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Tight Distance-Regular Graphs

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Abstract. We consider a distance-regular graph Γ with diameter $d \geq 3$ and eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_d$. We show the intersection numbers a_1, b_1 satisfy

$$\left(\theta_1 + \frac{k}{a_1+1}\right)\left(\theta_d + \frac{k}{a_1+1}\right) \geq -\frac{ka_1b_1}{(a_1+1)^2}.$$

We say Γ is *tight* whenever Γ is not bipartite, and equality holds above. We characterize the tight property in a number of ways. For example, we show Γ is tight if and only if the intersection numbers are given by certain rational expressions involving d independent parameters. We show Γ is tight if and only if $a_1 \neq 0$, $a_d = 0$, and Γ is 1-homogeneous in the sense of Nomura. We show Γ is tight if and only if each local graph is connected strongly-regular, with nontrivial eigenvalues $-1 - b_1(1+\theta_1)^{-1}$ and $-1 - b_1(1+\theta_d)^{-1}$. Three infinite families and nine sporadic examples of tight distance-regular graphs are given.

Keywords: distance-regular graph, equality, tight graph, homogeneous, locally strongly-regular parameterization

1. Introduction

Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$, and eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_d$ (see Section 2 for definitions). We show the intersection numbers a_1, b_1 satisfy

$$\left(\theta_1 + \frac{k}{a_1+1}\right)\left(\theta_d + \frac{k}{a_1+1}\right) \geq -\frac{ka_1b_1}{(a_1+1)^2}. \tag{1}$$

We define Γ to be *tight* whenever Γ is not bipartite, and equality holds in (1). We characterize the tight condition in the following ways.

Our first characterization is linear algebraic. For all vertices $x \in X$, let $\hat{x}$ denote the vector in $\mathbb{R}^X$ with a 1 in coordinate x, and 0 in all other coordinates. Suppose for the moment that $a_1 \neq 0$, let x, y denote adjacent vertices in X, and write $w = \sum \hat{z}$, where the sum is over all vertices $z \in X$ adjacent to both x and y. Let θ denote one of $\theta_1, \theta_2, \ldots, \theta_d$, and let E denote the corresponding primitive idempotent of the Bose-Mesner algebra. We say the edge xy is *tight with respect to* θ whenever $E\hat{x}$, $E\hat{y}$, Ew are linearly dependent. We show that if xy is tight with respect to θ, then θ is one of θ_1, θ_d. Moreover, we show the following are equivalent: (i) Γ is tight; (ii) $a_1 \neq 0$ and all edges of Γ are tight with respect to both θ_1, θ_d; (iii) $a_1 \neq 0$ and there exists an edge of Γ which is tight with respect to both θ_1, θ_d.

Our second characterization of the tight condition involves the intersection numbers. We show Γ is tight if and only if the intersection numbers are given by certain rational expressions involving d independent variables.

Our third characterization of the tight condition involves the concept of *1-homogeneous* that appears in the work of Nomura [14–16]. See also Curtin [7]. We show the following are equivalent: (i) Γ is tight; (ii) $a_1 \neq 0$, $a_d = 0$, and Γ is 1-homogeneous; (iii) $a_1 \neq 0$, $a_d = 0$, and Γ is 1-homogeneous with respect to at least one edge.

Our fourth characterization of the tight condition involves the local structure and is reminiscent of some results by Cameron, Goethals and Seidel [5] and Dickie and Terwilliger [8]. For all $x \in X$, let $\Delta(x)$ denote the vertex subgraph of Γ induced on the vertices in X adjacent to x. For notational convenience, define $b^+ := -1 - b_1(1+\theta_d)^{-1}$ and $b^- := -1 - b_1(1+\theta_1)^{-1}$. We show the following are equivalent: (i) Γ is tight; (ii) for all $x \in X$, $\Delta(x)$ is connected strongly-regular with nontrivial eigenvalues b^+, b^-; (iii) there exists $x \in X$ such that $\Delta(x)$ is connected strongly-regular with nontrivial eigenvalues b^+, b^-.

We present three infinite families and nine sporadic examples of tight distance-regular graphs. These are the Johnson graphs $J(2d, d)$, the halved cubes $\frac{1}{2}H(2d, 2)$, the Taylor graphs [19], four 3-fold antipodal covers of diameter 4 constructed from the sporadic Fisher groups [3, p. 397], two 3-fold antipodal covers of diameter 4 constructed by Soicher [18], a 2-fold and a 4-fold antipodal cover of diameter 4 constructed by Meixner [13], and the Patterson graph [3, Thm. 13.7.1], which is primitive, distance-transitive and of diameter 4.

2. Preliminaries

In this section, we review some definitions and basic concepts. See the books of Bannai and Ito [1] or Brouwer, Cohen, and Neumaier [3] for more background information.

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X, edge set R, path-length distance function ∂, and diameter $d := \max\{\partial(x, y) \mid x, y \in X\}$. For all $x \in X$ and for all integers i, we set $\Gamma_i(x) := \{y \in X \mid \partial(x, y) = i\}$. We abbreviate $\Gamma(x) := \Gamma_1(x)$. By the *valency* of a vertex $x \in X$, we mean the cardinality of $\Gamma(x)$. Let k denote a nonnegative integer. Then Γ is said to be *regular, with valency k*, whenever each vertex in X has valency k. Γ is said to be *distance-regular* whenever for all integers h, i, j $(0 \leq h, i, j \leq d)$, and for all $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{ij} := |\Gamma_i(x) \cap \Gamma_j(y)|$$

is independent of x and y. The constants p^h_{ij} are known as the *intersection numbers* of Γ. For notational convenience, set $c_i := p^i_{1i-1}$ $(1 \leq i \leq d)$, $a_i := p^i_{1i}$ $(0 \leq i \leq d)$, $b_i := p^i_{1i+1}$ $(0 \leq i \leq d-1)$, $k_i := p^0_{ii}$ $(0 \leq i \leq d)$, and define $c_0 = 0$, $b_d = 0$. We note $a_0 = 0$ and

$c_1 = 1$. From now on, $\Gamma = (X, R)$ will denote a distance-regular graph of diameter $d \geq 3$. Observe Γ is regular with valency $k = k_1 = b_0$, and that

$$k = c_i + a_i + b_i \quad (0 \leq i \leq d). \tag{2}$$

We now recall the Bose-Mesner algebra. Let $\mathrm{Mat}_X(\mathbb{R})$ denote the $\mathbb{R}$-algebra consisting of all matrices with entries in $\mathbb{R}$ whose rows and columns are indexed by X. For each integer i $(0 \leq i \leq d)$, let A_i denote the matrix in $\mathrm{Mat}_X(\mathbb{R})$ with x, y entry

$$(A_i)_{xy} = \begin{cases} 1, & \text{if } \partial(x, y) = i, \\ 0, & \text{if } \partial(x, y) \neq i \end{cases} \quad (x, y \in X).$$

A_i is known as the *ith distance matrix* of Γ. Observe

$$A_0 = I, \tag{3}$$
$$A_0 + A_1 + \cdots + A_d = J \qquad (J = \text{all 1's matrix}), \tag{4}$$
$$A_i^t = A_i \qquad (0 \leq i \leq d), \tag{5}$$
$$A_i A_j = \sum_{h=0}^{d} p_{ij}^h A_h \quad (0 \leq i, j \leq d). \tag{6}$$

We abbreviate $A := A_1$, and refer to this as the *adjacency matrix* of Γ. Let M denote the subalgebra of $\mathrm{Mat}_X(\mathbb{R})$ generated by A. We refer to M as the *Bose-Mesner algebra* of Γ. Using (3)–(6), one can readily show $A_0, A_1, \ldots, A_d$ form a basis for M. By [1, pp. 59, 64], the algebra M has a second basis $E_0, E_1, \ldots, E_d$ such that

$$E_0 = |X|^{-1} J, \tag{7}$$
$$E_0 + E_1 + \cdots + E_d = I, \tag{8}$$
$$E_i^t = E_i \qquad (0 \leq i \leq d), \tag{9}$$
$$E_i E_j = \delta_{ij} E_i \quad (0 \leq i, j \leq d). \tag{10}$$

The $E_0, E_1, \ldots, E_d$ are known as the *primitive idempotents* of Γ. We refer to E_0 as the *trivial* idempotent.

Let $\theta_0, \theta_1, \ldots, \theta_d$ denote the real numbers satisfying $A = \sum_{i=0}^{d} \theta_i E_i$. Observe $AE_i = E_i A = \theta_i E_i$ for $0 \leq i \leq d$, and that $\theta_0, \theta_1, \ldots, \theta_d$ are distinct since A generates M. It follows from (7) that $\theta_0 = k$, and it is known $-k \leq \theta_i \leq k$ for $0 \leq i \leq d$ [1, p. 197]. We refer to θ_i as the *eigenvalue* of Γ associated with E_i, and call θ_0 the *trivial* eigenvalue. For each integer i $(0 \leq i \leq d)$, let m_i denote the rank of E_i. We refer to m_i as the *multiplicity* of E_i (or θ_i). We observe $m_0 = 1$.

We now recall the cosines. Let θ denote an eigenvalue of Γ, and let E denote the associated primitive idempotent. Let $\sigma_0, \sigma_1, \ldots, \sigma_d$ denote the real numbers satisfying

$$E = |X|^{-1} m \sum_{i=0}^{d} \sigma_i A_i, \tag{11}$$

where m denotes the multiplicity of θ. Taking the trace in (11), we find $\sigma_0 = 1$. We often abbreviate $\sigma = \sigma_1$. We refer to σ_i as the *ith cosine* of Γ with respect to θ (or E), and call $\sigma_0, \sigma_1, \ldots, \sigma_d$ the *cosine sequence* of Γ associated with θ (or E). We interpret the cosines as follows. Let $\mathbb{R}^X$ denote the vector space consisting of all column vectors with entries in $\mathbb{R}$ whose coordinates are indexed by X. We observe $\mathrm{Mat}_X(\mathbb{R})$ acts on $\mathbb{R}^X$ by left multiplication. We endow $\mathbb{R}^X$ with the Euclidean inner product satisfying

$$\langle u, v\rangle = u^t v \quad (u, v \in \mathbb{R}^X), \tag{12}$$

where t denotes transposition. For each $x \in X$, let $\hat{x}$ denote the element in $\mathbb{R}^X$ with a 1 in coordinate x, and 0 in all other coordinates. We note $\{\hat{x} \mid x \in X\}$ is an orthonormal basis for $\mathbb{R}^X$.

Lemma 2.1 *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$. Let E denote a primitive idempotent of Γ, and let $\sigma_0, \sigma_1, \ldots, \sigma_d$ denote the associated cosine sequence. Then for all integers i $(0 \leq i \leq d)$, and for all $x, y \in X$ such that $\partial(x, y) = i$, the following* (i)–(iii) *hold.*

(i) $\langle E\hat{x}, E\hat{y}\rangle = m|X|^{-1}\sigma_i$, *where m denotes the multiplicity of E.*
(ii) *The cosine of the angle between the vectors $E\hat{x}$* and *$E\hat{y}$ equals σ_i.*
(iii) $-1 \leq \sigma_i \leq 1$.

Proof: Line (i) is a routine application of (10), (11), (12). Line (ii) is immediate from (i), and (iii) is immediate from (ii). □

Lemma 2.2 [3, Sect. 4.1.B] *Let Γ denote a distance-regular graph with diameter $d \geq 3$. Then for any complex numbers $\theta, \sigma_0, \sigma_1, \ldots, \sigma_d$, the following are equivalent.*

(i) *θ is an eigenvalue of Γ, and $\sigma_0, \sigma_1, \ldots, \sigma_d$ is the associated cosine sequence.*
(ii) $\sigma_0 = 1$, *and*

$$c_i\sigma_{i-1} + a_i\sigma_i + b_i\sigma_{i+1} = \theta\sigma_i \quad (0 \leq i \leq d), \tag{13}$$

where σ_{-1} and σ_{d+1} are indeterminates.

(iii) $\sigma_0 = 1$, $k\sigma = \theta$, *and*

$$c_i(\sigma_{i-1} - \sigma_i) - b_i(\sigma_i - \sigma_{i+1}) = k(\sigma - 1)\sigma_i \quad (1 \leq i \leq d), \tag{14}$$

where σ_{d+1} is an indeterminate.

For later use we record a number of consequences of Lemma 2.2.

Lemma 2.3 *Let Γ denote a distance-regular graph with diameter $d \geq 3$. Let θ denote an eigenvalue of Γ, and let $\sigma_0, \sigma_1, \ldots, \sigma_d$ denote the associated cosine sequence. Then* (i)–(vi) *hold below.*

(i) $kb_1\sigma_2 = \theta^2 - a_1\theta - k$.
(ii) $kb_1(\sigma - \sigma_2) = (k - \theta)(1 + \theta)$.

(iii) $kb_1(1-\sigma_2) = (k-\theta)(\theta+k-a_1)$.
(iv) $k^2 b_1(\sigma^2-\sigma_2) = (k-\theta)(k+\theta(a_1+1))$.
(v) $c_d(\sigma_{d-1}-\sigma_d) = k(\sigma-1)\sigma_d$.
(vi) $a_d(\sigma_{d-1}-\sigma_d) = k(\sigma_{d-1}-\sigma\sigma_d)$.

Proof: To get (i), set $i = 1$ in (13), and solve for σ_2. Lines (ii)–(iv) are routinely verified using (i) above and $k\sigma = \theta$. To get (v), set $i = d$, $b_d = 0$ in Lemma 2.2 (iii). To get (vi), set $c_d = k - a_d$ in (v) above, and simplify the result. □

In this article, the second largest and minimal eigenvalue of a distance-regular graph turn out to be of particular interest. In the next several lemmas, we give some basic information on these eigenvalues.

Lemma 2.4 [9, Lem. 13.2.1] *Let Γ denote a distance-regular graph with diameter $d \geq 3$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Let θ denote one of θ_1, θ_d and let $\sigma_0, \sigma_1, \ldots, \sigma_d$ denote the cosine sequence for θ.*
(i) *Suppose $\theta = \theta_1$. Then $\sigma_0 > \sigma_1 > \cdots > \sigma_d$.*
(ii) *Suppose $\theta = \theta_d$. Then $(-1)^i\sigma_i > 0 \quad (0 \leq i \leq d)$.*

Recall a distance-regular graph Γ is *bipartite* whenever the intersection numbers satisfy $a_i = 0$ for $0 \leq i \leq d$, where d denotes the diameter.

Lemma 2.5 *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$. Let θ_d denote the minimal eigenvalue of Γ, and let $\sigma_0, \sigma_1, \ldots, \sigma_d$ denote the associated cosine sequence. Then the following are equivalent*: (i) *Γ is bipartite*; (ii) $\theta_d = -k$; (iii) $\sigma_1 = -1$; (iv) $\sigma_2 = 1$. *Moreover, suppose* (i)–(iv) *hold. Then $\sigma_i = (-1)^i$ for $0 \leq i \leq d$.*

Proof: The equivalence of (i), (ii) follows from [3, Prop. 3.2.3]. The equivalence of (ii), (iii) is immediate from $k\sigma_1 = \theta_d$. The remaining implications follow from [3, Prop. 4.4.7]. □

Lemma 2.6 *Let Γ denote a distance-regular graph with diameter $d \geq 3$ and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Then* (i)–(iii) *hold below.*
(i) $0 < \theta_1 < k$.
(ii) $a_1 - k \leq \theta_d < -1$.
(iii) *Suppose Γ is not bipartite. Then $a_1 - k < \theta_d$.*

Proof: (i) The eigenvalue θ_1 is positive by [3, Cor. 3.5.4], and we have seen $\theta_1 < k$.
(ii) Let σ_1, σ_2 denote the first and second cosines for θ_d. Then $\sigma_2 \leq 1$ by Lemma 2.1 (iii), so $a_1 - k \leq \theta_d$ in view of Lemma 2.3 (iii). Also $\sigma_1 < \sigma_2$ by Lemma 2.4 (ii), so $\theta_d < -1$ in view of Lemma 2.3 (ii).
(iii) Suppose $\theta_d = a_1 - k$. Applying Lemma 2.3 (iii), we find $\sigma_2 = 1$, where σ_2 denotes the second cosine for θ_d. Now Γ is bipartite by Lemma 2.5, contradicting our assumptions. Hence $\theta_d > a_1 - k$, as desired. □

We mention a few results on the intersection numbers.

Lemma 2.7 *Let $\Gamma = (X, R)$ denote a nonbipartite distance-regular graph with diameter $d \geq 3$, let x, y denote adjacent vertices in X, and let E denote a nontrivial primitive idempotent of Γ. Then the vectors $E\hat{x}$ and $E\hat{y}$ are linearly independent.*

Proof: Let σ denote the first cosine associated to E. Then $\sigma \neq 1$, since E is nontrivial, and $\sigma \neq -1$, since Γ is not bipartite. Applying Lemma 2.1 (ii), we see $E\hat{x}$ and $E\hat{y}$ are linearly independent. □

Lemma 2.8 [3, Prop. 5.5.1] *Let Γ denote a distance-regular graph with diameter $d \geq 3$ and $a_1 \neq 0$. Then $a_i \neq 0$ $(1 \leq i \leq d-1)$.*

Lemma 2.9 [3, Lem. 4.1.7] *Let Γ denote a distance-regular graph with diameter $d \geq 3$. Then the intersection numbers satisfy*

$$p^1_{ii} = \frac{b_1 b_2 \dots b_{i-1}}{c_1 c_2 \dots c_i} a_i, \quad p^1_{i-1,i} = \frac{b_1 b_2 \dots b_{i-1}}{c_1 c_2 \dots c_{i-1}} \quad (1 \leq i \leq d).$$

For the remainder of this section, we describe a point of view we will adopt throughout the paper.

Definition 2.10 Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$, and fix adjacent vertices $x, y \in X$. For all integers i and j we define $D^j_i = D^j_i(x, y)$ by

$$D^j_i = \Gamma_i(x) \cap \Gamma_j(y). \tag{15}$$

We observe $|D^j_i| = p^1_{ij}$ for $0 \leq i, j \leq d$, and $D^j_i = \emptyset$ otherwise. We visualize the D^j_i as follows (figure 1).

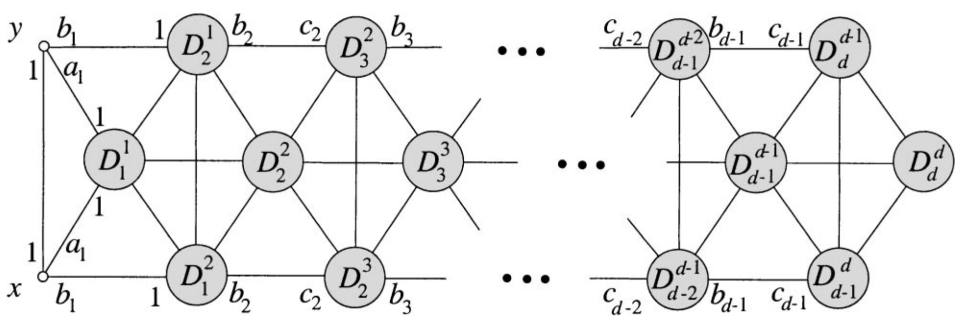


Figure 1. Distance distribution corresponding to an edge. Observe: $D^{i-1}_i \cup D^i_i \cup D^{i+1}_i = \Gamma_i(x)$ for $i = 1, \dots, d$. The number beside edges connecting cells D^j_i indicate how many neighbours a vertex from the closer cell has in the other cell, see Lemma 2.11.

Lemma 2.11 *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$. Fix adjacent vertices $x, y \in X$, and pick any integer i $(1 \leq i \leq d)$. Then with reference to Definition* 2.10*, the following* (i) *and* (ii) *hold.*

(i) *Each $z \in D^i_{i-1}$ (resp. D^{i-1}_i) is adjacent to*

(a)	*precisely*	c_{i-1}	*vertices in D^{i-1}_{i-2} (resp. D^{i-2}_{i-1}),*
(b)	*precisely*	$c_i - c_{i-1} - \lvert\Gamma(z) \cap D^{i-1}_{i-1}\rvert$	*vertices in D^{i-1}_i (resp. D^i_{i-1}),*
(c)	*precisely*	$a_{i-1} - \lvert\Gamma(z) \cap D^{i-1}_{i-1}\rvert$	*vertices in D^i_{i-1} (resp. D^{i-1}_i),*
(d)	*precisely*	b_i	*vertices in D^{i+1}_i (resp. D^i_{i+1}),*
(e)	*precisely*	$a_i - a_{i-1} + \lvert\Gamma(z) \cap D^{i-1}_{i-1}\rvert$	*vertices in D^i_i.*

(ii) *Each $z \in D^i_i$ is adjacent to*

(a)	*precisely*	$c_i - \lvert\Gamma(z) \cap D^{i-1}_{i-1}\rvert$	*vertices in D^i_{i-1},*
(b)	*precisely*	$c_i - \lvert\Gamma(z) \cap D^{i-1}_{i-1}\rvert$	*vertices in D^{i-1}_i,*
(c)	*precisely*	$b_i - \lvert\Gamma(z) \cap D^{i+1}_{i+1}\rvert$	*vertices in D^{i+1}_i,*
(d)	*precisely*	$b_i - \lvert\Gamma(z) \cap D^{i+1}_{i+1}\rvert$	*vertices in D^i_{i+1},*
(e)	*precisely*	$a_i - b_i - c_i + \lvert\Gamma(z) \cap D^{i-1}_{i-1}\rvert + \lvert\Gamma(z) \cap D^{i+1}_{i+1}\rvert$	*vertices in D^i_i.*

Proof: Routine. □

3. Edges that are tight with respect to an eigenvalue

Let $\Gamma = (X, R)$ denote a graph, and let Ω denote a nonempty subset of X. By the *vertex subgraph* of Γ *induced on* Ω, we mean the graph with vertex set Ω, and edge set $\{xy \mid x, y \in \Omega,\ xy \in R\}$.

Definition 3.1 Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$ and intersection number $a_1 \neq 0$. For each edge $xy \in R$, we define the scalar $f = f(x, y)$ by

$$f := a_1^{-1}\left|\left\{(z, w) \in X^2 \,\middle|\, z, w \in D^1_1,\ \partial(z, w) = 2\right\}\right|, \tag{16}$$

where $D^1_1 = D^1_1(x, y)$ is from (15). We observe f is the average valency of the complement of the vertex subgraph induced on D^1_1.

We begin with some elementary facts about f.

Lemma 3.2 *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$ and $a_1 \neq 0$. Let x, y denote adjacent vertices in X. Then with reference to* (15), (16), *lines* (i)–(iv) *hold below.*

(i) *The number of edges in R connecting a vertex in D^1_1 with a vertex in D^2_1 is equal to $a_1 f$.*
(ii) *The number of edges in the vertex subgraph induced on D^1_1 is equal to $a_1(a_1 - 1 - f)/2$.*
(iii) *The number of edges in the vertex subgraph induced on D^2_1 is equal to $a_1(b_1 - f)/2$.*
(iv) $0 \leq f,\ f \leq a_1 - 1,\ f \leq b_1$.

Proof: Routine. □

The following lemma provides another bound for f.

Lemma 3.3 *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$ and $a_1 \neq 0$. Let x, y denote adjacent vertices in X, and write $f = f(x, y)$. Then for each nontrivial eigenvalue θ of Γ,*

$$(k + \theta)(1 + \theta) f \leq b_1(k + \theta(a_1 + 1)). \tag{17}$$

Proof: Let $\sigma_0, \ldots, \sigma_d$ denote the cosine sequence of θ and let E denote the corresponding primitive idempotent. Set

$$w := \sum_{z \in D_1^1} \hat{z},$$

where $D_1^1 = D_1^1(x, y)$ is from (15). Let G denote the Gram matrix for the vectors $E\hat{x}$, $E\hat{y}$, Ew; that is

$$G := \begin{pmatrix} \|E\hat{x}\|^2 & \langle E\hat{x}, E\hat{y}\rangle & \langle E\hat{x}, Ew\rangle \\ \langle E\hat{y}, E\hat{x}\rangle & \|E\hat{y}\|^2 & \langle E\hat{y}, Ew\rangle \\ \langle Ew, E\hat{x}\rangle & \langle Ew, E\hat{y}\rangle & \|Ew\|^2 \end{pmatrix}.$$

On one hand, the matrix G is positive semi-definite, so it has nonnegative determinant. On the other hand, by Lemma 2.1,

$$\begin{aligned} \det(G) &= m^3|X|^{-3}\det \begin{pmatrix} \sigma_0 & \sigma_1 & a_1\sigma_1 \\ \sigma_1 & \sigma_0 & a_1\sigma_1 \\ a_1\sigma_1 & a_1\sigma_1 & a_1(\sigma_0 + (a_1 - f - 1)\sigma_1 + f\sigma_2) \end{pmatrix} \\ &= m^3 a_1 |X|^{-3}(\sigma - 1)((\sigma - \sigma_2)(1 + \sigma) f - (1 - \sigma)(a_1\sigma + 1 + \sigma)), \end{aligned}$$

where m denotes the multiplicity of θ. Since $a_1 > 0$ and $\sigma < 1$, we find

$$(\sigma - \sigma_2)(1 + \sigma) f \leq (1 - \sigma)(a_1\sigma + 1 + \sigma). \tag{18}$$

Eliminating σ, σ_2 in (18) using $\theta = k\sigma$ and Lemma 2.3(ii), and simplifying the result using $\theta < k$, we routinely obtain (17). □

Corollary 3.4 *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$ and $a_1 \neq 0$. Let x, y denote adjacent vertices in X, and let θ denote a nontrivial eigenvalue of Γ. Then with reference to Definition 2.10, the following are equivalent.*

(i) *Equality is attained in (17).*

(ii) *$E\hat{x}$, $E\hat{y}$, $\sum_{z \in D_1^1} E\hat{z}$ are linearly dependent.*

(iii) $\sum_{z\in D_1^1} E\hat{z} = \frac{a_1\theta}{k+\theta}(E\hat{x} + E\hat{y})$.

We say the edge xy is *tight with respect to* θ whenever (i)–(iii) hold above.

Proof: (i)⇔(ii) Let the matrix G be as in the proof of Lemma 3.3. Then we find (i) holds if and only if G is singular, if and only if (ii) holds.
(ii)⇒(iii) Γ is not bipartite since $a_1 \neq 0$, so $E\hat{x}$, and $E\hat{y}$ are linearly independent by Lemma 2. It follows

$$\sum_{z\in D_1^1} E\hat{z} = \alpha E\hat{x} + \beta E\hat{y} \tag{19}$$

for some $\alpha, \beta \in \mathbb{R}$. Taking the inner product of (19) with each of $E\hat{x}$, $E\hat{y}$ using Lemma 2.1, we readily obtain $\alpha = \beta = a_1\theta(k+\theta)^{-1}$.
(iii)⇒(ii) Clear. □

Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$, $a_1 \neq 0$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Pick adjacent vertices $x, y \in X$, and write $f = f(x, y)$. Referring to (17), we now consider which of $\theta_1, \theta_2, \ldots, \theta_d$ gives the best bounds for f. Let θ denote one of $\theta_1, \theta_2, \ldots, \theta_d$. Assume $\theta \neq -1$; otherwise (17) gives no information about f. If $\theta > -1$ (resp. $\theta < -1$), line (17) gives an upper (resp. lower) bound for f. Consider the partial fraction decompostion

$$b_1 \frac{k + \theta(a_1+1)}{(k+\theta)(1+\theta)} = \frac{b_1}{k-1}\left(\frac{ka_1}{k+\theta} + \frac{b_1}{1+\theta}\right).$$

Since the map $F : \mathbb{R}\backslash\{-k, -1\} \to \mathbb{R}$, defined by

$$x \mapsto \frac{ka_1}{k+x} + \frac{b_1}{1+x}$$

is strictly decreasing on the intervals $(-k, -1)$ and $(-1, \infty)$, we find in view of Lemma 2.6 that the least upper bound for f is obtained at $\theta = \theta_1$, and the greatest lower bound is obtained at $\theta = \theta_d$.

Theorem 3.5 *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$, $a_1 \neq 0$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. For all edges $xy \in R$,*

$$b_1 \frac{k + \theta_d(a_1+1)}{(k+\theta_d)(1+\theta_d)} \leq f(x, y) \leq b_1 \frac{k + \theta_1(a_1+1)}{(k+\theta_1)(1+\theta_1)}. \tag{20}$$

Proof: This is immediate from (17) and Lemma 2.6. □

Corollary 3.6 *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$, $a_1 \neq 0$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. For all edges $xy \in R$,*

(i) *xy is tight with respect to θ_1 if and only if equality holds in the right inequality of* (20),
(ii) *xy is tight with respect to θ_d if and only if equality holds in the left inequality of* (20),
(iii) *xy is not tight with respect to θ_i for $2 \le i \le d-1$.*

Proof: (i), (ii) Immediate from (17) and Corollary 3.4.
(iii) First suppose $\theta_i = -1$. We do not have equality for $\theta = \theta_i$ in (17), since the left side equals 0, and the right side equals b_1^2. In particular, xy is not tight with respect to θ_i. Next suppose $\theta_i \neq -1$. Then we do not have equality for $\theta = \theta_i$ in (17) in view of the above mentioned fact, that the function F is strictly decreasing on the intervals $(-k, -1)$ and $(-1, \infty)$. □

4. Tight edges and combinatorial regularity

Theorem 4.1 *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \ge 3$ and intersection number $a_1 \neq 0$. Let θ denote a nontrivial eigenvalue of Γ, and let $\sigma_0, \sigma_1, \ldots, \sigma_d$ denote its cosine sequence. Let x, y denote adjacent vertices in X. Then with reference to Definition* 2.10, *the following are equivalent.*
(i) *xy is tight with respect to θ.*
(ii) *For $1 \le i \le d$; both $\sigma_{i-1} \neq \sigma_i$, and for all $z \in D_{i-1}^i$*

$$\left|\Gamma_{i-1}(z) \cap D_1^1\right| = \frac{a_1}{1+\sigma}\frac{\sigma\sigma_{i-1} - \sigma_i}{\sigma_{i-1} - \sigma_i}, \tag{21}$$

$$\left|\Gamma_i(z) \cap D_1^1\right| = \frac{a_1}{1+\sigma}\frac{\sigma_{i-1} - \sigma\sigma_i}{\sigma_{i-1} - \sigma_i}. \tag{22}$$

Proof: (i)⇒(ii) Let the integer i be given. Observe by Corollary 3.6 that θ is either the second largest eigenvalue θ_1 or the least eigenvalue θ_d, so $\sigma_{i-1} \neq \sigma_i$ in view of Lemma 2.4. Pick any $z \in D_{i-1}^i$. Observe D_1^1 contains a_1 vertices, and each is at distance $i-1$ or i from z, so

$$\left|\Gamma_{i-1}(z) \cap D_1^1\right| + \left|\Gamma_i(z) \cap D_1^1\right| = a_1. \tag{23}$$

Let E denote the primitive idempotent associated to θ. By Corollary 3.4(iii), and since xy is tight with respect to θ,

$$\sum_{w \in D_1^1} E\hat{w} = \frac{a_1\sigma}{1+\sigma}(E\hat{x} + E\hat{y}). \tag{24}$$

Taking the inner product of (24) with $E\hat{z}$ using Lemma 2.1, we obtain

$$\sigma_{i-1}\left|\Gamma_{i-1}(z) \cap D_1^1\right| + \sigma_i\left|\Gamma_i(z) \cap D_1^1\right| = \frac{a_1\sigma}{1+\sigma}(\sigma_{i-1} + \sigma_i). \tag{25}$$

Solving the system (23), (25), we routinely obtain (21), (22).

(ii)⇒(i) We show equality holds in (17). Counting the edges between D_1^1 and D_1^2 using (21) (with $i = 2$), we find in view of Lemma 3.2(i) that

$$f(x, y) = b_1 \frac{\sigma^2 - \sigma_2}{(1+\sigma)(\sigma - \sigma_2)}. \tag{26}$$

Eliminating σ, σ_2 in (26) using $\theta = k\sigma$ and Lemma 2.3(ii), (iv), we readily find equality holds in (17). Now xy is tight with respect to θ by Corollary 3.4. □

Theorem 4.2 *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$ and $a_1 \neq 0$. Let θ denote a nontrivial eigenvalue of Γ, and let $\sigma_0, \sigma_1, \ldots, \sigma_d$ denote its cosine sequence. Let x, y denote adjacent vertices in X. Then with reference to Definition* 2.10, *the following are equivalent.*
(i) *xy is tight with respect to θ,*
(ii) *For $1 \leq i \leq d-1$; both $\sigma_i \neq \sigma_{i+1}$, and for all $z \in D_i^i$*

$$\left|\Gamma_{i+1}(z) \cap D_1^1\right| = \left|\Gamma_{i-1}(z) \cap D_1^1\right| \frac{\sigma_{i-1} - \sigma_i}{\sigma_i - \sigma_{i+1}} + a_1 \frac{1-\sigma}{1+\sigma} \frac{\sigma_i}{\sigma_i - \sigma_{i+1}}, \tag{27}$$

$$\begin{aligned}\left|\Gamma_i(z) \cap D_1^1\right| = -\left|\Gamma_{i-1}(z) \cap D_1^1\right| \frac{\sigma_{i-1} - \sigma_{i+1}}{\sigma_i - \sigma_{i+1}} + a_1 \frac{2\sigma}{1+\sigma} \\ -a_1 \frac{1-\sigma}{1+\sigma} \frac{\sigma_{i+1}}{\sigma_i - \sigma_{i+1}}.\end{aligned} \tag{28}$$

Suppose (i)–(ii) *above, and that $a_d \neq 0$. Then for all $z \in D_d^d$*

$$\left|\Gamma_{d-1}(z) \cap D_1^1\right| = -a_1 \frac{1-\sigma}{1+\sigma} \frac{\sigma_d}{\sigma_{d-1} - \sigma_d}, \tag{29}$$

$$\left|\Gamma_d(z) \cap D_1^1\right| = a_1 + a_1 \frac{1-\sigma}{1+\sigma} \frac{\sigma_d}{\sigma_{d-1} - \sigma_d}. \tag{30}$$

Proof: (i)⇒(ii) Let the integer i be given. Observe by Corollary 3.6 that θ is either the second largest eigenvalue θ_1 or the least eigenvalue θ_d, so $\sigma_i \neq \sigma_{i+1}$ by Lemma 2.4. Pick any $z \in D_i^i$. Proceeding as in the proof of Theorem 4.1(i)⇒(ii), we find

$$\left|\Gamma_{i-1}(z) \cap D_1^1\right| + \left|\Gamma_i(z) \cap D_1^1\right| + \left|\Gamma_{i+1}(z) \cap D_1^1\right| = a_1, \tag{31}$$

$$\sigma_{i-1}\left|\Gamma_{i-1}(z) \cap D_1^1\right| + \sigma_i\left|\Gamma_i(z) \cap D_1^1\right| + \sigma_{i+1}\left|\Gamma_{i+1}(z) \cap D_1^1\right| = \frac{2\sigma\sigma_i a_1}{1+\sigma}. \tag{32}$$

Solving (31), (32) for $|\Gamma_i(z) \cap D_1^1|$, $|\Gamma_{i+1}(z) \cap D_1^1|$, we routinely obtain (27) and (28).

(ii)⇒(i) Setting $i = 1$ in (27), and evaluating the result using (16), we find

$$f(x, y) = \frac{1-\sigma}{\sigma - \sigma_2} + a_1 \frac{1-\sigma}{1+\sigma} \frac{\sigma}{\sigma - \sigma_2}. \tag{33}$$

Eliminating σ, σ_2 in (33) using $\theta = k\sigma$ and Lemma 2.3 (ii), we find equality holds in (17). Now xy is tight with respect to θ by Corollary 3.4.
Now suppose (i)–(ii) hold above, and that $a_d \neq 0$. Pick any $z \in D_d^d$. Proceeding as in the proof of Theorem 4.1(i) ⇒(ii), we find

$$\left|\Gamma_{d-1}(z) \cap D_1^1\right| + \left|\Gamma_d(z) \cap D_1^1\right| = a_1, \tag{34}$$

$$\sigma_{d-1}\left|\Gamma_{d-1}(z) \cap D_1^1\right| + \sigma_d\left|\Gamma_d(z) \cap D_1^1\right| = \frac{2\sigma_d \sigma a_1}{1+\sigma}. \tag{35}$$

Observe $\sigma_{d-1} \neq \sigma_d$ by (ii) above, so the linear system (34), (35) has unique solution (29), (30). □

5. The tightness of an edge

Definition 5.1 Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$, intersection number $a_1 \neq 0$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. For each edge $xy \in R$, let $t = t(x, y)$ denote the number of nontrivial eigenvalues of Γ with respect to which xy is tight. We call t the *tightness* of the edge xy. In view of Corollary 3.6 we have:

(i) $t = 2$ if xy is tight with respect to both θ_1 and θ_d;
(ii) $t = 1$ if xy is tight with respect to exactly one of θ_1 and θ_d;
(iii) $t = 0$ if xy is not tight with respect to θ_1 or θ_d.

Theorem 5.2 *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$ and $a_1 \neq 0$. For all edges $xy \in R$, the tightness $t = t(x, y)$ is given by*

$$t = 3d + 1 - \dim(MH), \tag{36}$$

where M denotes the Bose-Mesner algebra of Γ, where

$$H = \text{Span}\left\{\hat{x},\ \hat{y},\ \sum_{z \in D_1^1(x,y)} \hat{z}\right\}, \tag{37}$$

and where MH means $\text{Span}\{mh \mid m \in M, h \in H\}$.

Proof: Since $E_0, E_1, \ldots, E_d$ is a basis for M, and in view of (10),

$$MH = \sum_{i=0}^{d} E_i H \quad \text{(direct sum)},$$

and it follows

$$\dim MH = \sum_{i=0}^{d} \dim E_i H.$$

Note that $\dim E_0H = 1$. For $1 \leq i \leq d$, we find by Lemma 2.7 and Corollary 3.4(ii) that $\dim E_iH = 2$ if xy is tight with respect to θ_i, and $\dim E_iH = 3$ otherwise. The result follows. □

6. Tight graphs and the fundamental bound

In this section, we obtain an inequality involving the second largest and minimal eigenvalue of a distance-regular graph. To obtain it, we need the following lemma.

Lemma 6.1 *Let Γ denote a nonbipartite distance-regular graph with diameter $d \geq 3$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Then*

$$\frac{k+\theta_1(a_1+1)}{(k+\theta_1)(1+\theta_1)} - \frac{k+\theta_d(a_1+1)}{(k+\theta_d)(1+\theta_d)} \tag{38}$$

$$= \Psi \frac{(a_1+1)(\theta_d-\theta_1)}{(1+\theta_1)(1+\theta_d)(k+\theta_1)(k+\theta_d)}, \tag{39}$$

where

$$\Psi = \left(\theta_1 + \frac{k}{a_1+1}\right)\left(\theta_d + \frac{k}{a_1+1}\right) + \frac{ka_1b_1}{(a_1+1)^2}. \tag{40}$$

Proof: Put (38) over a common denominator, and simplify. □

We now present our inequality. We give two versions.

Theorem 6.2 *Let Γ denote a distance-regular graph with diameter $d \geq 3$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Then* (i), (ii) *hold below.*

(i) *Suppose Γ is not bipartite. Then*

$$\frac{k+\theta_d(a_1+1)}{(k+\theta_d)(1+\theta_d)} \leq \frac{k+\theta_1(a_1+1)}{(k+\theta_1)(1+\theta_1)}. \tag{41}$$

(ii)
$$\left(\theta_1 + \frac{k}{a_1+1}\right)\left(\theta_d + \frac{k}{a_1+1}\right) \geq -\frac{ka_1b_1}{(a_1+1)^2}. \tag{42}$$

We refer to (42) *as the* Fundamental Bound.

Proof: (i) First assume $a_1 = 0$. Then the left side of (41) equals $(1+\theta_d)^{-1}$, and is therefore negative. The right side of (41) equals $(1+\theta_1)^{-1}$, and is therefore positive. Next assume $a_1 \neq 0$. Then (41) is immediate from (20).

(ii) First assume Γ is bipartite. Then $\theta_d = -k$ and $a_1 = 0$, so both sides of (42) equal 0. Next assume Γ is not bipartite. Then (42) is immediate from (i) above, Lemma 6.1, and Lemma 2.6. □

We now consider when equality is attained in Theorem 6.2. To avoid trivialities, we consider only the nonbipartite case.

Corollary 6.3 *Let Γ denote a nonbipartite distance-regular graph with diameter $d \geq 3$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Then the following are equivalent.*
(i) *Equality holds in* (41).
(ii) *Equality holds in* (42).
(iii) $a_1 \neq 0$ *and every edge of* Γ *is tight with respect to both* θ_1 *and* θ_d.
(iv) $a_1 \neq 0$ *and there exists an edge of* Γ *which is tight with respect to both* θ_1 *and* θ_d.

Proof: (i)⇔(ii) Immediate from Lemma 6.1.
(i),(ii)⇒(iii) Suppose $a_1 = 0$. We assume (42) holds with equality, so $(\theta_1 + k)(\theta_d + k) = 0$, forcing $\theta_d = -k$. Now Γ is bipartite by Lemma 2.5, contradicting the assumption. Hence $a_1 \neq 0$. Let xy denote an edge of Γ. Observe the expressions on the left and right in (20) are equal, so they both equal $f(x, y)$. Now xy is tight with respect to both θ_1, θ_d by Corollary 3.6(i),(ii).
(iii)⇒(iv) Clear.
(iv)⇒(i) Suppose the edge xy is tight with respect to both θ_1, θ_d. By Corollary 3.6(i),(ii), the scalar $f(x, y)$ equals both the expression on the left and the expression on the right in (20), so these expressions are equal. □

Definition 6.4 Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$. We say Γ is *tight* whenever Γ is not bipartite and the equivalent conditions (i)–(iv) hold in Corollary 6.3.

We wish to emphasize the following fact.

Proposition 6.5 *Let Γ denote a tight distance-regular graph with diameter $d \geq 3$. Then $a_i \neq 0$ $(1 \leq i \leq d - 1)$.*

Proof: Observe $a_1 \neq 0$ by Corollary 6.3(iii) and Definition 6.4. Now $a_2, \ldots, a_{d-1}$ are nonzero by Lemma 2.8. □

We finish this section with some inequalities involving the eigenvalues of tight graphs.

Lemma 6.6 *Let $\Gamma = (X, R)$ denote a tight distance-regular graph with diameter $d \geq 3$ and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Then* (i)–(iv) *hold below.*
(i) $\theta_d < \frac{-k}{a_1+1}$.
(ii) *Let ρ, ρ_2 denote the first and second cosines for θ_d, respectively. Then $\rho^2 < \rho_2$.*
(iii) *Let σ, σ_2 denote the first and second cosines for θ_1, respectively. Then $\sigma^2 > \sigma_2$.*
(iv) *For each edge xy of Γ, the scalar $f = f(x, y)$ satisfies $0 < f < b_1$.*

Proof: (i) Observe (42) holds with equality since Γ is tight, and $a_1 \neq 0$ by Proposition 6.5, so

$$\left(\theta_1 + \frac{k}{a_1+1}\right)\left(\theta_d + \frac{k}{a_1+1}\right) < 0.$$

Since $\theta_1 > \theta_d$, the first factor is positive, and the second is negative. The result follows.
(ii) By Lemma 2.3(iv),

$$k^2 b_1(\rho^2 - \rho_2) = (k - \theta_d)(k + \theta_d(a_1+1)). \tag{43}$$

The right side of (43) is negative in view of (i) above, so $\rho^2 < \rho_2$.
(iii) By Lemma 2.3(iv),

$$k^2 b_1(\sigma^2 - \sigma_2) = (k - \theta_1)(k + \theta_1(a_1+1)). \tag{44}$$

The right side of (44) is positive in view of Lemma 2.6(i), so $\sigma^2 > \sigma_2$.
(iv) Observe f equals the expression on the right in (20). This expression is positive and less than b_1, since θ_1 is positive. □

7. Two characterizations of tight graphs

Theorem 7.1 *Let Γ denote a nonbipartite distance-regular graph with diameter $d \geq 3$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Then for all real numbers α, β, the following are equivalent.*
(i) *Γ is tight, and α, β is a permutation of θ_1, θ_d.*
(ii) *$\theta_d \leq \alpha, \beta \leq \theta_1$, and*

$$\left(\alpha + \frac{k}{a_1+1}\right)\left(\beta + \frac{k}{a_1+1}\right) = -\frac{ka_1b_1}{(a_1+1)^2}. \tag{45}$$

Proof: (i)⇒(ii) Immediate since (42) holds with equality.
(ii)⇒(i) Interchanging α and β if necessary, we may assume $\alpha \geq \beta$. Since the right side of (45) is nonpositive, we have

$$0 \leq \alpha + \frac{k}{a_1+1} \leq \theta_1 + \frac{k}{a_1+1},$$
$$0 \geq \beta + \frac{k}{a_1+1} \geq \theta_d + \frac{k}{a_1+1}.$$

By (45), the above inequalities, and (42), we have

$$-\frac{ka_1b_1}{(a_1+1)^2} = \left(\alpha + \frac{k}{a_1+1}\right)\left(\beta + \frac{k}{a_1+1}\right)$$

$$\geq \left(\theta_1 + \frac{k}{a_1+1}\right)\left(\theta_d + \frac{k}{a_1+1}\right) \tag{46}$$

$$\geq -\frac{ka_1b_1}{(a_1+1)^2}. \tag{47}$$

Apparently we have equality in (46), (47). In particular (42) holds with equality, so Γ is tight. We mentioned equality holds in (46). Neither side is 0, since $a_1 \neq 0$ by Proposition 6.5, and it follows $\alpha = \theta_1$, $\beta = \theta_d$. □

Theorem 7.2 *Let $\Gamma = (X, R)$ denote a nonbipartite distance-regular graph with diameter $d \geq 3$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Let θ and θ' denote distinct eigenvalues of Γ, with respective cosine sequences $\sigma_0, \sigma_1, \ldots, \sigma_d$ and $\rho_0, \rho_1, \ldots, \rho_d$. The following are equivalent.*

(i) *Γ is tight, and θ, θ' is a permutation of θ_1, θ_d.*

(ii) *For $1 \leq i \leq d$,*

$$\frac{\sigma\sigma_{i-1} - \sigma_i}{(1+\sigma)(\sigma_{i-1} - \sigma_i)} = \frac{\rho\rho_{i-1} - \rho_i}{(1+\rho)(\rho_{i-1} - \rho_i)}, \tag{48}$$

and the denominators in (48) *are nonzero.*

(iii)
$$\frac{\sigma^2 - \sigma_2}{(1+\sigma)(\sigma - \sigma_2)} = \frac{\rho^2 - \rho_2}{(1+\rho)(\rho - \rho_2)}, \tag{49}$$

and the denominators in (49) *are nonzero.*

(iv) *θ and θ' are both nontrivial, and*

$$(\sigma_2\rho_2 - \sigma\rho)(\rho - \sigma) = (\sigma\rho_2 - \sigma_2\rho)(\sigma\rho - 1). \tag{50}$$

Proof: (i)⇒(ii) Recall $a_1 \neq 0$ by Proposition 6.5. Pick adjacent vertices $x, y \in X$, and let $D_1^1 = D_1^1(x, y)$ be as in Definition 2.10. By Corollary 6.3(iii), the edge xy is tight with respect to both θ, θ'; applying (21), we find both sides of (48) equal $a_1^{-1}|\Gamma_{i-1}(z) \cap D_1^1|$, where z denotes any vertex in $D_{i-1}^i(x, y)$. In particular, the two sides of (48) are equal. The denominators in (48) are nonzero by Lemma 2.4 and Lemma 2.5.

(ii)⇒(iii) Set $i = 2$ in (ii).

(iii)⇒(iv) θ is nontrivial; otherwise $\sigma = \sigma_2 = 1$, and a denominator in (49) is zero. Similarly θ' is nontrivial. To get (50), put (49) over a common denominator and simplify the result.

(iv)⇒(i) Eliminating $\sigma, \sigma_2, \rho, \rho_2$ in (50) using $\theta = k\sigma$, $\theta' = k\rho$, and Lemma 2.3(i), we routinely find (45) holds for $\alpha = \theta$ and $\beta = \theta'$. Applying Theorem 7.1, we find Γ is tight, and that θ, θ' is a permutation of θ_1, θ_d. □

8. The auxiliary parameter

Let Γ denote a tight distance-regular graph with diameter $d \geq 3$. We are going to show the intersection numbers of Γ are given by certain rational expressions involving d independent parameters. We begin by introducing one of these parameters.

Definition 8.1 Let Γ denote a tight distance-regular graph with diameter $d \geq 3$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Let θ denote one of θ_1, θ_d. By the *auxiliary parameter* of Γ associated with θ, we mean the scalar

$$\varepsilon = \frac{k^2 - \theta\theta'}{k(\theta - \theta')}, \tag{51}$$

where θ' denotes the complement of θ in $\{\theta_1, \theta_d\}$. We observe the auxiliary parameter for θ_d is the opposite of the auxiliary parameter for θ_1.

Lemma 8.2 *Let Γ denote a tight distance-regular graph with diameter $d \geq 3$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Let θ denote one of θ_1, θ_d, and let ε denote the auxiliary parameter for θ. Then* (i)–(iv) *hold below.*
(i) $\varepsilon > 0$ *if* $\theta = \theta_1$, *and* $\varepsilon < 0$ *if* $\theta = \theta_d$.
(ii) $1 < |\varepsilon|$.
(iii) $|\varepsilon| < k\theta_1^{-1}$.
(iv) $|\varepsilon| < -k\theta_d^{-1}$.

Proof: First assume $\theta = \theta_1$. By (51),

$$\varepsilon - 1 = (k + \theta_d)(k - \theta_1)(\theta_1 - \theta_d)^{-1}k^{-1} > 0,$$

so $\varepsilon > 1$. Recall $\theta_1 > 0$ and $\theta_d < 0$. By this and (51),

$$k\theta_1^{-1} - \varepsilon = \theta_d(k - \theta_1)(k + \theta_1)(\theta_d - \theta_1)^{-1}k^{-1}\theta_1^{-1} > 0,$$

so $\varepsilon < k\theta_1^{-1}$. Similarly

$$k\theta_d^{-1} + \varepsilon = \theta_1(k - \theta_d)(k + \theta_d)(\theta_1 - \theta_d)^{-1}k^{-1}\theta_d^{-1} < 0,$$

so $\varepsilon < -k\theta_d^{-1}$. We now have the result for $\theta = \theta_1$. The result for $\theta = \theta_d$ follows in view of the last line of Definition 8.1. □

Theorem 8.3 *Let Γ denote a nonbipartite distance-regular graph with diameter $d \geq 3$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Let θ and θ' denote any eigenvalues of Γ, with respective cosine sequences $\sigma_0, \sigma_1, \ldots, \sigma_d$ and $\rho_0, \rho_1, \ldots, \rho_d$. Let ε denote any complex scalar. Then the following are equivalent.*
(i) Γ *is tight*, θ, θ' *is a permutation of* θ_1, θ_d, *and* ε *is the auxiliary parameter for* θ.

(ii) *θ and θ' are both nontrivial, and*

$$\sigma_i \rho_i - \sigma_{i-1}\rho_{i-1} = \varepsilon(\sigma_{i-1}\rho_i - \rho_{i-1}\sigma_i) \tag{52}$$

for $1 \le i \le d$.

(iii) *θ and θ' are both nontrivial, and*

$$\sigma\rho - 1 = \varepsilon(\rho - \sigma), \quad \sigma_2\rho_2 - \sigma\rho = \varepsilon(\sigma\rho_2 - \rho\sigma_2). \tag{53}$$

Proof: (i)⇒(ii) It is clear θ, θ' are both nontrivial. To see (52), observe θ, θ' are distinct, so the equivalent statements (i)–(iv) in Theorem 7.2 hold. Putting (48) over a common denominator and simplifying using $\varepsilon = (1 - \sigma\rho)(\sigma - \rho)^{-1}$, we get (52).
(ii)⇒(iii) Set $i = 1$ and $i = 2$ in (52).
(iii)⇒(i) We first show $\theta \ne \theta'$. Suppose $\theta = \theta'$. Then $\sigma = \rho$, so the left equation of (53) becomes $\sigma^2 = 1$, forcing $\sigma = 1$ or $\sigma = -1$. But $\sigma \ne 1$ since θ is nontrivial, and $\sigma \ne -1$ since Γ is not bipartite. We conclude $\theta \ne \theta'$. Now $\sigma \ne \rho$; solving the left equation in (53) for ε, and eliminating ε in the right equation of (53) using the result, we obtain (50). Now Theorem 7.2(iv) holds. Applying Theorem 7.2, we find Γ is tight, and that θ, θ' is a permutation of θ_1, θ_d. Solving the left equation in (53) for ε, and simplifying the result, we obtain (51). It follows ε is the auxiliary parameter for θ. □

9. Feasibility

Let Γ denote a tight distance-regular graph with diameter $d \ge 3$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Let θ, θ' denote a permutation of θ_1, θ_d, with respective cosine sequences $\sigma_0, \sigma_1, \ldots, \sigma_d$ and $\rho_0, \rho_1, \ldots, \rho_d$. Let ε denote the auxiliary parameter for θ. Pick any integer i $(1 \le i \le d)$, and observe (52) holds. Rearranging terms in that equation, we find

$$\rho_i(\sigma_i - \varepsilon\sigma_{i-1}) = \rho_{i-1}(\sigma_{i-1} - \varepsilon\sigma_i). \tag{54}$$

We would like to solve (54) for ρ_i, but conceivably $\sigma_i - \varepsilon\sigma_{i-1} = 0$. In this section we investigate this possibility.

Lemma 9.1 *Let Γ denote a tight distance-regular graph with diameter $d \ge 3$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Let θ, θ' denote a permutation of θ_1, θ_d, with respective cosine sequences $\sigma_0, \sigma_1, \ldots, \sigma_d$ and $\rho_0, \rho_1, \ldots, \rho_d$. Let ε denote the auxiliary parameter for θ. Then for each integer i $(1 \le i \le d-1)$, the following are equivalent:* (i) $\sigma_{i-1} = \varepsilon\sigma_i$; (ii) $\sigma_{i+1} = \varepsilon\sigma_i$; (iii) $\sigma_{i-1} = \sigma_{i+1}$; (iv) $\rho_i = 0$. *Moreover, suppose* (i)–(iv) *hold. Then $\theta = \theta_d$ and $\theta' = \theta_1$.*

Proof: Observe Theorem 8.3(i) holds, so (52) holds.
(i)⇒(iv) Replacing σ_{i-1} by $\varepsilon\sigma_i$ in (52), we find $\sigma_i\rho_i(1 - \varepsilon^2) = 0$. Observe $\varepsilon^2 \ne 1$ by Lemma 8.2(ii). Suppose for the moment that $\sigma_i = 0$. We assume $\sigma_{i-1} = \varepsilon\sigma_i$, so $\sigma_{i-1} = 0$. Now $\sigma_{i-1} = \sigma_i$, contradicting Lemma 2.4. Hence $\sigma_i \ne 0$, so $\rho_i = 0$.

(iv)⇒(i) Setting $\rho_i = 0$ in (52), we find $\rho_{i-1}(\sigma_{i-1} - \varepsilon\sigma_i) = 0$. Observe $\rho_{i-1} \neq 0$, otherwise $\rho_{i-1} = \rho_i$, contradicting Lemma 2.4. We conclude $\sigma_{i-1} = \varepsilon\sigma_i$, as desired.
(ii)⇔(iv) Similar to the proof of (i)⇔(iv).
(i),(ii)⇒(iii) Clear.
(iii)⇒(i) We cannot have $\theta = \theta_1$ by Lemma 2.4(i), so $\theta = \theta_d$, $\theta' = \theta_1$. In particular $\rho_{i-1} \neq \rho_{i+1}$. Adding (52) at i and $i+1$, we obtain

$$\sigma_{i+1}\rho_{i+1} - \sigma_{i-1}\rho_{i-1} = \varepsilon(\sigma_i\rho_{i+1} - \sigma_{i+1}\rho_i + \sigma_{i-1}\rho_i - \sigma_i\rho_{i-1}).$$

Replacing σ_{i+1} by σ_{i-1} in the above line, and simplifying, we obtain

$$(\sigma_{i-1} - \varepsilon\sigma_i)(\rho_{i+1} - \rho_{i-1}) = 0.$$

It follows $\sigma_{i-1} = \varepsilon\sigma_i$, as desired.
Now suppose (i)–(iv). Then we saw in the proof of (iii)⇒(i) that $\theta = \theta_d$, $\theta' = \theta_1$. □

Definition 9.2 Let $\Gamma = (X, R)$ denote a tight distance-regular graph with diameter $d \geq 3$ and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Let $\sigma_0, \sigma_1, \ldots, \sigma_d$ denote any cosine sequence for Γ and let θ denote the corresponding eigenvalue. The sequence $\sigma_0, \sigma_1, \ldots, \sigma_d$ (or θ) is said to be *feasible* whenever (i) and (ii) hold below.

(i) θ is one of θ_1, θ_d.
(ii) $\sigma_{i-1} \neq \sigma_{i+1}$ for $1 \leq i \leq d-1$.

We observe by Lemma 2.4(i) that θ_1 is feasible.

We conclude this section with an extension of Theorem 8.3.

Theorem 9.3 *Let Γ denote a nonbipartite distance-regular graph with diameter $d \geq 3$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Let θ and θ' denote any eigenvalues of Γ, with respective cosine sequences $\sigma_0, \sigma_1, \ldots, \sigma_d$ and $\rho_0, \rho_1, \ldots, \rho_d$. Let ε denote any complex scalar. Then the following are equivalent.*
(i) *Γ is tight, θ is feasible, ε is the auxiliary parameter for θ, and θ' is the complement of θ in $\{\theta_1, \theta_d\}$.*
(ii) *θ' is not trivial,*

$$\rho_i = \prod_{j=1}^{i} \frac{\sigma_{j-1} - \varepsilon\sigma_j}{\sigma_j - \varepsilon\sigma_{j-1}} \quad (0 \leq i \leq d), \tag{55}$$

and denominators in (55) *are all nonzero.*

Proof: (i)⇒(ii) Clearly θ' is nontrivial. To see (55), observe Theorem 8.3(i) holds, so (52) holds. Rearranging terms in (52), we obtain

$$\rho_i(\sigma_i - \varepsilon\sigma_{i-1}) = \rho_{i-1}(\sigma_{i-1} - \varepsilon\sigma_i) \quad (1 \leq i \leq d). \tag{56}$$

Observe $\sigma_i \neq \varepsilon\sigma_{i-1}$ for $2 \leq i \leq d$ by Lemma 9.1(ii), and $\sigma \neq \varepsilon$ by Lemma 8.2(ii), so the coefficient of ρ_i in (56) is never zero. Solving that equation for ρ_i and applying induction, we routinely obtain (55).

(ii)⇒(i) We show Theorem 8.3(iii) holds. Observe θ is nontrivial; otherwise $\sigma = 1$, forcing $\rho = 1$ by (55), and contradicting our assumption that θ' is nontrivial. One readily verifies (53) by eliminating ρ, ρ_2 using (55). We now have Theorem 8.3(iii). Applying that theorem, we find Γ is tight, θ, θ' is a permutation of θ_1, θ_d, and that ε is the auxiliary parameter for θ. It remains to show θ is feasible. Suppose not. Then there exists an integer i $(1 \leq i \leq d-1)$ such that $\sigma_{i-1} = \sigma_{i+1}$. Applying Lemma 9.1, we find $\sigma_{i+1} = \varepsilon\sigma_i$. But $\sigma_{i+1} - \varepsilon\sigma_i$ is a factor in the denominator of (55) (with i replaced by $i+1$), and hence is not 0. We now have a contradiction, so θ is feasible. □

10. A parametrization

In this section, we obtain the intersection numbers of a tight graph as rational functions of a feasible cosine sequence and the associated auxiliary parameter. We begin with a result about arbitrary distance-regular graphs.

Lemma 10.1 *Let Γ denote a distance-regular graph with diameter $d \geq 3$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Let θ, θ' denote a permutation of θ_1, θ_d, with respective cosine sequences $\sigma_0, \sigma_1, \ldots, \sigma_d$ and $\rho_0, \rho_1, \ldots, \rho_d$. Then*

$$k = \frac{(\sigma - \sigma_2)(1-\rho) - (\rho - \rho_2)(1-\sigma)}{(\rho - \rho_2)(1-\sigma)\sigma - (\sigma - \sigma_2)(1-\rho)\rho}, \tag{57}$$

$$b_i = k\frac{(\sigma_{i-1} - \sigma_i)(1-\rho)\rho_i - (\rho_{i-1} - \rho_i)(1-\sigma)\sigma_i}{(\rho_i - \rho_{i+1})(\sigma_{i-1} - \sigma_i) - (\sigma_i - \sigma_{i+1})(\rho_{i-1} - \rho_i)} \quad (1 \leq i \leq d-1), \tag{58}$$

$$c_i = k\frac{(\sigma_i - \sigma_{i+1})(1-\rho)\rho_i - (\rho_i - \rho_{i+1})(1-\sigma)\sigma_i}{(\rho_i - \rho_{i+1})(\sigma_{i-1} - \sigma_i) - (\sigma_i - \sigma_{i+1})(\rho_{i-1} - \rho_i)} \quad (1 \leq i \leq d-1), \tag{59}$$

$$c_d = k\sigma_d\frac{\sigma - 1}{\sigma_{d-1} - \sigma_d} = k\rho_d\frac{\rho - 1}{\rho_{d-1} - \rho_d}, \tag{60}$$

and the denominators in (57)–(60) *are never zero.*

Proof: Line (60) is immediate from Lemma 2.3(v), and the denominators in that line are nonzero by Lemma 2.4. To obtain (58), (59), pick any integer i $(1 \leq i \leq d-1)$, and recall by Lemma 2.2(iii) that

$$c_i(\sigma_{i-1} - \sigma_i) - b_i(\sigma_i - \sigma_{i+1}) = k(\sigma - 1)\sigma_i, \tag{61}$$

$$c_i(\rho_{i-1} - \rho_i) - b_i(\rho_i - \rho_{i+1}) = k(\rho - 1)\rho_i. \tag{62}$$

To solve this linear system for c_i and b_i, consider the determinant

$$D_i := \det\begin{pmatrix} \sigma_{i-1} - \sigma_i & \sigma_i - \sigma_{i+1} \\ \rho_{i-1} - \rho_i & \rho_i - \rho_{i+1} \end{pmatrix}.$$

Using Lemma 2.4, we routinely find $D_i \neq 0$. Now (61), (62) has the unique solution (58), (59) by elementary linear algebra. The denominators in (58), (59) both equal D_i; in particular they are not zero. To get (57), set $i = 1$ and $c_1 = 1$ in (59), and solve for k. □

Theorem 10.2 *Let Γ denote a nonbipartite distance-regular graph with diameter $d \geq 3$, and let $\sigma_0, \sigma_1, \ldots \sigma_d, \varepsilon, h$ denote complex scalars. Then the following are equivalent.*

(i) *Γ is tight, $\sigma_0, \sigma_1, \ldots \sigma_d$ is a feasible cosine sequence for Γ, ε is the associated auxiliary parameter from (51), and*

$$h = \frac{(1-\sigma)(1-\sigma_2)}{(\sigma^2-\sigma_2)(1-\varepsilon\sigma)}. \tag{63}$$

(ii) $\sigma_0 = 1$, $\sigma_{d-1} = \sigma\sigma_d$, $\varepsilon \neq -1$,

$$k = h\frac{\sigma - \varepsilon}{\sigma - 1}, \tag{64}$$

$$b_i = h\frac{(\sigma_{i-1} - \sigma\sigma_i)(\sigma_{i+1} - \varepsilon\sigma_i)}{(\sigma_{i-1} - \sigma_{i+1})(\sigma_{i+1} - \sigma_i)} \quad (1 \leq i \leq d-1), \tag{65}$$

$$c_i = h\frac{(\sigma_{i+1} - \sigma\sigma_i)(\sigma_{i-1} - \varepsilon\sigma_i)}{(\sigma_{i+1} - \sigma_{i-1})(\sigma_{i-1} - \sigma_i)} \quad (1 \leq i \leq d-1), \tag{66}$$

$$c_d = h\frac{\sigma - \varepsilon}{\sigma - 1}, \tag{67}$$

and denominators in (64)–(67) *are all nonzero.*

Proof: Let $\theta_0 > \theta_1 > \cdots > \theta_d$ denote the eigenvalues of Γ.
(i)⇒(ii) Observe $\sigma_0 = 1$ by Lemma 2.2(ii), and $\varepsilon \neq -1$ by Lemma 8.2(ii). Let θ denote the eigenvalue associated with $\sigma_0, \sigma_1, \ldots, \sigma_d$, and observe by Definition 9.2 that θ is one of θ_1, θ_d. Let θ' denote the complement of θ in $\{\theta_1, \theta_d\}$, and let $\rho_0, \rho_1, \ldots, \rho_d$ denote the cosine sequence for θ'. Observe Theorem 9.3(i) holds. Applying that theorem, we obtain (55). Eliminating $\rho_0, \rho_1, \ldots, \rho_d$ in (57)–(60) using (55), we routinely obtain (64)–(67), and that $\sigma_{d-1} = \sigma\sigma_d$.
(ii) ⇒(i) One readily checks

$$c_i(\sigma_{i-1} - \sigma_i) - b_i(\sigma_i - \sigma_{i+1}) = k(\sigma - 1)\sigma_i \quad (1 \leq i \leq d),$$

where σ_{d+1} is an indeterminant. Applying Lemma 2.2(i),(iii), we find $\sigma_0, \sigma_1, \ldots, \sigma_d$ is a cosine sequence for Γ, with associated eigenvalue $\theta := k\sigma$. By (64), (65), and since

$k, b_1, \ldots, b_{d-1}$ are nonzero,

$$\sigma_j \neq \varepsilon\sigma_{j-1} \quad (1 \leq j \leq d).$$

Set

$$\rho_i := \prod_{j=1}^{i} \frac{\sigma_{j-1} - \varepsilon\sigma_j}{\sigma_j - \varepsilon\sigma_{j-1}} \quad (0 \leq i \leq d). \tag{68}$$

One readily checks $\rho_0 = 1$, and that

$$c_i(\rho_{i-1} - \rho_i) - b_i(\rho_i - \rho_{i+1}) = k(\rho - 1)\rho_i \quad (1 \leq i \leq d),$$

where ρ_{d+1} is an indeterminant. Applying Lemma 2.2(i),(iii), we find $\rho_0, \rho_1, \ldots, \rho_d$ is a cosine sequence for Γ, with associated eigenvalue $\theta' := k\rho$. We claim θ' is not trivial. Suppose θ' is trivial. Then $\rho = 1$. Setting $i = 1$ and $\rho = 1$ in (68) we find $\sigma - \varepsilon = 1 - \varepsilon\sigma$, forcing $(1 - \sigma)(1 + \varepsilon) = 0$. Observe $\sigma \neq 1$ since the denominator in (67) is not zero, and we assume $\varepsilon \neq -1$, so we have a contradiction. We have now shown θ' is nontrivial, so Theorem 9.3(ii) holds. Applying that theorem, we find Γ is tight, θ is feasible, and that ε is the auxiliary parameter of θ. To see (63), set $i = 1$ and $c_1 = 1$ in (66), and solve for h. □

Proposition 10.3 *With the notation of Theorem* 10.2, *suppose* (i), (ii) *hold, and let* $\theta_0 > \theta_1 > \cdots > \theta_d$ *denote the eigenvalues of* Γ. *If* $\varepsilon > 0$, *then*

$$\theta_1 = \frac{\sigma(\sigma - \varepsilon)(1 - \sigma_2)}{(1 - \varepsilon\sigma)(\sigma_2 - \sigma^2)}, \quad \theta_d = \frac{1 - \sigma_2}{\sigma_2 - \sigma^2}. \tag{69}$$

If $\varepsilon < 0$, *then*

$$\theta_1 = \frac{1 - \sigma_2}{\sigma_2 - \sigma^2}, \quad \theta_d = \frac{\sigma(\sigma - \varepsilon)(1 - \sigma_2)}{(1 - \varepsilon\sigma)(\sigma_2 - \sigma^2)}. \tag{70}$$

We remark that the denominators in (69), (70) *are nonzero.*

Proof: Let θ denote the eigenvalue of Γ associated with $\sigma_0, \sigma_1, \ldots, \sigma_d$. By Lemma 2.2(iii) and (64), we obtain

$$\begin{aligned}\theta &= k\sigma \\ &= \frac{\sigma(\sigma - \varepsilon)(1 - \sigma_2)}{(1 - \varepsilon\sigma)(\sigma_2 - \sigma^2)}.\end{aligned} \tag{71}$$

Observe $\theta \in \{\theta_1, \theta_d\}$ since $\sigma_0, \sigma_1, \ldots, \sigma_d$ is feasible. Let θ' denote the complement of θ in $\{\theta_1, \theta_d\}$, and let ρ denote the first cosine associated with θ'. Observe condition (i) holds in Theorem 9.3, so (55) holds. Setting $i = 1$ in that equation, we find

$$\rho = \frac{1 - \varepsilon\sigma}{\sigma - \varepsilon}. \tag{72}$$

By Lemma 2.2(iii), (64), and (72), we obtain

$$\begin{aligned}\theta' &= k\rho \\ &= \frac{1-\sigma_2}{\sigma_2-\sigma^2}. \end{aligned} \tag{73}$$

To finish the proof, we observe by Lemma 8.2(i) that $\theta = \theta_1$, $\theta' = \theta_d$ if $\varepsilon > 0$, and $\theta = \theta_d$, $\theta' = \theta_1$ if $\varepsilon < 0$. □

Theorem 10.4 *Let Γ denote a tight distance-regular graph with diameter $d \geq 3$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Then* (i) *and* (ii) *hold below.*
(i) $a_d = 0$.
(ii) *Let $\sigma_0, \sigma_1, \ldots, \sigma_d$ denote the cosine sequence for θ_1 or θ_d, and let ε denote the associated auxiliary parameter from* (51).
Then

$$a_i = g\frac{(\sigma_{i+1}-\sigma\sigma_i)(\sigma_{i-1}-\sigma\sigma_i)}{(\sigma_{i+1}-\sigma_i)(\sigma_{i-1}-\sigma_i)} \quad (1 \leq i \leq d-1), \tag{74}$$

where

$$g = \frac{(\varepsilon-1)(1-\sigma_2)}{(\sigma^2-\sigma_2)(1-\varepsilon\sigma)}. \tag{75}$$

Proof: (i) Comparing (64), (67), we see $k = c_d$, and it follows $a_d = 0$.
(ii) First assume $\sigma_0, \sigma_1, \ldots, \sigma_d$ is the cosine sequence for θ_1, and recall this sequence is feasible. Let h be as in (63). Then Theorem 10.2(i) holds, so Theorem 10.2(ii) holds. Evaluating the right side of $a_i = k - b_i - c_i$ using (64)–(66), and simplifying the result using (63), we obtain (74), (75). To finish the proof, let $\rho_0, \rho_1, \ldots, \rho_d$ denote the cosine sequence for θ_d, and recall by Definition 8.1 that the associated auxiliary parameter is $\varepsilon' = -\varepsilon$. We show

$$a_i = \frac{(\varepsilon'-1)(1-\rho_2)}{(\rho^2-\rho_2)(1-\varepsilon'\rho)}\frac{(\rho_{i+1}-\rho\rho_i)(\rho_{i-1}-\rho\rho_i)}{(\rho_{i+1}-\rho_i)(\rho_{i-1}-\rho_i)}. \tag{76}$$

By Theorem 7.2(ii) (with i replaced by $i+1$),

$$\frac{1}{1+\sigma}\frac{\sigma_{i+1}-\sigma\sigma_i}{\sigma_{i+1}-\sigma_i} = \frac{1}{1+\rho}\frac{\rho_{i+1}-\rho\rho_i}{\rho_{i+1}-\rho_i}. \tag{77}$$

Subtracting 1 from both sides of Theorem 7.2(ii), and simplifying, we obtain

$$\frac{1}{1+\sigma}\frac{\sigma_{i-1}-\sigma\sigma_i}{\sigma_{i-1}-\sigma_i} = \frac{1}{1+\rho}\frac{\rho_{i-1}-\rho\rho_i}{\rho_{i-1}-\rho_i}. \tag{78}$$

By (53),

$$\frac{(\varepsilon - 1)(1-\sigma_2)(1+\sigma)^2}{(\sigma^2-\sigma_2)(1-\varepsilon\sigma)} = \frac{(\varepsilon' - 1)(1-\rho_2)(1+\rho)^2}{(\rho^2-\rho_2)(1-\varepsilon'\rho)}. \tag{79}$$

Multiplying together (77)–(79) and simplifying, we obtain (76), as desired. □

We end this section with some inequalities.

Lemma 10.5 *Let Γ denote a tight distance-regular graph with diameter $d \geq 3$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Let θ denote one of θ_1, θ_d, and let $\sigma_0, \sigma_1, \ldots, \sigma_d$ denote the cosine sequence for θ. Suppose $\theta = \theta_1$. Then*
(i) $\sigma_{i-1} > \sigma\sigma_i \quad (1 \leq i \leq d-1)$,
(ii) $\sigma\sigma_{i-1} > \sigma_i \quad (2 \leq i \leq d)$.

Suppose $\theta = \theta_d$. Then

(iii) $(-1)^i(\sigma\sigma_i - \sigma_{i-1}) > 0 \quad (1 \leq i \leq d-1)$,
(iv) $(-1)^i(\sigma_i - \sigma\sigma_{i-1}) > 0 \quad (2 \leq i \leq d)$.

Proof: (i) We first show $\sigma_{i-1} - \sigma\sigma_i$ is nonnegative. Recall $a_1 \neq 0$ by Proposition 6.5, so Theorem 4.1 applies. Let x, y denote adjacent vertices in X, and recall by Corollary 6.3 that the edge xy is tight with respect to θ. Now Theorem 4.1(i) holds, so (22) holds. Observe the left side of (22) is nonnegative, so the right side is nonnegative. In that expression on the right, the factors $1+\sigma$ and $\sigma_{i-1} - \sigma_i$ are positive, so the remaining factor $\sigma_{i-1} - \sigma\sigma_i$ is nonnegative, as desired. To finish the proof, observe $\sigma_{i-1} - \sigma\sigma_i$ is a factor on the right in (74), so it is not zero in view of Proposition 6.5.
(ii)–(iv) Similar to the proof of (i) above. □

11. The 1-homogeneous property

In this section, we show the concept of tight is closely related to the concept of 1-homogeneous that appears in the work of Nomura [14–16].

Theorem 11.1 *Let $\Gamma = (X, R)$ denote a tight distance-regular graph with diameter $d \geq 3$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Let $\sigma_0, \sigma_1, \ldots, \sigma_d$ denote the cosine sequence associated with θ_1 or θ_d. Fix adjacent vertices $x, y \in X$. Then with the notation of Definition 2.10 we have the following: For all integers i $(1 \leq i \leq d-1)$, and for all vertices $z \in D_i^i$,*

$$\left|\Gamma_{i-1}(z) \cap D_1^1\right| = c_i \frac{(\sigma^2-\sigma_2)(\sigma_i - \sigma_{i+1})}{(\sigma-\sigma_2)(\sigma\sigma_i - \sigma_{i+1})}, \tag{80}$$

$$\left|\Gamma_{i+1}(z) \cap D_1^1\right| = b_i \frac{(\sigma^2-\sigma_2)(\sigma_{i-1} - \sigma_i)}{(\sigma-\sigma_2)(\sigma_{i-1} - \sigma\sigma_i)}. \tag{81}$$

Proof: First assume $\sigma_0, \sigma_1, \ldots, \sigma_d$ is the cosine sequence for θ_1, and let $\rho_0, \rho_1, \ldots, \rho_d$ denote the cosine sequence for θ_d. The edge xy is tight with respect to both θ_1, θ_d, so by Theorem 4.2(ii),

$$\left|\Gamma_{i+1}(z) \cap D_1^1\right| = \left|\Gamma_{i-1}(z) \cap D_1^1\right| \frac{\sigma_{i-1} - \sigma_i}{\sigma_i - \sigma_{i+1}} + a_1 \frac{1-\sigma}{1+\sigma} \frac{\sigma_i}{\sigma_i - \sigma_{i+1}}, \tag{82}$$

$$\left|\Gamma_{i+1}(z) \cap D_1^1\right| = \left|\Gamma_{i-1}(z) \cap D_1^1\right| \frac{\rho_{i-1} - \rho_i}{\rho_i - \rho_{i+1}} + a_1 \frac{1-\rho}{1+\rho} \frac{\rho_i}{\rho_i - \rho_{i+1}}. \tag{83}$$

Eliminating $\rho_0, \rho_1, \ldots, \rho_d$ in (83) using (55), we obtain

$$\begin{aligned}\left|\Gamma_{i+1}(z) \cap D_1^1\right| = {} & \left|\Gamma_{i-1}(z) \cap D_1^1\right| \frac{\sigma_{i-1} - \sigma_i}{\sigma_i - \sigma_{i+1}} \frac{\sigma_{i+1} - \varepsilon\sigma_i}{\sigma_{i-1} - \varepsilon\sigma_i} \\ & + a_1 \frac{(1-\sigma)(\sigma_{i+1} - \varepsilon\sigma_i)}{(1+\sigma)(1-\varepsilon)(\sigma_i - \sigma_{i+1})},\end{aligned} \tag{84}$$

where ε denotes the auxiliary parameter associated with θ_1. Solving (82), (84) for $|\Gamma_{i+1}(z) \cap D_1^1|$ and $|\Gamma_{i-1}(z) \cap D_1^1|$, and evaluating the result using (63), (65), (66), (74), we get (80), (81), as desired. To finish the proof observe by Theorem 7.2(ii), (iii) that

$$\frac{(\sigma^2 - \sigma_2)(\sigma_i - \sigma_{i+1})}{(\sigma - \sigma_2)(\sigma\sigma_i - \sigma_{i+1})} = \frac{(\rho^2 - \rho_2)(\rho_i - \rho_{i+1})}{(\rho - \rho_2)(\rho\rho_i - \rho_{i+1})}, \tag{85}$$

$$\frac{(\sigma^2 - \sigma_2)(\sigma_{i-1} - \sigma_i)}{(\sigma - \sigma_2)(\sigma_{i-1} - \sigma\sigma_i)} = \frac{(\rho^2 - \rho_2)(\rho_{i-1} - \rho_i)}{(\rho - \rho_2)(\rho_{i-1} - \rho\rho_i)}. \tag{86}$$

□

Theorem 11.2 *Let $\Gamma = (X, R)$ denote a tight distance-regular graph with diameter $d \geq 3$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Let $\sigma_0, \sigma_1, \ldots, \sigma_d$ denote the cosine sequence for θ_1 or θ_d. Fix adjacent vertices $x, y \in X$. Then with the notation of Definition* 2.10 *we have the following* (i), (ii).

(i) *For all integers i $(1 \leq i \leq d-1)$, and for all $z \in D_i^i$,*

$$\left|\Gamma(z) \cap D_{i-1}^{i-1}\right| = c_i \frac{(\sigma_i - \sigma_{i+1})(\sigma\sigma_{i-1} - \sigma_i)}{(\sigma_{i-1} - \sigma_i)(\sigma\sigma_i - \sigma_{i+1})}, \tag{87}$$

$$\left|\Gamma(z) \cap D_{i+1}^{i+1}\right| = b_i \frac{(\sigma_{i-1} - \sigma_i)(\sigma_i - \sigma\sigma_{i+1})}{(\sigma_i - \sigma_{i+1})(\sigma_{i-1} - \sigma\sigma_i)}. \tag{88}$$

(ii) *For all integers i $(2 \leq i \leq d)$, and for all $z \in D_{i-1}^i \cup D_i^{i-1}$,*

$$\left|\Gamma(z) \cap D_{i-1}^{i-1}\right| = a_{i-1} \frac{(1-\sigma)\left(\sigma_{i-1}^2 - \sigma_{i-2}\sigma_i\right)}{(\sigma_{i-1} - \sigma_i)(\sigma_{i-2} - \sigma\sigma_{i-1})}. \tag{89}$$

Proof: (i) To prove (87), we assume $i \geq 2$; otherwise both sides are zero. Let α_i denote the expression on the right in (80). Let N denote the number of ordered pairs uv such that

$$u \in \Gamma_{i-1}(z) \cap D_1^1, \quad v \in \Gamma(z) \cap D_{i-1}^{i-1}, \quad \partial(u, v) = i - 2.$$

We compute N in two ways. On one hand, by (80), there are precisely α_i choices for u, and given u, there are precisely c_{i-1} choices for v, so

$$N = \alpha_i c_{i-1}. \tag{90}$$

On the other hand, there are precisely $|\Gamma(z) \cap D_{i-1}^{i-1}|$ choices for v, and given v, there are precisely α_{i-1} choices for u, so

$$N = \left|\Gamma(z) \cap D_{i-1}^{i-1}\right| \alpha_{i-1}. \tag{91}$$

Observe by Lemma 2.4, Lemma 6.6, and (80) that $\alpha_{i-1} \neq 0$; combining this with (90), (91), we find

$$\left|\Gamma(z) \cap D_{i-1}^{i-1}\right| = c_{i-1}\alpha_i \alpha_{i-1}^{-1}.$$

Eliminating α_{i-1}, α_i in the above line using (80), we obtain (87), as desired. Concerning (88), first assume $i = d - 1$. We show both sides of (88) are zero. To see the left side is zero, recall $a_d = 0$ by Theorem 10.4, forcing $p_{dd}^1 = 0$ by Lemma 2.9, so $D_d^d = \emptyset$ by the last line in Definition 2.10. The right side of (88) is zero since the factor $\sigma_{d-1} - \sigma\sigma_d$ in the numerator is zero by Lemma 2.3(vi). We now show (88) for $i \leq d - 2$. Let β_i denote the expression on the right in (81). Let N' denote the number of ordered pairs uv such that

$$u \in \Gamma_{i+1}(z) \cap D_1^1, \quad v \in \Gamma(z) \cap D_{i+1}^{i+1}, \quad \partial(u, v) = i + 2.$$

We compute N' in two ways. On one hand, by (81), there are precisely β_i choices for u, and given u, there are precisely b_{i+1} choices for v, so

$$N' = \beta_i b_{i+1}. \tag{92}$$

On the other hand, there are precisely $|\Gamma(z) \cap D_{i+1}^{i+1}|$ choices for v, and given v, there are precisely β_{i+1} choices for u, so

$$N' = \left|\Gamma(z) \cap D_{i+1}^{i+1}\right| \beta_{i+1}. \tag{93}$$

Observe by Lemma 2.4, Lemma 6.6, and (81) that $\beta_{i+1} \neq 0$; combining this with (92), (93), we find

$$\left|\Gamma(z) \cap D_{i+1}^{i+1}\right| = b_{i+1}\beta_i \beta_{i+1}^{-1}.$$

Eliminating β_i, β_{i+1} in the above line using (81), we obtain (88), as desired.

(ii) Let γ_i denote the expression on the right in (21), and let δ_i denote the expression on the right in (87). Let N'' denote the number of ordered pairs uv such that

$$u \in \Gamma_{i-1}(z) \cap D_1^1, \quad v \in \Gamma(z) \cap D_{i-1}^{i-1}, \quad \partial(u, v) = i - 2.$$

We compute N'' in two ways. On one hand, by Theorem 4.1(ii), there are precisely γ_i choices for u. Given u, we find by (87) (with x and i replaced by u and $i-1$, respectively) that there are precisely $c_{i-1} - \delta_{i-1}$ choices for v; consequently

$$N'' = \gamma_i (c_{i-1} - \delta_{i-1}). \tag{94}$$

On the other hand, there are precisely $|\Gamma(z) \cap D_{i-1}^{i-1}|$ choices for v, and given v, there are precisely α_{i-1} choices for u, where α_{i-1} is from the proof of (i) above. Hence

$$N'' = \left|\Gamma(z) \cap D_{i-1}^{i-1}\right| \alpha_{i-1}. \tag{95}$$

Combining (94), (95),

$$\left|\Gamma(z) \cap D_{i-1}^{i-1}\right| = \gamma_i (c_{i-1} - \delta_{i-1}) \alpha_{i-1}^{-1}.$$

Eliminating α_{i-1}, γ_i, δ_{i-1} in the above line using (80), (21), (87), respectively, and simplifying the result using Theorem 10.4(ii), we obtain (89), as desired. □

Definition 11.3 Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$, and fix adjacent vertices $x, y \in X$.

(i) For all integers i, j we define the vector $w_{ij} = w_{ij}(x, y)$ by

$$w_{ij} = \sum_{z \in D_i^j} \hat{z}, \tag{96}$$

where $D_i^j = D_i^j(x, y)$ is from (15).

(ii) Let $\mathcal{L}$ denote the set of ordered pairs

$$\mathcal{L} = \{ij \mid 0 \leq i, j \leq d, \ p_{ij}^1 \neq 0\}. \tag{97}$$

We observe that for all integers i, j, $w_{ij} \neq 0$ if and only if $ij \in \mathcal{L}$.

(iii) We define the vector space $W = W(x, y)$ by

$$W = \text{Span}\{w_{ij} \mid ij \in \mathcal{L}\}. \tag{98}$$

Lemma 11.4 *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$, and assume $a_1 \neq 0$. Then*

(i) $\mathcal{L} = \{i-1, i \mid 1 \leq i \leq d\} \cup \{i, i-1 \mid 1 \leq i \leq d\} \cup \{ii \mid 1 \leq i \leq e\}$, *where $e = d-1$ if $a_d = 0$ and $e = d$ if $a_d \neq 0$.*

(ii) $$|\mathcal{L}| = \begin{cases} 3d & \text{if } a_d \neq 0, \\ 3d - 1 & \text{if } a_d = 0. \end{cases} \tag{99}$$

(iii) *Let x, y denote adjacent vertices in X, and let $W = W(x, y)$ be as in* (98). *Then*

$$\dim W = \begin{cases} 3d & \text{if } a_d \neq 0, \\ 3d - 1 & \text{if } a_d = 0. \end{cases} \tag{100}$$

Proof: Routine application of Lemma 2.8 and Lemma 2.9. □

Lemma 11.5 *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$, fix adjacent vertices $x, y \in X$, and let the vector space $W = W(x, y)$ be as in* (98). *Then the following are equivalent.*

(i) *The vector space W is A-invariant.*
(ii) *For all integers i, j, r, s ($ij \in \mathcal{L}$ and $rs \in \mathcal{L}$), and for all $z \in D_i^j$, the scalar $|\Gamma(z) \cap D_r^s|$ is a constant independent of z.*
(iii) *The following conditions hold.*
   (a) *For all integers i ($1 \leq i \leq d$), and for all $z \in D_i^i$, the scalars $|\Gamma(z) \cap D_{i-1}^{i-1}|$ and $|\Gamma(z) \cap D_{i+1}^{i+1}|$ are constants independent of z.*
   (b) *For all integers i ($2 \leq i \leq d$), and for all $z \in D_{i-1}^i \cup D_i^{i-1}$, the scalar $|\Gamma(z) \cap D_{i-1}^{i-1}|$ is a constant independent of z.*

Proof: (i)⇔(ii) Routine.
(ii)⇒(iii) Clear.
(iii)⇒(ii) Follows directly from Lemma 2.11. □

Definition 11.6 Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$. For each edge $xy \in R$, the graph Γ is said to be *1-homogeneous with respect to xy* whenever (i)–(iii) hold in Lemma 11.5. The graph Γ is said to be *1-homogeneous* whenever it is 1-homogeneous with respect to all edges in R.

Theorem 11.7 *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$. Then the following are equivalent.*

(i) *Γ is tight,*
(ii) *$a_1 \neq 0$, $a_d = 0$, and Γ is 1-homogeneous,*
(iii) *$a_1 \neq 0$, $a_d = 0$, and Γ is 1-homogeneous with respect to at least one edge.*

Proof: (i)⇒(ii) Observe $a_1 \neq 0$ by Proposition 6.5, and $a_d = 0$ by Theorem 10.4. Pick any edge $xy \in R$. By Theorem 11.2, we find conditions (iii)(a), (iii)(b) hold in Lemma 11.5, so Γ is 1-homogeneous with respect to xy by Definition 11.6. Apparently Γ is 1-homogeneous with respect to every edge, so Γ is 1-homogeneous.
(ii)⇒(iii) Clear.
(iii)⇒(i) Suppose Γ is 1-homogeneous with respect to the edge $xy \in R$. We show xy is tight with respect to both θ_1, θ_d. To do this, we show the tightness $t = t(x, y)$ from

Definition 5.1 equals 2. Consider the vector space $W = W(x, y)$ from (98), and the vector space H from (37). Observe W is A-invariant by Lemma 11.5, and W contains H, so it contains MH, where M denotes the Bose-Mesner algebra of Γ. The space W has dimension $3d - 1$ by (100), so MH has dimension at most $3d - 1$. Applying (36), we find $t \geq 2$. From the discussion at the end of Definition 5.1, we observe $t = 2$, and that xy is tight with respect to both θ_1, θ_d. Now Γ is tight in view of Corollary 6.3(iv) and Definition 6.4. □

12. The local graph

Definition 12.1 Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$. For each vertex $x \in X$, we let $\Delta = \Delta(x)$ denote the vertex subgraph of Γ induced on $\Gamma(x)$. We refer to Δ as the *local graph* associated with x. We observe Δ has k vertices, and is regular with valency a_1. We further observe Δ is not a clique.

In this section, we show the local graphs of tight distance-regular graphs are strongly-regular. We begin by recalling the definition and some basic properties of strongly-regular graphs.

Definition 12.2 [3, p. 3] A graph Δ is said to be *strongly-regular* with *parameters* $(\nu, \kappa, \lambda, \mu)$ whenever Δ has ν vertices and is regular with valency κ, adjacent vertices of Δ have precisely λ common neighbors, and distinct non-adjacent vertices of Δ have precisely μ common neighbors.

Lemma 12.3 [3, Thm. 1.3.1] *Let Δ denote a connected strongly-regular graph with parameters $(\nu, \kappa, \lambda, \mu)$, and assume Δ is not a clique. Then Δ has precisely three distinct eigenvalues, one of which is κ. Denoting the others by r, s,*

$$\nu = \frac{(\kappa - r)(\kappa - s)}{\kappa + rs}, \qquad \lambda = \kappa + r + s + rs, \qquad \mu = \kappa + rs. \tag{101}$$

The multiplicity of κ as an eigenvalue of Δ equals 1. The multiplicities with which r, s appear as eigenvalues of Δ are given by

$$mult_r = \frac{\kappa(s + 1)(\kappa - s)}{\mu(s - r)}, \qquad mult_s = \frac{\kappa(r + 1)(\kappa - r)}{\mu(r - s)}. \tag{102}$$

Theorem 12.4 *Let $\Gamma = (X, R)$ denote a tight distance-regular graph with diameter $d \geq 3$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Pick $\theta \in \{\theta_1, \theta_d\}$, let σ, σ_2 denote the first and second cosines for θ, respectively, and let ε denote the associated auxiliary parameter from* (51). *Then for any vertex $x \in X$, the local graph $\Delta = \Delta(x)$ satisfies* (i)–(iv) *below.*

(i) *Δ is strongly-regular with parameters (k, a_1, λ, μ), where k is the valency of Γ, and*

$$a_1 = -\frac{(1 - \sigma_2)(1 + \sigma)(1 - \varepsilon)}{(\sigma - \sigma_2)(1 - \varepsilon\sigma)}, \tag{103}$$

$$\lambda = a_1 \frac{2\sigma}{1+\sigma} - a_1 \frac{1-\sigma}{1+\sigma} \frac{\sigma_2}{\sigma - \sigma_2} - \frac{1-\sigma_2}{\sigma - \sigma_2}, \tag{104}$$

$$\mu = \frac{a_1}{1+\sigma} \frac{\sigma^2 - \sigma_2}{\sigma - \sigma_2}. \tag{105}$$

(ii) Δ *is connected and not a clique.*

(iii) *The distinct eigenvalues of* Δ *are* $a_1, r, s,$ *where*

$$r = \frac{a_1 \sigma}{1+\sigma}, \qquad s = -\frac{1-\sigma_2}{\sigma - \sigma_2}. \tag{106}$$

(iv) *The multiplicities of* r, s *are given by*

$$mult_r = \frac{(1+\sigma)(\sigma - \varepsilon)}{\sigma_2 - \sigma^2}, \qquad mult_s = -\frac{(1-\varepsilon)(1+\sigma)(\sigma_2 - \varepsilon\sigma)}{(\sigma_2 - \sigma^2)(1 - \varepsilon\sigma)}. \tag{107}$$

Proof: (i) Clearly Δ has k vertices and is regular with valency a_1. The formula (103) is from Theorem 10.4(ii). Pick distinct vertices $y, z \in \Delta$. We count the number of common neighbors of y, z in Δ. First suppose y, z are adjacent. By (28) (with $i = 1$) we find y, z have precisely λ common neighbors in Δ, where λ is given in (104). Next suppose y, z are not adjacent. By (21) (with $i = 2$), we find y, z have precisely μ common neighbors in Δ, where μ is given in (105). The result now follows in view of Definition 12.2.

(ii) We saw in Definition 12.1 that Δ is not a clique. Observe the scalar μ in (105) is not zero, since $a_1 \neq 0$ by Proposition 6.5, and since $\sigma^2 \neq \sigma_2$ by Lemma 6.6(ii),(iii). It follows Δ is connected.

(iii) The scalar a_1 is an eigenvalue of Δ by Lemma 12.3. Using (104), (105), we find the scalars r, s in (106) satisfy

$$\lambda = a_1 + r + s + rs, \qquad \mu = a_1 + rs.$$

Comparing this with the two equations on the right in (101), we find the scalars r, s in (106) are the remaining eigenvalues of Δ.

(iv) By (102) and (i) above,

$$mult_r = \frac{a_1(s+1)(a_1 - s)}{\mu(s - r)}, \qquad mult_s = \frac{a_1(r+1)(a_1 - r)}{\mu(r - s)}.$$

Eliminating a_1, μ, r, s in the above equations using (103), (105), (106), we routinely obtain (107). □

Definition 12.5 Let Γ denote a distance-regular graph with diameter $d \geq 3$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. We define

$$b^- := -1 - \frac{b_1}{1+\theta_1}, \qquad b^+ := -1 - \frac{b_1}{1+\theta_d}.$$

We recall $a_1 - k \leq \theta_d < -1 < \theta_1$ by Lemma 2.6, so $b^- < -1$, $b^+ \geq 0$.

Theorem 12.6 *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$. Then the following are equivalent.*

(i) *Γ is tight.*

(ii) *For all $x \in X$, the local graph $\Delta(x)$ is connected strongly-regular with eigenvalues a_1, b^+, b^-.*

(iii) *There exists $x \in X$ for which the local graph $\Delta(x)$ is connected strongly-regular with eigenvalues a_1, b^+, b^-.*

Proof: (i)⇒(ii) Pick any $x \in X$, and let $\Delta = \Delta(x)$ denote the local graph. By Theorem 12.4, the graph Δ is connected and strongly-regular. The eigenvalues of Δ other than a_1 are given by (106), where for convenience we take the eigenvalue θ involved to be θ_1. Eliminating σ, σ_2 in (106) using $\theta_1 = k\sigma$ and Lemma 2.3(i), and simplifying the results using equality in the fundamental bound (42), we routinely find $r = b^+$, $s = b^-$.
(ii)⇒(iii) Clear.
(iii)⇒(i) Since $\Delta = \Delta(x)$ is connected, its valency a_1 is not zero. In particular Γ is not bipartite. The graph Δ is not a clique, so (101) holds for Δ. Applying the equation on the left in that line, we obtain

$$k(a_1 + b^+b^-) = (a_1 - b^+)(a_1 - b^-). \tag{108}$$

Eliminating b^+, b^- in (108) using Definition 12.5, and simplifying the result, we routinely obtain equality in the fundamental bound (42). Now Γ is tight, as desired. □

13. Examples of tight distance-regular graphs

The following examples (i)–(xii) are tight distance-regular graphs with diameter at least 3. In each case we give the intersection array, the second largest eigenvalue θ_1, and the least eigenvalue θ_d, together with their respective cosine sequences $\{\sigma_i\}$, $\{\rho_i\}$, and the auxiliary parameter ε for θ_1. Also, we give the parameters and nontrivial eigenvalues of the local graphs.

(i) The *Johnson graph* $J(2d, d)$ has diameter d and intersection numbers $a_i = 2i(d-i)$, $b_i = (d-i)^2$, $c_i = i^2$ for $i = 0, \ldots, d$, cf. [3, p. 255]. It is distance-transitive, an antipodal double-cover, and Q-polynomial with respect to θ_1.
Each local graph is a *lattice graph* $K_d \times K_d$, with parameters $(d^2, 2(d-1), d-2, 2)$ and nontrivial eigenvalues $r = d-2$, $s = -2$, cf. [3, p. 256].

(ii) The *halved cube* $\frac{1}{2}H(2d, 2)$ has diameter d and intersection numbers $a_i = 4i(d-i)$, $b_i = (d-i)(2d-2i-1)$, $c_i = i(2i-1)$ for $i = 0, \ldots d$, cf. [3, p. 264]. It is distance-transitive, an antipodal double-cover, and Q-polynomial with respect to θ_1.
Each local graph is a Johnson graph $J(2d, 2)$, with parameters $(d(2d-1), 4(d-1), 2(d-1), 4)$ and nontrivial eigenvalues $r = 2d-4$, $s = -2$, cf. [3, p. 267].

(iii) The *Taylor graphs* are nonbipartite double-covers of complete graphs, i.e., distance-regular graphs with intersection array of the form $\{k, c_2, 1; 1, c_2, k\}$, where $c_2 < k-1$.

They have diameter 3, and are Q-polynomial with respect to both θ_1, θ_d. These eigenvalues are given by $\theta_1 = \alpha$, $\theta_d = \beta$, where

$$\alpha + \beta = k - 2c_2 - 1, \qquad \alpha\beta = -k,$$

and $\alpha > \beta$. See Taylor [19], and Seidel and Taylor [17] for more details.

Each local graph is strongly-regular with parameters (k, a_1, λ, μ), where $a_1 = k - c_2 - 1$, $\lambda = (3a_1 - k - 1)/2$ and $\mu = a_1/2$. We note both a_1, c_2 are even and k is odd. The nontrivial eigenvalues of the local graph are

$$r = \frac{\alpha - 1}{2}, \qquad s = \frac{\beta - 1}{2}.$$

(iv) The graph 3.$Sym(7)$ has intersection array $\{10, 6, 4, 1; 1, 2, 6, 10\}$ and can be obtained from a sporadic Fisher group, cf. [3, pp. 397–400]. It is sometimes called the Conway-Smith graph. It is distance-transitive, an antipodal 3-fold cover, and is not Q-polynomial.

Each local graph is a *Petersen graph*, with parameters (10, 3, 0, 1) and nontrivial eigenvalues $r = 1$, $s = -2$, see [11], [3, 13.2.B].

(v) The graph $3.O_6^-(3)$ has intersection array $\{45, 32, 12, 1; 1, 6, 32, 45\}$ and can be obtained from a sporadic Fisher group, cf. [3, pp. 397–400]. It is distance-transitive, an antipodal 3-fold cover, and is not Q-polynomial.

Each local graph is a *generalized quadrangle* $GQ(4, 2)$, with parameters (45, 12, 3, 3) and nontrivial eigenvalues $r = 3$, $s = -3$. See [3, p. 399].

(vi) The graph $3.O_7(3)$ has intersection array $\{117, 80, 24, 1; 1, 12, 80, 117\}$ and can be obtained from a sporadic Fisher group, cf. [3, pp. 397–400]. It is distance-transitive, an antipodal 3-fold cover, and is not Q-polynomial.

Each local graph is strongly-regular with parameters (117, 36, 15, 9), and nontrivial eigenvalues $r = 9$, $s = -3$. [3, 13.2.D].

(vii) The graph $3.Fi_{24}$ has intersection array $\{31671, 28160, 2160, 1; 1, 1080, 28160, 31671\}$ and can be obtained from a sporadic Fisher group, cf. [3, p. 397]. It is distance-transitive, an antipodal 3-fold cover, and is not Q-polynomial.

Each local graph is strongly-regular with parameters (31671, 3510, 693, 351) and nontrivial eigenvalues $r = 351$, $s = -9$. They are related to Fi_{23}.

(viii) The *Soicher1 graph* has intersection array $\{56, 45, 16, 1; 1, 8, 45, 56\}$, cf. [2], [4, 11.41], [18]. It is distance-transitive, an antipodal 3-fold cover, and is not Q-polynomial.

Each local graph is a *Gewirtz graph* with parameters (56, 10, 0, 2) and nontrivial eigenvalues $r = 2$, $s = -4$, [3, p. 372].

(ix) The *Soicher2 graph* has intersection array $\{416, 315, 64, 1; 1, 32, 315, 416\}$, cf. [18] [4, 13.8A]. It is distance-transitive, an antipodal 3-fold cover, and is not Q-polynomial.

Each local graph is strongly-regular with parameters (416, 100, 36, 20) and nontrivial eigenvalues $r = 20$, $s = -4$.

(x) The *Meixner1 graph* has intersection array $\{176, 135, 24, 1; 1, 24, 135, 176\}$, cf. [13] [4,12.4A]. It is distance-transitive, an antipodal 2-fold cover, and is Q-polynomial.

Each local graph is strongly-regular with parameters (176, 40, 12, 8) and nontrivial eigenvalues $r = 8$, $s = -4$.

(xi) The *Meixner2 graph* has intersection array $\{176, 135, 36, 1; 1, 12, 135, 176\}$, cf. [13] [4,12.4A]. It is distance-transitive, an antipodal 4-fold cover, and is not Q-polynomial.

Each local graph is strongly-regular with parameters (176, 40, 12, 8) and nontrivial eigenvalues $r = 8$, $s = -4$.

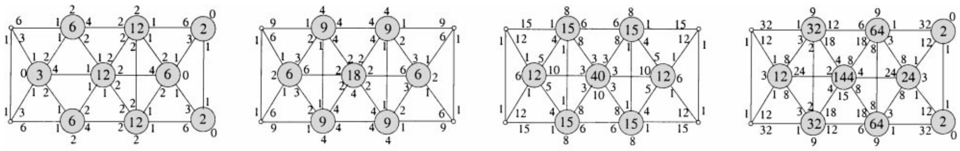
(xii) The *Patterson graph* has intersection array $\{280, 243, 144, 10; 1, 8, 90, 280\}$, and can be constructed from the Suzuki group, see [3, 13.7]. It is primitive and distance-transitive, but not Q-polynomial.

Each local graph is a generalized quadrangle $GQ(9,3)$ with parameters (280, 36, 8, 4) and nontrivial eigenvalues $r = 8$, $s = -4$, [3, Thm. 13.7.1].

Name	θ_1	θ_d	$\{\sigma_i\}$	$\{\rho_i\}$	ε
$J(2d, d)$	$d(d-2)$	$-d$	$\sigma_i = \frac{d-2i}{d}$	$\rho_i = \frac{(-1)^i \cdot 1 \cdot 2 \cdots i}{d(d-1)\cdots(d-i+1)}$	$\frac{d+2}{d}$
$\frac{1}{2}H(2d, 2)$	$(2d-1)(d-2)$	$-d$	$\sigma_i = \frac{d-2i}{d}$	$\rho_i = \frac{(-1)^i \cdot 1 \cdot 3 \cdots (2i-1)}{(2d-1)(2d-3)\cdots(2d-2i+1)}$	$\frac{d+1}{d-1}$
Taylor	α	β	$\left(1, \frac{\alpha}{k}, \frac{-\alpha}{k}, -1\right)$	$\left(1, \frac{\beta}{k}, \frac{-\beta}{k}, -1\right)$	$\frac{k+1}{\alpha-\beta}$
3.Sym(7)	5	-4	$\left(1, \frac{1}{2}, 0, \frac{-1}{4}, \frac{-1}{2}\right)$	$\left(1, \frac{-2}{5}, \frac{3}{10}, \frac{-2}{5}, 1\right)$	$\frac{4}{3}$
$3.O_6^-(3)$	15	-9	$\left(1, \frac{1}{3}, 0, \frac{-1}{6}, \frac{-1}{2}\right)$	$\left(1, \frac{-1}{5}, \frac{1}{10}, \frac{-1}{5}, 1\right)$	2
$3.O_7(3)$	39	-9	$\left(1, \frac{1}{3}, 0, \frac{-1}{6}, \frac{-1}{2}\right)$	$\left(1, \frac{-1}{13}, \frac{2}{65}, \frac{-1}{13}, 1\right)$	$\frac{5}{2}$
$3.Fi_{24}$	3519	-81	$\left(1, \frac{1}{9}, 0, \frac{-1}{18}, \frac{-1}{2}\right)$	$\left(1, \frac{-1}{391}, \frac{5}{17204}, \frac{-1}{391}, 1\right)$	$\frac{44}{5}$
Soicher1	14	-16	$\left(1, \frac{1}{4}, 0, \frac{-1}{8}, \frac{-1}{2}\right)$	$\left(1, \frac{-2}{7}, \frac{1}{7}, \frac{-2}{7}, 1\right)$	2
Soicher2	104	-16	$\left(1, \frac{1}{4}, 0, \frac{-1}{8}, \frac{-1}{2}\right)$	$\left(1, \frac{-1}{26}, \frac{1}{91}, \frac{-1}{26}, 1\right)$	$\frac{7}{2}$
Meixner1	44	-16	$\left(1, \frac{1}{4}, 0, \frac{-1}{4}, -1\right)$	$\left(1, \frac{-1}{11}, \frac{1}{33}, \frac{-1}{11}, 1\right)$	3
Meixner2	44	-16	$\left(1, \frac{1}{4}, 0, \frac{-1}{12}, \frac{-1}{3}\right)$	$\left(1, \frac{-1}{11}, \frac{1}{33}, \frac{-1}{11}, 1\right)$	3
Patterson	80	-28	$\left(1, \frac{2}{7}, \frac{1}{21}, \frac{-2}{63}, \frac{-1}{9}\right)$	$\left(1, \frac{-1}{10}, \frac{1}{45}, \frac{-1}{54}, \frac{5}{27}\right)$	$\frac{8}{3}$

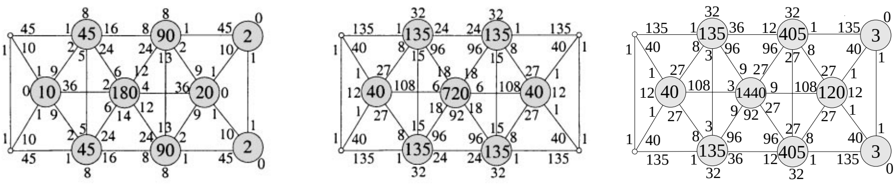
Appendix A: 1-homogeneous partitions of the known examples of the AT4 family and the Patterson Graph

In [21] a tight non bipartite antipodal distance-regular graph with diameter four was parameterized by the eigen values r and $-s$ of the local graphs and the size of its antipodal



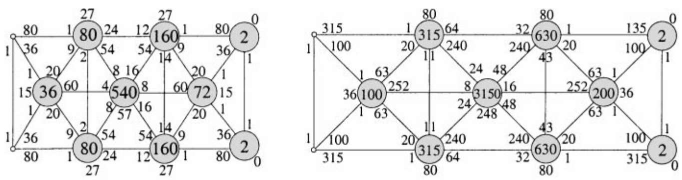
(a) AT4(1,2,3) (b) AT4(2,2,2) (c) AT4(4,2,2) (d) AT4(3,3,3)

Figure A.1. 1-homogeneous partition of (a) the Conway-Smith graph (b) the Johnson graph $J(8, 4)$, (c) the halved cube $\frac{1}{2}H(8, 2)$, and (d) the $3.O_6^-(3)$.



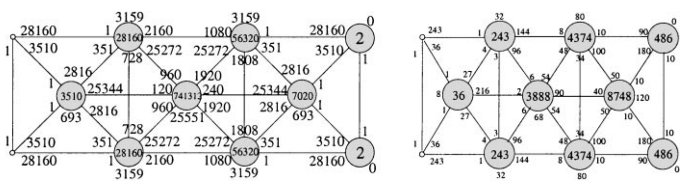
(e) AT4(2,4,3) (f) AT4(8,4,2) (g) AT4(8,4,4)

Figure A.2. 1-homogeneous partition of (e) the Soicherl graph, (f) the Meixner1 graph, (g) the Meixner2 graph.



(h) AT4(9,3,3) (i) AT4(20,4,3)

Figure A.3. 1-homogeneous partition of (h) the $3.O_7(3)$, (i) the Soicher2 graph.



(j) AT4(351,9,3) (j) AT4(351,9,3)

Figure A.4. 1-homogeneous partition of (j) the $3.Fi_{24}^-$ graph and (i) the Patterson graph.

classes. The graph was called an **antipodal tight graph** of diameter four and with Parameters (r, s, t) and denoted by AT4(r, s, t).

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