VIII. *Tidal Friction in Shallow Seas.*

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THE astronomical importance of the dissipation of energy that goes on in shallow seas has been shown by G. I. TAYLOR's recent estimate* of the amount in the Irish Sea, which is enough to account for about one-fiftieth of the secular acceleration of the moon. It also produces a considerable effect on the tides themselves, and there are probably many places where it must be taken into account before any satisfactory theory of the local tides, or even their empirical prediction, can be achieved. It is indeed very well known that there are bays and straits where the height of the tides, or the speed of the currents, or both, are greater than in the Irish Sea, and a careful examination of such places, with a view to finding the dissipation in them, is needed. There are other places where the dissipation for an equal area is less than in the Irish Sea, but which may actually contribute much more altogether on account of their greater size. The object of this paper is to discuss what regions are capable of producing notable parts of the secular acceleration; to estimate as accurately as possible from the data available the dissipation in these; and to compare this with that calculated from the secular acceleration, so as to find out whether it is necessary to assume the existence of any other important cause to account for the latter.

The horizontal force of the skin friction of water over the sea bottom is $0{\cdot}002\rho V^2$ dynes per square centimetre, where ρ is measured in grammes per cubic centimetre and V in centimetres per second. The difficulty of the problem is in the estimation of V. The available observations of the velocities of tidal currents are given in the Admiralty Sailing Directions; but they are never uniformly distributed, and are usually confined to the neighbourhood of the coasts, and they must be supplemented by theory before the velocities remote from the coast can be found. A few theoretical considerations that have been found useful in this process will now be mentioned.

Take first the case of a bay or strait long in comparison with its width, and consider a wave entering it whose period is much longer than the time needed for a

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wave of tidal type to cross. This time is known to be independent of the period of the disturbance, depending only on the width and depth. Such a wave is reflected from one side to the other again and again before it reaches the other end. It is also known that the transverse velocity at the sides is zero, since water cannot cross a rigid boundary. Thus if we compare two points on opposite sides of the channel, we know that the times of arrival of the wave at them differ by a small fraction of a period; and since the transverse velocity at both is zero, it cannot be great at any intermediate point, for that would contradict the hypothesis that the wave-length is much greater than the width of the channel. The transverse velocity may accordingly be neglected in problems of this class.

If the period of the entering wave is of the same order of magnitude as the time needed to cross the channel, we can no longer infer that the transverse velocity is much less than the longitudinal one. This case seldom or never arises. The velocity of a tidal wave is $(g\mathrm{D})^{\frac{1}{2}}$ where g is the intensity of gravity and D is the depth; and if the water was only 20 fathoms deep a tidal wave would in 12 hours travel 700 km., which is far greater than the width of almost any channel whose length is much greater than its width. Where the width is greater, the depth also is always greater, so that the above argument always holds in long channels of whatever size.

If now x be the distance of a point from the entrance to the channel, y the distance from the side, u the longitudinal velocity of a particle there, and η the height of the free surface above its undisturbed position, the equations of motion of the particle are

$$\frac{du}{dt} = -\frac{g\partial\eta}{\partial x} - \text{a term due to friction}$$

$$2\omega u = -g\frac{\partial\eta}{\partial y}$$

where ω is the component of the earth's angular velocity of rotation about the vertical at the point. The equation of continuity is

$$\frac{\partial}{\partial x}(\mathrm{D}u) = -\frac{\partial\eta}{\partial t}.$$

From this and the second equation of motion we deduce at once

$$\frac{\partial\eta}{\partial x} = \frac{\partial\eta_0}{\partial x} + \frac{2\omega}{g}\int\frac{1}{\mathrm{D}}\left(\frac{\partial\eta}{\partial t} + u\frac{\partial\mathrm{D}}{\partial x}\right)dy$$

where η_0 is the value of η at the side. In this we see from the conditions that the channel is narrow and the depth slowly varying along it that the first term is much greater than the others. Accordingly $\frac{\partial\eta}{\partial x}$ is the same for all particles in the same cross-section of the channel, and the first equation of motion then shows that the same is true of u if the friction is small. Thus in such cases the velocity is the same at all points in the same cross-section, so that observations made at the side will be correct for points in the middle. If friction is great this result must be modified, since for the same velocity the frictional force is independent of the depth of the water, whereas the mass affected is proportional to the depth. Thus friction has more influence in reducing the velocities in shallow water than in deep water. Hence when the channel is shallow at the sides and deep in the middle the velocity will be least at the sides. The ratio of the frictional term to the first term is $0{\cdot}002\ u/2D'\Omega$, where D' is the depth at the shallow part. When this is 10 fathoms and the velocity 1 knot the fraction is 0·25, so that this effect is then appreciable; and when the depths are less and the velocities greater, the influence of friction may increase to such an extent as to dominate the whole character of the motion. When this occurs, the velocity will always be in the direction of decreasing pressure, and inertia may be neglected.

The last result may appear to contradict the general principle that "still waters run deep." There is no real contradiction, however, for the problems referred to are different. The above argument deals with the differences between the velocities at places in the same cross-section of the channel, whereas the proverb concerns rivers whose depth varies *along* them, and in such cases the motion is naturally slowest where the depth is greatest, since the amount of water crossing any section in a given time must be the same. It also has an important and well-known application in bays of varying depth and width, such as the Bay of Fundy. If, for instance, the bay is very long, and these quantities change only by small fractions of themselves per wave-length, it can be shown* that the height of the wave at any point is proportional to $b^{-\frac{1}{2}}D^{-\frac{1}{4}}$, and the velocity of the water to $b^{-\frac{1}{2}}D^{-\frac{3}{4}}$, where b is the width at the surface. The rate of dissipation of energy across any section is proportional to bu^3 or to $b^{-\frac{1}{2}}D^{-\frac{9}{4}}$. It therefore increases slowly as the channel becomes narrower and much more quickly as it becomes shallower. When the depth and width vary much within a wave-length these results cease to be useful approximations, but the tendency for the height of the tide and the velocity of the tidal current to increase as the channel becomes narrower and shallower remains. Thus in such places we often find very high tides and strong tidal currents. Apparently, however, their limited area prevents the dissipation in them from being as great as that in larger places with less violent currents (at least, if the Bay of Fundy may be regarded as typical of them).

The widths of most actual bays are, however, comparable with their lengths, and in these it is generally a matter of some difficulty to settle whether we can treat the recorded currents as a fair sample of the whole. The amplitude of the tide in mid-ocean is only about a foot, but in the shallow water around the coasts it is magnified to several feet, and the tidal currents are increased correspondingly. Where the

* LAMB, 'Hydrodynamics,' p. 258.

shore is fairly open and regular in outline, like most of the coast of Africa, it is not possible to find the dissipation along it, for there are no data to show how far out the currents extend. In partly enclosed regions, however, it is frequently possible to interpolate between the records made on opposite sides. A serious difficulty may arise if the depth of the sea is very different at different points within it, for this may destroy the possibility of interpolation, and therefore we must always examine the soundings for any great variation. Ordinarily we should not expect much variation in the velocities, for such places are intermediate in character between narrow channels and the open shore, and therefore the currents in them may be expected to show some increase in shallow water, but not so much as would be caused by a proportional decrease in depth along a narrow bay or in approaching the open shore. In shallow water also friction may, and often does, neutralize the magnification that would occur in its absence.

One other fact may be noted. In shallow bays the difficulties of navigation may be great, and navigators avoid them if possible by choosing a harbour near the entrance. Thus observations of currents are most numerous about the entrances, and often at the very places where the currents, and consequently the dissipation, are greatest there are insufficient observations to give a satisfactory estimate.

Great care must be taken in dealing with observations among islands, straits, and shoals. When the passage of a tidal current is obstructed by a shallow of small horizontal extent, part of it goes round the shoal and part over the top. The influence of this on the main current is of course small, but on the top of the shoal and in its immediate neighbourhood the velocities may be much increased, for much the same reason as accounts for the greater speed of a river where it is shallow. On the other hand the increase in the influence of friction may greatly reduce the currents, and shoals often afford in this way an important shelter from tidal currents to the deeper water behind them. This is particularly noticeable at some points on the Korean side of the Yellow Sea. Thus observations of currents taken at lightships and buoys over shoals whose dimensions are all much smaller than those of the main bay or channel must be regarded as giving no reliable estimate of the main current. Small islands also require examination before the records obtained are accepted. If one is surrounded by a shoal they are of course untrustworthy; but if deep soundings are found within a few miles of it, they will probably give a very good idea of the main current, which will, especially in a wide channel, be fully as useful in our investigation as the results of observations at the sides. Straits are in a different position. When the tides in two seas or even oceans are in widely different phases, a large head may be produced between the two ends of a strait connecting them, so that a swift tidal current will flow along the strait. In no circumstances, however, can this give any indication of the currents in the seas, for it is produced by the tide heights, and not directly by the currents. Such currents may attain very great velocities, as in the

Magellan Strait and Smith Sound, and when the area of the strait is not insignificant the dissipation may be an important part of that in the seas as a whole.

European Seas.

1. *The Irish Sea.*

This sea has been discussed in detail by Taylor. The rate of dissipation is found to be 1040 ergs per square centimetre per second, or $4 \cdot 1 \times 10^{17}$ ergs per second in all, on an average at spring tides. This result is based on the law that the rate is proportional to the cube of the velocity. The Irish Sea is remarkable in that the maximum current occurs nearly at high water, whereas in ordinary places the water is nearly slack then; though other examples will be given later in this paper. This affords the most favourable conditions for an accurate estimate of the rate at which energy enters the sea; to this Taylor added the rate at which the moon's attraction does work on the sea, and from the fact that all this energy must be dissipated in the course of a period (for if it were not, there would be a continual increase of energy in the Irish Sea) he found that the mean rate of dissipation at spring tides was $6 \cdot 0 \times 10^{17}$ ergs per second. This estimate is probably more accurate than the other, as the data involved are obtained from observations in St. George's Channel, supplemented by an accurate theory; but the former is based on an average of the velocities in the Irish Sea itself, which are more difficult to determine.

2. *The English Channel.*

On an average the tidal currents in the English Channel at springs reach about 2·5 knots. The speed is greatest towards the Straits of Dover and least at the entrance to the Channel, and enough data are available in the Admiralty publication, 'The Tides and Tidal Streams of the British Isles' to give a very accurate estimate of the total dissipation if this were required in the present problem; but as the errors introduced by using only a rough approximation in this case are far less than those involved in the best data referring to regions with far larger dissipations, accuracy is not here required.

The rate of dissipation is $0 \cdot 002 \rho V^3$ ergs per square centimetre per second. Here ρ is practically 1; and one knot is 51·5 cm. per second. Thus the dissipation per square centimetre for a velocity of one knot is 274 ergs per second, and that per square kilometre is $2 \cdot 74 \times 10^{12}$ ergs per second. The area of the English Channel is about 60,000 sq. km., so that the dissipation when the currents are flowing fastest is $2 \cdot 74 \times 10^{12} \times 6 \times 10^4 \times 2 \cdot 5^3$, or $2 \cdot 5 \times 10^{18}$ ergs per second. This is of course a maximum, while the value obtained for the Irish Sea is the mean over a period; the average rates of dissipation in the two places are perhaps not very different.

3. *The North Sea.*

A satisfactory estimate of the dissipation in the North Sea is practically impossible. Velocities up to over 3 knots are recorded here and there, but all the observations are in the coastal region, which is very much complicated by shoals. The maximum in the outermost part of the Moray Firth is about 1·1 knot, and this is probably fairly typical of the whole of the North Sea. Taking the area to be 5×10^5 sq. km. and adopting the above value of the velocity, we see that the maximum dissipation is of the order of $1 \cdot 8 \times 10^{18}$ ergs per second.

4. *Other European Waters.*

In the Mediterranean there is probably little or no dissipation of tidal energy, for the Atlantic tidal wave can only enter through the very narrow Straits of Gibraltar, and partly for this reason and partly on account of the great length and considerable depth of the sea there is very little tidal movement in it. The same argument applies to the Baltic, for the entrance through the Kattegat is largely blocked up by the Danish islands, so that little water can enter to produce a tide. The Bay of Biscay is mostly too deep to have any important current, while the White Sea is too small and landlocked to give as much dissipation as the Irish Sea.

The average dissipation in a period is $4/3\pi$ of the maximum. If we find the maximum for the Irish Sea on this basis, we obtain for the total dissipation in European waters when the spring tide currents are flowing strongest about $6 \cdot 0 \times 10^{18}$ ergs per second; the average at spring tides is $2 \cdot 4 \times 10^{18}$ ergs per second.

Asiatic Seas.

It has already been pointed out that the tidal currents in mid-ocean are insufficient to give any important dissipation.* Accordingly we need consider only those places where the currents are very much magnified by great decreases in depth. On referring to a physical map of Asia it is at once seen that the places around the coast where the depth is less than 100 fathoms are the Straits of Malacca, the South China Sea (with the Java Sea), the Gulfs of Siam and Tongking, the Yellow Sea, the Persian Gulf, and parts of the Seas of Japan and Okhotsk and the Bering Sea. These regions will be dealt with separately. The Persian Gulf may be omitted at once, as its narrow entrance prevents the tide from being great.

1. *The South China Sea.*

This sea is in the form of a letter T. The middle stroke points north-east and lies between Annam and Southern China on the one side, and Borneo and the Philippines

* 'Philosophical Magazine,' May, 1920, vol. 39, pp. 578–586.

on the other. The two side-pieces are the Gulf of Siam and the Java Sea. The data for it are obtained from the 'China Sea Pilot,' volumes 3 and 4, except the depths, which are taken from the Admiralty Charts. The tides are affected by a large diurnal inequality due to the inclination of the Equator to the ecliptic. When the moon is north of the Equator it tends to raise two tidal protuberances in the ocean, one exactly below it and the other exactly opposite to it. Owing to the earth's rotation each of these moves round the earth once a day, keeping the same distance north or south of the Equator. Thus if a place is not on the Equator, they pass at different distances from it, so that the two tides in the lunar day are unequal in height. The variation in the level of the water thus caused can be described as a semi-diurnal change, on which a diurnal change is superposed.

Now the rate of travel of a tidal wave is practically independent of its period, but if the depth and the form of the coast are such that the waves starting from the north and south sides of the Equator take different times to reach the place of observation, their combined effect may be remarkable. In particular, if the wave from the south arrives a quarter of a lunar day after or before the other, the semi-diurnal part of the one wave will correspond to high water while that of the other corresponds to low water, and if the amplitudes are equal the two will neutralize each other. In other words, there will be a node of the semi-diurnal tide. The diurnal parts, however, will not neutralize each other, their phases being only a quarter of a period apart. Thus at such a place there will be a diurnal tide and no semi-diurnal tide. Several places are known where there is only one high water in each lunar day; among them are parts of the South China Sea, the Gulf of Carpentaria, and Bering Sea. The tides in these require special discussion before they can be considered in the present problem, because the diurnal tide depends essentially on the inclination of the Equator to the plane of the moon's orbit and would not exist if this were zero. The dissipation of energy in it must therefore arise from the motion of the moon in declination and not in right ascension, and will affect mainly the inclinations of the Equator and the moon's orbit to the ecliptic, while producing little effect on the earth's rotation and the motion of the moon in longitude. In discussing the secular acceleration of the moon it can therefore be ignored. If observations of the diurnal tide in the places where there are two high waters in the day were more numerous it might be possible to determine the dissipation in it, and from it the secular changes in the inclinations, but at present this is impossible.

In the Java Sea, between Borneo and Java, the tide is mainly diurnal; in fact according to the 'Eastern Archipelago Pilot,' part 3, the semi-diurnal tide is not appreciable on the north coast of Java till east of Surabaya, which is itself almost at the eastern end. On the south coast of Borneo the observations are not so numerous, but it seems clear that there also the tide is mainly or entirely diurnal. The tidal currents are described as weak. We can accordingly neglect the dissipation in the Java Sea. At its western end this sea is connected to the South China Sea by two rather wide straits, Carimata Strait and Gaspar Strait. In both of these the tide is mainly diurnal; in the latter, in fact, it is entirely so. At Pontianak, near the north-westerly point of Borneo, the tide is still diurnal; thus the part of the China Sea south of Pontianak and Singapore probably contributes little to the secular acceleration of the moon.

Apparently the main tidal stream from the China Sea strikes the Malay Peninsula somewhere near Cape Patani and spreads out from there; for on the coast north of this point the flood stream sets to the north, while south of it it sets to the south. The tide in this region is definitely semi-diurnal, though the heights of the two daily high waters may be unequal. The depths in the western part of the sea and in the Gulf of Siam are mostly about 30 fathoms, but there are many shoals around the coast where the depth is only a few fathoms, and it is therefore necessary to be very critical of the sites of observations of currents. The best results seem to be given at small islands with rapidly shelving sides, for the currents there are modified little by the form of the bottom and can be regarded as fairly typical of the general currents in the neighbourhood.

At the Anamba Islands (see map, fig. 1) the semi-diurnal tide appears to be usually much less than the diurnal one. The 'China Sea Pilot,' vol. 3, states that for a few days in each month, when the moon is near the Equator, there are two high tides in the day. It is easily seen that for two tides to occur in the day the amplitude of the semi-diurnal term must be at least a quarter of that of the diurnal term. It would not, however, vary much with the moon's declination, whereas that of the diurnal term vanishes when the moon is on the Equator; and the above fact shows that the semi-diurnal tide only attains this fraction of the diurnal tide when the latter is at its least. The true semi-diurnal tide at the Anamba Islands must therefore be insignificant. The same is evidently true of the currents, for the tidal streams take a day to run backwards and forwards.

In the Gulf of Siam also the tides are mainly diurnal. The oscillation in this is a forced one due to that in the South China Sea, and as the latter is diurnal so is that in the Gulf of Siam. Actually the only place where the semi-diurnal tide is considerable is Bangkok Harbour, at the head of the gulf. This tide seems to increase in relative importance towards Bangkok, for at Kamput on the eastern side and places in about the same latitude on the western side the tides are said to be very irregular, indicating the presence of some complicating influence. On the whole, therefore, it seems that the dissipation in the Gulf of Siam will not be underestimated if we assume that the semi-diurnal current reaches a maximum of one knot, this being one-third of the diurnal tidal current observed off Cape Patani, at all places north of the parallel of 11° N. The area of this region is about 70,000 sq. km., giving a maximum dissipation of 2×10^{17} ergs per second.

An estimate may be made of the dissipation of the energy of the diurnal tide in the same regions. At Pontianak there is a diurnal current of two knots when running strongest; and a similar velocity is recorded at the Burong Islands some distance to the north. As the depth of the passage between Borneo and Sumatra is fairly uniform, from 10 to 20 fathoms, these measures are probably typical of the whole. Quantitative estimates of the currents in the Gulf of Siam are few, and a reasonable guess at them would be difficult. The depth is mostly about 30 fathoms, but there are a deeper area in the middle and many shoals about the margins. As the current at the mouth of the gulf is across it, the tide in the gulf can arise only from

Fig. 1.

the reflection of this by the Malay coast, so that the general set of the current is across the gulf, and considerable magnification in the gulf is unlikely. Our estimate of the dissipation will probably be of the correct order of magnitude, if we suppose that everywhere west of a line joining Cape Datu to Cambodia Point the maximum current is two knots. The area of this region is 8×10^5 sq. km.; thus the maximum dissipation is $1 \cdot 7 \times 10^{19}$ ergs per second. The velocity is proportional to the sine of the hour angle of the moon increased by a constant; and as the dissipation is proportional to the cube of the velocity, the average dissipation is obtained by multiplying the maximum by the average arithmetical value of $\sin^3 \theta$ taken over a period, which is $4/3\pi$. It is not clear what declination of the moon the recorded currents refer to: if they refer to the maximum, the average for the month will be found by multiplying again by $4/3\pi$. Thus the average dissipation of energy in the diurnal tide in the western part of the China Sea is of the order of 3×10^{18} ergs per second.

We next proceed to examine the dissipation in the main part of the South China Sea, between the north-west coast of Borneo and the mainland. The currents are now semi-diurnal. Near Cape Sirik the flood runs for four hours and the ebb for eight hours, so that a considerable diurnal component exists, but not sufficient to preponderate over the semi-diurnal motion. The velocities here are from two to three knots. At Bruni the observations are very much interfered with by shoals and narrows, but in the offing the currents seem to be about two to three knots. About Tega, however, among shoals the velocities recorded are only 1·3 and 0·8 knots. The contours of the sea floor run roughly parallel to the coast, so that these currents may persist for some distance out to sea. Off the north end of Borneo the sea rapidly deepens, and in accordance with this there is scarcely any tidal current at Ulugan Bay, in Palawan. On the Asiatic side the tide is diurnal at Camran and Tourane, but the currents are weak.

A few shoals and islands in the middle of the sea have been made the localities of observations. At Rifleman Bank and Spratly Island there is only one tide in the day, and at the neighbouring island of Amboyna it is said that near neaps the stream reaches 1·4 knots. It is therefore clear that the semi-diurnal current of the Borneo coast does not extend half way across the sea, and its true extent is very doubtful. In the Gulf of Tongking also the currents appear to be diurnal. Thus in the whole of the South China Sea and its extensions there seems to be little semi-diurnal tide and little contribution to the secular acceleration of the moon, though there is a dissipation of the energy of the diurnal tide that may have a notable secular effect on the obliquity of the ecliptic.

2. *The Yellow Sea.*

This is a gulf about the size of Ireland, lying between Korea and the coast of China, and extending about as far south as the mouth of the Yang-tse-Kiang. It becomes very narrow where the Shan-tung peninsula projects into it, and north of this it forms the Gulfs of Pe-Chili and Liau-tung. Most of it is shallow, the depths in the main part of it being mostly about 30 fathoms, and those in the northern part about 15 fathoms. Around the shore the water is shallower, and in many places there are crowds of shoals. The data are obtained from the 'China Sea Pilot,' vol. 5.

The tidal phenomena are extremely complex. At the south end of the peninsula of Korea high water (full and change) is at about 11h., Greenwich time. As we advance up the Korean coast it occurs later and later, being practically 12 hours later at Port Arthur. On the opposite side of the Strait of Pe-Chili it is slightly earlier than at Port Arthur; then on the way round the Shan-tung peninsula and out of the sea again the tide again becomes steadily later. It therefore looks as if the tide enters up the coast of Korea, gradually passes up the sea, losing energy all the way, and a reflected wave from the Pe-Chili strait emerges down the Chinese coast. The tides and the currents on the Korean side are noticeably stronger than those on the Chinese side, and it does not seem likely that this is due wholly, or even largely, to the shoals on the former coast, for at islands in deep water, such as the Mackau, Myangoru, and Bate groups, velocities of 3 to 5 knots are recorded, while in

Fig. 2.

shallower water the velocities are not usually greater than these, though local strong streams exist. Apparently the effects of friction are great enough to counterbalance those of the diminution in depth. They are also seen in another respect. Among the islands of the Korean archipelago the tidal stream sets west from four hours before till two hours after high water, whereas in most places elsewhere there is little or no current at high water. Thus work is continually being done on the sea, and the energy entering is dissipated in it. An effect of the approximate agreement in phase between the tide and the velocity may be utilized to give an estimate of the currents and tide height in the entrance to the sea, far from the nearest land, and hence of the amount of energy entering. In any motion in a channel where, on account of its narrowness or for some other reason, there is little transverse motion, the velocity along the channel is related to the pressure gradient across it according to the equation

$$2\omega\rho u = -\partial p/\partial y,$$

where ρ is the density of the water,

ω is the component of the earth's angular velocity of rotation about the vertical at the point considered,

p is the pressure, and

y is the distance measured across the channel.

In the Yellow Sea the entrance is not narrow, but there seems reason to believe that the velocity across it is small, which is all that is required for the truth of the above equation. If now g denote the intensity of gravity, and η the elevation of the surface of the water above its mean position, then at any fixed point, at whatever depth, the variation of p is equal to that of $g\rho\eta$. Again, if Ω be the earth's angular velocity, and λ the latitude of the place,

$$\omega = \Omega \sin \lambda,$$

and we have on putting $\lambda = 35^\circ$; $\Omega = 7{\cdot}3 \times 10^{-5}/1$ sec.; $g = 981$ cm./sec.^{2};

$$u = -1{\cdot}17 \times 10^7 \frac{\partial \eta}{\partial y}$$

where C.G.S. units must now be used.

On the coast of Korea the tide has an amplitude of about 10 feet, or 300 cm. The velocity of the inward current is about 4 knots, or 200 cm./sec. Now suppose if possible that the current remained constant right into the middle of the entrance; then the above formula shows that at a distance of 176 km. from the side there would be little vertical movement of the surface, and further away still a huge tide with an amplitude of some 30 feet would exist. It is not reasonable that the tide in the middle should be greater than that at the side, though it may easily be smaller. The alternative hypothesis is therefore that the current decreases as we approach the middle, and is very small over most of the sea. This will be adopted in the forthcoming discussion. We shall suppose that the current at distance y from the coast is in the same phase as that at the coast, and is a linear function of y. Then put

$$u = (200 - ky) \cos \gamma t.$$

At the shore η is equal to 300 cos $(\gamma t - \alpha)$, where α is the difference in phase between the tide height and the current strength. For the semi-durnal tide it is twice the angle moved through by the moon relatively to the earth in one hour, or 29 degrees. In general

$$\eta = 300 \cos (\gamma t - \alpha) - 8{\cdot}6 \times 10^{-8} \cos \gamma t \,(200y - \tfrac{1}{2}ky^2).$$

The amplitude of η reaches a minimum where the coefficient of $\cos \gamma t$ vanishes. We shall have the least possible tide in the middle, thus satisfying our earlier assumptions, if we assume that this minimum is reached at the point where the velocity vanishes. In this way we shall underestimate the dissipation, but not by any great amount. The value of k that makes this coincidence possible is $6{\cdot}6 \times 10^{-6}/1$ cm., so that the current becomes zero 300 km. from the shore, practically in the middle of the entrance.

TAYLOR shows (*loc. cit.*, equation 15) that the average amount of energy crossing any line is the average over a period of $g\rho\int \mathrm{D}\eta u\, dy$ in my notation, where D is the depth of the water, in this case about 30 fathoms over most of the region in which the velocity is greatest, with a range in all of from 20 to 45 fathoms. We find easily

$$\eta = 147 \sin \gamma t - 4{\cdot}3 \times 10^{-8}\, k\,(y - 200/k)^2 \cos \gamma t,$$

and the average flux of energy across a parallel of latitude is found to be $1{\cdot}06 \times 10^{18}$ ergs/sec.

The northward flux on the Chinese side is more difficult to determine, as the direction of the currents is variable. Two phenomena are intermingled here. The issuing tide from the Yellow Sea comes down this coast, but there is also a definite tidal wave that travels into and out of the large bend in the coast whose extremities are Shan-tung promontory and Shanghai. This is shown by the fact that along the northern part of this bend, on which Tsing-tao stands, the current flows northwards while the tide is ebbing both along this coast and in the northern part of the Yellow Sea. Thus in this bay the main tide is the local tide of the bay itself and not the general tide of the Yellow Sea. The currents produced by these tides are rotary, probably an effect of the earth's rotation; and it seems that the northward component of the velocity is small. Further, it is probably nearly in a phase at right angles to the tide, as great divergences from this relation can be produced only by great dissipations and accordingly by great velocities in the vicinity. Hence for both reasons we infer that the northward flux of energy along the Chinese coast is small in comparison with that on the Korean side.

We also need to know the work done on the water by the moon. If η' is the height of the equilibrium tide, the work done by the moon in a period is $\iint g\eta' \frac{d\eta}{dt}\, dt\, d\mathrm{S}$, where $d\mathrm{S}$ is the element of horizontal surface and the integrals are to be taken, in the one case over a period, and in the other over the area considered. If the phase of η is β in advance of that of η', and the amplitudes be h and h', the average rate of doing work is $-\frac{1}{2}\, g\int hh'\gamma \sin\beta\, d\mathrm{S}$. In the present case h' naturally varies little; h decreases fairly steadily as we travel from the entrance to Port Arthur, where it has about half of its value at the entrance. On the other hand β varies a great deal. The longitude being about 120° E., the moon crosses the meridian at full and change at 3h. 43m. It is high water in Shoan harbour, at the southern end of Korea, at 10h. 33m., so that β is here $-199°$, or more conveniently $+161°$. The tide lags more and more on the way up the sea, and at Port Arthur it has lost practically a whole period, the high water at full and change occurring there at 11h. 7m., making $\beta = -216°$. Positive elements will be added to the integral by the places where β is from $161°$ to $0°$, negative elements where β is from $0°$ to $-180°$, and positive again where it is from $-180°$ to $-216°$. These values are however weighted according to the values of h and according to the extent of the areas for which they are correct. Now β is zero near the Conference Islands, only about 0·4 of the way to the narrowest part, so that on this account the area for which it is positive would appear to be less than that for which it is negative. The sea becomes much narrower farther north, however, which must reduce the ratio of the weights somewhat, and the tides are only about two-thirds of the height. Thus it seems that the weights to be attached to the positive and negative values of $\sin\beta$ are nearly equal, and its average over the sea is unlikely to be more than 0·2. The area of the sea as far as Port Arthur is about 300,000 sq. km.; taking the average of h as 240 cm., and that of h' as 20 cm., we find the energy imparted by the moon to be not greater in absolute magnitude than 2×10^{17} ergs per second on an average. That entering up the Korean coast is far greater.

In the discussion of the work done by the moon the Gulfs of Pe-Chili and Liau-tung have been ignored. There are two reasons for this: their united area is about a third of that of the main part of the sea, and the tides recorded at the sides are also about a third of those on the Korean coast. It seems to me, however, that these two gulfs afford an example of a special type of tidal problem different from any previously discussed. For, let us suppose if possible that the recorded amplitudes, of the order of 90 cm., were typical of the whole area. The average depth is about 2000 cm., and a tidal wave in water of such a depth would give rise to a current of maximum velocity about $h\,(g/\mathrm{D})^{\frac{1}{2}}$, which in this case is 65 cm./sec. The corresponding dissipation would be 550 ergs per square centimetre per second. On the other hand the average energy present is about $\frac{1}{2}g\rho h^2$, or 4×10^6 ergs per square centimetre. Thus if the above assumption were correct the whole energy of the tide would be dissipated in about 7000 seconds, or two hours. This is absurd, and we must suppose, in order to avoid the result that energy is dissipated faster than it enters the region, that the tides in the greater part of the gulfs are much less than the recorded ones. Their height may be only a few inches; while the recorded heights are the result of great magnification in the very shallow water around the edge. Accordingly the work done by the moon on this region may be neglected. The whole work done on the Yellow Sea by the moon is therefore small in comparison with the energy entering with the tide, and even its sign is uncertain. Thus the average dissipation in the whole of the sea is not very different from $1·1 \times 10^{18}$ ergs per second.

An alternative estimate may be deduced directly from the formula for the dissipation, with a suitable hypothesis on the distribution of velocity. At the entrance, when the current is flowing strongest, the dissipation in a strip a centimetre wide across it is $0{\cdot}002\rho\int u^3 dy$, and if the above distribution of velocity is correct this is $1{\cdot}2 \times 10^{11}$ ergs per second. Farther north the velocity is not so great; in fact, around the Shan-tung promontory it does not exceed one knot, and at the islands in Korea Bay opposite it is usually about two knots. Further, the width is here less than 300 km., so that on this account also the dissipation for a given velocity must be less; the amount in a strip a centimetre wide running east and west is therefore probably not more than one-sixteenth of that in a similar strip near the entrance. Even south of the narrow part off Shan-tung the currents appear to be slower than near the entrance; for about half the distance the velocity is about two knots, rising again to $3\frac{3}{4}$ knots at the Sir James Hall group, in the narrow region. If a proportional reduction takes place at all other distances from land, we must suppose that the above estimate of the dissipation per unit length is correct for the first 200 km., that the amount for the next 230 km. is an eighth of this, and that for the remaining 220 km., corresponding to Korea Bay, is a sixteenth. The Gulfs of Pe-Chili and Liau-tung may be ignored. The total dissipation, if the maximum velocity at every place occurred at the same time, would therefore be $(2 \times 10^7 + \frac{1}{8} \times 2{\cdot}3 \times 10^7 + \frac{1}{16} \times 2{\cdot}2 \times 10^7)\ 1{\cdot}2 \times 10^{11}$ ergs per second, or $2{\cdot}8 \times 10^{18}$ ergs per second. The average for each place is $4/3\pi$ of the maximum there, so that the average for the whole sea is $4/3\pi$ of this maximum, or $1{\cdot}2 \times 10^{18}$ ergs per second, which agrees with the previous estimate much more closely than the data would have led us to expect.

3. *The Sea of Japan.*

The Sea of Japan is an oval basin bounded on the eastern side by Japan and Sakhalin. It seems clear that the dissipation is small, for both the tide height and the current are small. Even in the comparatively narrow and shallow Korea Strait, through which the tide enters at the south end, the current only attains a speed of 1 or 2 knots; and when this opens into the sea the width suddenly increases to 900 km. and the depth to 400 fathoms. The currents in most of the sea must therefore be insignificant. They are appreciable at the gaps between the Japanese islands and in part of the narrow Gulf of Tartary, but the area affected is small and the currents only moderate (1 to 3 knots at most) so that the dissipation is small.

4. *The Sea of Okhotsk.*

In its essential features this resembles the Sea of Japan. The only shallow parts of it are narrow strips around the coast, while the tide enters through the shallow water of the straits between the Kurile Islands. As the tide in the sea depends on the supply of water to maintain it, the restriction on it imposed by the shallowness of the entrances causes the currents to be small. In the Gulfs of Ghijinsk and Penjinsk, in the north-east corner, the depth diminishes considerably, and the currents increase to $1\frac{1}{2}$ to 2 knots. Data are very scanty, but the area affected appears to be about 70,000 sq. km., leading to a dissipation when the currents are strongest of from 6×10^{17} to $1 \cdot 4 \times 10^{18}$ ergs per second. If we adopt the mean of these as giving roughly the actual dissipation, and apply the factor $4/3\pi$, we find the average dissipation in these gulfs to be about 4×10^{17} ergs per second. There is probably no important dissipation elsewhere in the Sea of Okhotsk.

5. *The Bering Sea.*

In the extreme north of the Pacific, between Siberia and Alaska, a chain of small islands, the Aleutian Islands, extends all the way across. The region north of these

Fig. 3.

has the shape of a quadrant and forms the Bering Sea. Between the islands the depth is great, and the tide of the Pacific seems to enter almost unhindered. Since the depth of more than half of the sea, mostly on the Alaskan side, is less than 40 fathoms, large currents are produced, especially in the three chief bays—the Gulf of Anadir, Norton Sound, and Bristol Bay. The dissipation must therefore be very great, but a reasonably accurate estimate of it is difficult to make on account of the form of the shallow portion, which has no narrow place that can be called an entrance. It is best to treat the main part of the sea and the bays separately.

In the south of the sea it is stated that the maximum rate of the water, when clear of the passes between the Aleutian Islands, is usually about $2\frac{1}{2}$ knots when the depth is less than 100 fathoms. In the region satisfying these conditions the depth is in most places about 80 fathoms, so that the current farther north, where the depth is often only 20 or 30 fathoms, may exceed this. On the other hand there seems to be little semi-diurnal tide in the extreme north. In Norton Sound the tide is diurnal, presumably because the waves from the south and from Bering Strait neutralize each other. At St. Lawrence Islands, near the entrance to the strait, the tide is only about a foot in height, confirming this suggestion. Farther south, however, the tidal wave from the Arctic must spread out and become inappreciable. At Pribilof Islands, 500 km. from the nearest land and surrounded by water 50 fathoms deep, the current reaches $2\frac{1}{2}$ knots. At St. Matthew Island, about midway between these and St. Lawrence Islands, in water 30 to 40 fathoms deep, and nearly as far from land, the current still reaches $2\frac{1}{2}$ knots. There are no other data given for islands far from shore, and it seems that we shall not be overestimating the dissipation if we take the maximum current to be $2\frac{1}{2}$ knots all over the shallow region bounded on the south by the Fox Islands and extending north till half-way between St. Matthew and St. Lawrence Islands. The size of this is 1,000 by 700 km., or 7×10^5 sq. km. The maximum dissipation is therefore $2{\cdot}74 \times 10^{12} \times 7 \times 10^5 \times (2{\cdot}5)^3$, or 3×10^{19} ergs per second, and the mean dissipation $1{\cdot}2 \times 10^{19}$ ergs per second.

In Bristol Bay the average velocity seems to be about 3 knots, though the observations are few. The corresponding dissipation is about $1{\cdot}5 \times 10^{18}$ ergs per second. In Norton Sound the dissipation is probably small, for it is mostly north of St. Lawrence Island, and the tide is diurnal. The Gulf of Anadir probably contributes about as much as Bristol Bay; for though its area is twice as great, its more northerly situation must reduce the current somewhat. In all, then, the average rate of dissipation in Bering Sea is about $1{\cdot}5 \times 10^{19}$ ergs per second. This estimate is of course subject to considerable error, for it depends wholly on a few observations, which may not give quite a fair sample of the whole of the sea. The depths around the localities considered are fairly typical of the sea as a whole, so that great error on this ground is not to be anticipated; but errors in observing the velocities may be greater, and both kinds of error are magnified in importance by the fact that the velocity must be cubed when the dissipation is calculated, so that if the true mean velocity were only 2 knots instead of $2\frac{1}{2}$ knots the dissipation would be almost halved. It does not appear that the velocity increases much towards the coast; in fact the velocities near the Alaskan coast seem to be rather smaller than those near the islands. Thus an underestimate on this ground is not probable.

6. *Malacca Strait.*

This is a narrow triangular area, about 800 km. in length, separating Sumatra from the Malay Peninsula. The tide of the Bay of Bengal enters at the north-west end, and gradually increases in height as it advances along the strait towards Singapore. At the south end, however, the part of the tide that has not been reflected or dissipated on the way through the strait is overwhelmed by the diurnal tide of the South China Sea. Ample observations of the tides and tidal currents on both sides are available. The currents as far south as Cape Medang (nearly due west of Malacca) seem to reach maxima of $1\frac{1}{2}$ to 3 knots, the average amplitude at springs being practically 2 knots. The area of this region is 100,000 sq. km., and the dissipation is accordingly found to be about 9×10^{17} ergs per second on an average.

An alternative determination can be made by finding the rate of inflow of energy. At Kumpei, on the Sumatran side and near the north end of the strait, it is high water, full and change, at noon, and the amplitude at springs is 120 cm. The flood tide outside the bar, in water about 20 fathoms deep, sets south-east from three hours before high water till three hours after it, so that it reaches its maximum speed of $1\frac{1}{2}$ knots at high water. At Penang, near the opposite shore, it is high water at 0h. 21m. and the current reaches its maximum velocity of $2\frac{1}{2}$ knots an hour before high water. As the strait is everywhere narrow in comparison to its length, and as these observations do not seem to have been taken on shoals, they are probably representative of that part of the strait. The amplitude of the tide at Penang is 100 cm. We can therefore take the average height of the tide along the section from Kumpei to Penang to be 110 cm., and the average current when flowing strongest to be 2 knots, reaching its maximum half an hour before high water. The average depth is about 30 fathoms. A modification must be made in the previous procedure to allow for the fact that the current flows along the strait, which is not quite at right angles to the line of the section; therefore, in finding the energy crossing the section, we must take for the length of the section, not the distance from Kumpei to Penang, but the projection of this on a line perpendicular to the strait, which is 230 km. The flux of energy is hence found to be, on an average, 7×10^{17} ergs per second. We also require the amount of this energy that emerges through the narrow part of the strait. Off Cape Medang, which marks the narrowest point of the strait away from the immediate neighbourhood of Singapore, the amplitude of the tide is 120 cm., high water occurring at 6h. 30m. The tidal current flows at an average speed of about $2\frac{1}{2}$ knots. There is no record of the tidal phenomena just opposite, but in Malacca Road it is high water at 7h. 30m., with an amplitude of 165 cm.; the current there reaches its maximum of 2 knots an hour before high water. At the eastern end of South Sands, which lies near the Malay side, north-west of Cape Medang, it is high water about 6h. 0m., and the tidal stream has a maximum speed of $1\frac{1}{2}$ knots an hour before high water. The width of the channel at Cape Medang is

36 km., and the depth 20 fathoms. The average flux of energy eastward past it is found to be $1 \cdot 0 \times 10^{17}$ ergs per second. Thus the average excess of the inflowing energy over the issuing energy is 6×10^{17} ergs per second. The work done by the moon is insignificant, for it crosses the meridian at full and change at 2h. 30m., which is nearly the average time of high water. Thus all the excess of energy just found is dissipated in the strait.

The area of the strait between the Kumpei-Penang section and the Medang section is 56,000 sq. km. If the average current in this had an amplitude of 2 knots the dissipation would be 5×10^{17} ergs per second on an average, in striking agreement with the estimate from the flux of energy, though the latter is more reliable. If in the final estimate the region north of the Kumpei-Penang section is to be included, we must add a fraction to the total to allow for it, making probably between 8×10^{17} and 12×10^{17} ergs per second in all.

In the part of the strait east of Medang there are few records of the currents, but the dissipation is probably small. In any case it could not exceed the 10^{17} ergs per second that pass Medang, and is probably less than this. The total dissipation in the Strait of Malacca is therefore $1 \cdot 1 \times 10^{18}$ ergs per second, subject to an uncertainty of a fifth of its amount.

Australian Waters.

Australia is surrounded by a belt of water less than 100 fathoms in depth; the width of this ranges from 10 to 200 miles, except at the Gulf of Carpentaria, where it extends right across to New Guinea. The tide in this neighbourhood is diurnal, like that in the South China Sea to the north of it. The contribution to the secular acceleration of the moon is accordingly very small. The tidal streams do not exceed 1 knot, and as the area is much less than that of the South China Sea the dissipation in the diurnal tide cannot be comparable with that already found for the larger sea.

African Waters.

The Mozambique Channel.

The channel between Madagascar and the mainland is mostly about 500 fathoms deep or more. There are few records of tidal currents in it; in fact the only record given in the 'African Pilot,' part 3, appears to be based on the statement of a single observer, that the tidal streams in the channel are comparable with the permanent current driven by the trade winds, which flows at about 2 knots. This cannot however be uniform all over the channel, for the following reason. The height of the tide along the African coast is about 12 feet, which is as usual measured relative to low water at ordinary springs, so that the vertical amplitude of the tide is 180 cm. Now the ordinary theory of tides in channels shows that the maximum velocity is of order $h\,(g/D)^{\frac{1}{2}}$; in a simple wave in a uniform channel it is exactly this. Taking the depth to be 90,000 cm., this makes the maximum velocity 19 cm./sec., or rather more than a third of a knot. Accordingly the currents with velocities of a knot or more must be confined to narrow coastal strips, and the dissipation is therefore small.

The only other partially enclosed regions around Africa are the Gulf of Aden and the Red Sea. The former is deep in the middle with narrow strips of shallow water on the margins, like the Mozambique Channel, and therefore the dissipation is small. The Red Sea is shallower, but can have no important currents, since the inlet at Aden is so narrow. The Mediterranean has already been dealt with. Thus the dissipation around the coasts of Africa is negligible.

North American Waters.

There are many partially enclosed bodies of water around North America, the chief of which are the Gulfs of Mexico and California, the Bay of Fundy, the Gulf of St. Lawrence, and the numerous straits and bays of the North-west Passage. Of these the Gulf of Mexico may be ruled out at once, for it is very deep and a large fraction of its entrance is blocked by Cuba. The Gulf of California is still deeper; and therefore the currents in these cannot be notable except in restricted localities.

1. *The Bay of Fundy.*

This bay requires to be considered separately in spite of its small size, for it is famous for possessing the largest tides in the world. It is fairly shallow, and the tides are much magnified in height by the diminution in both depth and width towards the head of the Bay. The currents are apparently not so great as would be expected from the height of the tides. The entrance is through the Grand Manan Channel, named after an island in it. The average current in the channel reaches about 1·8 knots, and that near St. John, half-way up the bay, reaches 1·7 knots. The area of the bay is 12,000 sq. km., so that the average dissipation for a maximum velocity of 1·8 knots all over would be $7{\cdot}7 \times 10^{16}$ ergs per second. It is likely that the currents farther up the bay are stronger, so that this must be regarded as a lower limit.

An alternative estimate may be obtained from the inflow of energy. The rise of the tide in Grand Manan Channel is at most places about 20 or 22 feet at springs, above low water ordinary springs. The amplitude is therefore about 320 cm. The current has an amplitude of 1·8 knots, and the depth of the channel is about 9000 cm. The average time of the turn of the current at three places near the south side of the channel (those numbered 14, 16 and 17, in the 'Nova Scotia and Bay of Fundy Pilot,' page 22) is 35 minutes after high water at St. John. This high water at full and change occurs at 11h. 21m., while at l'Etang, on the north side of the channel, it is at 11h. 18m., and the mean of the times at Westport, Petit Passage, and Digby Gut, which are nearly opposite, is 10h. 48m. Thus the mean time of high water in the channel must be about 11h. 3m., so that the current turns 53 minutes after the tide. The phase difference is therefore 25 degrees. The width of the channel is 83 km. Applying equation 15 of TAYLOR's paper, we find that the average rate at which energy enters is $4 \cdot 7 \times 10^{17}$ ergs per second.

The average rate at which the moon does work on the bay is $-\frac{1}{2}g \int hh'\gamma \sin \beta dS$, as was found for the Yellow Sea. In this case the latitude is 45° north, so that h' is 18 cm. In the lower half of the bay the amplitude of the tide is not greater than 12 feet, but in the upper half it rapidly increases, till in Minas Basin it reaches 25 feet and in Chignecto Bay 23 feet. The time of high water in the bay ranges from 11h. 3m. to 11h. 50m. The later value corresponds to the upper part, where the amplitude is greatest; but as this part is also the narrowest, the two times must receive about equal weights in finding the average. We therefore take the average time of high water to be 11h. 30m. The average amplitude of the tide is about 18 feet, or 540 cm. The longitude of the bay is 66° west, so that the moon crosses the meridian at full and change at 4h. 33m. The time of high water is more than 6h. 12m. later than this, so that the tide is falling when the moon is exerting its greatest upward pull, and the work done by the moon is therefore negative. The interval between transit and low water is 45 minutes, so that $\beta = 22°$. The area of the bay is $1 \cdot 2 \times 10^{14}$ sq. cm. The work done by the moon is therefore -3×10^{16} ergs per second. The total dissipation in the bay is $4 \cdot 4 \times 40^{17}$ ergs per second.

This estimate is six times as great as the earlier one based on the currents alone. It is much the more reliable, for the first depended on the assumption that the currents were equally great all the way up the bay, whereas actually they increase very much towards the head. Velocities up to 9 knots are recorded in Minas Basin, though the area in which these occur must be very restricted. The most serious source of error in the second estimate is the phase difference, for this is only an hour and would be affected to a considerable extent by an error in the determination of the time when the current turns. The observed time of turn does not vary much from place to place, however, and it does not seem likely that the estimate is wrong by more than a quarter of its amount. The second estimate will therefore be adopted. It will be noticed that it is rather less than the dissipation in the Irish Sea.

The Gulf of St. Lawrence gives very little dissipation. The narrow entrance through Cabot Strait prevents the tides from being considerable except in the estuary of the river itself and in Belle Island Strait, which separates Newfoundland from Labrador.

2. *The North-west Passage.*

The channel from the Atlantic to the Arctic between Canada and Greenland is blocked by a large number of islands of varying sizes, between which are narrow and shallow straits. The dissipation in several of these can be estimated from data in the 'Arctic Pilot,' vol. 3.

The chief of these channels is Davis Strait, with Baffin Bay to the north of it, which lies between Baffin Land and Greenland. It is about 1600 km. in length. The only tidal velocities recorded in it, except in fjords, are near Holstenborg, in Greenland, where the currents in the offing are said to reach a speed of two knots. This cannot, however, be general, for there is a shallow region off Holstenborg, some 150 km. long and 60 km. wide, with a depth of about 23 fathoms. Most of the strait is about 100 fathoms deep. This region is therefore a place where the main current of the strait is magnified by the form of the bottom, and there is no reason to believe that the current in the deep water is greater than half a knot, which is the observed velocity off the coast of Labrador. The dissipation in Davis Strait is therefore not great.

At the northern extremity of the strait there are several narrow passages into the Arctic. The dissipation in these must be important. In Smith Sound and Kennedy Channel, for instance, which separate the north-west coast of Greenland from Ellesmere Land, the current is said to be "nearer two figures than one." These straits are small in area, but if such currents exist over much of their extent we must take them into account. The data available at present are unfortunately too meagre.

The south end of Davis Strait is connected to Hudson Bay by Hudson Strait. The currents in this are described as "great enough to be dangerous," especially at the east end; but the recorded currents, even in the middle of the strait, are only about three-quarters of a knot. This makes the average dissipation about 5×10^{16} ergs per second. The danger arises mostly from drifting ice.

In the entrance to Hudson Bay the velocity increases to one and a half knots. The area over which this is true is about $3 \cdot 8 \times 10^{14}$ sq. cm., making the average dissipation $1 \cdot 5 \times 10^{17}$ ergs per second.

In Hudson Bay itself the currents are probably very small. Considerable velocities are recorded at Port Churchill, but there are no records in the middle of the bay. The depth in the middle is about 50 fathoms, which is about the same as at the entrance. The entering current must therefore spread out in the bay and undergo great diminution in strength. Near Port Churchill the depth is only 19 fathoms or less, so that the current there must be a local current magnified. The dissipation in Hudson Bay must therefore be small.

In Fox Strait, which runs northwards from the entrance to Hudson Bay, the depth is less, about 20 fathoms, and the current reaches one and a half knots. The area of this channel is 2×10^{15} sq. cm., and the appropriate average dissipation is about $1{\cdot}4 \times 10^{18}$ ergs per second. The remaining straits of the North-west Passage probably do not contribute nearly so much to the dissipation, for the energy of the entering wave must be mostly dissipated in the channels already dealt with, and partly through this and partly on account of the obstructive effect of the islands it is not likely that the straits farther north-west are very important, though this cannot be regarded as certain. The dissipation in the whole of the North-west Passage is thus about $1{\cdot}6 \times 10^{18}$ ergs per second on an average. Adding this to the amount found for the Bay of Fundy, we have for the whole of North America a total of 2×10^{18} ergs per second on an average.

Summary.

The mean rates of dissipation in the lunar semi-diurnal tide found in the foregoing investigation are as follows:—

	Ergs per Second.
European waters	$2{\cdot}4 \times 10^{18}$
Asiatic waters:	
South China Sea	Small.
Yellow Sea	$1{\cdot}1 \times 10^{18}$
Sea of Okhotsk	$0{\cdot}4 \times 10^{18}$
Bering Sea	$15{\cdot}0 \times 10^{18}$
Malacca Strait	$1{\cdot}1 \times 10^{18}$
North American Waters:	
Bay of Fundy	$0{\cdot}4 \times 10^{18}$
North-west Passage	$1{\cdot}6 \times 10^{18}$

The total thus accounted for is $2{\cdot}2 \times 10^{19}$ ergs per second. I have shown in a previous paper that the dissipation required to account for the secular acceleration of the moon (which amounts to $9''$ per century per century) is about $1{\cdot}4 \times 10^{19}$ ergs per second, so that it seems as if there is more dissipation than is required. If this was so it would be necessary to seek for a cause that could produce an appreciable secular retardation of the moon, and none such is known. A scrutiny of the results so far obtained is therefore desirable, with a view to finding out whether any of them have been over-estimated. One cause of such an over-estimate is easily seen. The data used for the Irish Sea, the English Channel, Malacca Strait and the Bay of Fundy refer definitely to spring tides alone when the currents are at a maximum. The height of the tide adopted in the calculation for the Yellow Sea was also that of the spring tide. In the other cases it is not stated whether the currents have average or spring values, but if they were determined at springs the requisite reduction is at once obtained. The theoretical ratio of the heights of the lunar and solar tides is $2{\cdot}3$ when inertial and frictional effects are neglected. This ratio is probably nearly correct in mid-ocean, for the periods of the two tides are not very different, so that inertia will affect their amplitudes in the same ratio. In shallow areas, however, the frictional force is not proportional to the velocity but to its square, and accordingly friction has more relative effect in reducing the tides when they are great than when they are smaller. The ratio of the solar to the lunar tide, as found from observations, is accordingly less than the theoretical value, since the ratio of the ranges at springs and neaps is reduced. This ratio is stated in the 'Admiralty Tide Tables for 1920' to be 1 : 2·73 on an average. If now θ be the phase of the lunar tide, let $(1-r)\theta$ be the phase of the solar tide, so that r is 1/29. Let A be the amplitude of the lunar tidal current and $A\nu$ that of the solar tidal current. The total current is

$$A\{\cos\theta+\nu\cos(1-r)\theta\} = A\,(1+2\nu\cos r\theta+\nu^2)^{\frac{1}{2}}\cos\left(\theta-\tan^{-1}\frac{\sin r\theta}{1+\nu\cos r\theta}\right)$$

which is now expressed as a simple harmonic motion with a slowly varying amplitude and period. The amplitude at springs is $A(1+\nu)$. The dissipation is proportional to the cube of the current, and therefore to the cube of the amplitude. The ratio of the mean dissipation to the dissipation at springs is therefore the average of $(1+2\nu\cos r\theta+\nu^2)^{\frac{3}{2}}/(1+\nu)^3$. If ν^6 be neglected, the numerator of this is $1+\frac{9}{4}\nu^2+\frac{9}{64}\nu^4$. Assuming that the ratio of the velocities is the same as that of the vertical ranges, we find that this fraction is equal to 0·51. Applying this correction to the spring tide dissipation, we find that the average dissipation is $1{\cdot}1\times10^{19}$ ergs per second, 80 per cent. of what is required. It would give a secular acceleration of the moon of 7″ per century per century.

The agreement between the dissipation in shallow seas and that necessary to account for the lunar secular acceleration is much closer than the data would entitle us to expect. Two-thirds of that found takes place in the Bering Sea, the estimate for which may be incorrect by half its amount. What we are entitled to assert, however, is that this dissipation is certainly enough to account for a large fraction of the secular acceleration, and that there is nothing to prove that it is incapable of accounting for the whole of it.

It is uncertain whether the dissipation in any other coastal regions is notable in comparison with those already considered. The only partly enclosed areas not treated here that are of considerable size are some of those in the North-west Passage. There is an extensive shallow region off the coast of Patagonia, but it is in no way enclosed, being perfectly open to the Atlantic. Thus it is difficult to make any reliable inference about the currents in it. Many records of tidal currents along the coast are given, some reaching several knots, but all of them seem to refer to currents up rivers or near their mouths, where the general currents must be magnified, or to currents over bars and shoals; there seem to be no data about the currents more than a few miles out to sea.

The dissipation over local shallows like shoals and bars and in narrow bays and straits has been systematically ignored in this paper, except where it has been automatically taken into account in the determination of the excess of the entering over the issuing energy. The chief reason for this is the utter impossibility of finding it. The fjords on the west coasts of Norway, Greenland, and North and South America are innumerable, and in many of them, perhaps in all, there is a strong tidal current, so that the dissipation per unit area in these places must very much exceed that in any of the areas here treated. On the other hand, the total area must be less, and it is uncertain whether the increase in velocity is enough to counterbalance the decrease in area and make the total dissipation in these places comparable with that here found for the larger shallow seas. The same is true of shoals; though the agreement between the results given by the two methods of finding the dissipation in shoaly waters, as in the Yellow Sea and the Strait of Malacca, indicates that the shoals at any rate do not contribute to the dissipation an amount overwhelmingly greater than the normal places, for one method necessarily includes the effect of the shoals and the other systematically omits it. Along the open shore again there must be some dissipation; the currents there do not usually extend many miles out to sea, but they exist along a very long stretch of coast, and the aggregate dissipation in them may be appreciable.

The hypothesis that the secular acceleration of the moon is due to dissipation of energy in the tides in shallow coastal regions therefore seems capable of satisfying all the quantitative demands on it, and it is also free from objections that have been urged against other attempted explanations.* It therefore occupies a strong position.

Appendix.

The Secular Change in the Obliquity of the Ecliptic.

In consequence of the dissipation of energy in the diurnal tides there must be a couple always acting on the earth so as to tend to resist its angular motion about an axis in the plane of the orbit of the moon or the sun, as the case may be. If δ be the declination of the moon, the angular velocity of the earth about the diameter that points to the moon is $\Omega \sin \delta$, and the angular momentum about it is $C\Omega \sin \delta$, where C is the earth's moment of inertia. Let L be the couple about this diameter. Then the rate of dissipation of energy in the diurnal tide is $L\Omega \sin \delta$. Also the rate of change in the inclination is given by

$$\frac{d}{dt}(C\Omega \sin \delta) = L.$$

Now L must contain $\Omega \sin \delta$ as a factor, since it depends for its existence on the existence of the diurnal tide, whose coefficient is proportional to $\sin \delta$, and whose speed

* *Cf.* 'Monthly Notices of R.A.S.,' vol. lxxx., 1920, pp. 309–317.

is equal to $\Omega - n$, which is sufficiently near to Ω for our present purpose. Hence the friction of the tide will have a damping effect on this component of the earth's rotation, which is not altered by the other couples acting, since none of these have a notable effect in diminishing the amplitude of the motion. If the average value of the dissipation is taken to be 5×10^{18} ergs per second, this being rather more than was found in the South China Sea, and we remember that the average of $\sin^2 \delta$ is $\frac{1}{2} \sin^2 i$, where i is the obliquity of the ecliptic, we find that the amplitude of $\Omega \sin \delta$ would be reduced to $1/e$ of its value in 2×10^9 years. This is of the same order as the probable age of the earth. If Ω remained constant this would show that the inclination of the Equator to the ecliptic would be reduced by $1''$ in about 2×10^4 years. Actually Ω is decreasing, so that if the rate were maintained it would be reduced to $1/e$ of its value in about 10^{10} years, a longer time than was found, on the assumption stated, to be enough for a similar reduction in the obliquity. Thus we can infer that the obliquity is at present diminishing, though there is no reason to believe that there has been any observable change in it in historic times. Even if there were as much dissipation in the diurnal tides as in the semi-diurnal ones this would hardly be possible.

[*Note added September 16.*—Mr. TAYLOR asks me to point out certain errata in his paper "Tidal Friction in the Irish Sea." On p. 2, $1 - \frac{\gamma}{\sqrt{r}}$ is twice written for $1 + \frac{\gamma}{\sqrt{r}}$; the correct form is used in equations (4) and (5). On p. 9, line 9, T_1 is Greenwich mean time of high water at full and change of the moon at the place considered, whereas the "establishment" is the *local* mean time of this event. In equation (16), $t + T_1$ should be $t - T_1$, and in equation (18), $t + T_0$ should be $t - T_0$. On pp. 19 and 20, $\sin^2 \phi_0$ is consistently written for $\sin 2\phi_0$; equation (33) is correct.

In this paper, as in TAYLOR's, integrals over a period are always determined as if the current velocity and the tide height varied harmonically. This could be strictly correct only if the frictional force was proportional to the velocity, which is not the case. It appears, however, that the departure from the harmonic variation is not enough to produce any great alteration in these integrals.

I wish to express my thanks to Mr. H. W. BRABY for drawing the maps in this paper.]