THROUGHPUT ANALYSIS IN WIDEBAND MIMO BROADCAST CHANNELS WITH PARTIAL FEEDBACK

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ABSTRACT

It has been recently shown that opportunistic transmit beamforming using partial channel state information (CSI) achieves the same throughput scaling obtained from dirty paper coding for a broadcast channel with fixed number of transmit antennas and many receivers [1]. In this paper, we study the generalization of this scheme to wideband broadcast channels. By using orthogonal frequency division multiplexing, an L-tap wideband channel can be decomposed to N parallel narrowband channels, where N is larger than L. Neighboring subchannels are therefore highly correlated, and it is intuitive to say that each group of neighboring subchannels (forming a cluster) can be characterized by one channel quality. We show in this paper that users need only feedback the best signal-to-noise-plus-interference ratio at the center of each cluster. Our results indicate that for cluster size of order $\frac{N}{L\sqrt{\log K}}$, where K is the number of users, this feedback scheme maintains the same throughput scaling as when full CSI is known. Simulation results show that larger cluster sizes ($\frac{N}{2L}$) can also be implemented for a small throughput hit.

1. INTRODUCTION

There has been growing interest in the study of the capacity region of multiple-input multiple-output (MIMO) broadcast channels [2, 3, 4] . Recently, it has been shown that dirty paper coding achieves the capacity region of the Gaussian MIMO broadcast channel [5]. This scheme assumes perfect channel state information (CSI) at the transmitter, and achieves throughput that scales linearly with the number of transmit antennas [6]. However, full channel knowledge is not always attainable or practical. It was therefore suggested in [1] that, using M random beams and partial feedback, opportunistic beamforming is performed at the trans-

mitter. Each user (receiver) need only feedback its maximum signal-to-noise-plus-interference ratio (SINR) and the index of the transmitting antenna at which this maximum value exists. This scheme requires less feedback and is computationally feasible, still it achieves the linear scaling in throughput that is achieved when full CSI is available. This has been shown for narrowband broadcast channels with a fixed number of transmit antennas and large number of users, which is a typical setting in practical cellular systems.

We investigate the generalization of this scheme to an L-tap wideband broadcast channel (with independent and identically distributed (i.i.d.) taps). Wideband channels are desireable due to the increased need to drive bit rates higher. Using orthogonal frequency division multiplexing (OFDM), an L-tap wideband channel can be decomposed to N parallel subchannels. However, neighboring subchannels are highly correlated (as N is bigger than L), and therefore with high probability they have similar channel qualities. Recently, it has been suggested in [7] that neighboring subchannels are grouped in a cluster. Then each user need only feedback the signal-to-noise ratio (SNR) values and indices of its strongest clusters to the transmitter. Based on this opportunistic reduced feedback, the transmitter sends one beam to the user with the highest SNR value per cluster. Using simulation results, [7] shows that this scheme can be implemented without significantly sacrificing performance. There have also been recent publications that looked at variations of this problem, for example [8, 9] among others.

In this paper, we look into the effects of partial feedback on the throughput of wideband broadcast channels. More specifically we address the question of how much feedback is necessary in order to maintain the same throughput scaling as when full CSI is available. The amount of feedback is inversely proportional to the cluster size, therefore this question also addresses the issue of cluster size. Through analysis and simulations of a wideband broadcast channel with M transmit antennas and K single-antenna users, we prove that for cluster size of order $\frac{N}{L\sqrt{\log K}}$, each user need only feedback its best SINR value at the center subcarrier

of each cluster in order for the transmitter to perform opportunistic beamforming (by constructing M random beams rather than a single beam). With this cluster size, this scheme achieves the same throughput scaling as when full CSI is available at the transmitter with less complexity and feedback. Our simulation results show that larger cluster size ($\frac{N}{2L}$) can also be implemented for a small loss in throughput.

2. PROBLEM FORMULATION

We start with a discrete-time baseband model of the channel. At time t and sampling at multiples of $\frac{1}{W}$, where W is the bandwidth of the input signal, the input-output relationship can be written as

$$y[t] = \sum_{l=0}^{L-1} h_l\, x[t-l] + w[t], \qquad (1)$$

where y is the sampled output at time t, x is the sampled input transmitted at time $t - l$ and w is additive white noise. h_l is the lth complex channel filter tap, where the number of taps L is dictated by the delay spread of the channel T_d ($L = T_d \times W$), and the coherence bandwidth $W_c = \frac{1}{2T_d}$. In practice, wideband channels can be represented by 4 or 5 channel taps [10]. We assume i.i.d channel taps, and form a vector of the circularly symmetric complex Gaussian random variables $H = [h_0\ h_1\ ...\ h_{L-1}]^T$. Using OFDM, data symbols modulate N subcarriers separated by $\frac{W}{N}$. The discrete Fourier transform of the L-tap channel gives the frequency response at frequency q as follows,

$$H(q) = \sum_{l=0}^{L-1} h_l\, e^{-j2\pi q \frac{l}{N}}, \quad q = 1,...,N. \qquad (2)$$

In matrix form, (2) can be written as $H(q) = V_q\, H$ where $V_q = [1\ e^{-j2\pi q \frac{1}{N}}\ ...\ e^{-j2\pi q \frac{(L-1)}{N}}]$.

The autocorrelation function depends solely on the frequency separation between subcarriers. It is therefore clear that a separation of W_c is required between subcarriers in order to de-correlate. Since the width of each subchannel is $\frac{W}{N}$, it is straightforward to show that there are roughly $\frac{N}{2L}$ neighboring subchannels that are highly correlated [10]. For this reason, we group neighboring subchannels in a cluster [7], and adopt a transmission scheme on each subchannel that requires partial feedback while maximizing the throughput [1].

We construct M random orthonormal beams ϕ_m ($M \times 1$) for $m = 1, ..., M$, and at time t, the mth vector is multiplied by the transmit symbol s_m^q. The transmitted signal on subchannel q is then,

$$S_q(t) = \sum_{m=1}^{M} s_m^q(t)\phi_m(t). \qquad (3)$$

The $L \times 1$ vector of channel gains at the mth beam is denoted by H_m. We assume that the kth receiver knows H_m^k for $m = 1, ..., M$ (can be readily available through training), and can therefore compute the following M $\text{SINR}_{q,m}^k$ values by assuming that s_m is the desired signal and the other s_i signals are interference as follows,

$$\text{SINR}_{q,m}^k = \frac{H_m^{k^*}\, V_q^*\, V_q\, H_m^k}{\frac{1}{\rho} + \sum_{i=1,i\neq m}^{M} H_i^{k^*}\, V_q^*\, V_q\, H_i^k}, \quad m = 1,...,M. \qquad (4)$$

ρ in (4) is the SNR for all the users, assuming a homogeneous network.

We consider a feedback scheme that requires one feedback per cluster for each user. We propose that each user feeds back the best SINR value at the center subcarrier of each cluster. This feedback can be expressed as (in addition to cluster and transmit antenna indices), $\max_{1\leq m\leq M} \text{SINR}_{q,m}^k$, where q here is the center subcarrier in the cluster. The transmitter assigns the mth beam to the user with the highest SINR; $\max_{1\leq k\leq K} \text{SINR}_{q,m}^k$. The throughput of each subchannel is estimated by,

$$R_q = E\left\{ \sum_{i=1}^{M} \log(1 + \max_{1\leq k\leq K} \text{SINR}_{q,m}^k) \right\}, \qquad (5)$$

and the throughput of the system is $R = \sum_{q=1}^{N} R_q$. Note that for $K \to \infty$, (5) is a tight estimate [1].

In order to further reduce the amount of feedback, we further assume that $\text{SINR}_{q,m}^k \geq \gamma$ where γ is a specified threshold of order $\log K$. We are able to set this threshold since we know that $\max_{1\leq k\leq K}\text{SINR}_{q,m}^k$ behaves like $(\log K)$ with high probability [1], and therefore we restrict feedback of only SINR values $\geq \gamma$. The amount of feedback from each user then reduces to the number of clusters $\times$ $\text{Pr}(\text{SINR}_q > \gamma)$.

In summary, we tackle two main questions in this paper. The first is concerned with justifying that the channel quality at the center subcarrier is indeed a valid representative of the channel qualities at (most of) the subcarriers in a cluster. And the second question is related to how big clusters should be (and subsequently how much feedback should we have) in order to maintain the same throughput scaling as when full CSI is available. We will answer these questions in the following section.

3. FEEDBACK SCHEME ANALYSIS

Due to correlations between adjacent subchannels, their corresponding adjacent SINR values are also correlated. This means that it should be satisfactory to have much less feedback than N per user in order to obtain sufficient channel

$$\Phi(P,Q) = \frac{e^{\frac{-1}{\rho}(jw_p\gamma_p + jw_q\gamma_q)}}{M^{LM} \quad \det\left[\frac{I_L}{M} - (jw_p V_p^* V_p + jw_q V_q^* V_q)\right] \quad \det\left[\frac{I_L}{M} + (jw_p\gamma_p V_p^* V_p + jw_q\gamma_q V_q^* V_q)\right]^{M-1}}$$

$$= \frac{e^{\frac{-1}{\rho}(jw_p\gamma_p + jw_q\gamma_q)}}{[1 - (jw_p + jw_q)LM - w_p w_q(L^2 - f)M^2][1 + (jw_p\gamma_p + jw_q\gamma_p)LM - w_p w_q\gamma_p\gamma_q(L^2 - f)M^2]^{M-1}}$$

Table 1. Joint Characteristic Function

knowledge at the transmitter. The fewer the number of clusters is, the lesser feedback the transmitter will need. However, the size of the cluster influences how correlated the subchannels are. Small cluster sizes guarantee that the subchannels are highly correlated and that their channel quality is almost the same, therefore resulting in higher throughput. Whereas wider cluster sizes will include less correlated subchannels especially toward the edges of the cluster, which will subsequently decrease the throughput. We consider that channel quality at the center subcarrier as a valid representative (and therefore sufficient feedback) for the quality of the channels at (most of) subcarriers in a cluster.

In fact we know that the best SINR over all users behaves like $\log K$. As the cluster size increases, the likelihood of the edge subcarrier (the farthest) being of order $\log K$ decreases. In this section, we show that if the size of the cluster is of order $\frac{N}{L\sqrt{\log K}}$, then with high probability the SINR at the edge subcarrier is also of order $\log K$. In other words, we show that the following probability,

$$\Pr\left\{ \text{SINR}_{p,m}^k \geq \gamma_p \mid \text{SINR}_{q,m}^k \geq \gamma_q \right\} \quad (6)$$

is close to one when $p - q = \frac{N}{L\sqrt{\log K}}$[1], M is fixed and K increasing, $\gamma_q = \alpha_q \log K$, $\gamma_p = \alpha_p \log K$ and $\alpha_q > \alpha_p$.

By making the following variable definitions,

$$P = H_m^{k^*} V_p^* V_p H_m^k - \gamma_p \sum_{i=1, i\neq m}^{M} H_i^{k^*} V_p^* V_p H_i^k - \frac{\gamma_p}{\rho}$$

$$Q = H_m^{k^*} V_q^* V_q H_m^k - \gamma_q \sum_{i=1, i\neq m}^{M} H_i^{k^*} V_q^* V_q H_i^k - \frac{\gamma_q}{\rho},$$

(6) can be rewritten as

$$\Pr\{P \geq 0 \mid Q \geq 0\} = \frac{\Pr\{P \geq 0, Q \geq 0\}}{\Pr\{Q \geq 0\}}. \quad (7)$$

In order to calculate (7), we start by evaluationg the joint probability. Using properties of the unit function, this joint

probability can be written as,

$$\Pr\{Q \geq 0, P \geq 0\} = \frac{1}{4} + \frac{1}{4\pi}\int \frac{\Phi(P)}{jw_p}dw_p \quad (8)$$
$$+ \frac{1}{4\pi}\int \frac{\Phi(Q)}{jw_q}dw_q$$
$$+ \frac{1}{4\pi^2}\int\int \frac{\Phi(P,Q)}{jw_p\, jw_q}dw_p dw_q,$$

where Φ stands for the characteristic function and the integration limits are from $-\infty$ to ∞. Note that the marginal terms (second and third terms in (8)) can be easily evaluated from the marginal distribution of SINR [1], and we can show that

$$\frac{1}{4\pi}\int \frac{\Phi(Q)}{jw_q}dw_q = -\frac{1}{4} - \frac{1}{2}\frac{e^{-\frac{\gamma_q}{\rho LM}}}{(1+\gamma_q)^{M-1}}$$
$$\frac{1}{4\pi}\int \frac{\Phi(P)}{jw_p}dw_p = -\frac{1}{4} - \frac{1}{2}\frac{e^{-\frac{\gamma_p}{\rho LM}}}{(1+\gamma_p)^{M-1}}. \quad (9)$$

However, these terms cancel out with terms that appear during the evaluation of the double integral in the fourth term of (8). Therefore, the main analysis of the joint probability lies in evaluating the joint characteristic funtion and the result of the double integral in (8).

3.1. Joint Characteristic Function

The joint characteristic function of P and Q is described as $\Phi(P,Q) = E\left(e^{jw_p P + jw_q Q}\right)$. We assume that the transmit antennas are far enough from each other to assume independent channels, and use properties of the joint pdf of Gaussian random variables [11] in order to obtain the result for the joint characteristic function in closed form as indicated in table 1, where $f = \frac{\sin^2(\pi(p-q)\frac{k}{N})}{\sin^2(\pi(p-q)\frac{1}{N})}$.

Note that when $p - q = \frac{N}{L}$, $f = 0$ which results in P and Q being independent. Also if we assume that the cluster size is 1 subcarrier, and therefore $p = q$, $(L^2 - f) = 0$, and the joint characteristic function reduces to a marginal function as in (9).

3.2. Complex Analysis

In order to evaluate (8), we need to obtain the double integral in the fourth term. Substituting jw_p with s_p and

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[1]In fact, in our analysis, we equivalently assume that the correlation coefficient of p and q is $1 - o(\frac{1}{\log K})$

$$\int \frac{-j}{2\pi} \frac{e^{-\frac{\gamma_q s_q}{\rho}} e^{-\frac{\gamma_p}{\rho} \frac{(LMs_q-1)}{s_q M^2(L^2-f)-LM}}}{s_q(LMs_q-1)\left[1+LM(\gamma_q s_q + \gamma_p(\frac{LMs_q-1}{s_q M^2(L^2-f)-LM}))+(\frac{LMs_q-1}{s_q M^2(L^2-f)-LM})s_q \gamma_p \gamma_q M^2(L^2-f)\right]^{M-1}} ds_q \quad (10)$$

$$\int \frac{-j}{2\pi} \frac{e^{-\frac{\gamma_q s_q}{\rho}} e^{-\frac{\gamma_p}{\rho} \frac{(LMs_q-1)}{s_q M^2(L^2-f)-LM}} \left[1-s_q \frac{M(L^2-f)}{L}\right]^{M-1}}{[1+\gamma_p]^{M-1} s_q \left[LM\frac{(\gamma_q-\gamma_p)}{1+\gamma_p}s_q+1+o(1)\right]^{M-1}(LMs_q-1)\left[\frac{-M(L^2-f)\gamma_q(1+\gamma_p)}{L(\gamma_q-\gamma_p)}s_q+1+o(1)\right]^{M-1}} ds_q \quad (11)$$

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jw_q with s_q, we perform contour integration where s_p and s_q go from $-\infty$ to ∞ on the complex axis, and the contour's direction is clock-wise. First, we integrate with respect to s_p using Cauchy integral formula [12]. The region where $s_p = 0$ results in a constant term that cancels out, therefore the only pole that we consider is a simple pole $s_p = \frac{LM\,s_q-1}{s_q M^2(L^2-f)-LM}$. The double integral then reduces to a single integral in terms of s_q as shown in (10). Solving this second integral is more involved than solving the first integral. In order to proceed, we need to obtain the roots of the polynomial (raised to the power $M-1$).

3.3. Asymptotic Analysis

For the rest of the analysis, we assume that the correlation coefficient of p and q is $1 - o(\frac{1}{\log K})$ as K increases. This implies that $(L^2 - f) \to 0$, or equivalently that $p - q = \frac{N}{L\sqrt{\log K}}$. We also assume that the threshold for the SINR at the center subcarrier γ_q behaves like $\alpha_q \log K$, while the edge subcarrier's threshold γ_p behaves like $\alpha_p \log K$ where α_q and α_p are constants and $\alpha_q > \alpha_p$. We solve for the roots of the polynomial in (10) and obtain (11). Now we have two poles inside contour s_q. The first pole $s_q = \frac{1}{LM}$ is simple, and the second $s_q = \frac{L(\gamma_q-\gamma_p)}{M(L^2-f)\gamma_q(1+\gamma_p)}$ is of order $M-1$. In order to use Cauchy integral formula [12], we expand the exponential as follows,

$$e^{-\frac{\gamma_p}{\rho} \frac{(LMs_q-1)}{s_q M^2(L^2-f)-LM}} = \sum_{a=0}^{\infty} \frac{(\frac{-\gamma_p}{\rho})^a}{a!}\left[\frac{LMs_q-1}{s_q M^2(L^2-f)-LM}\right]^a$$

and divide the summation into three regions. The first region is when $a = 0$, the second is when a is finite ($0 < a \le M-1$) and the last is when a goes to infinity. For the simplest case when $a = 0$, we first perform partial fraction expansion on the two poles inside contour s_q, and follow Cauchy integral formula at each pole to find the following result,

$$\frac{e^{-\frac{\gamma_q}{\rho LM}}}{(1+\gamma_q)^{M-1}}(1-o(1))^{M-1}+o\left(\gamma_q^{M-1}e^{-\frac{\gamma_q}{\rho}\frac{(\gamma_q-\gamma_p)}{(L^2-f)\gamma_q(1+\gamma_p)}}\right)$$

Following in the same fashion, we solve (11) for the finite and infinite a regions[2], and after simplifications we obtain

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[2] Both regions result in terms that go to 0 as $K \to \infty$. This is expected since the poles in the two regions inversely depend on (L^2-f), and there-

the result of (8) as,

$$\Pr\{P \ge 0, Q \ge 0\} = \frac{e^{-\frac{\gamma_q}{\rho ML}}}{(1+\gamma_q)^{M-1}}[1-o(1)]^{M-1} \quad (14)$$

and the conditional probability as,

$$\Pr\{P \ge 0 \mid Q \ge 0\} = 1 - o(1) \quad . \quad (15)$$

This proves that when SINR at the center subcarrier is of order $\log K$, and if we keep our cluster size as $\frac{N}{L\sqrt{\log K}}$, then with high probability SINR at the edge subcarrier is also of order $\log K$ for large K. This implies that all the subcarriers that are chosen for transmission have SINRs of order $\log K$. Therefore, we obtain throughput scaling as $MN \log \log K$ using our scheme with much less complexity and feedback. In fact, each user need only feedback its best SINR at the center of each cluster only if it is greater than some threshold γ. This threshold comes from the fact that we already know the order of the best SINR. This reduces the feedback to $L\sqrt{\log K} \times \Pr(\text{SINR}_{q,m} \ge \gamma)$ per user. Our simulation results show that wider cluster size is possible. This implies that $L^2 - f$ is a constant (not tending to zero), which we do not analyze in this paper.

4. SIMULATION RESULTS

To further verify our results, we conduct Monte Carlo simulations for a wideband broadcast channel with a three-antenna transmitter, single-antenna users, and 256 subcarriers. In this section, we discuss two simulation results. Figure (1) shows the throughput as a function of number of users for various cluster sizes. Due to subchannel correlation which depends on frequency separation, and due to the relationship between frequency coherence and channel delay spread, it is intuitive that cluster width of $\frac{N}{2L}$ achieves good performance. Indeed, figure (1) shows that cluster size $\frac{N}{2L}$ results in a small throughput loss, and we obtain throughput close to that when cluster size is 1 subcarrier (in which case the transmitter has information at each subchannel). However,

fore go to ∞ as $K \to \infty$. From the definition of our cluster, we predict that their residues are very close to 0.
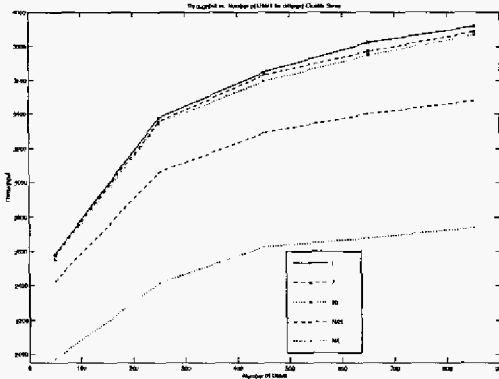
Fig. 1. Throughput vs. Number of users for different Cluster sizes (N=256, L=4, M=3, SNR=10).

Fig. 2. Throughput vs. Number of channel taps for different Cluster sizes (N=256, M=3, SNR=10, Number of users=1000).

as suggested by our analysis, figure (1) indicates that cluster sizes of 7 and 10 result in throughput scaling as with full CSI. This demonstrates that when we stay in the region of the highest SINR values (those that behave like $\log K$) and with $L^2 - f \rightarrow 0$, we preserve the throughput scaling as when we have full CSI. In figure (2), we plot the throughput versus the number of channel taps L for various cluster sizes. This figure indicates how much variation (spread) the channel is able to withstand (using a certain cluster size) without suffering in the throughput. Figure (2) also suggests that we can treat a small number of subcarriers as a single subcarrier, and still maintain flat channel response. This is an interesting result which requires further investigation.

5. CONCLUSIONS

In this paper we investigate the throughput of wideband multi-antenna broadcast channels. We prove that for cluster size $\frac{N}{L\sqrt{\log K}}$, each user need only feedback the best SINR value at the center subcarrier of each cluster so that the transmitter can perform opportunistic beamforming by constructing M beams on each subchannel and transmit to users with the best SINR values. This results in throughput scaling as when full CSI is available. Simulation results show that when cluster size is $\frac{N}{2L}$, the loss in throughput is very small.

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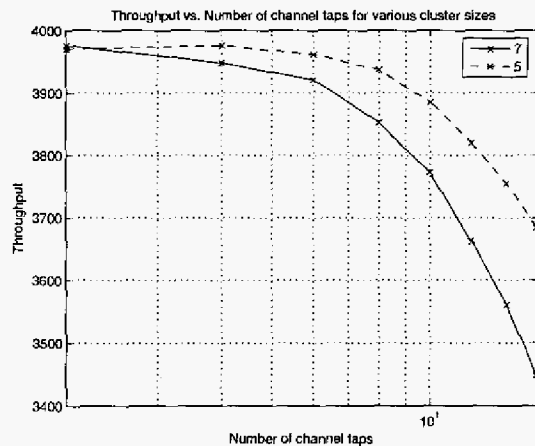
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