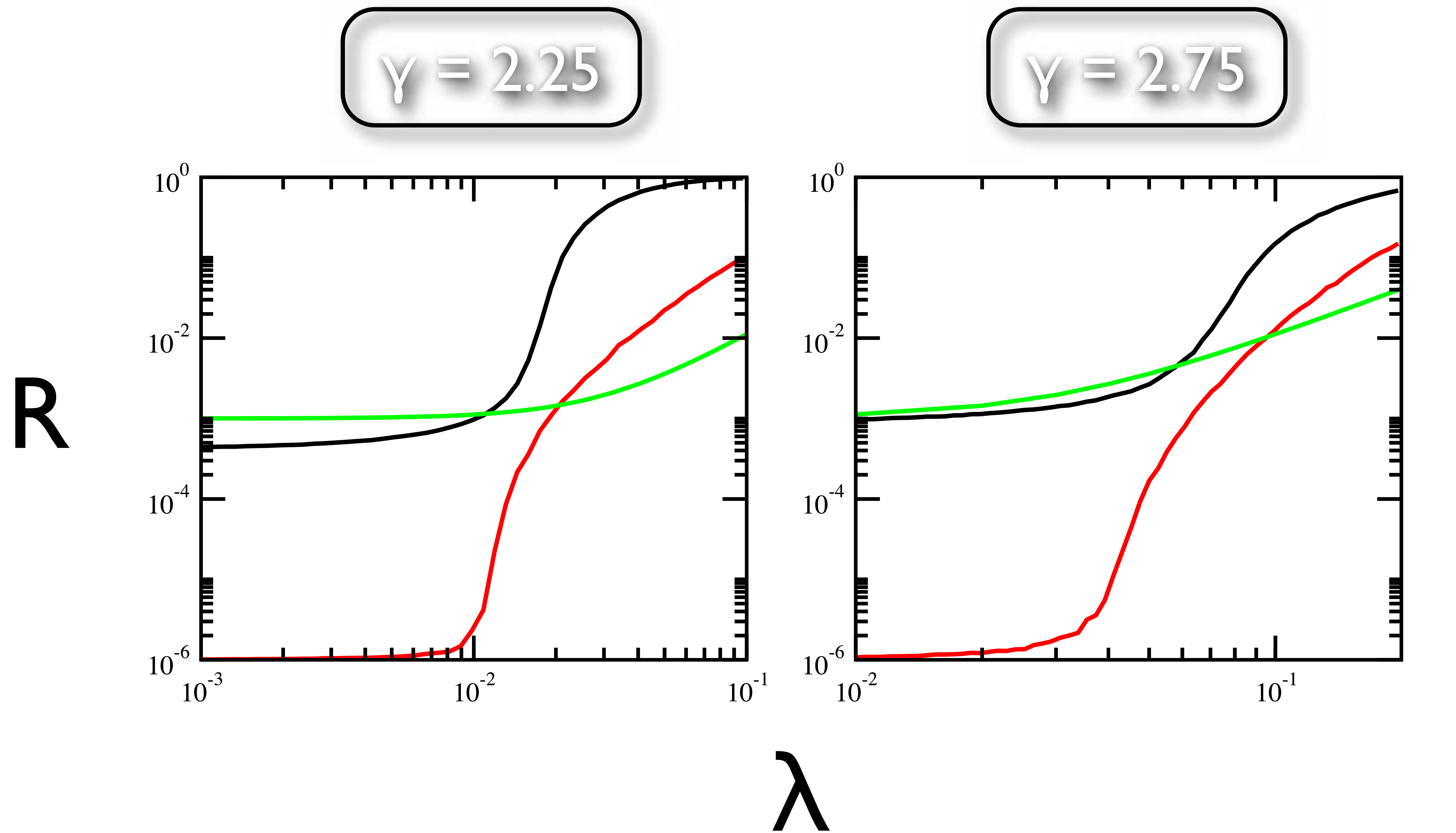
$$\lambda_c \sim 1/k_S \sim k_{max}^{\gamma-3} \sim \langle k \rangle / \langle k^2 \rangle$$

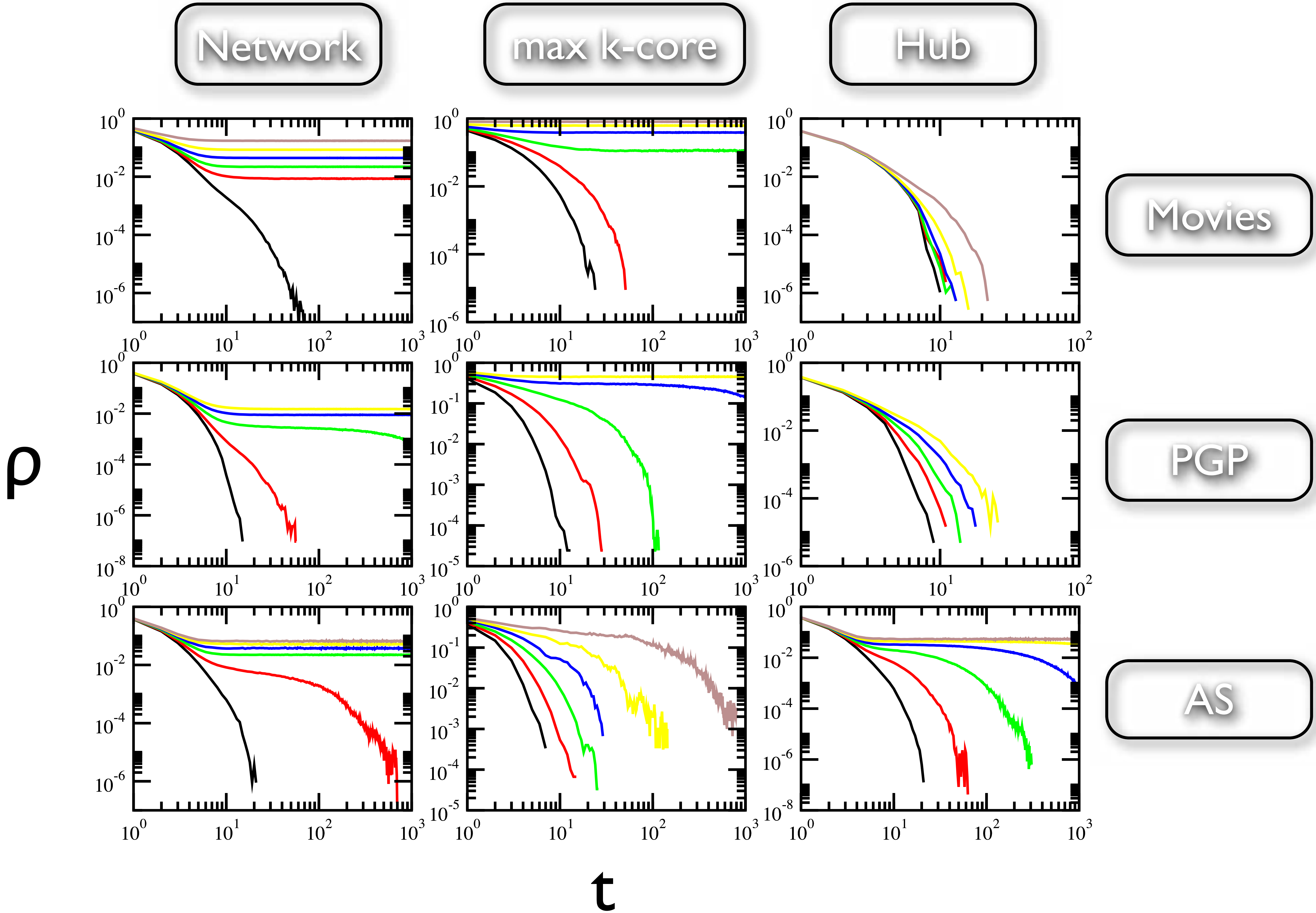
- On the star-graph centered around the hub

$$R = \lambda^2 + \frac{1+2\lambda}{k_{max}}$$

No dependence on k_{max} (when large)

The maximum k-core always governs the transition and sets the threshold


γ = 2.25
γ = 2.75
R
λ
10^0
10^-2
10^-4
10^-6
10^-3
10^-2
10^-1
10^0
10^-2
10^-4
10^-6
10^-2
10^-1


Network
max k-core
Hub
Movies
PGP
AS
ρ
t

Summary

- Zero epidemic threshold for SIS on scale-rich networks.
- Finite epidemic threshold for SIR on scale-rich networks.
- For SIS the transition can be governed either by the maximum k-core or by the largest hub.
- For SIR the maximum k-core always dominates.
- Correlations may change the picture

C. Castellano and R. Pastor-Satorras, PRL, 105, 218701 (2010)
C. Castellano and R. Pastor-Satorras, arXiv:1105.5545 (2011)