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Tidal Dynamics of the Water Table in Beaches

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Tidal motions of the water table height inside a sloping beach are investigated via field measurements and theoretical considerations. Only the movements forced by the tide are considered, so a beach with negligible wave activity was chosen for the field measurements. The data show that even in the absence of precipitation the time averaged inland water table stands considerably above the mean sea level. Also the water table at a fixed point inside the beach is far from sinusoidal even though its variation is forced by an essentially sinusoidal tide. This latter effect is due to the boundary condition along the sloping beach face which acts as a highly nonlinear filter. The observed behavior of the water table is explained in terms of perturbation extensions to the classical "deep aquifer solution." One extension deals with the nonlinearity in the interior, the other with the boundary condition at the sloping beach face.

1. Introduction

The height of the water table near the coast influences the stability of structures founded on soils or sands and it may be a limiting factor for agricultural land use because of saltwater intrusion. It has therefore been the object of a recent investigation carried out by the New South Wales Public Works Department. Field measurements have shown that even in the absence of significant rainwater input the time averaged (averaged over a tidal period) water table tends to be elevated significantly above the mean sea level. The overheight is largest on the open coast where wave runup may cause several meters of super elevation, but also protected beaches with negligible wave activity may exhibit overheights in excess of half a meter depending on tidal range, beach slope, and the drainage characteristics of the sand. The emphasis of the present paper is on the tidal mechanisms which have been measured on a protected beach north of Sydney, Australia.

2. Field Measurements

The water-table was measured every half hour for 25 hours on Barrenjoey Beach north of Sydney, Australia. At the time of the measurements (April 18–19, 1989) the tide was almost sinusoidal, semidiurnal with amplitude 0.516 m. The measurements were taken in 11 stilling wells placed at 2.5-m intervals along the normal to the beach. Figure 1 shows time series of the water table height in two of the wells together with the tide. Well number 7 (sand level 0.64 m) was the first well landward of the high water mark (sand level 0.516 m), and it shows three interesting characteristics.

First, the minimum water level is substantially higher than the low tide level (+0.09 m compared to −0.516 m), and similarly, the mean water level is 0.25 m higher than the mean sea level.

Second, the variation is far from sinusoidal. The rise is much steeper than the decline.

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Third, the maximum water level in Well 7 is a few centimeters higher than the high tide level.

The first two features result from the nature of tidal flow in and out of a sloping beach as explained in the following while the slight overheight of Well 7 above the offshore tide level at high tide must be due to wave activity. The wind waves arriving at the beach had a height of only 5–10 cm, but there might have been longer period oscillations (period 1–30 min) present as well with similar amplitude.

Well 11 was situated 10 m landward of Well 7, and by comparing the records from the two wells we see that the tidal water table wave is dampened considerably over the 10 m and Well 11 is lagging approximately 1 hour behind Well 7.

The amplitude and phase lag information from all 11 wells determined by harmonic analysis are shown in Table 1, and the complete data set is printed in Table 2. Please note that the expected absence of rainwater outflow at the test site is supported by the fact that the measured mean water levels approach a horizontal asymptote on the inland side.

3. Mathematical Modeling

Mathematically, the description of tidal water table motions in a beach with simple geometry contains two main challenges. First, the governing equation for horizontal flow below the water table in a beach which has uniform permeability above a horizontal impermeable boundary is nonlinear, and this nonlinearity results in a gradual rise of the time-averaged water table away from the beach. This effect has been studied theoretically by *Philip* [1973] and *Knight* [1981], and experimentally by *Smiles and Stokes* [1976]. Their work established an asymptotic result for the height of the time-averaged water table far from the beach. *Knight* [1981] showed further that although Philip's result was derived from *Boussinesq*'s [1903] equation, i.e., under the assumption of purely horizontal flow, it holds exactly also when the flow is two dimensional [$u = u(x, z, t)$]. Similar results hold for other flow characteristics. For example, the Dupuit-Forchheimer formula for seepage through a dam is exact although originally derived from Boussinesq's equation (see *Charnyi* [1951] for the original proof or *Knight* [1981]). These findings encourage the use of Boussinesq's equation for the initial study of new groundwater problems. Section 3.3.2 contains an approximate solution to this equa-

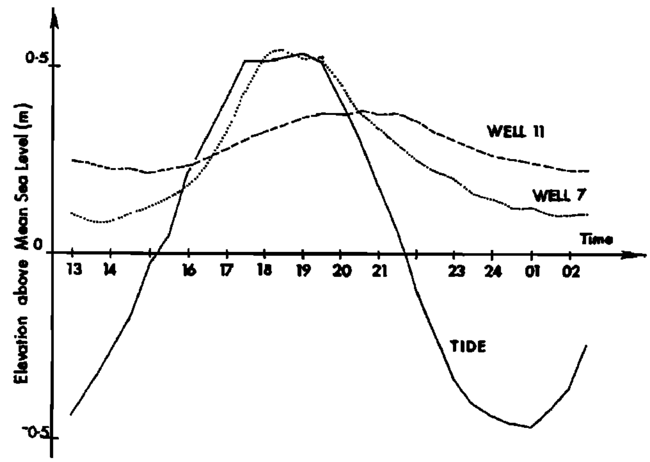

Fig. 1. Time series of water table heights in Wells number 7 and 11 together with the tide at Barrenjoey Beach, April 18–19, 1989. The distances from the intersection between the mean sea level and the beach to these wells are 6.6 m and 16.6 m, respectively.

tion for the tidal problem. It describes both time dependent and averaged effects of the nonlinearity, and the time-averaged results agree to the expected order with Philip's asymptotic result.

The second point of interest is the matching of the tidal variation along the inclined beach face. This problem has not previously been tackled analytically as far as the writer is aware. Previous authors have either contended with a solution which assumes a practically vertical beach face [e.g, *Dominick et al.*, 1971; *Philip*, 1973; *Knight*, 1981; *Smiles and Stokes*, 1976] or left the beach face as a black box while acknowledging its strong, nonlinear filtering effects on the tidal waves [e.g., *Lanyon et al.*, 1982]. These effects include a lifting of the time-averaged water table above the mean sea level and a skewing of the time dependance in such a way that the lag of low water table at a point behind low tide in the ocean is longer than the lag of high water table behind high tide. In other words, the rise of the water table occurs at a faster rate than its fall; see Figure 1. These effects of the sloping interface are treated in section 3.3.3.

3.1. *Governing Equations*

We consider the shore normal groundwater flow in a long straight beach as shown in Figure 2.

The origin of the x axis is at the intersection of the mean sea surface and the beach face, and x is positive landward. The beach face forms the angle β with the horizontal and it is assumed that the sand body is bounded by an impermeable layer at depth D below mean sea level.

It is assumed that the sand is homogeneous and isotropic with permeability K and porosity n. Further, we assume that the flow velocity $[u(x, t)]$ is essentially horizontal so that the pressure distribution is hydrostatic (Dupuit's assumption).

Under these assumptions, the governing equations for the groundwater flow and the local water table height h are Darcy's law,

$$u = -K\frac{\partial h}{\partial x} \tag{1}$$

and the continuity equation

$$\frac{\partial h}{\partial t} = -\frac{1}{n}\frac{\partial}{\partial x}(hu) \tag{2}$$

which combine into Boussinesq's equation

$$\frac{\partial h}{\partial t} = \frac{K}{n}\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right) \tag{3}$$

3.2. *Boundary Conditions*

In order to define uniquely the solution for (3) we need two boundary conditions. The first of these is obtained by requiring that all oscillations die out far from the beach, i.e.,

$$\frac{\partial h}{\partial t} \to 0 \qquad x \to \infty \tag{4}$$

Second, the water table is assumed to match the tide at the beach face unless the tide is dropping so quickly that the seepage point becomes separated from the shoreline by the development of a seepage face as in Figure 3. If this separation or decoupling occurs, analytical solution is probably impractical, but for the cases where it does not occur, analytical solutions are both simple and instructive.

With a given tidal variation $h_{\text{tide}}(t)$ the boundary condition at the beach face (slope angle β) is

$$h([h_{\text{tide}} - D] \cot \beta, t) = h_{\text{tide}} \tag{5}$$

provided the water table does not become decoupled by the formation of a seepage face. This general boundary condition is unusual and somewhat difficult to deal with because

TABLE 1. Harmonic Constants for Water Table Observations, Barrenjoey Beach April 18–19, 1989

	Well Number											
	1	2	3	4	5	6	7	8	9	10	11	Tide
Horizontal position (x), m	−8.4	−5.9	−3.4	−0.9	1.6	4.1	6.6	9.1	11.6	14.1	16.6	—
Sand level, m	−0.72	−0.54	−0.30	−0.11	0.13	0.38	0.64	0.87	1.12	1.45	1.78	—
Amplitude, m (period 12.25 hours)	0.516	0.510	0.429	0.337	0.272	0.228	0.192	0.135	0.107	0.087	0.075	0.516
Time lag [lag_1 (x)], hours	0	0.01	0.03	0.05	0.15	0.72	0.78	1.09	1.56	1.71	1.84	0
Amplitude, m (period 6.125 hours)	0.014	0.018	0.063	0.089	0.079	0.066	0.053	0.025	0.018	0.015	0.010	0.014
Time lag [$\text{lag}_2(2(x)$], hours	1.27	0.88	0.07	0.01	−0.10	0.21	0.26	0.73	1.07	1.37	1.51	1.27
Average water level, m	0	0.003	0.049	0.112	0.170	0.209	0.251	0.263	0.277	0.289	0.287	

The time lags [lag (x)] are defined in accordance with $h(x, t) = D + \text{Amp}_1(x) \cos \omega(t - \text{lag}_1(x)) + \text{Amp}_2(x) \cos 2\omega(t - \text{lag}_2(x))$ where ω is the radian frequency $2\pi/12.25$ radians per hour.

TABLE 2. Water Table Data From Barrenjoey Beach, April 18–19, 1989

		Well Number										
		11	10	9	8	7	6	5	4	3	2	1
Well top level, m 2.10		1.55	1.54	1.10	1.38	1.45	0.86	0.57	0.46	0.69	0.53	
Distance from Well 11		0.0	2.5	5.0	7.5	10.0	12.5	15.0	17.5	20.0	22.5	25.0
Sand level, m		1.79	1.46	1.12	0.87	0.64	0.38	0.14	−0.11	−0.30	−0.54	−0.71
		Water Table Heights Measured in Wells										
Time, EST	Tide Level	11	10	9	8	7	6	5	4	3	2	1
1200												
1230	−0.50		0.12	−0.05	−0.03	−0.13	−0.29	−0.51	−0.51			
1300	−0.47		0.12	0.00	−0.04	−0.13	−0.28	−0.47	−0.47			
1330	−0.43	0.24	0.23	0.12		0.11	0.01	−0.04	−0.13	−0.29	−0.43	−0.43
1400	−0.36	0.24	0.22	0.19	0.15	0.09	0.02	−0.05	−0.13	−0.28	−0.38	−0.37
1430	−0.27	0.23	0.21	0.19	0.15	0.09	0.02	−0.04	−0.13	−0.27	−0.31	−0.28
1500	−0.18	0.23	0.21	0.18	0.16	0.10	0.03	−0.02	−0.10	−0.16	−0.17	−0.19
1530	−0.03	0.22	0.20	0.18	0.17	0.12	0.05	0.02	−0.04	−0.03	−0.04	
1600	0.04	0.23	0.21	0.19	0.19	0.15	0.08	0.09	0.06	0.07	0.03	
1630	0.22	0.23	0.23	0.22	.22	0.19	0.13	0.21	0.21	0.21	0.21	
1700	0.31	0.26	0.25	0.24	0.25	0.24	0.22	0.31	0.31			
1730	0.42	0.29	0.28	0.28	0.30	0.34	0.34	0.41				
1800	0.52	0.31		0.31	0.35	0.44	0.43	0.51				
1830	0.50	0.33	0.33	0.34	0.40	0.53	0.50					
1900	0.53	0.35	0.35	0.37	0.43	0.54	0.53					
1930	0.54	0.37	0.37	0.39	0.45	0.53	0.54					
2000	0.52	0.38	0.39	0.41	0.45	0.53	0.53	0.51				
2030	0.42	0.38	0.40	0.42	0.44	0.46	0.47	0.41				
2100	0.31	0.39	0.40	0.41	0.41	0.38	0.39	0.30				
2130	0.18	0.38	0.39	0.39	0.38	0.34	0.31	0.18	0.17			
2200	0.07	0.38	0.37	0.37	0.34	0.30	0.25	0.12	0.06	0.07		
2230	−0.09	0.36	0.35	0.33	0.30	0.26	0.20	0.06	−0.06	−0.09		
2300	−0.21	0.33	0.31	0.29	0.27	0.22	0.16	0.04	−0.10	−0.20	−0.22	
2330	−0.33	0.31	0.29	0.27	0.24	0.20	0.13	0.02	−0.10	−0.27	−0.34	
0000	−0.40	0.29	0.27	0.25	0.22	0.17	0.11	0.00	−0.10	−0.28	−0.40	−0.41
0030	−0.43	0.27	0.26	0.24	0.20	0.15	0.09	−0.01	−0.12	−0.29	−0.46	−0.44
0100	−0.45	0.26	0.25	0.22	0.19	0.13	0.07	−0.02	−0.13	−0.29	−0.46	−0.46
0130	−0.46	0.25	0.23	0.21	0.17	0.12	0.05	−0.04	−0.13	−0.29	−0.39	−0.47
0200	−0.42	0.24	0.22	0.20	0.16	0.10	0.04	−0.03	−0.13	−0.28	−0.43	−0.43
0230	−0.36	0.23	0.22	0.19	0.16	0.10	0.03	−0.04	−0.13	−0.29	−0.39	−0.37
0300	−0.24	0.23	0.21	0.19	0.16	0.10	0.03	−0.03	−0.12	−0.25	−0.25	
0330	−0.15	0.23	0.21	0.18	0.16	0.11	0.05	0.00	−0.08	−0.15		
0400	−0.10	0.22	0.21	0.19	0.17	0.13	0.07	0.06	−0.00	−0.10		
0430	0.09	0.23	0.21	0.19	0.19	0.16	0.11	0.13	0.08			
0500	0.24	0.25	0.24	0.23	0.23	0.18	0.18	0.22	0.23			
0530	0.32	0.27	0.26	0.25	0.27	0.27	0.28	0.31				
0600	0.38	0.28	0.28	0.30	0.30	0.33	0.34	0.37				
0630	0.46	0.31	0.31	0.31	0.35	0.42	0.42	0.45				
0700	0.50	0.33	0.33	0.33	0.38	0.46	0.46	0.49				
0730	0.48	0.34	0.34	0.36	0.40	0.48	0.47	0.47				
0800	0.44	0.35	0.36	0.37	0.40	0.45	0.45	0.43				
0830	0.33	0.35	0.37	0.38	0.39	0.39	0.40	0.32				
0900	0.22	0.36	0.37	0.37	0.37	0.35	0.33	0.21	0.21			
0930	0.14	0.36	0.36	0.35	0.34	0.31	0.28	0.16	0.13			
1000	−0.04	0.34	0.33	0.32	0.30	0.26	0.20	0.08	−0.04	−0.04		
1030	−0.18	0.32	0.31	0.29	0.27	0.23	0.17	0.06	−0.08	−0.15	−0.19	
1100	−0.29	0.30	0.28	0.27	0.24	0.20	0.13	0.02	−0.10	−0.26	−0.31	−0.29
1130	−0.40	0.28	0.26	0.25	0.21	0.17	0.11	0.01	−0.10	−0.28	−0.41	−0.41
1200	−0.49	0.26	0.24	0.23	0.18	0.14	0.07	−0.02	−0.13	−0.27	−0.50	−0.50
1230	−0.55	0.25	0.24	0.21	0.17	0.11	0.09	−0.04	−0.13	−0.27	−0.52	−0.56
1300	−0.55	0.23	0.22	0.20	0.15	0.09	0.03	−0.06	−0.15	−0.29	−0.52	−0.56
1330	−0.58	0.22	0.20	0.18	0.14	0.08	0.02	−0.07	−0.15	−0.29	−0.52	−0.59
1400	−0.51	0.21	0.19	0.17	0.13	0.06	−0.00	0.07	−0.17	−0.29	−0.52	−0.52

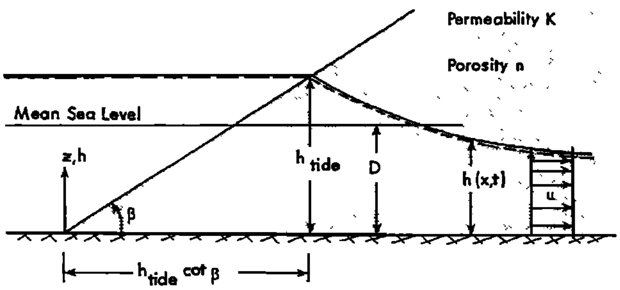

Fig. 2. Definition sketch.

the point at which h is prescribed does not have a fixed abscissa unless the beach is vertical ($\cot \beta = 0$).

3.3. *Analytical Solutions*

Analytical solutions to (3) with the boundary conditions (4) and (5) can only be obtained under further simplifying assumptions, and the aim of the following sections is to derive some such solutions and discuss their merits and limitations.

3.3.1. Vertical beach and small tidal amplitude. Consider first the very simplest situation where the beach is assumed to be practically vertical. That is, $A \cot \beta / L \ll 1$, where A is the tidal amplitude and L is the length of the water table wave inside the sand. In that case the boundary condition (5) becomes

$$h(0, t) = h_{\text{tide}} \tag{6}$$

We assume further that the tidal amplitude is small compared to D so that Boussinesq's equation becomes the diffusion equation

$$\frac{\partial h}{\partial t} = \frac{KD}{n} \frac{\partial^2 h}{\partial x^2} \tag{7}$$

For this highly simplified system a textbook solution is available (see, for example, *Kovacs* [1981]). For $h_{\text{tide}} = D + A \cos \omega t$ the solution is

$$h(x, t) = D + A \cos (\omega t - kx) e^{-kx} + Bx \tag{8}$$

i.e., essentially a landward traveling wave with exponentially narrowing envelope. The linear term Bx corresponds to any time-averaged flux of water out of the beach caused by rainfall, etc. The wave number k is given by

$$k = \left(\frac{n\omega}{2KD} \right)^{0.5} \tag{9}$$

in terms of the beach parameters and the radian frequency ω of the tide. For a more complicated tidal variation with the Fourier series,

$$h_{\text{tide}} = D + \Sigma A_j \cos (j\omega t - \varphi_j) \tag{10}$$

the corresponding solution is

$$h(x, t) = D + \Sigma A_j \cos (j\omega t - \varphi j - \sqrt{j}kx) e^{-\sqrt{j}kx} + Bx \tag{11}$$

An evaluation of the solution (8) can be obtained from the observed amplitudes and phase lags of the tidal harmonics listed in Table 1. The solution predicts that damping and phase lag in radians should both grow as kx. However, least squares fits to the data yield different growth rates, i.e., different values for k from lags and damping of the semidiurnal component. The damping yields $k_{1d} = 0.093\ \text{m}^{-1}$ with the goodness of fit parameter $r = 0.987$, while the phase lags yield $k_{1p} = 0.056\ \text{m}^{-1}$ with $r = 0.971$, thus the semidiurnal component of the measurements does not behave quite as prescribed by (8). Also, the damping and lags of the second harmonic ($T = 6.125$ hours) are significantly larger than the predicted $\sqrt{2}$ times those of the fundamental mode. In fact, they are both very close to twice the values for the fundamental mode. For the damping of the second harmonic we find $k_{2d} = 0.154\ \text{m}^{-1}$ with $r = 0.970$ and for the corresponding phase lags in radians $k_{2p} = 0.129\ \text{m}^{-1}$ with $r = 0.982$. The fact that the damping and lag of the second harmonic are closer to 2 than to $\sqrt{2}$ times those of the fundamental mode indicate that the second harmonic is more like a forced wave than a free wave with the form of (8). That is, the deviations of the observations from the predictions from (8) are most likely due to the nonlinear nature of the problem. The regression analysis was based only on the five wells landward of the high water mark.

3.3.2. Vertical beach and finite tidal amplitude. It is the purpose of the present section to try and explain the observations above by investigating the magnitude and character of the differences between the solutions to Boussinesq's equation (3) and to the diffusion equation (7) for typical beach conditions.

When the tidal amplitude is more than a negligible fraction of the mean depth D, the nonlinear term of Boussinesq's equation start to play a role and the simple solution (8) is of course no longer exact. However, it is still useful to use it as a starting point and then approximate the exact solution with a perturbation expansion in the parameter A/D.

Apart from perturbation solutions an exact, asymptotic solution for the steady part $\overline{h(x)}$ of $h(x, t)$ can be found by using a result from *Philip* [1973]. Philip pointed out that if the process is periodic so that

$$\int_t^{t+T} \frac{\partial h}{\partial t}\, dt = 0 \tag{12}$$

then it follows from Boussinesq's equation that the mean square value $\overline{h^2(x, t)}$ must be a linear function of x because

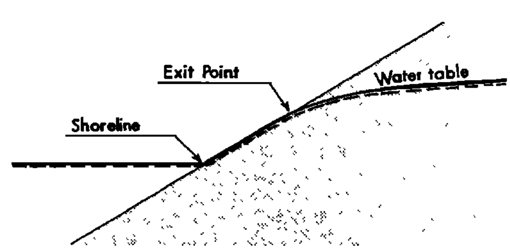

Fig. 3. For flat beaches and/or large tidal range the water table can become decoupled at low tide, i.e., the water table emerges at the exit point some distance above the shoreline. When this occurs there is no longer direct coupling between the water table and the tide.

$$\frac{\partial^2}{\partial x^2} \overline{h^2(x, t)} = 0 \tag{13}$$

Hence, for example, with $h_{\text{tide}} = A \cos \omega t + D$ we must have

$$\overline{h^2(x, t)} = \overline{(h_{\text{tide}})^2} + Bx = \overline{(D + A \cos \omega t)^2} + Bx \tag{14}$$

where again, the linear term Bx corresponds to any time-averaged flux of water out of the beach. Far from the beach where all oscillations have died out so that $h(x, t) = h(x)$ we thus get

$$h^2(x) = \overline{h(x, t)^2} = D^2 + A^2/2 + Bx \tag{15}$$

and hence

$$h(x) \approx D + A^2/4D + Bx/2D \qquad kx \gg 1 \tag{16}$$

Thus Philip's argument shows that the nonlinearity of Boussinesq's equation leads to an overheight of the water table inside the beach even in the absence of a net flux of water and that in the absence of such net flux the asymptotic inland overheight is approximately $A^2/4D$.

This is a useful result, but it only provides information about the time average of $h(x, t)$, and only about its asymptotic value. In order to investigate the general behavior of $h(x, t)$, a perturbation approach is adopted. We start with writing $h(x, t)$ as an expansion in the relative tidal amplitude A/D,

$$h(x, t) = D + \frac{A}{D} h_1(x, t) + \left(\frac{A}{D}\right)^2 h_2(x, t) + \cdots \tag{17}$$

then by inserting this into Boussinesq's equation and separating terms of equal order in A/D we get the following system of equations,

Order A/D

$$\frac{\partial h_1}{\partial t} = \frac{KD}{n} \frac{\partial^2 h_1}{\partial x^2} \tag{18}$$

Order $(A/D)^2$

$$\frac{\partial h_2}{\partial t} = \frac{KD}{n} \frac{\partial^2 h_2}{\partial x^2} + \frac{K}{n} h_1 \frac{\partial^2 h_1}{\partial x^2} + \frac{K}{n} \left(\frac{\partial h_1}{\partial x}\right)^2 \tag{19}$$

For simplicity, let us just consider the case with no net flux out of the beach ($B = 0$) in which the boundary conditions are

$$\frac{\partial h}{\partial x} = \frac{\partial h}{\partial t} = 0 \qquad x = \infty \tag{20}$$

$$h(0, t) = h_{\text{tide}} = D + A \cos \omega t \tag{21}$$

(the beach is still assumed vertical). In this case the solution to (18) and (19) is

$$h_1(x, t) = D \cos(\omega t - kx)e^{-kx} \tag{22}$$

$$h_2(x, t) = \frac{D}{4} [1 - 2 \cos^2 (\omega t - kx)e^{-2kx} + \cos(2\omega t - \sqrt{2}kx)e^{-\sqrt{2}kx}] \tag{23}$$

or in other terms

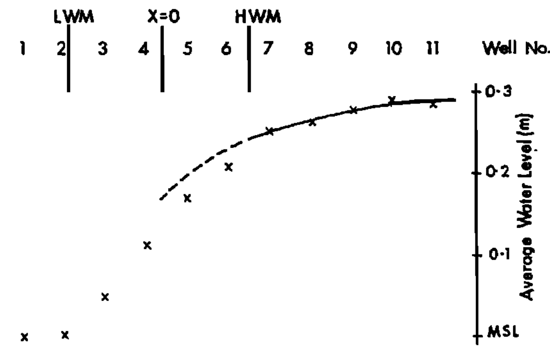

Fig. 4. Fit of the expression (25) to the shape of the time-averaged water table landward of the high water mark.

$$h(x, t) = D + A \cos(\omega t - kx)e^{-kx} + \frac{A^2}{4D} [1 - 2 \cos^2 (\omega t - kx)e^{-2kx} + \cos(2\omega t - \sqrt{2}kx)e^{-\sqrt{2}kx}] \tag{24}$$

We note that the asymptotic value of $h(x)$ from this expression agrees (to the order $[A/D]^2$) with the result (16) which was based on *Philip*'s [1973] argument.

Let us now briefly compare the results of this section with the measurements listed in Tables 1 and 2. Based on the form of the fundamental mode (22), we can estimate the value of the wave number k from the damping and phase lags of those wells which are above the high water mark. This was done at the end of section 3.3.1. We adopt the average $k = 0.075$ m^{-1} and compare the time average of (24):

$$h(x) = D + \frac{A^2}{4D} [1 - e^{-2kx}] \tag{25}$$

with the observed rises in average water levels between the wells landward of the high water mark (HWM; see Figure 4). We find the best fit value of $A^2/4D$ to be 0.13 m corresponding to $D \approx 0.51$ m with $A = 0.516$ m.

This value (0.13 m) is less than half of the observed overheight of the inland asymptote above mean sea level (0.288 m), and only about 0.05 m is actually built up landward of the HWM where (25) is strictly valid. Most of the observed overheight is accrued between the low water mark and the high water mark, where additional and equally strong mechanisms to the nonlinearity treated in this section must be active. Those mechanisms are the subject of the following section.

3.3.3. Sloping beach and small tidal amplitude. In the following it will be shown that the rise of the average water table height between the low water mark and the high water mark could be due to the asymmetry of the tidal infiltration/draining process for a sloping beach. In popular terms: It is much easier for the water to pour into a sloping beach at high tide than to drain away at low tide. The same asymmetry also results in timewise skewing of the water table variation, that is, it makes the rise steep and the decline flat (see Figure 1).

The difference between the "vertical beach solution" (8) and the more realistic solution which is pursued in the present section is outlined in Figure 5. While the simple solution (8) matches the tide level along the vertical line $x =$

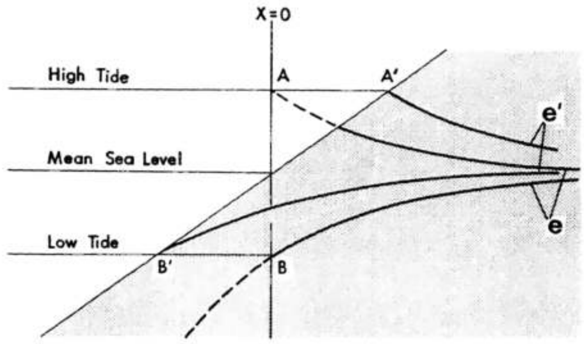

Fig. 5. The solution pursued in the present section is brought to match the tide along the beach face between the high and low water marks A' and B', that is, it satisfies the boundary condition (5). The "vertical beach solution" considered in the previous section fulfils the simpler condition (6). The difference between the solutions is illustrated by the envelopes e and e'.

0 and thus matches the high tide level at A and the low tide at B, respectively, the new solution is brought to match the tide level along the beach face and hence to match high tide at point A' and low tide at B'. The curves e and e' are the corresponding envelopes.

For the analysis, assume again that the tidal variation is given by

$$h_{\text{tide}} = D + A \cos \omega t \tag{26}$$

and the beach forms the angle β with the horizontal. We assume further that $A \ll D$ and hence look for solutions of the form

$$h(x, t) = \Sigma A_j \cos (j\omega t - \varphi_j - \sqrt{jk}x)e^{-\sqrt{jk}x} \tag{27}$$

which correspond to the textbook solutions (8) and (11) to the diffusion equation. The coefficients A_j must be determined so that the water table follows the tide along the sloping beach face as expressed by the boundary condition (5). Inserting the form of the solution (27) and the assumed tidal variation (26) into (5) yields

$$\Sigma A_j \cos (j\omega t - \varphi_j - \sqrt{jk}A \cot \beta \cos \omega t)e^{-\sqrt{jk}A \cot \beta \cos \omega t} = A \cos \omega t \tag{28}$$

where we introduce the perturbation parameter ε given by

$$\varepsilon = kA \cot \beta \tag{29}$$

and get

$$\Sigma A_j \cos (j\omega t - \varphi_j - \sqrt{j\varepsilon} \cos \omega t)e^{-\sqrt{j\varepsilon} \cos \omega t} = A \cos \omega t \tag{30}$$

After writing the exponential as a Taylor expansion in $\sqrt{j}\ \varepsilon \cos \omega t$ and separating terms of equal order in ε we get

$$h(x, t) = D + A \cos (\omega t - kx)e^{-kx} + \varepsilon A\left[\frac{1}{2} + \frac{\sqrt{2}}{2} \cos (2\omega t + \pi/4 - \sqrt{2}kx)e^{-\sqrt{2}kx}\right] \tag{31}$$

which is correct to the first order in ε. We see that the first-order effects of the finite beach slope are a lifting of the mean water table the distance $0.5\varepsilon A$ above mean sea level and a skewing of the variation with time via the $\cos 2\omega t$ term. If terms of order ε^2 are included, the result is

$$\begin{aligned} h(x, t) &= D + A \cos (\omega t - kx)e^{-kx} \\ &+ \varepsilon A\left[\frac{1}{2} + \frac{\sqrt{2}}{2} \cos (2\omega t + \pi/4 - \sqrt{2}kx)e^{-\sqrt{2}kx}\right] \\ &+ \varepsilon^2 A\left(\frac{1}{4} - \frac{\sqrt{2}}{2}\right)[\sin (\omega t - kx)e^{-kx} \\ &+ \sin (3\omega t - \sqrt{3}kx)e^{-\sqrt{3}kx}] \end{aligned} \tag{32}$$

we note that the second-order effects include further steepening of the rise via the $\sin 3\omega t$ term but no extra overheight for the time average $h(x)$. Thus the average overheight predicted by the first-order solution is correct to order ε^2.

The approximation (32) is very efficient with respect to satisfying the sloping beach boundary condition (5) for a beach where shoreline and exit point coincide (no seepage face). This is illustrated by Figure 6 where (8), (31) and (32) calculated at the shoreline ($x = x_s = A \cos \omega t \cot \beta$) are compared to the exact tidal elevation $A \cos \omega t$. The parameters were chosen to match the present field study, i.e., $A = 0.516$ m, $k = 0.075$ m^{-1} and $\tan \beta = 0.1$. This corresponds to $\varepsilon = 0.387$, and we see that even for this fairly large value of the perturbation parameter, the deviation is at most 0.032 m occurring at low tide.

The solutions (8), (31) and (32) are compared to the measurements from Well 7 in Figure 7. The parameters used in the solutions are the same as used in connection with (25) and Figures 4 and 6.

We see that the improvement obtained by using the first-order solution (31) instead of the "vertical beach solution" (8) is considerable although neither the overheight nor the steepening of the rise are strong enough. Addition of the second-order terms leads to further though fairly modest improvement.

3.4. *Decoupling of the Water Table by Formation of a Seepage Face*

From the example shown in Figure 7 we see that the greatest discrepancies occur at low tide where the measurements stay around 0.1 m above MSL while the models predict values down to about −0.20 m. The difference cannot be explained by the nonlinear effects discussed in section 3.3.2. Qualitatively, these are different in that they are not concentrated at low tide like the discrepancies in Figure 7. Also the nonlinear effects are not big enough. Addition of the nonlinear terms of (24) leads to an increase of only 0.085 m of the low tide level at Well 7.

The deviation is therefore mainly due to the fact that the water table did get decoupled from the tide at low tide in the field experiment while the models assume that it did not, i.e., (32) is based on the boundary condition (5) which assumes that no seepage face is formed. Figure 8 shows the measurements of the water table with the exit point about 0.3 m above the shoreline at low tide (2300 hours, April 18, 1989).

The dynamics of a decoupled water table are, as far as the writer is aware, unresolved, and the literature on the subject is confused. For example, *Bear* [1972, p. 262] gives a "proof" that the water table must be tangent to the seepage

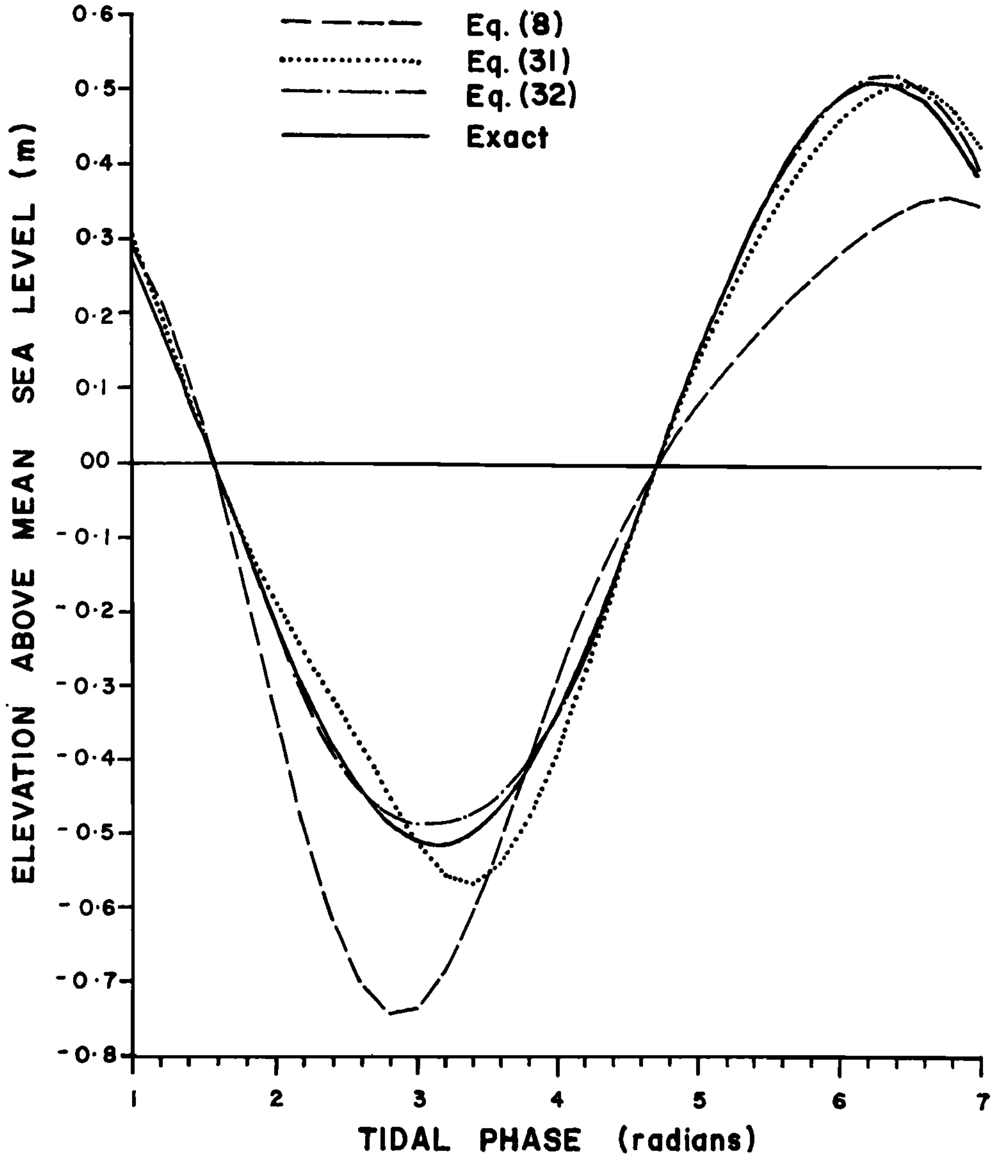

Fig. 6. Shoreline elevations $h(x, t)$ from "the vertical beach solution" (8), the first-order sloping beach solution (31) and the second-order version (32) compared to the exact values for $\varepsilon = 0.387$, corresponding to the parameters of the present field study. Considerable improvement is achieved by the first-order version (31), and the second-order solution is almost perfect.

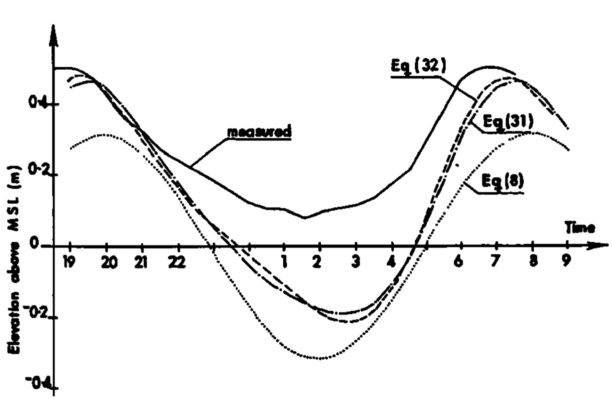

Fig. 7. Comparison of the solutions (8), (31) and (32) to the measurements from Well 7.

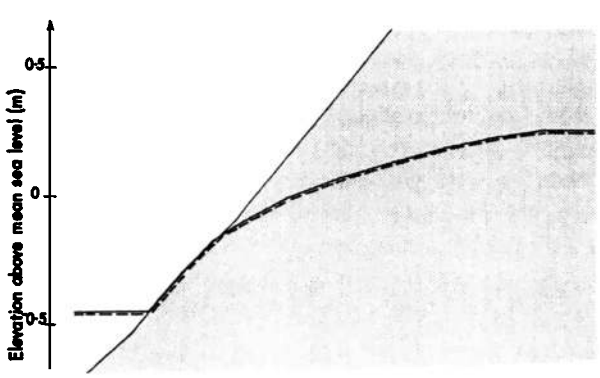

Fig. 8. During the field experiments a seepage face developed approximately 2.5 hours before low tide. The speed of the falling tide was about 7×10^{-5} m/s when this happened.

face in the unsteady as well as in the steady case, while the data (photographs) by *Dracos* [1963] clearly show that it is not. Thus theoretical knowledge about the behavior of the exit point is shaky. That is, not much is known beyond the fact that the pressure must be constant along the seepage face.

Dracos proposes a model for the movements of the exit point based entirely on the dynamics of an isolated water particle on the seepage face, but he does not consider the periodic (tidal) problem. Dracos does, however, suggest a maximum vertical velocity v_{max} for the exit point

$$v_{max} = \frac{K}{n} \sin^2 \beta \quad (33)$$

and thus decoupling is predicted if the speed of the falling tide exceeds v_{max}. However, this formula predicts for the present field conditions a value of v_{max} which is considerably higher than the speed of the falling tide observed just before the water table became decoupled. In order to apply Dracos' formula to the present field conditions we derive a value of K/n for the field site by inserting $k = 0.075$ m^{-1} and $D = 0.51$ m into

$$k = \left(\frac{n\omega}{2KD}\right)^{0.5} \quad (34)$$

and get $K/n = 0.024$ m/s. Hence Dracos' formula with $\sin \beta = 0.1$ gives $v_{max} = 2.4 \times 10^{-4}$ m/s which is a factor 3.5 more than the observed value. While it must be acknowledged that the value of K/n used for this comparison is subject to some uncertainty, the data does indicate that Dracos criterion for decoupling is conservative, i.e., it would not predict the decoupling which occurred for the conditions of the present field study.

From the remarks above it is clear that further research is required into the behavior of the exit point on a sloping seepage face. However, since this represents a major task of its own, it will not be attempted here.

4. CONCLUSIONS

Field measurements have been made of water table heights inside a beach which had no significant wave activity or outflow of rainwater and thus exhibited the effects of tide in a pure form. The measurements show that the inland average water table was elevated 0.29 m above mean sea level for a tidal amplitude of 0.516 m.

This superelevation is due to three mechanisms: formation of a seepage face around low tide, asymmetry of the boundary condition at the sloping beach face, and finally the nonlinearity of the governing equation in the interior.

While resolution of the behavior of a dynamic exit point and seepage face has been left for further study, a clear insight into the nature and magnitude of the two other effects has been achieved.

The inland overheight due to beach slope is of the order of magnitude $0.5kA \cot \beta$, and this result, although being part of the first-order solution, is correct to the second order in $\varepsilon = kA \cot \beta$.

The asymptotic inland overheight due to the nonlinearity of Boussinesq's equation is of the order $A^2/4D$, and this value is approached asymptotically as e^{-2kx} (see (25)).

Apart from these time-averaged features, the time dependence of the water table height at a point has been investigated, and it has been shown that the sloping beach face acts as highly nonlinear filter which causes the water table to rise abruptly and drop off slowly compared to the near-sinusoidal tide which drives it. This effect is quantified by the results in section 3.3.3.

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