# THRUST BEARING WITH ROUGH SURFACES LUBRICATED BY AN ELLIS FLUID

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In the paper the influence of bearing surfaces roughness on the pressure distribution and load-carrying capacity of a thrust bearing is discussed. The equations of motion of an Ellis pseudo-plastic fluid are used to derive the Reynolds equation. After general considerations on the flow in a bearing clearance and using the Christensen theory of hydrodynamic rough lubrication the modified Reynolds equation is obtained. The analytical solutions of this equation for the cases of a squeeze film bearing and an externally pressurized bearing are presented. As a result one obtains the formulae expressing pressure distribution and load-carrying capacity. A thrust radial bearing is considered as a numerical example.

**Key words:** pseudo-plastic fluid, Ellis model, thrust bearing, Christensen roughness.

## 1. Introduction

Steady state radial flows and time-dependent squeezing flows of viscoplastic fluids are encountered in a variety of fields (Covey and Stanmore, 1981; Dai and Bird, 1981; Lipscomb and Denn, 1984). These flows are found in fabrication operations such as stamping, injection moulding, and sheet forming. Material properties of highly viscous fluids are measured with a device called the 'plastometer' which incorporates a parallel-disk squeeze flow geometry (Covey and Stanmore, 1981). In addition, such flows are encountered in lubrication systems, and there is a considerable interest as to the degree in which viscoplastic additives enhance the load-bearing capacity of a lubricant. Finally, the study of these flows can provide a rigorous test of rheological constitutive equations since the flows involve a combination of elongation and shearing effects.

The flows of Newtonian fluids in the clearance of a thrust bearing with impermeable surfaces have been examined theoretically. The bearing walls have been modeled as two disks, two conical or spherical surfaces. A bearing formed by two surfaces of revolution constitutes a more general case of the problem (Walicka, 1989).

In recent years, a considerable amount of tribology research has been devoted to the study of the effect of surface roughness or geometric imperfections on hydrodynamic lubrication because the bearings surfaces, in practice, are all rough and the height of the roughness asperities may have the same order as the mean bearing clearance. Under these conditions, the surface roughness affects the bearing performance considerably.

The work in this area has mainly been confined to impermeable surfaces. The well-established stochastic theory of hydrodynamic lubrication of rough surfaces developed by Christensen (1970) formed the basis of this paper. In a series of works (Bujurke *et al*., 2007; Lin, 2000; 2001; Prakash and Tiwari, 1985;

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Walicka 2009; 2012; Walicka and Walicki, 2002a; 2002b) the model was applied to the study of the surface roughness of various geometrical configurations.

The modified Reynolds equation (Gurujan and Prakash, 1999) applicable to two types of directional roughness structure were used by Walicka and Walicki (2002a; 2002b) to find bearing parameters for the squeeze film between two curvilinear surface.

In this paper the Ellis fluid model is used to describe the pseudo-plastic behaviour of a lubricant. The modified Reynolds equation is derived and its solution for the curvilinear thrust bearing is presented. According to the Christensen stochastic model (1970), different forms of Reynolds equations are derived to take account of various types of surface roughness. Analytical solutions for the film pressure are presented for the longitudinal and circumferential roughness patterns.

## 2. Derivation of the modified Reynolds

It may be assumed that lubricating oils and some greases, with a viscosity index improver added, exhibit the same characteristics as pseudo-plastic fluids. Rotem and Shinnar (1961) proposed a method for expressing empirically the relation between the stress and the shear rate as

$$\frac{d\gamma}{dt} = \frac{\tau}{\mu}\left(1 + \sum_{i=-1}^{n} k_i \tau^{2i}\right). \tag{2.1}$$

Similar relations were proposed by Whorlow (1992)

$$\mu \frac{d\gamma}{dt} = \sum_{i=0}^{n} k_i \tau^{2i+1} \tag{2.2}$$

or

$$\tau = \sum_{i=0}^{n} k_i \left(\frac{d\gamma}{dt}\right)^{2i+1} \tag{2.3}$$

known as power series models.

Retaining only the first and the last order terms the relations (2.1) or (2.2) reduce to

$$\mu \frac{d\gamma}{dt} = \tau + k_n \tau^n , \tag{2.4}$$

the model known as the Ellis model (1927).

In what follows we will use the original notation given by Ellis in the form

$$\frac{d\gamma}{dt} = \frac{\tau}{\mu_N} = \frac{\tau}{\mu}\left[1 + \left|\frac{\tau}{\tau_{1/2}}\right|^{n-1}\right] \tag{2.5}$$

where $\tau_{1/2}$ is a characteristic shear stress at which $\mu = 2\mu_N$ ($\mu_N$ - Newtonian viscosity) and $n$ is an identical parameter in the model.

The three-dimensional notation of Eq.(2.5) may be expressed as (Walicka, 2002a; 2002b)

$$\mu A_1 = \Lambda \left\{ 1 + \frac{1}{\tau_{1/2}^{n-1}} \Lambda^{n-1} \right\} \quad \text{where} \quad \Lambda = \left[ \frac{1}{2} \operatorname{tr}\left( \Lambda^2 \right) \right]^{\frac{1}{2}} \tag{2.6}$$

is the magnitude of the second-order shear stress tensor $\Lambda$, but $A_1$ is the first Rivlin-Ericksen kinematic tensor.
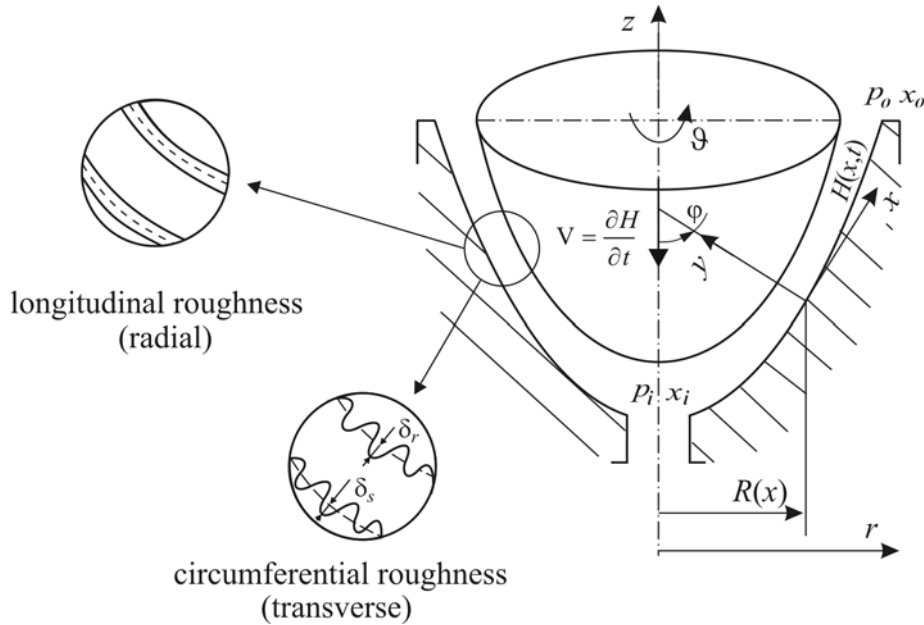


Fig.1. Geometry of a curvilinear thrust bearing.

Let us consider a thrust bearing with a curvilinear profile of the working surfaces shown in Fig.1. The lower surface is described by the function $R(x)$ which denotes the radius of this surface. The nominal bearing clearance thickness is given by the function $H(x,t)$.

The expression for the film thickness is considered to be made up of two parts.

$$H = h(x,t) + h_s(x, \vartheta, \xi) \tag{2.7}$$

where $h(x,t)$ represents the nominal smooth part of the film geometry, while $h_s = \delta_r + \delta_s$ denotes the random part resulting from the surface roughness asperities measured from the nominal level, $\xi$ describes a random variable which characterizes the definite roughness arrangement. An intrinsic curvilinear orthogonal coordinate system $x, \vartheta, y$ linked with the lower bearing surface is also presented in Fig.1.

Taking into account the considerations of the works (Walicka, 2002; Walicki, 2005) one may present the equation of continuity and the equations of motion of an Ellis fluid for axial symmetry in the form

$$\frac{1}{R} \frac{\partial (R \upsilon_x)}{\partial x} + \frac{\partial \upsilon_y}{\partial y} = 0, \tag{2.8}$$

$$\frac{\partial \Lambda_{xy}}{\partial y} = \frac{\partial p}{\partial x}, \qquad \frac{\partial p}{\partial y} = 0. \tag{2.9}$$

$$\mu \frac{\partial \upsilon_x}{\partial y} = \Lambda_{xy} \left[ 1 + \frac{1}{\tau_{1/2}^{n-1}} \left| \Lambda_{xy} \right|^{n-1} \right] \tag{2.10}$$

The problem statement is complete after specification of boundary conditions. These conditions for the velocity component are stated as follows:

$$\upsilon_x(x,0,t) = 0, \qquad \upsilon_x(x,H,t) = 0, \tag{2.11}$$

$$\upsilon_y(x,0,t) = 0, \qquad \upsilon_y(x,h,t) = \frac{\partial H}{\partial t} = \dot{H}, \tag{2.12}$$

Solving the equations of motion (2.8), (2.9) and taking into account the constitutive equation (2.10) one obtains the modified Reynolds equation:

$$\frac{1}{R} \frac{\partial}{\partial x} RH^3 \left( -\frac{\partial p}{\partial x} \right) \left[ 1 + \frac{3}{n+2} \left( \frac{H}{2\tau_{1/2}} \right)^{n-1} \left( -\frac{\partial p}{\partial x} \right)^{n-1} \right] = -12\mu \frac{\partial h}{\partial t} \tag{2.13}$$

for a lubricating pseudo-plastic fluid of Ellis type. At $n = 1$ – taking into consideration that $\mu = 2\mu_N$ – the above equation stands identical to the Reynolds equation for a Newtonian fluid (Walicki, 1977).

If the film thickness is regarded as a random quantity, a height distribution function must be associated. Many real bearing surfaces show a roughness height distribution which is closely Gaussian, at least up to three standard deviations. From a practical point of view, the Gaussian distribution is rather inconvenient and therefore a polynomial form of its approximation is chosen. Following Christensen (1970; 1971; 1973) such a probability density function is

$$f(h_s) = \begin{cases} \dfrac{35}{32c^7} \left( c^2 - h_s^2 \right)^3, & -c \le h_s \le +c \\ 0, & \text{elsewhere} \end{cases} \tag{2.14}$$

where $c$ is the half total range of the random film thickness variable. The function terminates at $c = \pm 3\sigma$, where $\sigma$ is the standard deviation.

Inserting expected values in Eq.(2.13) we get the general form of the stochastic Reynolds equation

$$\frac{1}{R} \frac{\partial}{\partial x} \left( E \left\{ RH^3 \left( -\frac{\partial p}{\partial x} \right) \left[ 1 + \frac{3}{n+2} \left( \frac{H}{2\tau_{1/2}} \right)^{n-1} \left( -\frac{\partial p}{\partial x} \right)^{n-1} \right] \right\} \right) = -12\mu \frac{\partial E(H)}{\partial t} \tag{2.15a}$$

or

$$\frac{1}{R} \frac{\partial}{\partial x} R \left( E(H^3) \left[ -\frac{\partial (Ep)}{\partial x} \right] \left\{ 1 + \frac{3E(H^{n-1})}{(n+2)(2\tau_{1/2})^{n-1}} \left[ -\frac{\partial (Ep)}{\partial x} \right]^{n-1} \right\} \right) = -12\mu \frac{\partial E(H)}{\partial t} \tag{2.15b}$$

where $E(\bullet)$ is the expectancy operator defined by

$$E(\bullet) = \int\limits_{-c}^{+c} (\bullet) f(h_s) dh_s \qquad (2.16)$$

The values of the exponent $n$ for the Ellis fluid is included in the interval (Whorlow, 1992)

$$0.25 \le n \le 4.0 .$$

The problem is now reduced to devising means of evaluating the left-hand side of Eq.(2.15) subject to the specific model of roughness. The calculation of the mean film pressure distribution would require the evaluation of the expected value of various film thickness functions; these functions are given in the Appendix.

## 3. Solutions to the modified Reynolds equation

Consider the case of an Ellis fluid for which there is

$$\frac{3}{n+2}\left(\frac{H}{2\tau_{1/2}}\right)^{n-1}\left(-\frac{\partial\overline{p}}{\partial x}\right)^{n-1} \ll 1 ; \qquad (3.1)$$

this relation allows us to search the solutions of the Reynolds equation (2.15) in the form of a sum

$$Ep = Ep^{(0)} + Ep^{(1)}. \qquad (3.2)$$

which arrives at two linearized equations

$$\frac{1}{R}\frac{\partial}{\partial x}R\left\{E\left(H^3\right)\left[-\frac{\partial\left(Ep^{(0)}\right)}{\partial x}\right]\right\} = -12\mu\frac{\partial E(H)}{\partial t}, \qquad (3.3)$$

and

$$\frac{1}{R}\frac{\partial}{\partial x}R\left\{E\left(H^3\right)\left[-\frac{\partial\left(Ep^{(1)}\right)}{\partial x}\right]\right\} = -\frac{3}{n+2}\frac{1}{R}\frac{\partial}{\partial x}R\left\{\frac{E\left(H^{n+2}\right)}{\left(2\tau_{1/2}\right)^{n-1}}\left[-\frac{\partial\left(Ep^{(0)}\right)}{\partial x}\right]^n\right\}. \qquad (3.4)$$

In this study two types of roughness will be of interest: the longitudinal (radial) one-dimensional roughness pattern, having the form of long narrow ridges and valleys running in the $x$ direction, and the circumferential (transverse) one-dimensional roughness pattern, having the form of long narrow ridges and valleys running in the $\vartheta$ direction (Walicka and Walicki, 2002a; 2002b; Walicka, 2009). For the longitudinal one-dimensional roughness

$$H = h(x,t) + h_s(\vartheta, \xi) \tag{3.5}$$

the stochastic Reynolds equation (3.3) and (3.4) remain without change, but for the circumferential one-dimensional roughness

$$H = h(x,t) + h_s(x, \xi) \tag{3.6}$$

the stochastic Reynolds equations (3.3) and (3.5) take the form

$$\frac{1}{R}\frac{\partial}{\partial x}R\left\{\frac{1}{E(H^{-3})}\left[-\frac{\partial\left(Ep^{(0)}\right)}{\partial x}\right]\right\} = -12\mu\frac{\partial E(H)}{\partial t}, \tag{3.7}$$

and

$$\frac{1}{R}\frac{\partial}{\partial x}R\left\{\frac{1}{E(H^{-3})}\left[-\frac{\partial\left(Ep^{(1)}\right)}{\partial x}\right]\right\} =$$

$$= -\frac{3}{(n+2)\left(2\tau_{1/2}\right)^{n-1}}\frac{1}{R}\frac{\partial}{\partial x}R\left\{\frac{1}{E(H^{-n-2})}\left[-\frac{\partial\left(Ep^{(0)}\right)}{\partial x}\right]^{n}\right\} \tag{3.8}$$

Introducing the following relations

$$H_j^{(3)} = \begin{cases} E(H^3) & \text{for} \quad j = l, \\ \dfrac{1}{E(H^{-3})} & \text{for} \quad j = c, \end{cases} \qquad H_j^{(n+2)} = \begin{cases} E\left(H^{(n+2)}\right) & \text{for} \quad j = l, \\ \dfrac{1}{E(H^{-n-2})} & \text{for} \quad j = c; \end{cases} \tag{3.9}$$

we may present Eqs (3.3), (3.4) and Eqs (3.7), (3.8) in one common form as follows

$$\frac{1}{R}\frac{\partial}{\partial x}\left[RH_j^{(3)}\left(-\frac{\partial\overline{p}^{(0)}}{\partial x}\right)\right] = -12\mu\frac{\partial h}{\partial t}, \tag{3.10}$$

$$\frac{1}{R}\frac{\partial}{\partial x}\left[RH_j^{(3)}\left(-\frac{\partial\overline{p}^{(1)}}{\partial x}\right)\right] = -\frac{3}{(n+2)\left(2\tau_{1/2}\right)^{n-1}}\frac{1}{R}\frac{\partial}{\partial x}\left[RH_j^{(n+2)}\left(-\frac{\partial\overline{p}^{(0)}}{\partial x}\right)^{n}\right], \tag{3.11}$$

where

$$\overline{p}^{(0)} = Ep^{(0)}, \qquad \overline{p}^{(1)} = Ep^{(1)}, \qquad E(H) = h \tag{3.12}$$

and the lower index $j = l$ refers to the one-dimensional longitudinal roughness, but $j = c$ refers to the one-dimensional circumferential roughness.

The boundary conditions for pressure are:

- for a squeeze film bearing $\left( \dfrac{\partial h}{\partial t} \neq 0 \right)$

$$\left. \frac{\partial \overline{p}^{(0)}}{\partial x} \right|_{x=0} = 0, \quad \overline{p}^{(0)}(x_o) = p_o, \quad \left. \frac{\partial \overline{p}^{(1)}}{\partial x} \right|_{x=0} = \overline{p}^{(1)}(x_o) = 0, \tag{3.13}$$

- for a externally pressurized bearing $\left( \dfrac{\partial h}{\partial t} = 0 \right)$

$$\overline{p}^{(0)}(x_i) = p_i, \quad \overline{p}^{(0)}(x_o) = p_o, \quad \overline{p}^{(1)}(x_i) = \overline{p}^{(1)}(x_o) = 0. \tag{3.14}$$

The solutions of Eqs (3.10) and (3.11) are given, respectively, as follows

$$\overline{p}(x,t) = p_o - \left[ F_o^{(n)} - F^{(n)}(x,t) \right] \tag{3.15}$$

and

$$\overline{p} = \alpha^{(n)} G^{(n)}(x) + \frac{\left[ A(x) - A_o \right]\left( p_i - \alpha^{(n)} G_i^{(n)} \right) - \left[ A(x) - A_i \right]\left( p_o - \alpha^{(n)} G_o^{(n)} \right)}{A_i - A_o} \tag{3.16}$$

where

$$I(x,t) = 12\mu \int \frac{\int R\dot{h}dx}{RH_j^{(3)}} dx,$$

$$J^{(n)}(x,t) = \frac{3(12\mu)^n}{(n+2)(2\tau_{1/2})^{n-1}} \int \frac{H_j^{(n+2)}}{R^n \left( H_j^{(3)} \right)^{n+1}} \left( -\int R\dot{h}dx \right)^n dx,$$

$$F^{(n)}(x,t) = I(x,t) + J^{(n)}(x,t), \qquad F_o = F(x_o,t); \tag{3.17}$$

$$A(x) = \int \frac{dx}{RH_j^{(3)}}, \qquad A_i = A(x_i), \qquad A_o = A(x_o),$$

$$C = \frac{p_i - p_o}{A_i - A_o}, \qquad \alpha^{(n)} = \frac{3}{n+2} \frac{(-C)^n}{(2\tau_{1/2})^{n-1}},$$

$$G^{(n)}(x) = \int \frac{H_j^{(n+2)}dx}{R^n \left( H_j^{(3)} \right)^{n+1}}, \qquad G_i^{(n)} = G^{(n)}(x_i), \qquad G_o^{(n)} = G^{(n)}(x_o).$$

The load carrying capacity is defined, respectively, by

$$N = 2\pi \int_0^{x_o} \left( \overline{p} - p_o \right) R \cos \varphi \, dx$$
(3.18a)

or

$$N = \pi R_i^2 p_i + 2\pi \int_{x_i}^{x_o} \overline{p} R \cos \varphi \, dx \,;$$
(3.18b)

where the sense of angle $\varphi$ arises from Fig.1.

## 4. Radial thrust squeeze film bearing

Let us consider the radial thrust squeeze film bearing presented in Fig.2.
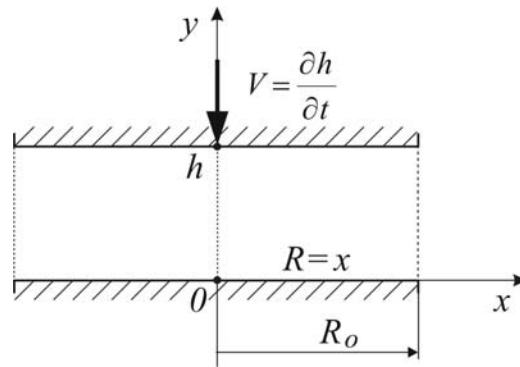


Fig.2. Radial squeeze film bearing.

Introducing the following parameters

$$\tilde{x} = \frac{x}{R_o}, \quad x = R, \quad \tilde{R} = \frac{R}{R_o}, \quad \tilde{h} = \frac{h}{h_o} = e(t), \quad e(t) = 1 - \varepsilon(t), \quad c^* = \frac{c}{h_o},$$
(4.1)

$$\tilde{p} = \frac{(\overline{p} - p_o)}{\mu \dot{\varepsilon}} \left( \frac{h_o}{x_o} \right)^2, \quad \dot{\varepsilon} = \frac{d\varepsilon}{dt}, \quad \tilde{N} = \frac{N h_o^2}{\mu \dot{\varepsilon} x_o^4}, \quad \lambda^{(n)} = \left( \frac{\mu \dot{\varepsilon} x_o}{\tau_{1/2} h_o} \right)^{n-1};$$

we have

$$\tilde{p} = \frac{3}{\tilde{H}_j^{(3)}} \left[ 1 - \tilde{x}^2 - \frac{2 \cdot 3^n \lambda^{(n)}}{(n+1)(n+2)} \frac{\tilde{H}_j^{(n+2)}}{\left( \tilde{H}_j^{(3)} \right)^n} \left( 1 - \tilde{x}^{n+1} \right) \right],$$
(4.2)

$$\tilde{N} = \frac{3\pi}{2\tilde{H}_j^{(3)}} \left[ 1 - \frac{4 \cdot 3^n \lambda^{(n)}}{(n+2)(n+3)} \frac{\tilde{H}_j^{(n+2)}}{\left( \tilde{H}_j^{(3)} \right)^n} \right],$$
(4.3)

To illustrate graphically these expressions let us take the values $n = 2$ and $n = 4$; introducing the notations

$$\tilde{H}_j^{(3)} = \begin{cases} e^3\left[1+\dfrac{1}{3}\left(\dfrac{c^*}{e}\right)^2\right] & \text{for} \quad j = l \\[4mm] \left(\dfrac{1}{e^3}\left[1+\dfrac{2}{3}\left(\dfrac{c^*}{e}\right)^2\right]\right)^{-1} & \text{for} \quad j = c \end{cases}$$

$$\tilde{H}_j^{(4)} = \begin{cases} e^4\left[1+\dfrac{2}{3}\left(\dfrac{c^*}{e}\right)^2\right] & \text{for} \quad j = l \\[4mm] \left(\dfrac{1}{e^4}\left[1+\dfrac{10}{9}\left(\dfrac{c^*}{e}\right)^2\right]\right)^{-1} & \text{for} \quad j = c \end{cases} \tag{4.4}$$

$$\tilde{H}_j^{(6)} = \begin{cases} e^6\left[1+\dfrac{5}{93}\left(\dfrac{c^*}{e}\right)^2\right] & \text{for} \quad j = l \\[4mm] \left(\dfrac{1}{e^6}\left[1+\dfrac{7}{3}\left(\dfrac{c^*}{e}\right)^2\right]\right)^{-1} & \text{for} \quad j = c \end{cases}$$

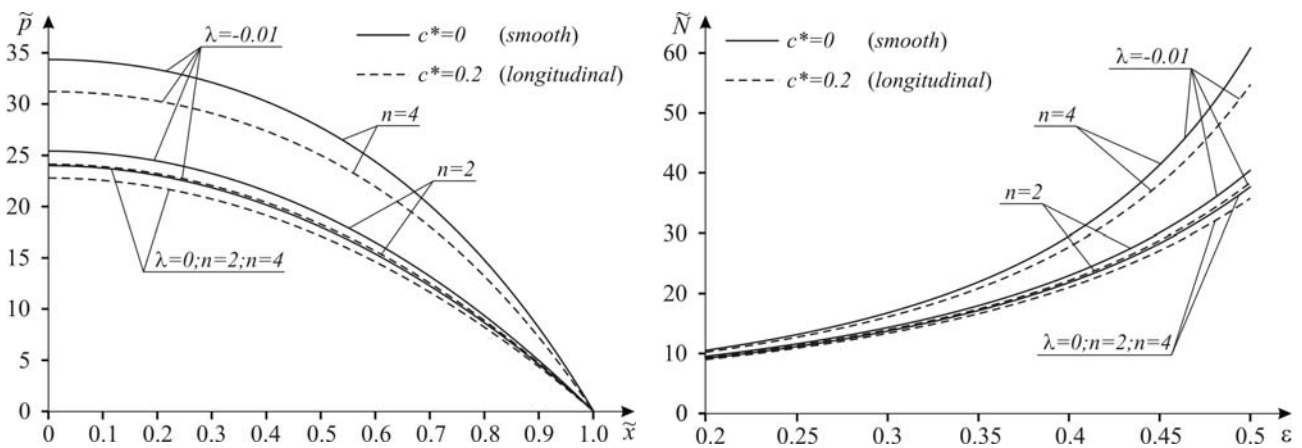we will present in Figs 3-6 the graphs of the pressure distributions and the load-carrying capacities.



Fig.3. Dimensionless mechanical parameters of the thrust bearing with rough surfaces for longitudinal roughness for $\lambda = -0.01$ and $\lambda = 0$ and $\varepsilon = 0.5$; (a) pressure distribution and (b) load-carrying capacity.
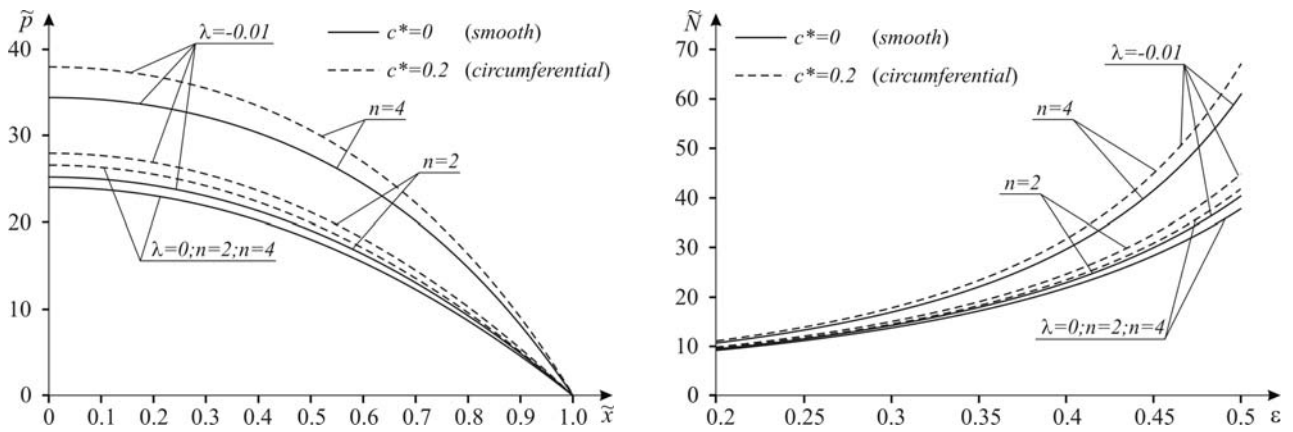
Fig.4.   Dimensionless mechanical parameters of the thrust bearing with rough surfaces for circumferential roughness for $\lambda = -0.01$ and $\lambda = 0$ and $\varepsilon = 0.5$; (a) pressure distribution and (b) load-carrying capacity.
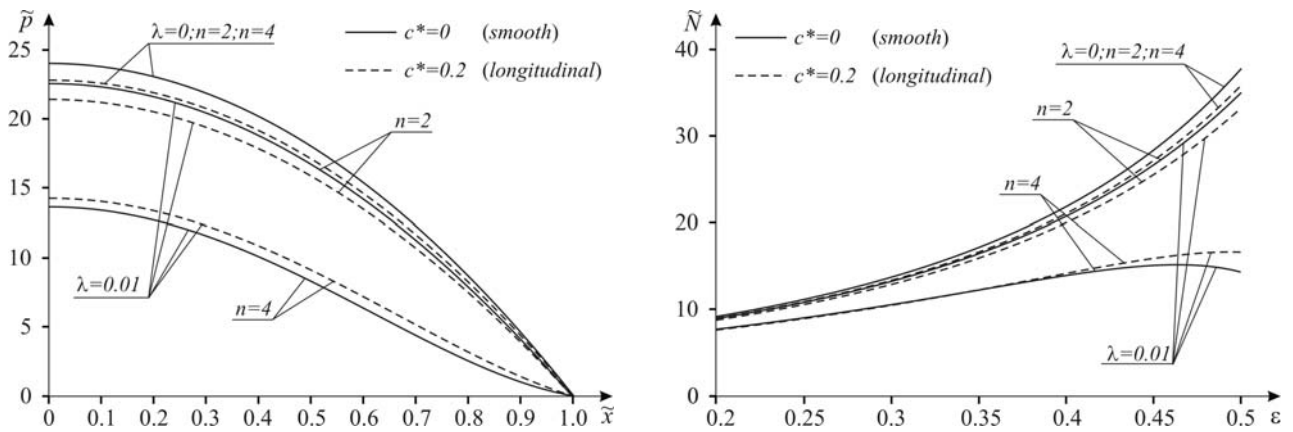


Fig.5.   Dimensionless mechanical parameters of the thrust bearing with rough surfaces for longitudinal roughness for $\lambda = 0$ and $\lambda = 0.01$ and $\varepsilon = 0.5$; (a) pressure distribution and (b) load-carrying capacity.
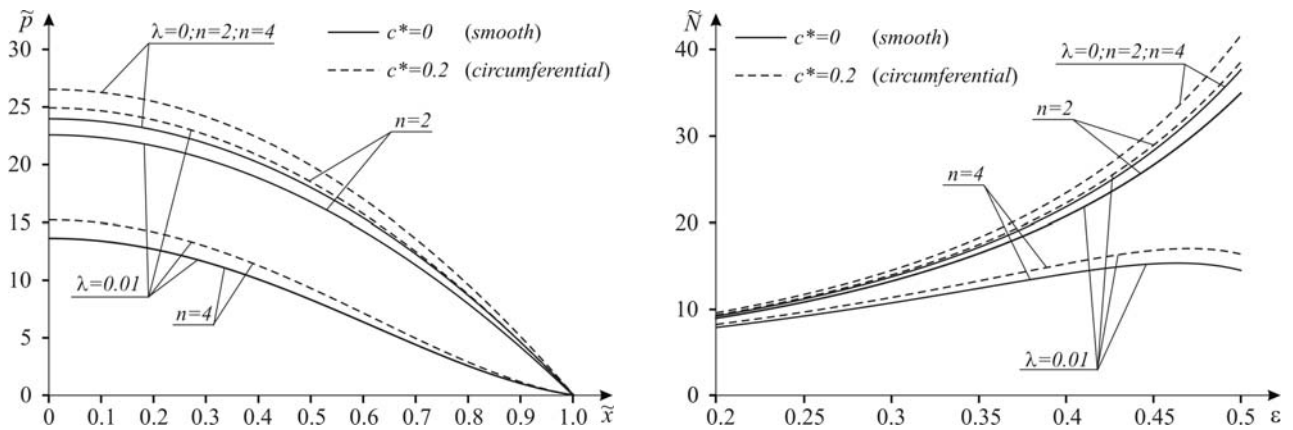


Fig.6.   Dimensionless mechanical parameters of the thrust bearing with rough surfaces for circumferential roughness for $\lambda = 0$ and $\lambda = 0.01$ and $\varepsilon = 0.5$; (a) pressure distribution and (b) load-carrying capacity.

## 5. Conclusions

The modified Reynolds equation for an Ellis type of pseudo-plastic lubricants flowing in a clearance of a thrust curvilinear bearing with rough surfaces is derived. As a result the general formulae for pressure distributions and the load-carrying capacity are obtained. It follows from the calculations and their graphic presentations that both the magnitudes depend on the rheological parameters $\tau_{1/2}$ and $n$ or $\lambda$ and $n$. For a squeeze film radial bearing the pressures and load-carrying capacities increase with a decrease of the $\lambda$ values with respect to the values of Newtonian lubricants $(\lambda = 0)$. These bearing parameters increase with an increase of $n$ for negative values of $\lambda$ and decrease for positive values of $\lambda$. The bearing surface roughness expressed by $c^*$, results in some small increase of the values of the bearing parameters for circumferential roughness and in a small decrease for longitudinal roughness.

## Nomenclature

| | | |
|---|---|---|
| $A_1$ | – | the first Rivlin-Ericksen kinematic tensor |
| $c$ | – | maximum asperity deviation |
| $c^*$ | – | nondimensional roughness parameter |
| $E(\bullet)$ | – | expectancy operator |
| $e(t)$ | – | bearing eccentricity or bearing squeezing |
| $f(h_s)$ | – | probability density distribution function |
| $H$ | – | film thickness |
| $h(x,t)$ | – | nominal film thickness |
| $h_s(x,\vartheta,\xi)$ | – | random deviation of film thickness |
| $N$ | – | load-carrying capacity |
| $p$ | – | pressure |
| $R, R(x)$ | – | local radius of the lower bearing surface |
| $r$ | – | radius |
| $\upsilon_x, \upsilon_y$ | – | velocity components |
| $x, y$ | – | orthogonal coordinate |
| $\varepsilon$ | – | squeezing ratio |
| $\vartheta$ | – | angular coordinate |
| $\mu$ | – | coefficient of viscosity |
| $\xi$ | – | random variable |
| $\rho$ | – | fluid density |

## Appendix

For the probability density function given by Eq.(2.14) we have – for integer exponents – the following expected values of film thickness functions

$$E(H) = h, \quad E(H^2) = h^2\left(1 + \frac{1}{9}Y^2\right), \quad E(H^3) = h^3\left(1 + \frac{1}{3}Y^2\right),$$

$$E(H^4) = h^4\left(1 + \frac{2}{3}Y^2 + \frac{1}{33}Y^4\right), \quad E(H^5) = h^5\left(1 + \frac{10}{9}Y^2 + \frac{5}{33}Y^4\right),$$

$$E\left(H^6\right) = h^6\left(1 + \frac{5}{93}Y^2 + \frac{5}{11}Y^4 + \frac{5}{429}Y^6\right),$$

$$E\left(H^{-1}\right) = \frac{1}{h}\left\{\frac{35}{32}\frac{1}{Y^7}\left[\left(Y^2 - 1\right)\ln\frac{1+Y}{1-Y} - \frac{2}{15}Y\left(15 - 40Y^2 + 33Y^4\right)\right]\right\},$$

$$E\left(H^{-2}\right) = \frac{1}{h^2}\left\{\frac{35}{32}\frac{1}{Y^7}\left[6\left(Y^2 - 1\right)\ln\frac{1+Y}{1-Y} - \frac{4}{5}Y\left(15 - 25Y^2 + 8Y^4\right)\right]\right\},$$

$$E\left(H^{-3}\right) = \frac{1}{h^3}\left\{\frac{35}{32}\frac{1}{Y^7}\left[3\left(5 - Y^2\right)\left(Y^2 - 1\right)\ln\frac{1+Y}{1-Y} + 2Y\left(15 - 13Y^2\right)\right]\right\},$$

$$E\left(H^{-4}\right) = \frac{1}{h^4}\left\{\frac{35}{32}\frac{1}{Y^7}\left[4\left(5 - 3Y^2\right)\ln\frac{1+Y}{1-Y} - \frac{8}{3}Y\left(15 - 4Y^2\right)\right]\right\},$$

$$E\left(H^{-5}\right) = \frac{1}{h^5}\left\{\frac{35}{32}\frac{1}{Y^7}\left[3\left(Y^2 - 5\right)\ln\frac{1+Y}{1-Y} + \frac{2Y}{1-Y^2}\left(15 - 13Y^2\right)\right]\right\},$$

$$E\left(H^{-6}\right) = \frac{1}{h^6}\left\{\frac{35}{32}\frac{1}{Y^7}\left[3\ln\frac{1+Y}{1-Y} - \frac{2Y}{5\left(1-Y^2\right)}\left(15 - 25Y^2 + 8Y^4\right)\right]\right\}$$

where

$$Y = \frac{c}{h}.$$

The expected values of film thickness functions $E\left(H^{-i}\right), i = 1, 2, ..., 6,$ are not suitable for numerical purposes for small $Y$ values, therefore Taylor expansions in powers of $Y$ are proposed

$$E\left(H^{-1}\right) = \frac{1}{h}\left\{1 + \sum_{n=1}^{\infty}\frac{105 Y^{2n}}{\left(2n+1\right)\left(2n+3\right)\left(2n+5\right)\left(2n+7\right)}\right\},$$

$$E\left(H^{-2}\right) = \frac{1}{h^2}\left\{1 + \sum_{n=1}^{\infty}\frac{105 Y^{2n}}{\left(2n+3\right)\left(2n+5\right)\left(2n+7\right)}\right\},$$

$$E\left(H^{-3}\right) = \frac{1}{h^3}\left\{1 + \sum_{n=1}^{\infty}\frac{105\left(n+1\right)Y^{2n}}{\left(2n+3\right)\left(2n+5\right)\left(2n+7\right)}\right\},$$

$$E\left(H^{-4}\right) = \frac{1}{h^4}\left\{1 + \sum_{n=1}^{\infty}\frac{35\left(n+1\right)Y^{2n}}{\left(2n+5\right)\left(2n+7\right)}\right\},$$

$$E\left(H^{-5}\right) = \frac{1}{h^5}\left\{1 + \sum_{n=1}^{\infty} \frac{35(n+1)(n+2)Y^{2n}}{2(2n+5)(2n+7)}\right\},$$

$$E\left(H^{-6}\right) = \frac{1}{h^6}\left\{1 + \sum_{n=1}^{\infty} \frac{7(n+1)(n+2)Y^{2n}}{2(2n+7)}\right\}.$$

The first three formulae $(i = 1,2,3)$ were given by Gururajan and Prakash (1999), the others − by Walicka (2012).

For fractional exponents we have

$$E\left(H^m\right) = h^m\left\{1 + \sum_{n=1}^{\infty} \frac{105 Y^{2n}}{2n!(2n+1)(2n+3)(2n+5)(2n+7)}\prod_{i=1}^{2n}(m-i+1)\right\},$$

$$E\left(H^{-m}\right) = \frac{1}{h^m}\left\{1 + \sum_{n=1}^{\infty} \frac{105 Y^{2n}}{2n!(2n+1)(2n+3)(2n+5)(2n+7)}\prod_{i=1}^{2n}(m+i-1)\right\}.$$

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