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TIGHT BOUNDS ON THE EXPONENTIAL APPROXIMATION OF SOME AGING DISTRIBUTIONS

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Tight bounds on the sup metric between the exponential distribution and new better (worse) than used in expectation (NBUE, NWUE) distributions are established in terms of the proximity of the second moments of the distributions concerned. Real variable methods are used to identify the extremal distributions that attain the bounds. Similar methods establish similar results for the harmonic NBUE and NWUE classes of distributions.

1. Introduction and results. The exponential distribution is often used in applied probability models as an approximation to some unknown distribution for nonnegative random variables depicting "age." The adequacy or otherwise of such an approximation can be described by various criteria. This paper is concerned with establishing the precise nature of one particular measure of approximation, namely, the supremum metric, for distributions belonging to certain classes of aging distributions. For further details on motivation, see, e.g., Brown and Ge (1984) or Barlow and Proschan (1975).

We study the classes of the distribution functions (d.f.'s) F of nonnegative random variables X with unit mean and specified second moment, with

$$\rho \equiv |EX^2/2 - 1|, \tag{1.1}$$

and which are either new better or new worse than used in expectation (NBUE, NWUE), defined by

$$E(X - y|X > y) \le (\ge) EX = 1, \tag{1.2}$$

respectively. The restriction to unit mean is purely for convenience: without it we should merely have the more general definition of ρ as

$$\rho = \begin{cases} 1 - EX^2/2(EX)^2 & (F \text{ NBUE}), \\ EX^2/2(EX)^2 - 1 & (F \text{ NWUE}). \end{cases} \tag{1.1'}$$

Writing

$$\Delta(F_1, F_2) \equiv \sup_x |F_1(x) - F_2(x)|$$

for the sup metric of d.f.'s, and letting Exp denote the unit exponential d.f., the main aim is to establish best possible inequalities for $\Delta(F, \text{Exp})$ in terms of ρ when F is either NBUE or NWUE.

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THEOREM 1. *When the d.f. F is NBUE as above,*

$$\Delta(F,\mathrm{Exp}) \le \Delta(F_B,\mathrm{Exp}) = 1 - \exp\left(-(2\rho)^{1/2}\right), \tag{1.3}$$

where F_B is NBUE and is given by

$$\overline{F}_B(x) = \begin{cases} 1, & x < (2\rho)^{1/2}, \\ \left\{1 - (2\rho)^{1/2}\right\}\exp\left(-\left(x - (2\rho)^{1/2}\right)\right), & x > (2\rho)^{1/2}. \end{cases} \tag{1.4}$$

THEOREM 2. *When the d.f. F is NWUE as above,*

$$\Delta(F,\mathrm{Exp}) \le \Delta(F_W,\mathrm{Exp}) = \left(\rho^2 + 2\rho\right)^{1/2} - \rho \equiv p_\rho, \tag{1.5}$$

where F_W is NWUE and is given by

$$\overline{F}_W(x) = (1 - p_\rho)\exp\left(-\max\left(0, x - p_\rho/(1 - p_\rho)\right)\right), \qquad x \ge 0. \tag{1.6}$$

It is easily checked from (1.3) and (1.5) that for all ρ, and in particular, for small ρ,

$$\Delta(F_B,\mathrm{Exp}) \le (2\rho)^{1/2}, \qquad \Delta(F_W,\mathrm{Exp}) \le (2\rho)^{1/2}, \tag{1.7}$$

and that

$$\Delta(F_B,\mathrm{Exp}) = (2\rho)^{1/2}(1 + o(1)) = \Delta(F_W,\mathrm{Exp}), \tag{1.8}$$

for $\rho \downarrow 0$. Brown and Ge (1984) used Fourier methods to establish inequalities similar to (1.7) with $(2\rho)^{1/2}$ replaced by $(4\sqrt{6})\rho^{1/2}/\pi \approx 3.119\rho^{1/2}$; our results, prompted by reading their paper, give the best possible bounds for the classes NBUE and NWUE. Their paper contains references to inequalities on the sup metric for other classes of aging distributions, and outlines the motivation for seeking such inequalities in terms of convergence of d.f.'s to the exponential within such classes of aging d.f.'s.

Both NBUE and NWUE d.f.'s are most easily discussed via the d.f. G corresponding to the integral of $\overline{F}(x) \equiv 1 - F(x)$, namely,

$$G(x) = \int_0^x \overline{F}(u)\,du. \tag{1.9}$$

The basic inequalities at (1.2) are then expressible as

$$\overline{G}(x) \equiv 1 - G(x) \le \overline{F}(x) \qquad (F \text{ NBUE}), \tag{1.10a}$$

$$\overline{G}(x) \ge \overline{F}(x) \qquad (F \text{ NWUE}). \tag{1.10b}$$

As intermediate steps in proving our theorems we have the following results also.

PROPOSITION 1. $\Delta(F,G) \le \Delta(F_B,G_B) = (2\rho)^{1/2}$ $\quad$ (*F NBUE*).

PROPOSITION 2. $\Delta(F,G) \le \Delta(F_W,G_W) = \Delta(F_W,\mathrm{Exp}) = p_\rho$ $\quad$ (*F NWUE*).

We also consider the larger classes of distributions for which $\overline{G}(x) \le (\ge) e^{-x}$, called by Rolski (1975) HNBUE (HNWUE), where H stands for "harmonic" on account of the characterization that F is HNBUE as above if and only if

$$x^{-1}\int_0^x \left(\overline{G}(u)/\overline{F}(u)\right)^{-1} du \geq 1, \quad \text{all } x > 0.$$

Klefsjö (1983) gives more references on HNBUE and HNWUE distributions. For distributional inequalities see Stoyan (1983).

THEOREM 3. *Within the HNBUE class as above,*

$$\Delta(F, \text{Exp}) \leq \Delta(F_{\text{HB}}, \text{Exp}) = 1 - e^{-\gamma}, \tag{1.11}$$

where

$$\gamma = 1 - \theta e^{-\theta}/(1 - e^{-\theta}), \tag{1.12}$$

$$2\rho = 1 - e^{-\theta} - \theta^2 e^{-\theta}/(1 - e^{-\theta}), \tag{1.13}$$

for some $\theta > 0$, *and* F_{HB} *is HNBUE and is given by*

$$\overline{F}_{\text{HB}}(x) = \begin{cases} 1, & x < \gamma, \\ e^{-\theta}, & \gamma \leq x < \theta, \\ e^{-x}, & \theta \leq x. \end{cases} \tag{1.14}$$

THEOREM 4. *Within the HNWUE class as above,*

$$\Delta(F, \text{Exp}) \leq \Delta(F_{\text{HW}}, \text{Exp}) = \left(e^{-\theta} - (1 - \theta)\right)/\theta, \tag{1.15}$$

where

$$2\rho = \theta(1 + e^{-\theta}) - 2(1 - e^{-\theta}), \tag{1.16}$$

for some $\theta > 0$, *and* F_{HW} *is HNWUE and is given by*

$$\overline{F}_{\text{HW}}(x) = \begin{cases} (1 - e^{-\theta})/\theta, & 0 \leq x < \theta, \\ e^{-x}, & x \geq \theta. \end{cases} \tag{1.17}$$

It is not difficult to check from these expressions that as $\rho \to 0$ we have

$$\Delta(F_{\text{HB}}, \text{Exp}) = (3\rho)^{1/3}(1 + o(1)), \tag{1.18}$$

$$\Delta(F_{\text{HW}}, \text{Exp}) = (3\rho/2)^{1/3}(1 + o(1)). \tag{1.19}$$

Indeed, with a little more algebra and calculus these results can be strengthened to

$$\Delta(F_{\text{HB}}, \text{Exp}) \leq 1 - \exp\left(-(3\rho)^{1/3}\right) \leq (3\rho)^{1/3}, \tag{1.20}$$

$$\Delta(F_{\text{HW}}, \text{Exp}) \leq (3\rho/2)^{1/3}. \tag{1.21}$$

The converse question of whether knowing $\Delta(F, \text{Exp})$ implies an upper bound on ρ is discussed in the final section of the paper.

When the results and methods of this paper are compared with those of Brown and Ge (1984), they can be seen as furnishing yet another example of inequalities established to the right order (here, $\rho^{1/2}$), via Fourier methods, but with a constant around double the size of the best possible constant [see, e.g., Daley (1980) for a similar note and references concerning the renewal function].

The proofs below are given verbally. Mostly, it is simplest to argue from a diagram, sketching the exponential function, the tails $\bar{F}$ and $\bar{G}$ of distribution functions that are, respectively, monotonic, and monotonic and convex, and considering areas of appropriate regions that are bounded at least in part by a straight line or lines. Detailed notes that expand the proofs sketched in Sections 5 and 6 are available from the author.

2. Proofs of the propositions. We start by observing that since the (right-hand) derivative of $-\log \bar{G}(x)$ equals $\bar{F}(x)/\bar{G}(x)$, we necessarily have by integration on (x, y) for $0 \le x < y < \infty$ that

$$\bar{G}(x)/e^{-x} \ge \bar{G}(y)/e^{-y} \qquad (F \text{ NBUE}), \tag{2.1a}$$

$$\bar{G}(y)/e^{-x} \le \bar{G}(y)/e^{-y} \qquad (F \text{ NWUE}). \tag{2.1b}$$

In particular, as is well known, $\bar{G}(x) \le$ or $\ge e^{-x}$ (all x) according as F is NBUE or NWUE, respectively.

Consider first the case that F is NWUE, and suppose that for some given $z \ge 0$,

$$\bar{G}(z) = \bar{F}(z) + p,$$

for some $p > 0$. Then since ρ equals the area between the curves $\bar{F}(x)$ and $\bar{G}(x)$ over $0 \le x < \infty$, $\bar{G}(z)$ has slope $-\bar{F}(x)$ at z, $\bar{G}(x)$ is convex on $x \ge 0$, and $\bar{F}(x)$ is nonincreasing in x, the right-angled triangle with two sides parallel to the axes through $(z, \bar{F}(z))$ and hypotenuse the tangent to $\bar{G}(x)$ at z, has area equal to $(p/2)(p/\bar{F}(z))$ which cannot exceed ρ, i.e.,

$$p^2 \le 2\rho(\bar{F}(z)) = 2\rho(\bar{G}(z) - p),$$

so the quadratic inequality yields

$$p \le -\rho + \left(\rho^2 + 2\rho\bar{G}(z)\right)^{1/2} \le p_\rho,$$

since $\bar{G}(z) \le 1$ (all z). It is readily checked that for F equal to F_W,

$$1 - \bar{F}(0+) = \bar{G}(0+) - \bar{F}(0+) = p_\rho,$$

which establishes Proposition 2.

Suppose now that F is NBUE, and that for some z we have

$$\bar{F}(z-) = \bar{G}(z) + q.$$

This time consider the right-angled triangle with two sides parallel to the axes through $(z, \bar{F}(z-))$ and hypotenuse the intercept of the line from $(0, 1)$ to $(z, \bar{G}(z))$. Since $\bar{F}(x)$ is nondecreasing as x decreases and $\bar{G}(x)$ is convex with $\bar{G}(0) = 1$, the area of the triangle, equal to $(q/2)z(q/(1 - \bar{G}(z)))$, is again bounded above by ρ, i.e.,

$$\rho \ge q^2 z/2(1 - \bar{G}(z)).$$

Now $1 - \bar{G}(z) = \int_0^z \bar{F}(u)\,du \le z$, whence by substitution in the inequality for ρ it follows that $q \le (2\rho)^{1/2}$. It is readily checked that equality holds here for $F = F_B$ and $z = (2\rho)^{1/2}$, establishing Proposition 1.

3. Proof of Theorem 1. As an aside, we start by noting that if for some $z > 0$ we have

$$e^{-x} \geq \bar{F}(x) \geq \bar{G}(x), \quad \text{for } 0 \leq x \leq z, \tag{3.1}$$

then from $1 - e^{-z} \geq \int_0^z \bar{F}(u)\,du = 1 - \bar{G}(z)$ we have $\bar{G}(z) \geq e^{-z}$, implying that if (3.1) holds, then it holds with equality throughout, and that if $\bar{F}(z) < e^{-z}$ for some $z > 0$, then for some z' in $0 < z' < z$ we have $\bar{F}(z') > e^{-z'}$.

Let F be NBUE as in Section 1. Suppose that for some z we have

$$\bar{F}(z-) = e^{-z} + q,$$

for some $q > 0$. As in Section 2 we consider a right-angled triangle with sides parallel to the axes intersecting at $(z, \bar{F}(z-))$ and hypotenuse the tangent to the unit exponential function e^{-x}, where it intersects the level $\bar{F}(z-)$. This intersection point is at a distance $y(q)$, say, from $(z, \bar{F}(z-))$, being determined by

$$\bar{F}(z-) = e^{-(z-y(q))} = e^{-z} + q. \tag{3.2}$$

The area of the triangle is bounded by ρ because $e^{-x} \geq \bar{G}(x)$ when F is NBUE, and $\bar{G}(x)$ being convex, it lies always below the hypotenuse for $z - y(q) \leq x \leq z$. Thus,

$$\rho \geq (y(q)/2)y(q)\bar{F}(z-),$$

i.e., $y(q) \leq (2\rho/\bar{F}(z-))^{1/2}$. Then from (3.2) we have

$$q = \bar{F}(z-)(1 - e^{-y(q)}) \leq \bar{F}(z-)\left\{1 - \exp\left(-(2\rho/\bar{F}(z-))^{1/2}\right)\right\}.$$

It is a matter of simple calculus to check that $\xi^2(1 - e^{-\alpha/\xi})$ increases for $0 \leq \xi \leq 1$, so

$$q \leq 1 - \exp\left(-(2\rho)^{1/2}\right). \tag{3.3}$$

The d.f. F_B at (1.4) is NBUE and for this function $\bar{F}_B(z-) - e^{-z}$ equals q as at (3.3) when $z = (2\rho)^{1/2}$.

It remains to consider how large $e^{-z} - \bar{F}(z)$ may be for NBUE F. Denoting it by q, we have from (2.1a) that for $y \geq z$,

$$\bar{G}(y) \leq \bar{G}(z)e^{-(y-z)} \leq \bar{F}(z)e^{-(y-z)} = (e^{-z} - q)e^{-(y-z)}$$
$$= e^{-y} - qe^{-(y-z)},$$

so that

$$\rho = \int_0^\infty (e^{-u} - \bar{G}(u))\,du$$
$$\geq \int_z^\infty (e^{-u} - \bar{G}(u))\,du$$
$$\geq \int_z^\infty qe^{-(u-z)}\,du = q.$$

Now $1 - \exp(-(2\rho)^{1/2}) \geq (2\rho)^{1/2} - \rho$, which in turn exceeds ρ for $0 < 2\rho < 1$ as holds for NBUE F. Thus, $e^{-z} - \bar{F}(z)$ is again bounded by the right-hand side of (1.3), and Theorem 1 is proved.

4. Proof of Theorem 2. Supposing now that F is NWUE and that $e^{-z} > \bar{F}(z) = e^{-z} - q$ for some $q > 0$ and some $z > 0$, it follows from $\bar{G}(x) \geq e^{-x}$ (all x) that $q \leq \bar{G}(z) - \bar{F}(z)$, and thus, from Proposition 2, that $q \leq (\rho^2 + 2\rho)^{1/2} - \rho$. Since this bound is attained for F_W as $1 - F_W(0+)$, it remains only to consider the possibility that for some z, $\bar{G}(z) \geq \bar{F}(z) > e^{-z} = \bar{F}(z) - q$.

From (2.1b) it follows that

$$\rho = \int_0^\infty (\bar{G}(u) - e^{-u})\, du$$
$$\geq \int_z^\infty (\bar{G}(u) - e^{-u})\, du$$
$$\geq \bar{G}(z) - e^{-z} \geq q,$$

and since $(\rho^2 + 2\rho)^{1/2} - \rho$ increases with ρ and exceeds ρ when $\rho < 2/3$, it suffices to show that we always have such $q < 2/3$. Writing

$$\bar{G}(z) - e^{-z} = \int_0^z (e^{-u} - \bar{F}(u))\, du$$
$$\leq \int_0^z (e^{-u} - e^{-z} + q)\, du$$
$$= 1 - e^{-z} - z(e^{-z} + q),$$

it follows that $q \leq (1+z)^{-1} - e^{-z}$, and since the right-hand side here always $\leq 1/4$ in $z > 0$, the theorem is finally proved.

5. Proof of Theorem 3. We sketch the key steps in the proof. Suppose that for some z we have either $\bar{F}(z-) - e^{-z} > 0$ or $e^{-z} - \bar{F}(z) > 0$. In either event, the area between $\bar{G}(x)$ and e^{-x} must be at least that bounded by the exponential function and two straight-line segments, intersecting at $(z, \bar{G}(z))$ and tangential to the exponential function at $z - \xi$ and $z + \eta$, say, respectively. Since this area is bounded by ρ, an upper bound on $|e^{-z} - \bar{F}(z)|$ is thereby implied, with the greatest such bound being achieved by maximizing the area between $\bar{G}(x)$ and e^{-x} into a "triangular" region with vertices at $(\alpha, e^{-\alpha})$, $(\alpha + \theta, e^{-\alpha-\theta})$, and the intersection at $(\alpha + \gamma, \lambda)$, say, of the tangents to the exponential function at the other two vertices. Immediately, for such $\bar{G}(x)$ and, therefore, F,

$$\lambda = e^{\alpha} - \gamma e^{-\alpha} = (1 - \gamma)e^{-\alpha},$$
$$\Delta \equiv \Delta(F, \mathrm{Exp}) = \max(e^{-\alpha} - e^{-\alpha-\gamma}, e^{-\alpha-\gamma} - e^{-\alpha-\theta}),$$

so

$$\Delta e^{\alpha} = \max(1 - e^{-\gamma}, e^{-\gamma} - e^{-\theta}),$$

and after computing areas,

$$\gamma = 1 - \theta e^{-\theta}/(1 - e^{-\theta}),$$
$$2\rho e^{\alpha} = 2(1 - e^{-\theta}) - \theta(1 + e^{-\theta}) + \gamma(e^{-\theta} - (1 - \theta))$$
$$= 1 - e^{-\theta} - \theta^2 e^{-\theta}/(1 - e^{-\theta}).$$

Show that

$$\max(1 - e^{-\gamma}, e^{-\gamma} - e^{-\theta}) = 1 - e^{-\gamma}$$

by establishing

$$\gamma \equiv 1 - \theta e^{-\theta}/(1 - e^{-\theta}) \geq -\log[(1 + e^{-\theta})/2],$$

by differentiating in θ, and finding that $d\gamma/d\theta$ exceeds the derivative of the right-hand side. Next, noting that for fixed ρ we can regard θ as a function of α, γ as a function of θ, and thus Δ a function of α, we have

$$(\Delta + d\Delta/d\alpha)e^{\alpha} = e^{-\gamma}\, d\gamma/d\alpha = e^{-\gamma}(d\gamma/d\theta)(d\theta/d\alpha).$$

Substituting $e^{-\theta} = (1 - \gamma)/(1 + \theta - \gamma)$, deduce that $d\Delta/d\alpha$ has the same sign as $-1 + \gamma e^{-\gamma}(1 + 1/(\theta - \gamma))$, which is shown to be negative via $e^{\gamma} > \gamma + \gamma/(\theta - \gamma)$ and $\gamma < \theta - \gamma$, i.e., $\theta > 2\gamma$ (in this last step, expand $e^{-\theta}$ to the term in θ^5). Thus, $\alpha = 0$ and (1.11)–(1.14) follow.

(1.18) is proved by noting from (1.11)–(1.13) that

$$\gamma = (\theta/2)(1 + O(\theta)), \qquad 1 - e^{-\gamma} = (\theta/2)(1 + O(\theta)),$$
$$\left[(1 - e^{-\theta})^2 - \theta^2 e^{-\theta}\right]/(1 - e^{-\theta}) = (\theta^3/12)(1 + O(\theta)).$$

To prove (1.20), the right-hand inequality is simply $1 - e^{-x} < x$, $x > 0$, while for the rest, it is enough to show [cf. (1.11) and (1.20)] that $\gamma^3 \leq 3\rho$. Since both sides vanish at $\theta = 0$, it is enough to show that $\gamma^2 d\gamma/d\theta \leq d\rho/d\theta$ (all θ), which on simplification reduces to $2\gamma^2 \leq \theta - 1 + e^{-\theta}$. Again, both sides vanish at $\theta = 0$, so it is enough that $4\gamma\, d\gamma/d\theta \leq 1 - e^{-\theta}$, which reduces to $4\gamma(\theta - \gamma) \leq e^{\theta}(1 - e^{-\theta})^2$, which from $4x(1 - x) \leq 1$ means that it is enough that $\theta e^{-\theta/2} \leq 1 - e^{-\theta}$, or equivalently, that $\theta \leq e^{\theta/2} - e^{-\theta/2}$ for $\theta > 0$, which is a standard inequality.

6. Proof of Theorem 4. Again, we merely sketch the proof. Supposing $q \equiv e^{-z} - \overline{F}(z) > 0$ for some $z > 0$, we proceed (cf. the proofs of the results for the NWUE case) by noting that the area between $\overline{G}(x)$ and e^{-x}, which in total equals ρ, is at least as large as the area between the unit exponential function and a straight line intersecting the exponential at z and $z + \xi$, say, with slope $-\overline{F}(z) = -e^{-z} + q$. Similarly, if $q = \overline{F}(z-) - e^{-z} > 0$ for some $z > 0$, then the area between $\overline{G}(x)$ and e^{-x} is at least as large as that between a straight line with slope $-\overline{F}(z-)$ and the unit exponential between intercepts at (z, e^{-z}) and $(z - \eta, e^{-z+\eta})$, say. In either case, for given z, the quantity q is maximized by having the area between the straight line and the exponential function equal to ρ.

Consider then such a straight line with intercepts with the exponential at $(\alpha, e^{-\alpha})$, $(\alpha + \theta, e^{-\alpha-\theta})$ for some $\alpha > 0$, $\theta > 0$. Computing areas, we have

$$\Delta = e^{-\alpha} - \overline{F}(\alpha+),$$
$$2\rho e^{\alpha} = \theta(1 + e^{-\theta}) - 2(1 - e^{-\theta}),$$

with

$$\theta\overline{F}(\alpha+) = e^{-\alpha}(1 - e^{-\theta}),$$

so

$$\Delta e^{\alpha} = 1 - (1 - e^{-\theta})/\theta.$$

As in Section 5, $d\Delta/d\alpha$ has the same sign as $-\Delta + 2\rho/\theta^2$, and thus as

$$\theta(1 + e^{-\theta}) - 2(1 - e^{-\theta}) - \theta^2 + \theta(1 - e^{-\theta}) = -2(1 - \theta + \theta^2/2 - e^{-\theta}) < 0,$$

so Δ is maximized at $\alpha = 0$. (1.15)–(1.17) follow.

Expanding (1.16) and (1.17) in powers of θ yields (1.19).

To prove (1.21) examine the sign of the derivative in θ of $3\rho - 2\Delta^3$. It has the same sign as $\theta^2 - 4\Delta^2$, and thus of $\theta - 2\Delta$, which is positive for $\theta > 0$.

7. The complementary lower-bound problem. Since $\Delta(F, \text{Exp}) = 0$ if and only if F is an exponential distribution, a question which then arises is the following. Given $\rho > 0$, is there some $\Delta_\rho > 0$ such that

$$\Delta(F, \text{Exp}) \geq \Delta_\rho, \tag{7.1}$$

whenever F is in some specified class of distributions with ρ as in (1.1)? Or is it the case that for some classes we can find a sequence $\{F_n\}$ of distributions in the class for which

$$\Delta(F_n, \text{Exp}) \to 0, \tag{7.2}$$

as $n \to \infty$?

THEOREM 5. *Within the HNBUE class as in Section* 1,

$$\Delta(F, \exp) \geq \Delta_\rho > 0, \tag{7.3}$$

where the lower bound is attained by a NBUE distribution with an atom of size $\delta + e^{-\gamma}$ *at* θ *and unit exponential density on* (α, γ), *where* $\delta = 1 - e^{-\alpha}$ *and, with* ψ *the root of*

$$\alpha - \delta + \delta(\psi - \alpha) = e^{-\psi}, \tag{7.4}$$

$\theta = \gamma = \psi$ *when* $e^{-\psi} < \delta$ *and*

$$2\rho = \alpha(2\theta - \alpha) + \delta(\theta - \alpha)^2, \tag{7.5}$$

and otherwise $\gamma = -\log \delta$,

$$\theta = 1 + (\alpha + \gamma)/2 - \alpha/2\delta, \tag{7.6}$$

$$2\rho = 2\delta\theta - 2\delta\theta^2 + \gamma^2 e^{-\gamma} - \alpha^2 e^{-\alpha}. \tag{7.7}$$

Within the HNWUE class, (7.2) holds rather than (7.1).

THEOREM 6. *For any given* ρ, *there exist HNWUE distributions* $\{F_n\}$ *for which* (7.2) *holds.*

Proof of Theorem 5. Suppose that F is HNBUE with $\Delta(F, \mathrm{Exp}) = \delta > 0$, so that for all x,

$$\max(0, e^{-x} - \delta) \leq \overline{F}(x) \leq \min(1, e^{-x} + \delta).$$

From the second inequality, writing $\alpha = -\log(1 - \delta)$, we have

$$G(x) \leq \begin{cases} x, & 0 \leq x \leq \alpha, \\ \alpha + 1 - \delta - e^{-x} + \delta(x - \alpha), & x \geq \alpha, \end{cases}$$

so that

$$e^{-x} - \overline{G}(x) \leq \begin{cases} e^{-x} - (1 - x), & 0 \leq x \leq \alpha, \\ \alpha - \delta + \delta(x - \alpha), & x \geq \alpha. \end{cases} \tag{7.8}$$

From the first inequality, writing $\beta = -\log \delta$, we likewise obtain

$$e^{-x} - \overline{G}(x) \leq \begin{cases} e^{-x}, & x \geq \beta, \\ \delta + \delta(\beta - x), & 0 \leq x \leq \beta. \end{cases} \tag{7.9}$$

It is easily checked that (7.8) is smaller for small x and (7.9) for large x, with the change-over point being the root ψ of (7.4) if $e^{-\psi} < \delta$, and otherwise θ as at (7.6). In either case, substituting the appropriate bound for $e^{-x} - \overline{G}(x)$ leads to retention of equality in

$$\rho = \left(\int_0^\alpha + \int_\alpha^\theta + \int_\theta^\gamma + \int_\gamma^\infty\right)(e^{-x} - \overline{G}(x))\, dx,$$

and the rest of the proof is a matter of algebra. □

It should be noted that for sufficiently small δ, the distribution detailed in the theorem is not NBU.

Proof of Theorem 6. We sketch the idea, and leave the algebraic detail to the reader. First, notice that a unit exponential distribution perturbed by adding atoms Δ at 0 and δ at (large) θ and deleting an interval (α, β) interior to $(0, \theta)$ of mass $\Delta + \delta = e^{-\alpha} - e^{-\beta}$ is HNWUE (a sketch graph shows this easily). Its first moment is unchanged at one if

$$\delta\theta = (\Delta + \delta)\eta,$$

for a certain η with $\alpha < \eta < \beta$, and its second moment is approximately

$$2 + 2\rho \cong 2 + \delta\theta^2 = (\Delta + \delta)\eta^2 = 2 + \delta\theta^2 - \delta\theta\eta.$$

Choose $\delta \cong 2\rho/\theta^2$, $\Delta = 2\rho/\theta$, so $\eta \cong 1$, and $\Delta(F, \mathrm{Exp}) = 2\rho/\theta$, $\to 0$ as $\theta \to \infty$.
Such a distribution is not NWUE because $\overline{F}(x) = \overline{G}(x) = e^{-x}$ for $x > \theta$. □

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