Tight Lower Bounds for the Size of Epsilon-Nets

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Abstract

According to a well known theorem of Haussler and Welzl (1987), any range space of bounded VC-dimension admits an ε-net of size $O\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)$. Using probabilistic techniques, Pach and Woeginger (1990) showed that there exist range spaces of VC-dimension 2, for which the above bound is sharp. The only known range spaces of small VC-dimension, in which the ranges are geometric objects in some Euclidean space and the size of the smallest ε-nets is superlinear in $\frac{1}{\varepsilon}$, were found by Alon (2010). In his examples, the size of the smallest ε-nets is $\Omega\left(\frac{1}{\varepsilon}g(\frac{1}{\varepsilon})\right)$, where g is an extremely slowly growing function, closely related to the inverse Ackermann function.

We show that there exist geometrically defined range spaces, already of VC-dimension 2, in which the size of the smallest ε-nets is $\Omega\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)$. We also construct range spaces induced by axis-parallel rectangles in the plane, in which the size of the smallest ε-nets is $\Omega\left(\frac{1}{\varepsilon}\log\log\frac{1}{\varepsilon}\right)$. By a theorem of Aronov, Ezra, and Sharir (2010), this bound is tight.

1 Introduction

Let X be a *finite* set and let $\mathcal{R}$ be a system of subsets of an underlying set which contains X. In computational geometry, the pair $(X, \mathcal{R})$ is usually called a *range space*. The elements of X and $\mathcal{R}$ are said to be the *points* and the *ranges* of the range space, respectively. Consider a subset $A \subseteq X$. A is called *shattered* if for every subset $B \subseteq A$, one can find a range $R_B \in \mathcal{R}$ with $R_B \cap A = B$. The size of the largest shattered subset of points, $A \subseteq X$, is said to be the *Vapnik-Chervonenkis dimension* (or *VC-dimension*) of the range space $(X, \mathcal{R})$.

In their seminal paper [VaC71], Vapnik and Chervonenkis proved that, from the point of view of random sampling, all range spaces whose VC-dimensions are bounded

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by a constant behave very nicely. In particular, for any $\varepsilon > 0$, a randomly selected "small" subset of X, whose number of elements depends only on the VC-dimension d and ε, will "hit" every range containing at least $\varepsilon|X|$ points of X, with large probability. A set of points in X with the property that every range $R \in \mathcal{R}$ with $|R \cap X| \geq \varepsilon|X|$ contains at least one of its elements is called an *ε-net* for the range space $(X, \mathcal{R})$. Note that these sets are often called *strong* ε-nets in the literature, to distinguish them from the so-called *weak ε-nets*, which may also contain points from $\cup\mathcal{R} \setminus X$, but must still hit all ranges that contain at least $\varepsilon|X|$ elements of X. In this paper, we will consider only strong ε-nets, apart from some remarks in the last section.

The ideas of Vapnik and Chervonenkis have been adapted by Haussler and Welzl [HaW87] to show that the minimum number $f = f_d(\varepsilon)$ such that every range space of VC-dimension d admits an ε-net of size at most f satisfies $f_d(\varepsilon) = O\left(\frac{d}{\varepsilon}\log\frac{d}{\varepsilon}\right)$. They asked whether the logarithmic factor can be removed in this formula. Pach and Woeginger [PaW90] proved that while $f_1(\varepsilon) = \max(2, \lceil\frac{1}{\varepsilon}\rceil - 1)$, the logarithmic factor is needed for every $d \geq 2$. Moreover, it was shown by Komlós et al. [KoPW92, PaA95] that for any $d \geq 2$,

$$(d - 2 + \frac{1}{d+2} + o(1))\frac{1}{\varepsilon}\ln\frac{1}{\varepsilon} \leq f_d(\varepsilon) \leq (d + o(1))\frac{1}{\varepsilon}\ln\frac{1}{\varepsilon},$$

as ε tends to 0. (Here ln denotes the natural logarithm.)

Haussler and Welzl discovered that the above results apply to many geometrically defined range spaces. Roughly speaking, the VC-dimension is bounded by a constant for any set of ranges with bounded *description complexity*, that is if the ranges can be described in terms of a bounded number of parameters. This observation has far reaching consequences. The construction of small epsilon-nets has become one of the most powerful general techniques in computational geometry (see [Ch00, EvRS05]).

In a number of basic geometric scenarios it was possible to improve on the above bounds. For instance, for any finite set of points in the plane, one can find an ϵ-net of size linear in $1/\varepsilon$, where the ranges are half-planes, translates of a convex polygon, disks or certain kind of pseudo-disks. Similar results hold in three-dimensional space for half-space ranges [PaW90, MaSW90, Ma92, PyR08]. We state two results here.

Theorem A. (Matoušek, Seidel, Welzl [MaSW90, Ma92]) *All range spaces $(X, \mathcal{R})$, where X is a finite set of points in $\mathbb{R}^3$ and $\mathcal{R}$ consists of half-spaces, admit ε-nets of size $O(1/\varepsilon)$.*

Theorem B. (Aronov, Ezra, Sharir [ArES10]) *All range spaces $(X, \mathcal{R})$, where X is a finite set of points in $\mathbb{R}^2$ (or $\mathbb{R}^3$) and $\mathcal{R}$ consists of axis-parallel rectangles (boxes), admit ε-nets of size $O\left(\frac{1}{\varepsilon}\log\log\frac{1}{\varepsilon}\right)$.*

Aronov et al. have also established a similar result for "fat" triangular ranges in the place of axis-parallel rectangles. For weak ε-nets, Ezra [Ez10] extended Theorem B to higher dimensions.

In algorithmic applications, it is often natural to consider the dual range space, in which the roles of points and ranges are swapped [BrG95, PaA95]. Given a finite family $\mathcal{R}$ of ranges in $\mathbb{R}^m$, the *dual range space induced* by them is defined as a set system (hypergraph) on the underlying set $\mathcal{R}$, consisting of the sets $\mathcal{R}_x := \{R \mid x \in R \in \mathcal{R}\}$, for all $x \in \mathbb{R}^m$. (Note that $\mathcal{R}_x$ and $\mathcal{R}_y$ may coincide for $x \neq y$.) It is easy to see (cf. [PaA95]) that if the VC-dimension of the range space $(X, \mathcal{R})$ is less than d for every $X \subset \mathbb{R}^m$, then the VC-dimension of the dual range space induced by any subset of $\mathcal{R}$ is less than 2^d.

Clarkson and Varadarajan [ClV07] found a simple and beautiful connection between the complexity of the boundary of the union of n members of $\mathcal{R}$ and the size of the smallest epsilon-net in the dual range space. If the complexity of the boundary is $o(n \log n)$, then the dual range space admits ε-nets of size $o\left(\frac{1}{\varepsilon} \log \frac{1}{\varepsilon}\right)$. This connection has been further explored and improved in [Va09, ArES10, ArES11]. In particular, it was shown that dual range spaces of "fat" triangles in the plane admit ε-nets of size $O\left(\frac{1}{\varepsilon} \log \log^* \frac{1}{\varepsilon}\right)$, where $\log^*$ stands for the iterated logarithm function.

In most range spaces $(X, \mathcal{R})$, one can find roughly $1/\varepsilon$ pairwise disjoint ranges $R \in \mathcal{R}$ such that the sets $R \cap X$ are of size at least $\varepsilon|X|$. In these cases, the size of any ε-net is $\Omega(1/\varepsilon)$. For the last two decades, "the prevailing conjecture" was that in "geometric scenarios," this bound is essentially tight: there always exists an ε-net of size $O(1/\varepsilon)$ (see, e.g., [MaSW90, ArES10]). This conjecture had to be revised after Alon [Al10] discovered some geometric range spaces of small VC-dimension, in which the ranges are straight lines, rectangles or infinite strips in the plane, and which do not admit ε-nets of size $O(1/\varepsilon)$. Alon's construction is based on the density version of the Hales-Jewett theorem [HaJ63], due to Furstenberg and Katznelson [FuK89, FuK91], and recently improved by participants of the Polymath blog project [Po09]. However, Alon's lower bound is only barely superlinear: $\Omega\left(\frac{1}{\varepsilon} g(\frac{1}{\varepsilon})\right)$, where g is an extremely slowly growing function, closely related to the inverse Ackermann function.

1.1 New lower bounds

The main aim of this note is to prove that the $O\left(\frac{1}{\varepsilon} \log \frac{1}{\varepsilon}\right)$ general upper bound for the size of the smallest ε-nets in range spaces of bounded VC-dimension is tight even in simple geometric scenarios.

Our first theorem claims that there exist dual range spaces induced by finite families of axis-parallel rectangles in which the size of the smallest ε-nets is $\Omega\left(\frac{1}{\varepsilon} \log \frac{1}{\varepsilon}\right)$. More precisely, we have the following. In the sequel log always denotes the binary logarithm.

Theorem 1. *For any $\varepsilon > 0$ and for any sufficiently large integer $n > n_0(\varepsilon)$, there exists a dual range space Σ^* of VC-dimension 2, induced by n axis-parallel rectangles in $\mathbb{R}^2$, in which the size of any ε-net is at least $\frac{1}{9\varepsilon} \log \frac{1}{\varepsilon}$.*

From Theorem 1 it is not hard to deduce the following results for primal range spaces.

Theorem 2. *For any $\varepsilon > 0$ and for any sufficiently large integer $n > n_0(\varepsilon)$, there exists a (primal) range space $\Sigma = (X, \mathcal{R})$ of VC-dimension 2, where X is a set of n points in $\mathbb{R}^4$, $\mathcal{R}$ consists of axis-parallel boxes with one of their vertices at the origin (or axis-parallel orthants), and in which the size of any ε-net is at least $\frac{1}{9\varepsilon} \log \frac{1}{\varepsilon}$.*

Theorem 3. *For any $\varepsilon > 0$ and for any sufficiently large integer $n > n_0(\varepsilon)$, there exists a (primal) range space $\Sigma = (X, \mathcal{R})$ of VC-dimension 2, where X is a set of n points in $\mathbb{R}^4$, $\mathcal{R}$ consists of half-spaces, and in which the size of the smallest ε-net is at least $\frac{1}{9\varepsilon} \log \frac{1}{\varepsilon}$.*

Theorems 2 and 3 show that Theorems B and A cannot be generalized to 4-dimensional space. It also follows, by a standard duality argument, that there exist *dual* range spaces induced by half-spaces in $\mathbb{R}^4$, for which the size of the smallest ε-net is $\Omega\left(\frac{1}{\varepsilon} \log \frac{1}{\varepsilon}\right)$.

Our next result shows that Theorem B of Aronov, Ezra, and Sharir is tight.

Theorem 4. *For any $\varepsilon > 0$ and for any sufficiently large integer $n > n_0(\varepsilon)$, there exists a (primal) range space $\Sigma = (X, \mathcal{R})$ of VC-dimension 2, where X is a set of n points in the plane, $\mathcal{R}$ consists of axis-parallel rectangles, and in which the size of any ε-net is at least $C\frac{1}{\varepsilon} \log\log \frac{1}{\varepsilon}$. Here $C > 0$ is an absolute constant.*

The proof of Theorem 1 is based on a construction reminiscent of the one described and studied in [PaT10] in connection with a hypergraph coloring problem. In fact, we could use precisely the same construction, but this would require a more complicated analysis. For the proof of Theorem 4, we use a randomly selected set of roughly $\frac{1}{\varepsilon} \log\log \frac{1}{\varepsilon}$ points in the unit square. In case VC-dimension 3 suffices we can choose the points uniformly randomly. Some related properties of this set were already established in [ChPS09]. Our paper is self-contained: we do not rely on any material from [PaT10] or [ChPS09]

1.2 Organization

In Section 2, we present the proofs of Theorems 1, 2, and 3, based on an explicit construction of systems of axis-parallel rectangles, described in [PaT10]. Section 3 contains a similar proof of Theorem 4, based on randomized construction from Chen et al. [ChPS09]. In the final section, we make some concluding remarks and mention some open problems.

2 Boxes and half-spaces—Proofs of Theorems 1-3

Theorems 2 and 3 are corollaries of Theorem 1, so we start with the proof of Theorem 1. The proof is based on an explicit construction of systems of rectangles, similar to the one described and analyzed in [PaT10]. In order to describe this construction, we have to introduce some notations.

Let d be a fixed positive integer. For any integers $a, b \geq 0$ and $0 \leq i \leq d$, let $R^i_{a,b}$ denote the open axis-parallel rectangle defined as the cross-product of two half-open intervals:

$$R^i_{a,b} = [a2^i, (a+1)2^i) \times [b2^{d-i}, (b+1)2^{d-i}).$$

Let

$$\overline{\mathcal{R}} = \{R^i_{a,b} \ \mid \ 0 \leq i \leq d,\ 0 \leq a < 2^{d-i},\ 0 \leq b < 2^i\}.$$

The elements of $\overline{\mathcal{R}}$ are called *canonical rectangles.* For each $i, 0 \leq i \leq d$, there are precisely 2^d canonical rectangles $R^i_{a,b}$, and they form a tiling of the square $[0, 2^d)^2$. That is, we have $|\overline{\mathcal{R}}| = (d+1)2^d$. Note that we use half-open boxes to simplify a presentation in Section 3, here open or closed boxes would work just as well.

Consider the set of rectangles

$$\mathcal{R} := \left\{R^i_{a,b} \in \overline{\mathcal{R}} \ \mid \ a, b \text{ are even } \right\}.$$

Clearly, we have

$$|\mathcal{R}| = (d+3)2^{d-2}.$$

We claim that the dual range space Σ^* induced by the elements of $\mathcal{R}$ meets the requirements of Theorem 1 for $\varepsilon \approx 2^{-d}$. Recall that a subset $\mathcal{S} \subset \mathcal{R}$ is an ε-net in Σ^* if and only if every point in the plane that belongs to at least $\varepsilon|\mathcal{R}|$ elements of $\mathcal{R}$ is covered by at least one element of $\mathcal{S}$.

The heart of the proof is the following statement.

Lemma 2.1. *Let d be a positive integer, let $\mathcal{R}$ and Σ^* be defined as above and let $0 < \varepsilon < 1$. If $\mathcal{S} \subseteq \mathcal{R}$ is an ε-net in Σ^*, then we have*

$$|\mathcal{S}| > (1 - 2^{d-1}\varepsilon)|\mathcal{R}| = (1 - 2^{d-1}\varepsilon)(d+3)2^{d-2}.$$

Proof. Let $\mathcal{S}$ be a fixed ε-net in Σ^*. Assign to $\mathcal{S}$ a collection of canonical rectangles $\mathcal{T} = \mathcal{T}(\mathcal{S}) \subset \overline{\mathcal{R}}$, as follows. Let

$$\mathcal{T} := \{R^i_{a,b} \ \mid \ R^i_{2\lfloor a/2\rfloor, 2\lfloor b/2\rfloor} \in \mathcal{S} \text{ and } a \not\equiv b, \text{ or } R^i_{2\lfloor a/2\rfloor, 2\lfloor b/2\rfloor} \notin \mathcal{S} \text{ and } a \equiv b\}.$$

Here "≡" is taken modulo 2.

It follows from the definition that for each i, precisely half of the canonical rectangles $R^i_{a,b} \in \overline{\mathcal{R}}$ belong to $\mathcal{T}$. It is also clear that $\mathcal{S}$ and $\mathcal{T}$ are disjoint, moreover, every element of $\mathcal{R} \setminus \mathcal{S}$ belongs to $\mathcal{T}$.

Notice that the elements of $\mathcal{T}$ can be decomposed into 2^{d-1} disjoint "cliques" R^0, R^1, ..., R^d, where each R^i is a $2^i \times 2^{d-i}$ canonical rectangle, and $\cap_{i=0}^d R^i \neq \emptyset$. Indeed, by our construction, for every $2^0 \times 2^d$ rectangle $R^0 \in \mathcal{T}$, there is precisely one $2^1 \times 2^{d-1}$ rectangle $R^1 \in \mathcal{T}$ that intersects it. Analogously, there is precisely one $2^2 \times 2^{d-2}$ rectangle $R^2 \in \mathcal{T}$ that intersects R^1, and this rectangle must also intersect $R^0 \cap R^1$. Proceeding like this, starting with a fixed $R^0 \in \mathcal{T}$, we obtain a uniquely determined clique of size $d+1$ whose elements have a point in common. There are 2^{d-1} possible choices for R^0, and each element of $\mathcal{T}$ belongs to precisely one of the resulting cliques. Note that any point in the plane is contained in at most $d+1$ canonical rectangles, so a point in the intersection of the rectangles forming a clique is not covered by any canonical rectangle outside the clique.

Since all elements of $\mathcal{R} \setminus \mathcal{S}$ belong to $\mathcal{T}$, but $\mathcal{T}$ is disjoint from $\mathcal{S}$, it follows from the above clique decomposition that there is a point $x \in \mathbb{R}^2$ contained

1. in at least $\frac{|\mathcal{R} \setminus \mathcal{S}|}{2^{d-1}}$ elements of $\mathcal{R}$, and
2. in no element of $\mathcal{S}$.

Since $\mathcal{S}$ is an ε-net we must have

$$\frac{|\mathcal{R} \setminus \mathcal{S}|}{2^{d-1}} < \varepsilon |\mathcal{R}|$$

proving the lemma. □

We also need the following simple property. Let $\overline{\Sigma}^*$ denote the dual range space induced by all canonical rectangles in $\overline{\mathcal{R}}$.We let $\overline{\Sigma}$ denote the (primal) range space dual to $\overline{Sigma}^*$. According to our somewhat unorthodox terminology, the precise definition of Σ is the following. The rectangles in $\overline{\mathcal{R}}$ divide the plane into finitely many cells. Two points belong to the same cell if they are contained in the same rectangles. Pick a point in each cell, and let X denote the set of points we picked. The range space $\overline{\Sigma}$ is the pair $(X, \overline{\mathcal{R}})$.

Lemma 2.2. *All of Σ, Σ^* and $\overline{\Sigma}^*$ have VC-dimension 2.*

Before turning to the proof of the lemma, we introduce a partial order on the family of axis-parallel rectangles in the plane. For any two axis-parallel rectangles R and R', we write $R \prec R'$ if the orthogonal projection of R on the x-axis is contained in the orthogonal projection of R' on the x-axis, and the orthogonal projection of R on the

y-axis contains the orthogonal projection of R' on the y-axis. Obviously, this is a partial order.

Proof of Lemma 2.2. Clearly, we have VC-dim$(\Sigma) \geq 2$ and VC-dim$(\Sigma^*) \geq 2$.

Observe first any two intersecting rectangles in $\overline{\mathcal{R}}$ are comparable by $\prec$.

Assume for contradiction that Σ, Σ^* or $\overline{\Sigma}^*$ has VC-dimension 3 or more. The existence of a shattered 3-element set would imply that there are three distinct points p_1, p_2, and p_3 in the plane and three rectangles $R_1, R_2, R_3 \in \overline{\mathcal{R}}$ with $\{p_1, p_2, p_3\} \setminus R_i = \{p_i\}$ for $i = 1, 2, 3$. The rectangles R_i pairwise intersect, and hence must be linearly ordered by $\prec$. Suppose without loss of generality $R_1 \prec R_2 \prec R_3$. Then $R_1 \cap R_3 \subseteq R_2$, contradicting our assumption that p_2 is contained in the left-hand side but not in the right. □

Proof of Theorem 1. Let us choose the positive integer d and $1/3 \leq \alpha \leq 2/3$ with $\varepsilon = \alpha/2^{d-1}$. For this we assume (without loss of generality) that $\varepsilon \leq 2/3$. According to Lemmas 2.2 and 2.1, the dual range space Σ^* defined for this d has VC-dimension 2 and it does not admit an ε-net of size smaller than $\frac{\alpha(1-\alpha)}{2}(d+3)\frac{1}{\varepsilon}$. Here $d + 3 > \log \frac{1}{\varepsilon}$ and $\frac{\alpha(1-\alpha)}{2} \geq \frac{1}{9}$ proving that Σ^* satisfies the statement of the theorem. Note that if $\log \frac{1}{\varepsilon}$ is an integer the constant $\frac{1}{9}$ can be replaced by $\frac{1}{8}$ in the bound.

This example is very special: for every ε, we have defined a single dual range space Σ^*, induced by $\Theta\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)$ rectangles. However, from one small example we can easily construct arbitrarily large ones, as required by the theorem. Keep ε fixed, and choose a large integer t. Replace each rectangle $R \in \mathcal{R}$ by a chain of rectangles $R_1 \prec R_2 \prec \cdots \prec R_t$, where $\prec$ denotes the ordering relation defined after Lemma 2.2, and each R_i differs only very little from R. Let $\mathcal{R}_t$ denote the resulting family of rectangles. It is not difficult to see that this transformation can be carried out keeping the property that intersecting rectangles are comparable by $\prec$ and therefore the VC-dimension of the dual range space Σ_t^* induced by $\mathcal{R}_t$, as well as the VC-dimension of the corresponding "primal" space remains 2.

We have $|\mathcal{R}_t| = t|\mathcal{R}|$, and the size of the smallest ε-net for Σ_t^* is at least as large as it was in Σ^*. Suppose to the contrary that there is a smaller set $\mathcal{S}'$ of rectangles in $\mathcal{R}_t$ that form an ε-net in Σ_t^*. Let $\mathcal{S}''$ be the set of rectangles in $\mathcal{R}$ that were replaced by the elements of $\mathcal{S}'$. Since $|\mathcal{S}''| \leq |\mathcal{S}'|$, the rectangles in $\mathcal{S}''$ do not form an ε-net in Σ^*. Thus, there is a point in the plane contained in at least $\varepsilon|\mathcal{R}|$ elements of $\mathcal{R}$, which is not covered by any element of $\mathcal{S}''$. We can choose such a point lying not too close to the boundaries of the rectangles in $\mathcal{R}$, and then it is contained in at least $t\varepsilon|\mathcal{R}| = \varepsilon|\mathcal{R}_t|$ elements of $\mathcal{R}_t$, none of which belongs to $\mathcal{S}'$, a contradiction. □

Proof of Theorem 2. The statement follows from Theorem 1 by a standard duality argument (see, e.g., [KaRS08]). We assume without loss of generality the the rectangles are closed and lie in the first quadrant of the plane. We assign to each rectangle $R = [x_1, x_2] \times [y_1, y_2]$ the point $p(R) = (x_1, 1/x_2, y_1, 1/y_2) \in \mathbb{R}^4$. Now a point $q = (a, b)$ of the first quadrant lies in R if and only if $x_1 \leq a \leq x_2$ and $y_1 \leq b \leq y_2$, that is, if and only if the point $p(R)$ is contained in the 4-dimensional box

$$B(q) = [0, a] \times [0, 1/a] \times [0, b] \times [0, 1/b]. \quad \square$$

Theorem 3 is an immediate corollary of Theorem 2 and the following lemma.

Lemma 2.3. *Let P be a finite set of points in the positive orthant of $\mathbb{R}^d$. To each $p \in P$, we can assign a point p' in the positive orthant of $\mathbb{R}^d$ so that the set $P' = \{p' \mid p \in P\}$ satisfies the following condition.*

For any axis-parallel box $B \subset \mathbb{R}^d$ that contains the origin, there is a half-space $H(B) \subset \mathbb{R}^d$ which contains the origin and for which

$$\{p' \mid p \in B \cap P\} = P' \cap H(B).$$

Proof. Let $x_1, x_2, \ldots, x_d$ denote the orthogonal coordinates in $\mathbb{R}^d$. Observe that from the point of view of intersections with axis-parallel boxes, the actual values of the coordinates do not matter: we need to know only the order of the x_i-coordinates of the points of P for each i. For every i $(1 \leq i \leq d)$, let $0 < \xi_{i,1} < \xi_{i,2} < \xi_{i,3} < \ldots$ denote the sequence of different values of the x_i-coordinates of the elements of P. Every such sequence is of length at most $|P|$. By rescaling the coordinates if necessary, we can assume that $\xi_{i,j+1}/\xi_{i,j} > d$ holds for every i and j.

Consider now an axis-parallel box B, which contains the origin and intersects P in at least one element. We can shrink B if necessary, without changing its intersection with P, so that we can suppose without loss of generality that B is of the form

$$B = [0, b_1] \times [0, b_2] \times \ldots \times [0, b_d],$$

where each b_i is equal to ξ_{ij_i} for a suitable j_i.

We claim that $B \cap P$ is equal to the intersection of P with the half-space $H(B)$ defined by

$$\frac{x_1}{b_1} + \frac{x_2}{b_2} + \ldots + \frac{x_d}{b_d} \leq d.$$

For every point in B, each term of the above sum is at most 1, so that we have $B \subset H(B)$, and hence $B \cap P \subseteq H(B) \cap P$. Suppose now that p is a point of P that does not belong to B. Then one of its coordinates, $x_i(p)$, say, is more than d times larger than b_i. Therefore, the i-th term in the above sum is already larger than d, which implies that $p \notin H(B)$. $\square$

3 Proof of Theorem 4

Except for the bound on the VC-dimension, Theorem 4 is an easy consequence of the following result on a set of randomly selected points in the unit square. Note that the VC-dimension of all axis-parallel rectangles in the plane is 4. The rectangles considered here have diadic projections to the x axis but arbitrary projections to the y axis, this makes their VC-dimension 3. Although it is higher than the optimal value claimed in Theorem 4 we still consider this lemma interesting as it deals with a uniform random point set from the unit square.

Lemma 3.1. *Let $n > 2$, $r = \lceil \log\log n/5 \rceil$ be integers, where* log *stands for the binary logarithm, and let and $\varepsilon = r/n$. Let X be a set of n randomly and uniformly selected points in the unit square, and let $\mathcal{R}$ denote the family of all axis-parallel rectangles of the form $[j/2^t, (j+1)/2^t) \times [a, b]$, where j, t are nonnegative integers, and $a < b$ are reals.*

Then, with probability tending to 1, the range space $(X, \mathcal{R})$ does not admit an ε-net of size at most $n/2$.

A similar property of random point sets with respect to axis-parallel rectangles was established in Chen et al. [ChPS09] (see Theorem 9). In their setting, r was a constant, $\varepsilon = r/n$, and it was shown that every ε-net contains all but a very small fraction of point set. Here we allow r to slowly tend to infinity. For the proof of this lemma we refer to the conference version of this paper [PaT11]. Here we concentrate on the proof of Theorem 4 based on very similar ideas but using a non-uniform random distribution for the points.

Proof of Theorem 4. Let us fix an integer $d \geq 1$. To obtain VC-dimension 2 as desired we consider the family $\overline{\mathcal{R}}$ of cannonical rectangles determined by d and introduced in the preceding section and use Lemma 2.2.

Let r be a positive integer. We select a random set X of $r2^d$ points from $[0, 2^d)^2$ as follows. We set $X = \{p_j \mid j = 0, 1, \dots r2^d - 1\}$, where the x-coordinate of p_j is set deterministically to $x_j = j/r$, while the y coordinate y_j of p_j is chosen randomly. One way to describe the choice of the y coordinates is saying that for each j we choose an integer $0 \leq y_j < 2^d$ making $|R \cap X| = r$ for each canonical rectangle $R \in \overline{\mathcal{R}}$ and we choose uniformly among the possibilities satisfying this requirement. But it is more convenient to consider the coordinates y_j in binary and choosing the binary digits in stages. Note that the process described below yields the uniform distribution on the point sets satisfying the requirements above.

We write y_j in binary form: $y_j = \sum_{i=1}^{d} y_j^{(i)} 2^{d-i}$. We choose the binary digits $y_j^{(i)} \in \{0, 1\}$ in stages: in stage i we choose the digits $y_j^{(i)}$ for all j. Note that before stage i the set $S_{a,b}^{i'} = \{0 \leq j < r2^d \mid p_i \in R_{a,b}^{i'}\}$ is determined for all $i' < i$ and $R_{a,b}^{i'} \in \overline{\mathcal{R}}$. In particular we have that $S_{a,0}^{0} = \{ar, ar+1, \dots, ar+r-1\}$ holds deterministically for any $0 \leq a < 2^d$. In stage i ($1 \leq i \leq d$) we consider the $2r$ element set $S^{i-1}_{2a,b} \cup S^{i-1}_{2a+1,b}$ and partition it uniformly randomly to two r element sets T and T' and set $y^{(i)}_j = 0$ for $j \in T$ and $y^{(i)}_j = 1$ for $j \in T'$. This makes $S^i_{a,2b} = T$ and $S^i_{a,2b+1} = T'$, so maintains that all these sets are of size r. We do this partitioning independently for $0 \leq a < 2^{d-i}$ and $0 \leq b < 2^{i-1}$.

Let us set $\varepsilon = 2^{-d}$, then any ε-net S of the range space $(X, \overline{\mathcal{R}})$ has to intersect all canonical rectangles. Let us fix a set $I \in \{0, 1, \ldots, r2^d - 1\}$ of size at most $r2^{d-2}$ and calculate the probability that $S = \{p_i \mid i \in I\}$ is an ε-net. S is *not* an ε-net if at some stage i we partition some set $T_0 = S^{i-1}_{2a,b} \cup S^{i-1}_{2a+1,b}$ in an "unlucky" way bringing all the indices in S into the same part. This has probability larger than 2^{-2r} unless $|T_0 \cap I| > r$ (in which case there is no unlucky partition of T_0). At any stage we partition independently 2^{d-1} pairwise disjoint sets, so (from the bound on the size of I) at least half of them contains at most r elements of I making the probability of no unlucky partition at a fixed stage be at most $(1 - 2^{-2r})^{2^{d-2}}$. This is true in any stage independent of the outcome of earlier stages, thus S is an ε-net with probability at most $(1 - 2^{-2r})^{d2^{d-2}}$. Finally, as there are less than 2^{r2^d} choices for I, the probability that $(X, \overline{\mathcal{R}})$ has an ε-net of size at most $r2^{d-2}$ is less than $2^{r2^d}(1 - 2^{-2r})^{d2^{d-2}}$.

Whenever $d \geq 4r4^r$ this last probability is less than 1, so there exists a choice of X such that the size of any 2^{-d}-net of $(X, \overline{\mathcal{R}})$ is at least $r2^{d-2}$.

To choose the parameters d and r for a given ε we set $d = \lfloor \log \frac{1}{\varepsilon} \rfloor$ (this ensures that any ε-net is a 2^{-d}-net) and choose r to be maximal among the integers satisfying $4r4^r \leq d$. The last paragraph claims the existence of a VC-dimension 2 range space $(X, \overline{\mathcal{R}})$ of axis-parallel rectangles with the size of any ε-net being at least $r2^{d-2} > \frac{r}{8\varepsilon} > (\frac{1}{16} - o(1))\frac{1}{\varepsilon} \log \log \frac{1}{\varepsilon}$.

Once we have one example of a range space $\Sigma = (X, \overline{\mathcal{R}})$ that admits no small ε-net for a given value of ε, we can create arbitrarily large examples with the same property, by replacing each point $p \in X$ with t new points, contained in the same ranges of $\overline{\mathcal{R}}$. (The same trick was applied in [Al10] and in the proof of Theorem 1.) This completes the proof of Theorem 4. □

4 Concluding remarks

1. It was shown in [PaW90] that any range space $(X, \mathcal{R})$, where X is a finite point set in the plane and $\mathcal{R}$ consists of half-planes, admits ε-nets of size at most $\lceil 2/\varepsilon \rceil - 1$, and that this bound is tight up to an additive constant at most 1. The corresponding result on the line is almost trivial. Consequently, Theorem A holds in any dimension $d \leq 3$, and our Theorem 3 shows that it is false for $d > 3$.

The epsilon-net problem for half-spaces (containing the origin) is self-dual. That is, any *dual* range space induced by half-spaces in $\mathbb{R}^d$ admits an ε-net of size $O(1/\varepsilon)$ if $d \leq 3$, and this statement is false whenever $d > 3$.

2. Recall that a *weak ε-net* for a range space $(X, \mathcal{R})$ is a set of elements of $\cup_{R \in \mathcal{R}} R$ (not necessarily in X) such that every range $R \in \mathcal{R}$ with $|R \cap X| \geq \varepsilon|X|$ contains at least one of them. In [Ez10], Ezra proved that if X is any finite set of points in $\mathbb{R}^d$ and $\mathcal{R}$ consists of all axis-parallel boxes, then $(X, \mathcal{R})$ admits a weak ε-net of size $O\left(\frac{1}{\varepsilon} \log \log \frac{1}{\varepsilon}\right)$. This implies that our Theorem 2 cannot be strengthened by requiring that the constructed range spaces do not admit *weak* ε-nets of size smaller than $\frac{1}{\varepsilon} \log \frac{1}{\varepsilon}$, provided that $\varepsilon > 0$ is sufficiently small.

It is easy to see that the analogue of Theorem 3 is also false for *weak* ε-nets instead of strong ones. Indeed, any finite system of half-spaces in $\mathbb{R}^d$ can be hit by $d+1$ points, so that in (primal or dual) half-space range spaces there always exist weak ε-nets of size $O(1)$.

However, we have been unable to decide whether the analogue of Theorem 4 holds for weak ε-nets in place of strong ones.

3. Let X be a finite or infinite set and let $\mathcal{R}$ be a family of "ranges" of a certain type in $\mathbb{R}^d$ (e.g., lines, balls, half-spaces, axis-parallel boxes). We say that a subfamily $\mathcal{S} \subset \mathcal{R}$ forms a *k-fold covering* of X if every point of X belongs to at least k members of $\mathcal{S}$. It is an old problem in discrete geometry to decide whether every k-fold covering selected from a family $\mathcal{R}$ can be decomposed into two or more coverings [PaTT09]. For example, it was shown by Gibson and Varadarajan [GiV09] that every k-fold covering of the plane with translates of a convex polygon can be decomposed into $\Omega(k)$ coverings.

There is an intimate relationship between epsilon-net problems and problems about decomposition of multiple coverings. If we know that every k-fold covering $\mathcal{S} \subset \mathcal{R}$ with $|\mathcal{S}| = n$ splits into at least ck coverings for some absolute constant $c > 0$, then one of these coverings contains at most $n/(ck)$ sets. Setting $k = \varepsilon n$, we find a covering consisting of at most $1/(c\varepsilon)$ members of $\mathcal{S}$. This means that the *dual* range space Σ^* induced by the members of $\mathcal{S}$ admits an ε-net of size $O(1/\varepsilon)$. Therefore, if the dual range space does not always admit an ε-net of size $O(1/\varepsilon)$, then it cannot be true that every k-fold covering with ranges from $\mathcal{R}$ splits into $\Omega(k)$ coverings.

In particular, Alon [Al10] proved that there are n-element point sets $X \subset \mathbb{R}^2$ and straight-line ranges that do not admit ε-nets of size $O(1/\varepsilon)$. The standard duality between points and lines preserves incidences. Switching to the dual, we obtain dual range spaces induced by sets of n lines in the plane that do not admit ε-nets of size $O(1/\varepsilon)$. According to the argument in the previous paragraph, this implies that it cannot be true that every k-fold covering of a finite set of points in $\mathbb{R}^2$ with straight lines splits into $\Omega(k)$ coverings. This consequence of Alon's theorem had been proved earlier, using the Hales-Jewett theorem [PaTT09]. Alon [Al10] proved that the same example also disproves that all range spaces consisting of straight-line ranges in the plane admit ε-nets of size $O(1/\varepsilon)$.

4. If we replace Lemma 3.1 by a slightly weaker statement (Theorem 9) in [ChPS09], we obtain a weaker version of Theorem 4, resulting in an $\Omega\left(\frac{1}{\varepsilon}\log\log\frac{1}{\varepsilon}/\log\log\log\frac{1}{\varepsilon}\right)$ bound on the size of the ε-nets. Similarly, if we replace Lemma 2.1 by a slightly weaker statement (Theorem 3) in [PaT10] we obtain a weaker version of Theorem 1 (and hence Theorems 2 and 3) with an $\Omega\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}/\log\log\frac{1}{\varepsilon}\right)$ bound on the size of the ε-nets.

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