Mon. Not. R. astr. Soc. (1977) **181**, 441–454

Tidal torques on accretion discs in close binary systems

J. Papaloizou and J. E. Pringle *Institute of Astronomy, Madingley Road, Cambridge, CB3 0HA*

Received 1977 April 28; in original form 1977 March 15

Summary. We calculate the transfer of angular momentum between an accretion disc and orbital motion in a close binary system. If the dissipative process in the accretion disc can transport angular momentum (e.g. shear viscosity) the disc fills its Roche lobe and tidal torques always dominate the transport process in the outer parts of the Roche lobe. If not, the disc does not expand, all the transferred material is accreted and the inflow rate in the disc is controlled solely by tidal processes. We show that in an optically thick disc, the presence of a shear viscosity gives rise to a bulk viscosity of comparable magnitude.

1 Introduction

We consider the effects of tidal perturbations on an accretion disc around the primary star, mass M_1, in a close binary system due to the presence of the secondary, mass M_2. We calculate (Section 2) the perturbations to the disc flow in the absence of dissipation using the approximation that the perturbation is small. We find that the results of our calculations are valid for all reasonable mass ratios. In the absence of dissipative processes the perturbed quantities, in particular the density perturbations, are in phase with the secondary. If, however, a small amount of dissipation is introduced, a correspondingly small phase lag occurs in the density perturbations. There is therefore a net torque on the secondary due to the disc. Assuming that the disc and the binary motion are coplanar and that all angular velocities have the same sign, the effect of this torque is to transfer angular momentum (relative to the primary) from the disc into orbital motion. In this way the disc can get rid of its excess angular momentum and accrete on to the primary. The importance of this process has been stressed by Börner *et al.* (1973) and evidence of its presence presented by Lin & Pringle (1976).

We demonstrate (Appendix 2) that, provided the amount of dissipation is small, the tidal torque can be calculated without explicit knowledge of the phase lag of the density perturbations. Using this, we calculate (Section 2) the dissipation and tidal torques for the two types of viscosity, bulk and shear.

In Section 3 we show that in an optically thick accretion disc in which the dominant pressure is ordinary gas pressure and the dominant opacity is of Kramer's type, the presence of a shear viscosity gives rise via radiative processes to a bulk viscosity of comparable magnitude. In Section 4, we summarize and discuss our results.

2 Calculations

Consider a disc of gas orbiting around a central gravitating mass M_1. We ignore the effects of pressure on the flow and assume that dissipation terms are small so that initially the gas flows in circular orbits around M_1 with velocity $\mathbf{u}_0 \equiv (u_r, u_\theta) = [0, r\Omega(r)]$ where r is the radial distance from M_1, θ is the corresponding azimuthal coordinate and $\Omega^2 = GM_1/r^3$. We now suppose that this flow field is perturbed by the presence of another gravitating body of mass M_2 orbiting M_1 at a distance R with angular velocity $\omega = [G(M_1 + M_2)/R^3]^{1/2}$. We write $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1$ where $\mathbf{u}_1^2 \ll \mathbf{u}_0^2$ and $\mathbf{u}_1 \equiv (u_r', u_\theta')$. Linearizing the equations of motion we find

$$\frac{\partial u_r'}{\partial t} + \Omega \frac{\partial u_r'}{\partial \theta} - 2\Omega u_\theta' = F_r \tag{2.1a}$$

$$\frac{\partial u_\theta'}{\partial t} + \Omega \frac{\partial u_\theta'}{\partial \theta} + \tfrac{1}{2}\Omega u_r' = F_\theta \tag{2.1b}$$

where we have used $\Omega \propto r^{-3/2}$. We use coordinates (r, θ) centred on M_1 and moving (but not rotating) with it. We then find

$$F_r = \frac{GM_2[R\cos(\omega t - \theta) - r]}{[R^2 + r^2 - 2Rr\cos(\omega t - \theta)]^{3/2}} - \frac{GM_2\cos(\omega t - \theta)}{R^2} \tag{2.2a}$$

$$F_\theta = \frac{GM_2 R\sin(\omega t - \theta)}{[R^2 + r^2 - 2Rr\cos(\omega t - \theta)]^{3/2}} - \frac{GM_2\sin(\omega t - \theta)}{R^2}. \tag{2.2b}$$

The last term in each equation corresponds to the centrifugal force experienced due to the motion of M_1 (see Appendix 1). The density perturbations may be found from the linearized continuity equation.

The equations may be solved for the Fourier components

$$(u_{rn}', u_{\theta n}') = \int_0^{2\pi} (u_r', u_\theta') \exp(-in\theta)\, d\theta \tag{2.3}$$

by taking Fourier transforms with respect to time t. The solutions are

$$u_{rn}' = \exp(-in\omega t)\, \frac{[in(\Omega - \omega) f_{rn} + 2\Omega f_{\theta n}]}{\Omega^2 - n^2(\Omega - \omega)^2} \tag{2.4a}$$

$$u_{\theta n}' = \exp(-in\omega t)\, \frac{[in(\Omega - \omega) f_{\theta n} - \tfrac{1}{2}\Omega f_{rn}]}{\Omega^2 - n^2(\Omega - \omega)^2} \tag{2.4b}$$

where

$$f_{rn} = \frac{\pi GM_2}{R^2}\left(\frac{db_{1/2}^{(n)}}{d\alpha} - \delta_{1n}\right) \tag{2.5a}$$

$$f_{\theta n} = \frac{in\pi GM_2}{R^2}\,(b_{1/2}^{(n)}\alpha^{-1} - \delta_{1n}). \tag{2.5b}$$

Here δ_{1n} is the Kronecker δ symbol, $\alpha = r/R$ and $b_s^{(n)}(\alpha)$ is the Laplace coefficient defined by Hagihara (1972)

$$\tfrac{1}{2} \sum_{n=-\infty}^{\infty} b_s^{(n)} \cos n\psi = (1 + \alpha^2 - 2\alpha \cos \psi)^{-s}. \tag{2.6}$$

We now introduce a small amount of dissipation into the flow and consider the additional dissipation due to tidal perturbations to first order of smallness in the dissipation parameter. More specifically we consider two distinct modes of dissipation which for a Newtonian fluid would be shear viscosity ν and bulk viscosity ζ. The average dissipation per unit radius in the disc due to bulk viscosity may be written

$$D_1(\alpha) = \Sigma\zeta \int_0^{2\pi} (\operatorname{div} \mathbf{u})^2 \, d\theta$$

$$= \pi\Sigma\zeta \left(\frac{q}{1+q}\right)^2 \omega^2 \sum_{n=1}^{\infty} n^2 F_n^2(\alpha)$$

where $F_n(\alpha) = \omega(U_n - V_n)$, $q = M_2/M_1$

$$V_n = \frac{n^2(\Omega - \omega)\, b_{1/2}^{(n)} \alpha^{-1} + \tfrac{1}{2}\Omega\, (db_{1/2}^{(n)}/d\alpha)}{[\Omega^2 - n^2(\Omega - \omega)^2] \cdot \alpha}$$

and

$$U_n = \frac{1}{\alpha} \frac{d}{d\alpha} (\alpha W_n)$$

where

$$W_n = \frac{(\Omega - \omega)\,(db_{1/2}^{(n)}/d\alpha) + 2\Omega\, b_{1/2}^{(n)} \alpha^{-1}}{[\Omega^2 - n^2(\Omega - \omega)^2]}. \tag{2.7}$$

We note that Σ is the surface density in the disc. We write

$$S_1(\alpha) = \sum_{n=1}^{\infty} n^2 F_n^2(\alpha).$$

Provided the assumption of linearization is valid throughout the disc (this can always be arranged by making it small enough), the effective torque induced per unit radius is given by (see Appendix 2)

$$T_1(\alpha) = \pi\Sigma\zeta\omega \frac{q^2}{(1+q)^2} S_1(\alpha) \left(\frac{1}{\alpha^{3/2}(1+q)^{1/2}} - 1\right)^{-1} \tag{2.8}$$

and the rate at which angular momentum is transferred from the disc into orbital motion is

$$\frac{dJ}{dt} = -R^2 \int_{\text{disc}} T_1(\alpha)\, \alpha \, d\alpha. \tag{2.9}$$

A similar calculation can be made for shear viscosity, ν, for which the total dissipation per unit radius is

$$D_{\text{tot}}(\alpha) = 2\Sigma\nu \int_0^{2\pi} E_{ij} E_{ij} \, d\theta \tag{2.10}$$

where $E_{ij} = e_{ij} - \frac{1}{3}\delta_{ij}$ div **u** and e_{ij} is the usual rate of strain tensor. To estimate the torque due to tidal perturbations alone we consider only the additional dissipation due to non-axisymmetric tidal perturbations, that is harmonics with $n \geqslant 1$ (see Appendix 2). For these we obtain

$$D_2(\alpha) = \pi \Sigma \nu \omega^2 \frac{q^2}{(1+q)^2} S_2(\alpha). \tag{2.11}$$

$S_2(\alpha)$ is a more complicated function than $S_1(\alpha)$ and we have plotted it in Fig. 1 for the case $q = 1$. In fact, since the dominant contribution to the dissipation rate comes from the term $\partial u_r'/\partial r$ in both cases, the function $S_1(\alpha)$ is practically identical (to within 20 per cent) to $S_2(\alpha)$ for the range of α shown in Fig. 1. We see that S_2 and S_1 are strongly increasing functions of α. This is partly because (from equation 2.4) the velocity perturbations resonate with the orbital motion at radii given, for $n \geqslant 1$, by

$$\Omega^2 = n^2(\Omega - \omega)^2 \tag{2.12}$$

that is

$$\alpha_n = [1 - (1/n)]^{2/3}(1+q)^{-1/3}. \tag{2.13}$$

For $n = 1$, the resonant radius is $\alpha_1 = 0$ and for $n = 2$, $q = 1$ the radius is $\alpha_2 = 0.5$. Thus, close to the origin ($\alpha \lesssim 0.05$) the $n = 1$ harmonic provides the dominant contribution, and elsewhere within the Roche lobe of M_1, the $n = 2$ harmonic dominates.

The flow in the mean potential gives rise to an axisymmetric ($n = 0$) dissipation rate $D_0(\alpha)$ and corresponding $S_0(\alpha)$, which does not result in angular momentum transfer from the disc to the secondary. When $M_2 = 0$, we have the familiar accretion disc dissipation rate

$$D_0(\alpha) = \frac{9\pi}{2} \Sigma \nu \omega^2 (1+q)^{-1} \alpha^{-3}. \tag{2.14}$$

In Fig. 1 we plot $S_0(\alpha)$ for the case $M_2 = M_1$.

To obtain a rough comparison of the significance of tidal and ordinary viscous torques when there is only a shear viscosity acting, we remark that if a thin ring of matter is placed orbiting at a given radius α, the viscous torque exerted at the density maximum is approximately $D_0(\alpha)/(3\Omega)$. Thus tidal and viscous torques are comparable at this maximum if α is such that $D_0(\alpha) \sim 3D_2(\alpha)$.

In Fig. 1 we also indicate the mean radius α_s, outside which the perturbed orbits intersect each other. This radius is given by the condition

$$\frac{\partial}{\partial r} \int u_r' \, dt + 1 = 0 \tag{2.15}$$

where the time integral is taken following a particle orbit. The intersection always occurs along the line of centres of the two stars. We tabulate α_s for various values of q in Table 1. These estimates of α_s compare favourably with the values of the intersection radius obtained by Paczynski (1977) who computes the particle orbits directly. We note that the condition (2.15) can be approximated by $|\partial u_r'/\partial r| = \Omega$. It is for this reason that near α_s, zero order ($n = 0$) and tidal ($n \geqslant 1$) effects become comparable (see Fig. 1). Just inside this radius linear theory breaks down. In fact, if viscosity is treated as a perturbation, one would derive infinite dissipation and loss of angular momentum at this radius, even in non-linear theory, because of the development of infinite gradients there.

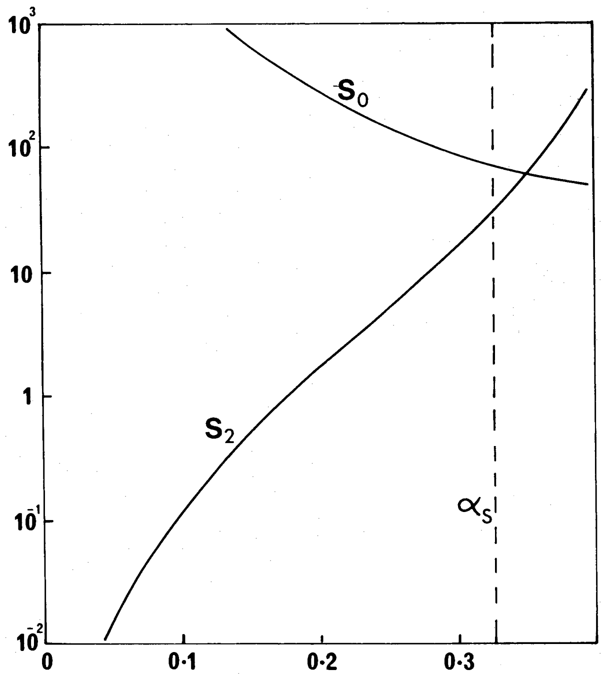

Figure 1. The dimensionless dissipation per unit radius $S_2(\alpha)$ due to tidal effects and shear viscosity is drawn as a function of dimensionless radius for $M_2 = M_1$. The dissipation $S_1(\alpha)$ due to bulk viscosity is virtually identical. In the same units $S_0(\alpha)$, the axially symmetric dissipation due to shear viscosity is also shown. We indicate the radius, α_S, at which the perturbed orbits start to intersect. It is evident that this is also the radius at which tidal and non-tidal dissipative effects become comparable.

3 Viscosities

In standard, axisymmetric, accretion discs the viscous mechanisms usually invoked are turbulent viscosity and magnetic viscosity (Shakura & Sunyaev 1973; Eardley & Lightman 1975). Naturally, the major consideration of these processes has been in terms of a shear-type viscosity, although the same processes probably give rise to a bulk-type viscosity of comparable magnitude. However, even if these processes gave rise to negligible bulk viscosity, we expect an effective bulk viscosity to arise in an accretion disc because of variable radiative cooling. This process of dissipation is exactly analogous to the process of radiative damping of ordinary stellar pulsations. We estimate the size of this effect below.

The energy equation may be written in the form

$$\frac{\partial p}{\partial t} - \frac{\gamma p}{\rho}\frac{\partial \rho}{\partial t} = (\gamma - 1)\left(\rho \epsilon_\nu - \frac{\sigma T_e^4}{H}\right) \tag{3.1}$$

where p is the pressure, ρ the density, T_e the effective temperature of the disc, γ the adiabatic index, H the scale height of the disc and ϵ_ν the energy dissipation per unit mass by

Table 1. The radius α_S, outside which tidal torques dominate, and α_L, the mean Roche-lobe radius, are tabulated as functions of $q = M_2/M_1$.

M_2/M_1	0.2	0.4	0.6	0.8	1.0
α_S	0.45	0.41	0.37	0.35	0.33
α_L	0.52	0.46	0.42	0.40	0.38
M_2/M_1	2.0	4.0	6.0	8.0	10.0
α_S	0.28	0.23	0.21	0.19	0.18
α_L	0.32	0.26	0.23	0.22	0.20

shear viscosity (assumed axisymmetric for our present purposes). Perturbing this equation in the form $p \rightarrow p + p'$ etc., and linearizing with regard to the perturbed quantities we obtain for the nth harmonic

$$p' - \frac{\gamma p}{\rho}\rho' = \left[\!\left[(\gamma-1)\left[\rho'\epsilon_\nu - \frac{4\sigma T_e^3 T_e'}{H} + \frac{\sigma T_e^4 H'}{H^2}\right]\right]\!\right] \Big/ [in(\omega-\Omega)]. \tag{3.2}$$

For an optically thick disc we may approximate the central temperature T at a given radius by $T^4 = \tau T_e^4$ where τ is the optical depth. Hydrostatic equilibrium perpendicular to the disc under the assumption that gas pressure dominates, yields, approximately,

$$p \equiv \frac{\mathscr{R}\rho T}{\mu} = \frac{GM_1 H\tau}{\kappa r^3} \tag{3.3}$$

where κ is the opacity, $\mathscr{R}$ the gas constant and μ the mean molecular weight. Using these, and assuming an opacity law of the form $\kappa = \kappa_0 \rho^\alpha T^{-\beta}$, the energy equation becomes

$$p' = \frac{\gamma p}{\rho}\rho' + \frac{(\gamma-1)\rho\epsilon_\nu}{i(\omega-\Omega)}\left\{\frac{T'}{T}(\beta+3) - \frac{\rho'}{\rho}(\alpha+2)\right\}. \tag{3.4}$$

The linearized continuity equation is

$$\rho' = -\frac{\rho \operatorname{div}\mathbf{u}}{in(\omega-\Omega)}. \tag{3.5}$$

Assuming as a first approximation that

$$\frac{T'}{T} = (\gamma-1)\frac{\rho'}{\rho} \tag{3.6}$$

and substituting equations (3.5) and (3.6) into equation (3.4) we obtain, to first order in the effective bulk viscosity ζ,

$$p' = \frac{\gamma p}{\rho}\rho' - \rho\zeta \operatorname{div}\mathbf{u} \tag{3.7}$$

where

$$\zeta = \frac{(\gamma-1)\epsilon_\nu}{n^2(\omega-\Omega)^2}[(\gamma-1)(\beta+3)-(\alpha+2)]. \tag{3.8}$$

Bearing in mind that

$$\epsilon_\nu \sim \nu r^2\left(\frac{d\Omega}{dr}\right)^2 \tag{3.9}$$

and taking $\gamma = 5/3$, we see that for electron scattering opacity $\zeta = 0$ but that for Kramer's opacity ($\alpha = 1, \beta = 3.5$) $\zeta \sim \nu$.

4 Discussion

We have shown that if the dissipative process is of a type that is much more able to remove energy from dilatation rather than from shearing motions, an accretion disc formed within the Roche lobe around M_1 will eventually collapse on to M_1, losing all the necessary angular momentum via tidal effects alone. We note that an accretion disc consisting of material transferred through the inner Lagrangian point is formed at a radius $\alpha_h \ll \alpha_s$ (Flannery 1974). In this case *all* the mass transferred from M_2 can be accreted by M_1. If there is some form of shear viscosity present in the disc we have shown that the additional tidal-induced dissipation becomes comparable to that due to the mean flow at a radius α_s which is about 0.85–0.9 of the mean Roche lobe radius α_L (Table 1). We have also shown that in a disc in which gas pressure dominates radiation pressure and in which electron scattering opacity is negligible, the presence of the shear viscosity together with radiative processes give rise to a bulk viscosity of comparable magnitude. Outside α_s, the perturbed particle orbits intersect, and, strictly, pressure effects and non-linear effects should not be neglected. In all probability, standing shocks form in the gas flow outside this radius, giving rise to a large increase in tidal dissipation and hence in the rate of angular momentum loss from the disc. We indicate in Appendix 2 that the same principles governing angular momentum loss and dissipation apply in both the linear and non-linear regimes. There is no *a priori* reason to associate the radius α_s with the outside edge of the disc (contrary to the suggestion by Pacynski 1977) since even an infinitesimal amount of viscosity ensures that the flow streamlines do not intersect.

In the absence of tidal torques, the fraction, f, of matter transferred from M_2 to M_1 (in a system in which an accretion disc is formed) that has to be lost from the Roche lobe in order to allow the remaining matter to accrete is $f \sim (\alpha_h/\alpha_L)^{1/2} \sim 0.3$–$0.5$ (Prendergast & Burbidge 1968; Lin & Pringle 1976). The effect of tidal torques is to reduce the fraction f. If there is no shear viscosity we have seen that $f = 0$. If there is a shear viscosity the outer edge of the disc expands beyond α_s (Lynden-Bell & Pringle 1974). Since we cannot precisely calculate the torque beyond α_s, we cannot provide a precise value for f in this case. However, as the disc expands into the non-linear regime outside α_s, and close to α_L, we expect the dissipation rate to increase until it becomes of the order of the orbital kinetic energy (around M_1) per orbital cycle, $P(\alpha_s) \approx 2\pi/\Omega(\alpha_s)$. Accordingly, in order that the mass orbiting in the region of α_L can absorb the angular momentum flux from a disc which is secularly evolving on a timescale $\tau_\nu \equiv NP$, where $N \gg 1$, we require of order N^{-1} of the mass of the disc to be in the neighbourhood of the Roche lobe. As a corollary, if shear-viscosity still acts in the neighbourhood of the Roche lobe we expect the mass loss rate at the outer edge of the disc to be $\sim N^{-1}$ times the accretion rate on to M_1. That is we expect $f \sim P(\alpha_s)/\tau_\nu$.

In observed systems such as dwarf novae, where τ_ν may be estimated from the decay timescale of the outbursts (Bath *et al.* 1974) we expect f to be of the order of a few per cent. In the numerical simulations of viscous gas flow in binary systems by Lin & Pringle (1976), tidal effects are clearly evident, see Fig. 5(c), and f is found to be of the order of a few per cent. In this connection we remark that the fact that the above authors did not consider a fluid is of no relevance. Any system that can provide the required dissipation should yield the same results for f.

Throughout our discussion we have neglected the effect on the fluid flow of the incoming stream from the inner Lagrangian point. In an equilibrium situation in which the disc evolves on a secular (viscous) timescale, the density in the disc is greater than that in the stream, and so the stream does not affect the fluid flow in the disc to any great extent. It does, however, give rise to considerably enhanced dissipation at the place where it strikes the disc and may lead to additional mass loss from the collision that occurs there. We should also note that although the dissipation due to tidal effects occurs in a non-axisymmetric manner, this does not *necessarily* imply that those parts of the disc where tidal effects dominate radiate particularly non-axisymmetrically. This is only true if the cooling time in the disc is sufficiently short that $\tau_{\rm cool}\Omega \ll 1$.

Acknowledgments

It is a pleasure to thank D. Lynden-Bell and D. N. C. Lin for stimulating our interest in this problem. JEP is grateful to IBM for financial support. We thank the referee for helping to improve the clarity of this paper.

References

Bath, G. T., Evans, W. D., Papaloizou, J. & Pringle, J. E., 1974. *Mon. Not. R. astr. Soc.*, **169**, 447.
Börner, G., Meyer, F., Schmidt, H. U. & Thomas, H.-C., 1973. *Mitt. der. astr. Ges.*, **32**, 237.
Eardley, D. & Lightman, A. P., 1975. *Astrophys. J.*, **200**, 187.
Flannery, B. P., 1974. *Mon. Not. R. astr. Soc.*, **170**, 325.
Hagihara, Y., 1972. *Celestial mechanics*, Vol. II, Part I, Chapter 7, The MIT Press, London.
Lin, D. N. C. & Pringle, J. E., 1976. *Proc. IAU Symp. 73*, p. 237, eds Eggleton, P. P. *et al.*, Reidel, Dordrecht.
Lynden-Bell, D. & Pringle, J. E., 1974. *Mon. Not. R. astr. Soc.*, **168**, 603.
Paczynski, B., 1977. *Astrophys. J.*, in press.
Prendergast, K. H. & Burbridge, G. R., 1968. *Astrophys. J. Lett.*, **151**, L83.
Shakura, N. I. & Sunyaev, R. A., 1973. *Astr. Astrophys.*, **29**, 179.

Appendix 1: the equations of motion

In an inertial frame based on the centre of mass of the system, the Eulerian equation of motion can be written

$$\frac{\partial \mathbf{u}_G}{\partial t} + \mathbf{u}_G \cdot \nabla \mathbf{u}_G = -\frac{1}{\rho} \nabla p - \nabla \psi + \boldsymbol{\chi} \tag{A1}$$

where $\boldsymbol{\chi}$ is a general viscous force and ψ, the total gravitational potential can be written

$$\psi = -\frac{GM_1}{r} - GM_2/[r^2 + R^2 - 2rR\cos(\theta - \omega t)]^{1/2} \tag{A2}$$

where r, θ are cylindrical polar coordinates based on M_1. These coordinates are defined in a system which is based on M_1 but retains its orientation relative to the original system based on the centre of mass. The relation between these coordinates is

$$\begin{aligned} x &= -r\cos\theta = x_G - qR\cos\omega t/(1+q) \\ y &= -r\sin\theta = y_G - qR\sin\omega t/(1+q). \end{aligned} \tag{A3}$$

We define velocities relative to M_1 by

$$\begin{aligned} u_x &= u_{Gx} + qR(1+q)^{-1}\omega\sin\omega t \\ u_y &= u_{Gy} - qR(1+q)^{-1}\omega\cos\omega t. \end{aligned} \tag{A4}$$

It is then readily verified that the operator $\partial/\partial t + \mathbf{u}_G \cdot \nabla$ transforms to $\partial/\partial t + \mathbf{u} \cdot \nabla$. The equations of motion thus become

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p - \nabla \Psi + \boldsymbol{\chi} \tag{A5}$$

where Ψ is the modified potential

$$\Psi = -\frac{GM_1}{r} - \frac{GM_2}{[r^2 + R^2 - 2Rr\cos(\omega t - \theta)]^{1/2}} + \frac{qR\omega^2 r\cos(\theta - \omega t)}{(1+q)}$$

or (A6)

$$\Psi = -\frac{GM_1}{r} - \frac{GM_2}{[r^2 + R^2 - 2Rr\cos(\omega t - \theta)]^{1/2}} + \frac{GM_2 r\cos(\theta - \omega t)}{R^2}.$$

On account of the invariance of the operators $\partial/\partial t + \mathbf{u}\cdot\nabla$, $\partial/\partial x$ and $\partial/\partial y$, other equations such as the equation of continuity retain invariant form.

Appendix 2: the perturbed equations and the net loss of angular momentum from the disk

2.1 THE LINEAR REGIME

The total modified potential can be Fourier analysed in the form

$$\Psi = \Psi_0 + \sum_{n\neq 0} -\frac{GM_2}{R}\left(b_{1/2}^{(n)}(r/R) - \frac{r\delta_{1n}}{R}\right)\cos n(\theta - \omega t)$$

or (A7)

$$\Psi = \Psi_0 + \Psi'$$

where Ψ_0 contains a symmetric part of the perturbing potential which modifies the Keplerian orbits slightly but which produces no tidal effects. The effect of this can be ignored throughout. The zero order disc with no tides can then be seen to satisfy

$$\frac{u_r}{r}\frac{d}{dr}(r^2\Omega) = \chi_{\theta_0}$$

and (A8)

$$u_r\frac{du_r}{dr} - r\Omega^2 = -\frac{d\Psi_0}{dr} + \chi_{r_0}.$$

Thus u_r is first order in the viscosity.

We take into account the non-symmetric part of the potential by expanding linearly the equations of motion, we then find that

$$\frac{\partial u_r'}{\partial t} + \Omega\frac{\partial u_r'}{\partial\theta} - 2\Omega u_\theta' + u_r\frac{\partial u_r'}{\partial r} + u_r'\frac{\partial u_r}{\partial r} = -\frac{\partial\Psi'}{\partial r} + \chi_r'$$

(A9)

$$\frac{\partial u_\theta'}{\partial t} + \Omega\frac{\partial u_\theta'}{\partial\theta} + \frac{u_r'}{r}\frac{\partial(r^2\Omega)}{\partial r} + \frac{u_r}{r}\frac{\partial(ru_\theta')}{\partial r} = -\frac{1}{r}\frac{\partial\Psi'}{\partial\theta} + \chi_\theta'$$

From (A9) we note that because the viscosity (and hence u_r) is small u_r', χ_r', and $\partial\Psi'/\partial\theta$, are approximately in phase, while u_θ', χ_θ' and $\partial\Psi'/\partial r$ differ from these in phase by $\pi/2$.

This means that we can keep terms of the same order in the viscosity and with the same phase together by defining modified viscous forces by

$$\bar{\chi}'_r = \chi'_r - u_r \frac{\partial u'_r}{\partial r} - u'_r \frac{\partial u_r}{\partial r}$$
$$\bar{\chi}'_\theta = \chi'_\theta - \frac{u_r}{r} \frac{\partial (r u'_\theta)}{\partial r}. \tag{A10}$$

This effectively eliminates u_r from the problem. The equations then become

$$\frac{\partial u'_r}{\partial t} + \Omega \frac{\partial u'_r}{\partial \theta} - 2\Omega u'_\theta = -\frac{\partial \Psi'}{\partial r} + \bar{\chi}'_r$$
$$\frac{\partial u'_\theta}{\partial t} + \Omega \frac{\partial u'_\theta}{\partial \theta} + \frac{u'_r}{r} \frac{\partial (r^2 \Omega)}{\partial r} = -\frac{1}{r} \frac{\partial \Psi'}{\partial \theta} + \bar{\chi}'_\theta. \tag{A11}$$

The solutions for the nth harmonic can then be written down as

$$u'_{rn} = Q_{rn} \sin n(\theta - \omega t) + q_{rn} \cos n(\theta - \omega t)$$
$$u'_{\theta n} = Q_{\theta n} \cos n(\theta - \omega t) + q_{\theta n} \sin n(\theta - \omega t) \tag{A12}$$

where

$$Q_{rn} = \left(\frac{2n\Omega}{r} D_n + n(\Omega - \omega) \frac{dD_n}{dr}\right) \Big/ d$$

$$q_{rn} = [2\Omega \bar{\chi}'_{n\theta} + n(\Omega - \omega) \bar{\chi}'_{nr}]/d$$

$$Q_{\theta n} = \left(\frac{1}{r} \frac{d}{dr} (r^2 \Omega) \frac{dD_n}{dr} + \frac{n^2(\Omega - \omega)}{r} D_n\right) \Big/ d$$

$$q_{\theta n} = \left(-\frac{1}{r} \frac{d}{dr} (r^2 \Omega) \bar{\chi}'_{nr} - \bar{\chi}'_{n\theta} n(\Omega - \omega)\right) \Big/ d$$

$$D_n = -\frac{GM_2}{R} \left(b_{1/2}^{(n)}\left(\frac{r}{R}\right) - \frac{r}{R} \delta_{1n}\right)$$

$$d = \frac{2\Omega}{r} \frac{d}{dr} (r^2 \Omega) - n^2 (\Omega - \omega)^2.$$

For comparison with Section 2 we note that

$$D_n = \frac{i f_{\theta n} R}{\alpha \pi n}$$

and

$$\frac{dD_n}{dr} = -\frac{f_{rn}}{\pi}.$$

The loss of angular momentum can now be calculated from

$$\frac{dJ}{dt} = -\int_{\text{disc}} \rho \frac{\partial \Psi'}{\partial \theta} d\tau. \tag{A13}$$

If

$$\rho = \rho_0 + \sum_{n>0} \rho_n' \cos n(\theta - \omega t) + \sum_{n>0} \rho_n'' \sin n(\theta - \omega t) \tag{A14}$$

then

$$\frac{dJ}{dt} = \tfrac{1}{2} \int_{\text{disc}} \sum_n \rho_n'' \, n D_n \, d\tau. \tag{A15}$$

But from the equation of continuity

$$\rho_n'' n(\Omega - \omega) = -\left(\frac{1}{r}\frac{d}{dr}(r\rho_0 q_{rn}) + \frac{n q_{\theta n} \rho_0}{r}\right) \tag{A16}$$

and so after an integration by parts

$$\frac{dJ}{dt} = \tfrac{1}{2} \int_{\text{disc}} \left\{\frac{-D_n n q_{\theta n} \rho_0}{r(\Omega - \omega)} + \rho_0 q_{rn} \frac{d}{dr}\left(\frac{D_n}{\Omega - \omega}\right)\right\} d\tau \tag{A17}$$

substituting for $(q_{rn}, q_{\theta n})$ one finally obtains

$$\frac{dJ}{dt} = \sum_{n>0} \frac{dJ_n}{dt}$$

where

$$\frac{dJ_n}{dt} = \tfrac{1}{2} \int_{\text{disc}} \left\{\frac{\rho_0 \bar{\chi}_{nr}' Q_{rn}}{\Omega - \omega} + \rho_0 \bar{\chi}_{n\theta}' \left[\frac{Q_{\theta n}}{\Omega - \omega} - \frac{Q_{rn}\, r(d\Omega/dr)}{n(\Omega - \omega)^2}\right]\right\} d\tau \tag{A18}$$

But if $\xi_{nr}, \xi_{n\theta}$ are the Lagrangian displacements associated with the nth harmonic then

$$\begin{aligned} \left(\frac{\partial \xi_{nr}}{\partial t}\right)_{\text{L}} &= Q_{rn} \sin n(\theta - \omega t) + O(\nu) \\ \left(\frac{\partial \xi_{n\theta}}{\partial t}\right)_{\text{L}} &= \left[Q_{\theta n} - \frac{r(d\Omega/dr)\, Q_{rn}}{n(\Omega - \omega)}\right] \cos n(\theta - \omega t) + O(\nu) \end{aligned} \tag{A19}$$

so that,

$$\frac{dJ_n}{dt} = \int_{\text{disc}} \rho_0 \bar{\boldsymbol{\chi}}_n' \cdot \left(\frac{\partial \boldsymbol{\xi}_n}{\partial t}\right)_{\text{L}} (\Omega - \omega)^{-1} d\tau \tag{A20}$$

where the bar indicates a time average. It is instructive to consider the Lagrangian equations for the perturbations. These are

$$\begin{aligned} \frac{\partial^2 \xi_r}{\partial t^2} - 2\Omega \frac{\partial \xi_\theta}{\partial t} + 2r\xi_r \Omega \frac{d\Omega}{dr} &= \bar{\chi}_r' - \frac{\partial \Psi'}{\partial r} \\ \frac{\partial^2 \xi_\theta}{\partial t^2} + 2\Omega \frac{\partial \xi_r}{\partial t} &= \bar{\chi}_\theta' - \frac{1}{r}\frac{\partial \Psi'}{\partial \theta}. \end{aligned} \tag{A21}$$

The energy equation can then be written

$$\frac{\partial}{\partial t}\int_{\text{disc}} \tfrac{1}{2}\rho_0 \left\{\left(\frac{\partial \xi_r}{\partial t}\right)^2 + \left(\frac{\partial \xi_\theta}{\partial t}\right)^2 + 2r\Omega\Omega'\xi_r^2 + \xi_r \frac{\partial \Psi'}{\partial r} + \frac{\xi_\theta}{r}\frac{\partial \Psi'}{\partial \theta}\right\} d\tau$$

$$= \int_{\text{disc}} \rho_0 \bar{\chi}' \cdot \frac{\partial \xi}{\partial t} d\tau + \int_{\text{disc}} \rho_0 \left(\xi_r \frac{\partial}{\partial r} + \frac{\xi_\theta}{r}\frac{\partial}{\partial \theta}\right)\frac{\partial \Psi'}{\partial t} d\tau. \quad \text{(A22)}$$

This shows that the angular momentum and energy loss from the perturbations are related per unit mass by the factor $1/(\Omega - \omega)$. We note that for the simple case of bulk viscosity, ζ,

$$\bar{\chi}' = \frac{1}{\rho}\nabla(\zeta\rho \operatorname{div}\mathbf{u}), \quad \text{(A23)}$$

on account of the fact that to zero order in the viscosity

$$\operatorname{div}\left\{\frac{\partial \xi_n}{\partial t}\Big/(\Omega - \omega)\right\} = \frac{\operatorname{div}\mathbf{u}_n}{\Omega - \omega} \quad \text{(A24)}$$

and

$$\frac{dJ_n}{dt} = -\int_{\text{disc}} \frac{\rho_0 \zeta(\operatorname{div}\mathbf{u}_n')^2}{\Omega - \omega} d\tau. \quad \text{(A25)}$$

For other viscous forces the situation is more complicated but if the initial shear is neglected (i.e. terms with Ω'), we can always write

$$\frac{dJ}{dt} = -\int_{\text{disc}} \frac{\rho\epsilon_\nu d\tau}{\Omega - \omega} \quad \text{(A26)}$$

where ϵ_ν is the viscous heat production per unit mass, as a result of the perturbations. For comparison, equation (2.10) can be written

$$D_{\text{tot}}(\alpha) = \int_0^{2\pi} \rho\epsilon_\nu d\theta.$$

This is a good approximation when tidal effects start to become important as the major terms which contribute come from the radial derivatives of the velocity which are much more rapidly varying than Ω.

2.2 THE NON-LINEAR REGIME

We show now that the formula for the net angular momentum loss from the disc extends into the non-linear regime in the case of bulk viscosity and provided that certain approximations, which should be reasonable, are adhered to. This allows for possible shock waves, which can be adequately supported by bulk viscosity.

We write the equation of motion in Lagrangian form as

$$\frac{d^2\mathbf{r}}{dt^2} = -\nabla\Psi - \nabla\Psi' + \frac{\nabla\Pi}{\rho}.$$

We include an isothermal equation of state by taking

$$\Pi = -\rho c_s^2 + \zeta\rho \operatorname{div}\mathbf{u}.$$

From this we find the equations of conservation of energy and angular momentum in the form

$$\frac{de}{dt} = + \frac{\partial \Psi'}{\partial t} - \zeta(\operatorname{div}\mathbf{u})^2 + \frac{1}{\rho}\operatorname{div}(\Pi \mathbf{u})$$

$$\frac{dh}{dt} = -\frac{\partial \Psi'}{\partial \theta} + \frac{1}{\rho}\frac{\partial \Pi}{\partial \theta}.$$

Here

$$e = \tfrac{1}{2}u^2 + \Psi + \Psi' + c_s^2.$$

Using the fact that

$$\frac{\partial \Psi'}{\partial \theta} = -\frac{1}{\omega}\frac{\partial \Psi'}{\partial t}$$

we obtain

$$\frac{d}{dt}(e - \omega h) = -\zeta(\operatorname{div}\mathbf{u})^2 - \frac{\omega}{\rho}\frac{\partial \Pi}{\partial \theta} + \frac{1}{\rho}\operatorname{div}(\Pi \mathbf{u}).$$

If we integrate over the outer edge of the disc, containing the non-linear regime, we find

$$\frac{\partial}{\partial t}\left\{\int e\,dm - \omega \int h\,dm\right\} = -\int \zeta(\operatorname{div}\mathbf{u})^2 dm - \int \Pi \mathbf{u}\cdot d\mathbf{S} + S_\mathrm{e} - \omega S_\mathrm{h}$$

where the last three terms represent fluxes of energy and angular momenta from the inner boundary. In practice we may take the inner boundary to be at about 80 per cent of the lobe radius. We can then ignore the flux arising from the stress tensor. We define $\bar{\Omega}$, by

$$\frac{d}{dt}\int e\,dm = \bar{\Omega}\frac{d}{dt}\int h\,dm$$

and if

$$\int h\,dm = J,$$

and

$$S_\mathrm{e} = \Omega_b S_h;$$

we find

$$\frac{dJ}{dt} = -\frac{\int \zeta(\operatorname{div}\mathbf{u})^2 dm}{\bar{\Omega} - \omega} + \frac{S_h(\Omega_b - \omega)}{\bar{\Omega} - \omega}.$$

From the general equations we see that the rate of loss of angular momentum is the order of the ratio of the rate of dissipation of energy arising from the perturbed potential to the rotation frequency. For bulk viscosity, this roughly corresponds to the total energy dissipation rate. Thus, as the non-linear region is probably not extensive, $\bar{\Omega}$, and Ω_b should both roughly correspond to the mean rotation rate in that region, which acts as if it has a mean loss rate per unit mass equivalent to

$$\frac{dh}{dt} = -\frac{\zeta\,(\mathrm{div}\,\mathbf{u})^2}{\bar{\Omega} - \omega}$$