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TIDAL RECTIFICATION IN LATERAL VISCOUS BOUNDARY LAYERS OF A SEMI-ENCLOSED BASIN

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The rectified flow, induced by divergence of the vorticity flux in lateral oscillatory viscous boundary layers along the side-walls of a semi-enclosed basin, is studied as a function of the Strouhal number, κ, equivalent to the Reynolds number. It is shown that for small Strouhal numbers the ratio of the rectified flow and the tidal current amplitude is proportional to κ, but for larger κ values the behaviour is exponential. The latter conclusion is reached at by using a global renormalization of the vorticity equation.

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1. Introduction

In many shallow seas, where the tidal current amplitude is larger than about 0.5 ms^{-1}, it is known that if the local velocity field is averaged over one or more tidal cycles the result is unequal to zero. Many mechanisms may generate such a constant flow (Zimmerman 1981). Here we will study the effect of lateral frictional boundary layers. As shown by Yasuda (1980), in a semi-enclosed tidal basin they induce a rectified mean circulation with an intense outward flow along the boundary and a weak inward flow in the central region. The mechanism, essentially, can be understood as a divergence of the tidal averaged flux of vorticity, produced by viscous friction along the side-walls, ultimately balanced by viscous vorticity diffusion (Zimmerman 1981). The crucial parameter, on which the strength of the rectification depends, is the Reynolds number based on the ratio of longitudinal vorticity advection and lateral vorticity diffusion.

Let U be a velocity scale, L the length of the semi-enclosed basin, r the horizontal (turbulent) viscosity and σ the basic frequency of the tidal flow. Then the lateral boundary layer thickness due to oscillatory flow along the side-walls is

$$\delta = (r / \sigma)^{1/2} . \tag{1-1}$$

Hence the Reynolds number reads

$$\mathrm{Re} = \frac{U^2/L}{rU/\delta^2} = \frac{U\delta^2}{rL} = \frac{U}{\sigma L} , \tag{1-2}$$

the latter equality following from (1-1). Thus for the dynamics concerned, the Reynolds number is equivalent to the ratio of the tidal excursion $\frac{U}{\sigma}$ and the basin-length L, being the Strouhal number κ (Zimmerman 1981). Yasuda (1980) studied the rectified flow for small Strouhal numbers and found that in that case the ratio of the rectified current to the tidal velocity amplitude depends linearly on κ. In another context however, viz. the generation of rectified flow over varying bottom topography, where also the Strouhal number is the crucial parameter, it has been shown (Zimmerman 1978, 1980, see also Huthnance 1981) that extrapolating results for small Strouhal numbers to $O(1)$ or larger can be quite misleading. It appeared that for Strouhal numbers much larger than 1 the rectified current, relative to the tidal velocity amplitude, becomes inversely proportional to the Strouhal number, implying a resonance peak for moderate Strouhal numbers. One of the motivations for the present paper,therefore, is to see whether an extrapolation of Yasuda's (1980) results to large Strouhal numbers also shows qualitative deviations from the results for the small parameter regime.

To reach for that we start from the shallow water equations for a homogeneous fluid with corresponding boundary conditions, properly scaled in chapter 2. In passing we note that after scaling it appears that Yasuda's (1980) solution is incomplete in that an additional rectification mechanism, of the same strength as the one discussed by him, cannot be neglected, viz. lateral vorticity advection. We therefore recalculate the rectified current velocity field in chapter 3.

In analyzing the model for larger Strouhal numbers we note a problem, viz. that in contrast to the topographical rectification mechanism mentioned above, the present mechanism is a locally strong non-linear interaction as the velocity perturbations induced by viscous friction in the side-wall boundary layers are necessarily of the same order as the undisturbed velocity in the region outside the side-walls. For large Strouhal numbers the primitive perturbation procedure of Yasuda (1980) breaks down. In order to overcome this difficulty we have devised in chapter 4 a global renormalization procedure, making use of the fact that the global (i.e. laterally integrated) vorticity flux is independent of the detailed structure of the boundary layer. This procedure enables us to deal with the strength of the rectified flow for large Strouhal numbers, albeit at the expense of not being able to reproduce exactly the detailed lateral structure of the rectified flow. With those provisos the conclusion is that for large Strouhal numbers the ratio of the rectified current to the tidal velocity amplitude depends, in an asymptotic sense, exponentially on the Strouhal number rather than linearly. This result is derived in chapter 5. Thus our conclusion is that the large Strouhal number solution noticeably deviates from the asymptotic solution, although we admit that the means to arrive at that conclusion has the character of a "brute force."

2. Scaling of the basic equations.

We consider a semi-enclosed basin having a uniform equilibrium depth, H, a length L and a width $2B$. The shallow water equations of motion for a rotating homogeneous fluid read

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial \zeta}{\partial x} + r\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right], \tag{2-1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial \zeta}{\partial y} + r\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right]. \tag{2-2}$$

The continuity equation is given by

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x}\left[(H+\zeta)u\right] + \frac{\partial}{\partial y}\left[(H+\zeta)v\right] = 0. \tag{2-3}$$

Here u and v are the horizontal components of the velocity vector, assumed to be vertically uniform as we have left out vertical turbulent momentum transfer. From Yasuda (1980) it is clear that the inclusion of the latter process does not add anything of importance to the dynamics we are concerned with here, which is mainly the generation of the vertical vorticity component by side-wall friction, represented by the (turbulent) viscosity coefficient r in the right hand side of (2-1) - (2-2). Furthermore f is the coriolisparameter, g the acceleration due to gravity and ζ height of the surface of the fluid above the reference level.

In looking for the dimensionless form of (2-1) - (2-3) we scale x and y by L, t by σ^{-1} (σ the tidal frequency), u and v with a velocity scale U (the velocity amplitude in the middle of the open boundary, say), whereas the continuity equation suggests to scale ζ with $UH/\sigma L$. We a priori assume that the ratios $\frac{B}{L}$ and $\frac{\sigma}{f}$ are of the order 1. Defining the following nondimensional parameters;

$$\left.\begin{aligned}
&\kappa \text{ (Strouhal number)} = \frac{U}{\sigma L},\\
&F \text{ (Froude number)} = \frac{U}{\sqrt{gH}},\\
&\lambda = \frac{\text{basin-length}}{2\pi\cdot\text{wave-length}} = \frac{\sigma L}{\sqrt{gH}} = F\kappa^{-1},\\
&E = \frac{\text{viscous boundary layer width}}{\text{basin-length}} = \frac{\delta}{L},
\end{aligned}\right\} \tag{2-4}$$

the equations of motion and the continuity equation read

$$\frac{\partial u}{\partial t} + \kappa\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] - \frac{f}{\sigma}v = -\frac{1}{\lambda^2}\frac{\partial \zeta}{\partial x} + E^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right], \tag{2-5}$$

$$\frac{\partial v}{\partial t} + \kappa\left[u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right] + \frac{f}{\sigma}u = -\frac{1}{\lambda^2}\frac{\partial \zeta}{\partial y} + E^2\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right], \tag{2-6}$$

$$\frac{\partial \zeta}{\partial t} + \kappa\left[u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y}\right] = -(1+\kappa\zeta)\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right]. \tag{2-7}$$

All variables have to be understood as being dimensionless and scaled according to the scheme given above. We search for nontransient solutions, satisfying the boundary conditions

$$\left.\begin{aligned}
&u = \sin t \quad \text{at } x = 0\\
&u = 0 \quad \text{at } x = 1,\ y = 0 \text{ and } y = \frac{2B}{L},\\
&v = 0 \quad \text{at } x = 0, x = 1, y = 0 \text{ and } y = \frac{2B}{L}.
\end{aligned}\right\} \tag{2-8}$$

From hereon we shall assume that

$$E \ll 1\,, \tag{2-9}$$

giving rise to a singular perturbation problem as E multiplies the highest order derivatives in (2-5) - (2-7). Furthermore we consider basins having a characteristic length scale which is much smaller than the tidal wave-length. From (2-4) it then follows

$$\lambda \ll 1\,, \tag{2-10}$$

and as a consequence rotation effects will not be of importance.

After substitution of the regular expansions

$$\left.\begin{aligned} u(x,y,t) &= U_0(x,y,t) + EU_1(x,y,t) + \ldots.., \\ v(x,y,t) &= V_0(x,y,t) + EV_1(x,y,t) + \ldots.., \\ \zeta(x,y,t) &= Z_0(x,y,t) + EZ_1(x,y,t) + \ldots.., \end{aligned}\right\} \tag{2-11}$$

it follows that the zeroth order momentum equations can be linearized for any Strouhal number κ, since

$$\kappa\lambda^2 = \lambda F \ll 1\,, \tag{2-12}$$

the estimate following from (2-10) and the assumption that the Froude number is small in order to prevent breaking tidal waves.

Evidently the zeroth order equation in the regular expansion read

$$\left.\begin{aligned} \frac{\partial Z_0}{\partial x} &= 0\,, \\ \frac{\partial Z_0}{\partial y} &= 0\,, \\ \frac{\partial Z_0}{\partial t} &= -(1+\kappa Z_0)\left[\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial y}\right]. \end{aligned}\right\} \tag{2-13}$$

Note that we write capitals for the regular expansions. The solution of (2-13), subject to the slip boundary conditions

$$\left.\begin{aligned} &U_0 = sint \quad \text{at } x = 0\,, \\ &U_0 = 0 \quad \text{at } x = 1\,, \\ &V_0 = 0 \quad \text{at } y = 0 \text{ and } y = \frac{2B}{L}, \end{aligned}\right\} \tag{2-14}$$

reads

$$\left.\begin{aligned} &U_0 = (1-x)sint \quad ; \quad V_0 = 0 \quad ; \\ &Z_0 = \frac{-1+\exp(-\kappa cost)}{\kappa}. \end{aligned}\right\} \tag{2-15}$$

This is the well-known expression for a standing shallow water gravity wave, which is valid under the conditions (2-9) and (2-10). In the same way the first order regular system can be solved. For the lateral velocity component we obtain

$$V_1 = 0\,, \tag{2-16}$$

which will be used later on.

As the solution (2-14) does not include the viscous side-wall layers, necessary to bring the tangential velocity components to zero along the walls, we have to correct the velocity field near the side-walls by introducing boundary layers. Let

$$y' = \frac{L}{\delta} y \tag{2-17}$$

be a lateral stretched coordinate near the boundary $y = 0$, based on the already dimensionless coordinate y (scaled by L), and assume that we have to rescale the lateral velocity component by $\frac{\delta}{L}U$, as suggested by mass balance in the viscous boundary layer. Then we have the expansions

$$\left.\begin{aligned} u &= U_0(x,t) + u_0(x,y',t) + E\{U_1(x,y,t) + u_1(x,y',t)\} + ..., \\ v &= Ev' = E \qquad v_1(x,y',t) + ..., \\ \zeta &= Z_0(t) + \zeta_0(y',t) + E\{Z_1(x,y,t) + \zeta_1(x,y',t)\} + ..., \end{aligned}\right\} \tag{2-18}$$

where capitals refer to the variables in the regular expansion and small characters to the boundary layer corrections. These series should be substituted in the rescaled equations of motion:

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} + \kappa\left[u\frac{\partial u}{\partial x} + v'\frac{\partial u}{\partial y'}\right] - E\frac{f}{\sigma}v' = -\frac{1}{\lambda^2}\frac{\partial \zeta}{\partial x'} + E^2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y'^2}, \\ &E^2\{\frac{\partial v'}{\partial t} + \kappa\left[u\frac{\partial v'}{\partial x} + v'\frac{\partial v'}{\partial y'}\right]\} + E\frac{f}{\sigma}u = -\frac{1}{\lambda^2}\frac{\partial \zeta}{\partial y'} + E^2\{E^2\frac{\partial^2 v'}{\partial x^2} + \frac{\partial^2 v'}{\partial y'^2}\}, \\ &\frac{\partial \zeta}{\partial t} + \kappa\left[u\frac{\partial \zeta}{\partial x} + v'\frac{\partial \zeta}{\partial y'}\right] = -(1+\kappa\zeta)\left[\frac{\partial u}{\partial x} + \frac{\partial v'}{\partial y'}\right]. \end{aligned}\right\} \tag{2-19}$$

To zeroth order in E we find, after some manipulations and use of the zeroth order regular equations (2-13), that

$$\frac{\partial u_0}{\partial x} + \frac{\partial v_1}{\partial y'} = 0, \tag{2-20}$$

showing that to this approximation the rectified velocity field is free of divergence, hence $\zeta_0(y',t) = 0$. Thus a streamfunction ψ may be introduced, such that

$$u_0 = -\frac{\partial \psi}{\partial y'} \quad ; \quad v_1 = \frac{\partial \psi}{\partial x}. \tag{2-21}$$

and the dynamics will be governed by a vorticity equation.

First we note that the inviscid regular field (2-15) is free of rotation. Obviously vorticity arises only by the presence of frictional boundary layers. Its dimensionless form (scaled by U/δ) reads

$$w = E^2\frac{\partial v'}{\partial x} - \frac{\partial u}{\partial y'}. \tag{2-22}$$

From the rescaled momentum equation a vorticity equation can be derived. Substitution of the expansions (2-18) gives in zeroth order

$$\frac{\partial w_0}{\partial t} + \kappa\left[U_0 + u_0\right]\frac{\partial w_0}{\partial x} + \kappa v_1\frac{\partial w_0}{\partial y'} + \ + \kappa w_0\frac{\partial U_0}{\partial x} = \frac{\partial^2 w_0}{\partial y'^2}, \tag{2-23}$$

where use have been made of (2-16) and (2-20). Writing w_0 in terms of the streamfunction by means of (2-21) and (2-22) and substituting (2-15) for U_0, we finally obtain

$$\frac{\partial}{\partial t}\frac{\partial^2\psi}{\partial y'^2} + \kappa(1-x)\,sint\,\frac{\partial}{\partial x}\frac{\partial^2\psi}{\partial y'^2} + \kappa\,J(\psi,\frac{\partial^2\psi}{\partial y'^2}) - \kappa\,sint\frac{\partial^2\psi}{\partial y'^2} = \frac{\partial^4\psi}{\partial y'^4}. \tag{2-24}$$

The Jacobian J has its usual meaning. It describes the advection of vorticity by the boundary layer flow and is the cause of the principal nonlinearity of the vorticity equation.

In order to have the problem of solving (2-24) fully posed we finally introduce the following boundary conditions:

$$\left.\begin{aligned} &\psi = 0 && \text{at } y' = 0, \\ &\frac{\partial \psi}{\partial y'} = (1-x)\sin t && \text{at } y' = 0, \\ &\psi = 0 && \text{at } y' = \frac{B}{\delta} = b, \\ &\frac{\partial^2 \psi}{\partial y'^2} = 0 && \text{at } y' = \frac{B}{\delta} = b. \end{aligned}\right\} \tag{2-25}$$

The first one is an obvious choice. The second one is in fact the no-slip condition as the regular velocity U_0 and the boundary layer correction together must vanish at the side-wall. The third and fourth condition naturally arise from the symmetry of the flow about the mid basin-axis at $y' = b$.

3. Residual flow for small Strouhal numbers.

Obviously equation (2-24) describes the generation of vorticity by the no-slip conditions (2-25) at the side-walls. Since the latter are periodic in time, the resulting streamfunction will also have a periodic character, but, as can be seen in (2-24), all terms proportional to κ may produce higher harmonics as well as a rectified time-independent solution. For small κ the latter can be obtained in an approximative way by expanding the solution in κ :

$$\psi = \psi_0 + \kappa\, \psi_1 + \kappa^2 \psi_2 + \ldots\ldots . \tag{3-1}$$

To zeroth order we then have

$$\frac{\partial}{\partial t} \frac{\partial^2 \psi_0}{\partial y'^2} - \frac{\partial^4 \psi_0}{\partial y'^4} = 0\,, \tag{3-2}$$

subject to the same boundary conditions for ψ_0 as for ψ in (2-25). The solution is straightforward as in fact (3-2) describes a nondimensional diffusion of vorticity with periodic boundary conditions. We find

$$\psi_0(x,y',t) = (1-x)\, \phi_0(y',t)\,, \tag{3-3}$$

where

$$\left.\begin{aligned} &\phi_0 = \hat{\phi}_0 e^{it} + \hat{\phi}_0{}^* e^{-it} \;; \\ &\hat{\phi}_0 = \frac{-\sinh\,[\sqrt{i}\,(b-y')] + (1-\frac{y'}{b})\sinh(\sqrt{i}\,b)}{2i\,\{\sqrt{i}\,\cosh\,(\sqrt{i}\;b) - \frac{1}{b}\sinh\,(\sqrt{i}\;b)\}}\,, \end{aligned}\right\} \tag{3-4}$$

and the asterix denotes complex conjugation.

Rectification now arises to first order in κ. To this order the streamfunction obeys

$$\frac{\partial}{\partial t} \frac{\partial^2 \psi_1}{\partial y'^2} - \frac{\partial^4 \psi_1}{\partial y'^4} = -\{(1-x)\sin t\, \frac{\partial}{\partial x} \frac{\partial^2 \psi_0}{\partial y'^2} + J\,(\psi_0, \frac{\partial^2 \psi_0}{\partial y'^2}) - \sin t\, \frac{\partial^2 \psi_0}{\partial y'^2}\}\,, \tag{3-5}$$

subject to the first-, third- and fourth boundary condition in (2-24) for ψ_0, as well as

$$\frac{\partial \psi_1}{\partial y'} = 0 \quad at \quad y' = 0, \tag{3-6}$$

as ψ_0 already satisfies the second boundary condition of (2-25).

The time-independent, rectified, part of the solution of (3-5) can be obtained by applying a time-averaging operator to (3-5). Let this operator be denoted by a bar :

$$\overline{(\,.\,)} = \frac{1}{2\pi} \int_0^{2\pi} (\,.\,) dt. \tag{3-7}$$

Then (3-5) reads

$$\frac{\partial^4 \overline{\psi}_1}{\partial y'^4} = (1-x)\, \overline{sint \frac{\partial}{\partial x} \frac{\partial^2 \psi_0}{\partial y'^2}} + \overline{J(\psi_0, \frac{\partial^2 \psi_0}{\partial y'^2})} - \overline{sint \frac{\partial^2 \psi_0}{\partial y'^2}}\,. \tag{3-8}$$

Substituting (3-3) and (3-4) in (3-8), performing the time averaging, integrating over y' and using the boundary conditions, we find a rather complicated solution of (3-8), involving hyperbolic- and trigoniometric functions of y'. However, this can be simplified if it is assumed that the width of the bay is sufficiently greater than the viscous boundary layer width. Then the result is

$$\overline{\psi}_1 = (1-x)\, \overline{\phi}_1\,, \tag{3-9}$$

where

$$\begin{aligned} \overline{\phi}_1 = \; & C_1(1-\frac{y'}{b})^3 + C_2(1-\frac{y'}{b}) + \sqrt{2}\, \{-\tfrac{1}{8}e^{-\sqrt{2}y'} + \\ & -\tfrac{1}{2}\left[\sin(\tfrac{1}{2}\sqrt{2}\,y') + \cos(\tfrac{1}{2}\sqrt{2}\,y')\right] e^{-\frac{1}{2}\sqrt{2}\,y'} + \\ & -\frac{3}{2\sqrt{2}b} \sin(\tfrac{1}{2}\sqrt{2}\,y')e^{-\frac{1}{2}\sqrt{2}y'} + \tfrac{1}{4}(1-\frac{y'}{b})\left[\sin(\tfrac{1}{2}\sqrt{2}\,y') - \cos(\tfrac{1}{2}\sqrt{2}\,y')\right]e^{-\frac{1}{2}\sqrt{2}\,y'}\}, \end{aligned} \tag{3-10}$$

and

$$C_1 = \frac{6b - 11\sqrt{2}}{16} \quad ; \quad C_2 = \frac{25\sqrt{2} - 6b}{16}\,. \tag{3-11}$$

The corresponding residual current is given by

$$\begin{aligned} \overline{u} = \kappa(1-x)\, \{ & \frac{3C_1}{b}(1-\frac{y'}{b})^2 + \frac{C_2}{b} - \tfrac{1}{4}e^{-\sqrt{2}y'} - \sin(\tfrac{1}{2}\sqrt{2}y')e^{-\frac{1}{2}\sqrt{2}y'} + \\ & + \frac{1}{\sqrt{2}b}\left[\cos(\tfrac{1}{2}\sqrt{2}\,y') - \sin(\tfrac{1}{2}\sqrt{2}y')\right] e^{-\frac{1}{2}\sqrt{2}\,y'} + \\ & -\tfrac{1}{2}(1-\frac{y'}{b}) \cos(\tfrac{1}{2}\sqrt{2}\,y')e^{-\frac{1}{2}\sqrt{2}\,y'}\}. \end{aligned} \tag{3-12}$$

It appears that as soon as b is larger than about 3,25 the difference between the approximative- and exact solution is less than 1 %.

Obviously the intensity of the residual current is proportional to the Strouhal number as long as κ is small, and for that matter the residual current velocity in dimensional form is proportional to the square of the undisturbed tidal velocity amplitude, a result already derived by Yasuda (1980). However there is a qualitative disagreement between our solution and Yasuda's, viz. that an additional term is present in (3-12). This can be traced back to the basic equation used here, viz. equation (2-24), and to the one used by Yasuda (1980). It appears that the latter author only takes the longitudinal advection of vorticity into account. However, as our scaling shows, lateral vorticity advection in the boundary layer is of the same order as the former term, and thus has to be taken into account.

In figure 1 the lateral profiles of both the solution (3-12) and the Yasuda solution are shown for $b = 15$ (a characteristic value for tidal basins). They have the same qualitative behaviour, viz. an outward flux near the side-wall, an inward flux just outside the boundary layer and again an outflux in the central region. The difference between them is entirely due to the lateral advection term. Including this contribution makes that vorticity is advected laterally over the bay, obviously resulting in a weaker outflux in the boundary layer and stronger fluxes in the central region.

4. Global renormalization

The result of the chapter before, showing that the residual velocity is proportional to the Strouhal number, is only valid for small Strouhal numbers, for which the perturbation series (3-1) applies. For arbitrary values of κ it is not possible to solve the basic problem (2-24) - (2-25) exactly. This is mainly due to the full nonlinear contributions, represented by the Jacobian in (2-24). We may expect to find an approximative solution of the problem, provided that this term were to vanish or could be neglected in the first instance altogether. Although the first is not to be expected, it is the purpose of this chapter to provide a formal scheme for the exploitation of the second possibility.

Basically we wish to simplify the complicated equation (2-24) by means of physical constraints, such that the resulting equation can be solved. In this case the motivation to get rid of the nonlinear advection term follows from consideration of the laterally integrated vorticity advection due to the boundary flow, i.e.

$$\int_0^b J(\psi, \frac{\partial^2\psi}{\partial y'^2})\, dy' = -\int_0^b \{u_0 \frac{\partial^2 u_0}{\partial x \partial y'} + v_1 \frac{\partial^2 u_0}{\partial y'^2}\} dy' , \tag{4-1}$$

the latter equality following from (2-21). By means of partial integration and use of the boundary conditions, it follows that

$$\int_0^b J(\psi, \frac{\partial^2\psi}{\partial y'^2})\, dy' = 0. \tag{4-2}$$

Thus on a global scale there is only advection of vorticity due to the inviscid regular velocity field. This suggests that if we consider the global (i.e. laterally integrated) vorticity balance we may be able to approximate the total, nonlinear advection by introducing a renormalized y'-independent time-varying longitudinal velocity.

In order to set up our renormalization scheme we introduce a formal expansion parameter ϵ and a renormalized advection velocity amplitude $\tilde{U}$ for the outer velocity, so that (2-24) is recasted as

$$\frac{\partial}{\partial t}\frac{\partial^2\psi}{\partial y'^2} + \kappa\,\{\tilde{U}(1-x)\, sint\, \frac{\partial}{\partial x}\frac{\partial^2\psi}{\partial y'^2} + \epsilon\, J(\psi, \frac{\partial^2\psi}{\partial y'^2}) - sint\, \frac{\partial^2\psi}{\partial y'^2}\} = \frac{\partial^4\psi}{\partial y'^4} , \tag{4-3}$$

where now

$$\left.\begin{aligned} \tilde{U} &= \tilde{U}_0 + \epsilon\tilde{U}_1 + \epsilon^2\tilde{U}_2 + \ldots\ldots \equiv 1 , \\ \psi &= \psi_0 + \epsilon\,\psi_1 + \epsilon^2\,\psi_2 + \ldots\ldots . \end{aligned}\right\} \tag{4-4}$$

Of course the variables $\psi_1, \psi_2, \ldots\ldots$ are different from those in expansion (3-1) of the streamfunction for small Strouhal numbers. Note that if we put $\epsilon = 1$ our fully nonlinear basic equation (2-24) appears, whereas for $\epsilon = 0$ its linearized form arises. Now to zeroth order in ϵ we have

$$\frac{\partial}{\partial t}\frac{\partial^2\psi_0}{\partial y'^2} + \kappa\left[\tilde{U}_0\,(1-x)\, sint\, \frac{\partial}{\partial x}\frac{\partial^2\psi_0}{\partial y'^2} - sint\, \frac{\partial^2\psi_0}{\partial y'^2}\right] - \frac{\partial^4\psi_0}{\partial y'^4} = 0 , \tag{4-5}$$

which contains an as yet unknown velocity amplitude $\tilde{U}_0$. To first order in ϵ we have

$$\begin{aligned} &\frac{\partial}{\partial t}\frac{\partial^2\psi_1}{\partial y'^2} + \kappa\left[\tilde{U}_0(1-x)\, sint\, \frac{\partial}{\partial x}\frac{\partial^2\psi_1}{\partial y'^2} - sint\, \frac{\partial^2\psi_1}{\partial y'^2}\right] - \frac{\partial^4\psi_1}{\partial y'^4} = \\ &\qquad = -\kappa\left[\tilde{U}_1\,(1-x)\, sint\, \frac{\partial}{\partial x}\frac{\partial^2\psi_0}{\partial y'^2} + J\,(\psi_0, \frac{\partial^2\psi_0}{\partial y'^2})\right]. \end{aligned} \tag{4-6}$$

If we now truncate our renormalized expansion at this lowest nontrival order and set $\epsilon = 1$, (4-4) gives

$$\tilde{U}_1 = 1 - \tilde{U}_0 , \tag{4-7}$$

which may be substituted in (4-6).

In order now to solve for $\tilde{U}_0$ we introduce a global renormalization condition, stating that the right hand side of equation (4-6) should vanish after integration over the half width of the basin. Application of (4-1) and substitution of (4-7) gives

$$(1-\tilde{U}_0)\,(1-x)\,sint\,\frac{\partial}{\partial x}\int_0^b \frac{\partial^2\psi_0}{\partial y'^2}\,dy' = 0\,, \tag{4-8}$$

which is satisfied for

$$\tilde{U}_0 = 1. \tag{4-9}$$

Hence if we neglect the details of the boundary layer profile, controlled by the right hand side of (4-6), we can investigate the global character of the rectified flow as a function of κ by solving (4-5) with $\tilde{U}_0 = 1$, which is equivalent to solving (2-24) for arbitrary κ, neglecting the Jacobian term. Physically this means that, for all values of κ, we can investigate the global vorticity influx, necessary to produce a residual circulation cell in each of the half-width sections of the basin. The overall strength of the circulation in these cells is clearly independent of the detailed structure of the lateral velocity profile.

5. Approximate solution for arbitrary Strouhal numbers.

We now look for a solution of

$$\frac{\partial}{\partial t}\frac{\partial^2\psi}{\partial y'^2} + \kappa\left[(1-x)\,sint\,\frac{\partial}{\partial x}\frac{\partial^2\psi}{\partial y'^2} - sint\,\frac{\partial^2\psi}{\partial y'^2}\right] = \frac{\partial^4\psi}{\partial y'^4}\,, \tag{5-1}$$

subject to the boundary conditions (2-25). Note that (5-1) is the zeroth order equation in the renormalized perturbation procedure, but that we have dropped the subscript. Furtheron the subscript will be used for denoting a harmonic order.

We look again for a solution of the form

$$\psi(x,y',t) = (1-x)\,\phi\,(y',t)\,. \tag{5-2}$$

Substitution in (5-1) gives

$$\frac{\partial}{\partial t}\frac{\partial^2\phi}{\partial y'^2} - 2\kappa\,sint\,\frac{\partial^2\phi}{\partial y'^2} = \frac{\partial^4\phi}{\partial y'^4}\,. \tag{5-3}$$

This can be transformed into a diffusion equation by setting

$$\phi(y',t) = \chi(y',t).\,\exp\,(-2\kappa\,cost). \tag{5-4}$$

so that χ obeys

$$\frac{\partial}{\partial t}\frac{\partial^4\chi}{\partial y'^2} - \frac{\partial^4\chi}{\partial y'^4} = 0\,, \tag{5-5}$$

subject to the first-, third- and fourth boundary condition in (2-24), as for ψ, but with a modified no-slip condition at the wall:

$$\frac{\partial\chi}{\partial y'} = sint\,.\,\exp\,(2\kappa cost)\text{ at }y' = 0\,. \tag{5-6}$$

Evidently we again have a linear vorticity diffusion equation driven by a periodic boundary condition. But in contrast to the fully linearized equation (3-2) and its boundary conditions, the vorticity advection due to the prescribed inviscid regular velocity field gives after transformation an advectionless diffusion equation driven by all harmonics of the basic tidal frequency, as can be seen from (5-6) when the right hand side is expanded as a Fourier series:

$$\frac{\partial\chi}{\partial y'} = sint\sum_{m=-\infty}^{\infty} I_m(2\kappa)e^{imt} =$$

$$= \sum_{m=-\infty}^{\infty}\frac{1}{2i}\left[I_{m-1}(2\kappa)-I_{m+1}(2\kappa)\right]e^{imt}\quad at\quad y' = 0\,, \tag{5-7}$$

where I_m denotes a m'th order modified Bessel function of the first kind. This suggests expanding χ as a Fourier series as well:

$$\chi = \sum_{m=-\infty}^{\infty} \chi_m(y')\, e^{imt} \;, \tag{5-8}$$

which gives after substitution in (5-5) :

$$im \frac{d^2\chi_m}{dy'^2} - \frac{d^4\chi_m}{dy'^4} = 0 \,. \tag{5-9}$$

The general solution of (5-9) for $m = 0$ reads

$$\chi_0 = A_0 y'^3 + B_0 y'^2 + C_0 y' + D_0 \,, \tag{5-10}$$

and for $m \neq 0$:

$$\chi_m = A_m\, e^{\sqrt{im}y'} + B_m e^{-\sqrt{im}y'} + C_m\, y' + D_m. \tag{5-11}$$

The boundary conditions now read

$$\left.\begin{aligned}
\chi_m &= 0 && \text{at } y' = 0 \;; \\
\frac{d\chi_m}{dy'} &= \frac{1}{2i}\,\{I_{m-1}(2\kappa) - I_{m+1}(2\kappa)\} && \text{at } y' = 0 \;; \\
\chi_m &= 0 && \text{at } y' = b \;; \\
\frac{d^2\chi_m}{dy'^2} &= 0 && \text{at } y' = b \;.
\end{aligned}\right\} \tag{5-12}$$

For $m = 0$ only the trival solution $\chi_0 = 0$ obeys (5-12). Thus we are left with solving for $\chi_m (m \neq 0)$. The solution that satisfies (5-12) reads

$$\chi_m = \frac{I_{m-1}(2\kappa) - I_{m+1}(2\kappa)}{2i\,\{\sqrt{im}\,\cosh(\sqrt{im}\; b) - \frac{1}{b}\sinh(\sqrt{im}\; b)\}} \times$$
$$\times \left\{ -\sinh[\sqrt{im}\;(b - y')] + (1 - \frac{y'}{b}) \sinh(\sqrt{im}\; b) \right\}. \tag{5-13}$$

By (5-8), transforming back by (5-4), our final result reads

$$\phi(y',t) = \sum_{p=-\infty}^{\infty} \left\{ \sum_{m=-\infty}^{\infty} (-1)^m\, I_m\,(2\kappa)\, \chi_{m+p}\,(y') \right\} e^{ipt} \;, \tag{5-14}$$

where the quantity between brackets is the y'- dependent Fourier component of the p'^{th} harmonic mode.

We first consider the basic tidal frequency. This can be found by adding the modes $p = -1$ and $p = 1$ in (5-14). By application of (5-13) it can be shown that the corresponding streamfunction is given by

$$\psi^{(1)} = (1 - x) \left\{ \phi_1 e^{it} + \phi_1^* e^{-it} \right\}, \tag{5-15}$$

where

$$\phi_1 = \sum_{m=-\infty}^{\infty} (-1)^{m-1}\, I_{m-1}(2\kappa).\left\{ \frac{I_{m-1}(2\kappa) - I_{m+1}(2\kappa)}{2i[\sqrt{im}\,\cosh(\sqrt{im}\; b) - \frac{1}{b}\sinh(\sqrt{im}\; b)]} \right\} \times$$

$$\times\{ -\sinh[\sqrt{im}\,(b-y')] + (1-\frac{y'}{b})\sinh(\sqrt{im}\;b)\}. \qquad (5\text{-}16)$$

Use of the definition of the modified Bessel functions

$$I_m(z) = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+m+1)} (\frac{z}{2})^{2n+m} \qquad (5\text{-}17)$$

shows that in the small κ limit (5-15) - (5-16) reduces to the solution (3-4), obtained by a primitive perturbation procedure. This is to be expected, since in this limit both ψ in (3-1) and $\psi^{(1)}$ in (5-15) are solutions to the same problem, viz. a diffusion equation with periodic boundary conditions. Therefore (5-15) may be conceived as a renormalized extention of the primitive perturbation solution.

The residual current profile can be obtained from (5-13) by differentiation of the zeroth order streamfunction Fourier mode. Writing out the complex functions we again find a rather complicated solution, which may be simplified for large values of b. The result is

$$u_0 = (1-x) \sum_{m=1}^{\infty} (-1)^m\, I_m(2\kappa).\{I_{m-1}(2\kappa) - I_{m+1}(2\kappa)\} \times \qquad (5\text{-}18)$$

$$\times\left\{ \sin(\tfrac{1}{2}\sqrt{2m}\;y')\, e^{-\frac{1}{2}\sqrt{2m}\,y'} - \frac{1}{\sqrt{2m}\;b}\left[\sin(\tfrac{1}{2}\sqrt{2m}y') - \cos(\tfrac{1}{2}\sqrt{2m}y')\right] e^{-\frac{1}{2}\sqrt{2m}y'} - \frac{1}{\sqrt{2m}b} \right\}.$$

The asymptotic form for small Strouhal numbers reads

$$u_0 = \kappa(1-x)\left\{ -\sin(\tfrac{1}{2}\sqrt{2}y')\, e^{-\frac{1}{2}\sqrt{2}y'} + \frac{1}{\sqrt{2}b}\left[\sin(\tfrac{1}{2}\sqrt{2}y') - \cos(\tfrac{1}{2}\sqrt{2}y')\right] e^{-\frac{1}{2}\sqrt{2}y'} + \frac{1}{\sqrt{2}b}\right\}. \qquad (5\text{-}19)$$

In the next chapter this result will be compared with the exact solution for small κ.

6. Summary / Conclusions

The dynamics of tidal rectification due to lateral viscous boundary layers in a rectangular basin are governed by the vorticity equation (2-24) with boundary conditions (2-25), as long as the parameter E, defined in (2-4), is small. For small Strouhal numbers a solution can be constructed as a power series in κ. The solution for the basic tidal frequency and the residual current profile are given in (3-3) and (3-12) respectively. For larger values of the Strouhal number no general solution of (2-24) can be found, due to the full nonlinear terms which describe vorticity advection by the viscous correction field. However, as argued in chapter 4, these contributions may be neglected on a global scale, resulting in the renormalized vorticity equation (5-1). Its solutions for the first harmonic and for the residual current profile are presented in (5-15) - (5-16) and (5-18).

We now compare the renormalized solution in the small κ limit with the exact solution. In figure 2 the lateral profile of both the primitive- and renormalized residual current in this limit are plotted for $b=15$, showing differences in their detailed structure. This can be understood from the governing vorticity equations. In fact the renormalized model is a gross simplification of the basic equation (2-24), vorticity now only being advected by the regular inviscid velocity field without lateral structure. But in the full model this is counteracted by the longitudinal advection due to the viscous correction field, which is locally of the same order in the boundary layer. The situation becomes even more complicated due to the lateral vorticity advection; see also figure 1 and the discussion in chapter 3.

As a consequence both solutions show an outward flux near the side-wall, but the renormalized flux is stronger. Furthermore, since the laterally integrated mass transport must be zero, the outward flux has to be compensated outside the boundary layer. In this region the renormalized solution shows a weak inward flux and a residual current tending to zero at $y' = b$ for large b. Actually however, the neglected advective contributions cause an inward flux just outside the boundary layer and again an outward flux towards the central axis of the basin, with a nonvanishing residual current at the axis, even if b becomes large.

However, apart from the detailed differences, both solutions show the same qualitative behaviour. As far as the peak of the outward residual current velocity is concerned, both in its position and the order of its strength, we may trust the renormalized solution as a first approximation. Actually the result can in principle be improved on by adding the first order solution of the renormalized expansion. The procedure, however, is so cumbersome that we have not attempted to pursue that path here.

In order to study the behaviour of the residual current for large values of the Strouhal number we consider (5-18) as a function of κ, since (3-12) must break down. An exact asymptotic result for $\kappa \rightarrow \infty$ is difficult to obtain because of the summation over m. But the well-known asymptotic property of the modified Bessel functions of the first kind,

$$I_m(z) \rightarrow \frac{e^z}{\sqrt{z}} \quad ; \; z \rightarrow \infty \quad , \tag{6-1}$$

suggests an exponential rather than a linear dependency. This is indeed confirmed by figure 3, where we have plotted the ratio of the residual current to the tidal current amplitude at a fixed position ($x = 0.5, y' = b / 40$, where $b = 15$), where an inward velocity is present, for both the primitive perturbation solution and the renormalized solution as a function of the Strouhal number. The linear versus the exponential character of the solutions is immediately clear.

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1 Figure 1. Lateral profiles of the nondimensional longitudinal residual current solution (3-12) (solid line), and the Yasuda solution (dashed line); $b = 15$.

2 Figure 2. Lateral profiles of the nondimensional longitudinal residual current, obtained by a primitive perturbation technique (solid line), and the renormalized nondimensional longitudinal residual current solution in the small κ limit (dashed line) ; $b = 15$.

3 Figure 3. Ratio of the longitudinal residual current to the tidal amplitude at $y' = b/40 (b = 15)$ as a function of the Strouhal number obtained from the renormalized model (solid line), and the primitive perturbation model. (dashed line).

Figure 1

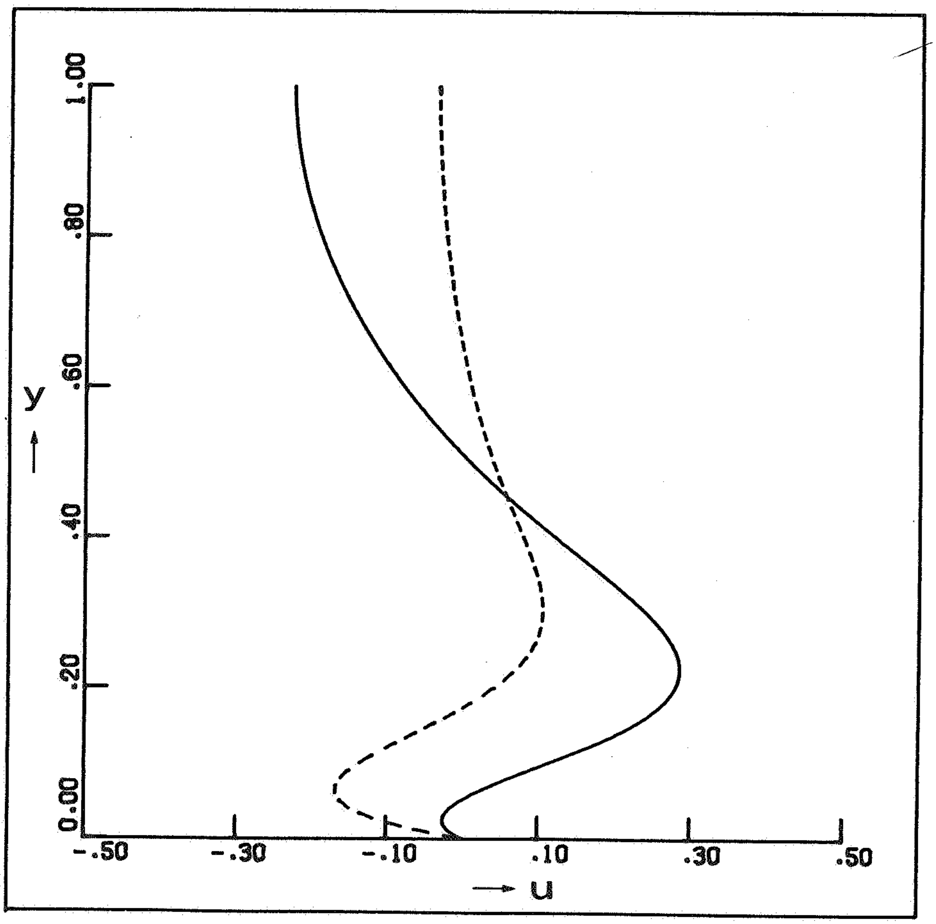

Figure 2

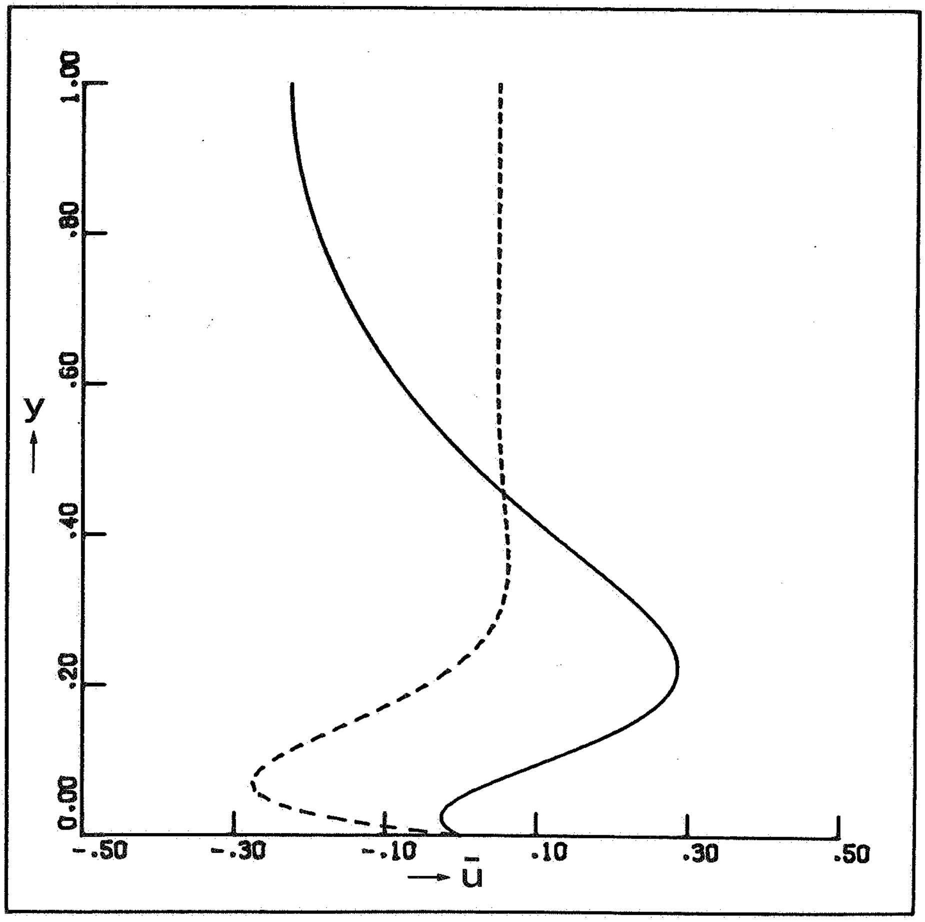

Figure 3