

# TIGHT REPRESENTATIONS OF SEMILATTICES AND INVERSE SEMIGROUPS

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By a *Boolean inverse semigroup* we mean an inverse semigroup whose semilattice of idempotents is a Boolean algebra. We study representations of a given inverse semigroup  $\mathcal{S}$  in a Boolean inverse semigroup which are *tight* in a certain well defined technical sense. These representations are supposed to preserve as much as possible any trace of *Booleannes* present in the semilattice of idempotents of  $\mathcal{S}$ . After observing that the Vagner–Preston representation is not tight, we exhibit a canonical tight representation for any inverse semigroup with zero, called the *regular tight representation*. We then tackle the question as to whether this representation is faithful, but it turns out that the answer is often negative. The lack of faithfulness is however completely understood as long as we restrict to *continuous* inverse semigroups, a class generalizing the  $E^*$ -unitaries.

## 1. Introduction.

We shall say that an inverse semigroup  $\mathcal{S}$  is a *Boolean inverse semigroup*, if  $E(\mathcal{S})$ , the semilattice of idempotents of  $\mathcal{S}$ , admits the structure of a Boolean algebra whose order coincides with the usual order on  $E(\mathcal{S})$ .

Boolean inverse semigroups are quite common, a well known example being the semigroup  $\mathcal{I}(X)$  of all partially defined bijections on  $X$ . The semilattice of idempotents of  $\mathcal{I}(X)$  coincides with the Boolean algebra  $\mathcal{P}(X)$  of all subsets of  $X$ .

Given an inverse semigroup  $\mathcal{S}$  one might like to study how far it is from being a Boolean inverse semigroup by considering homomorphisms

$$\beta : \mathcal{S} \rightarrow \mathcal{B},$$

into some Boolean inverse semigroup  $\mathcal{B}$ . Simply requiring  $\beta$  to be a semigroup homomorphism completely sidesteps the issue since, in case  $\mathcal{S}$  itself happens to be a Boolean inverse semigroup, a mere semigroup homomorphism has no reason to respect the Boolean algebra structures involved.

To deal with this situation we propose to consider a special class of homomorphisms called *tight representations* (see Definition (6.1)), which applies to every inverse semigroup with zero. In case  $\mathcal{S}$  is a Boolean inverse semigroup we prove in Proposition (6.2) that tight representations are precisely those which restrict to a homomorphism  $\beta : E(\mathcal{S}) \rightarrow E(\mathcal{B})$  in the category of Boolean algebras.

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One of the most important homomorphisms from an inverse semigroup  $\mathcal{S}$  to a Boolean inverse semigroup is the so called Vagner–Preston map [7]

$$\gamma : \mathcal{S} \rightarrow \mathcal{I}(X),$$

which shows, among other things, that every inverse semigroup is a subsemigroup of some  $\mathcal{I}(X)$ . However  $\gamma$  is never a tight representation, even in case  $\mathcal{S}$  is a Boolean inverse semigroup. For example  $\gamma(0)$  is never equal to the zero of  $\mathcal{I}(X)$ , namely the empty function. In fact this is not the only flaw presented by  $\gamma$  from the point of view of tight representations, as explained below.

It is the main purpose of this work to introduce a canonical tight representation

$$\lambda : \mathcal{S} \rightarrow \mathcal{I}(\Omega),$$

where  $\Omega$  is a certain space of *filters*, which we call the *regular tight representation*. See Theorem (6.16).

Unlike the Vagner–Preston representation, the regular tight representation is not always faithful, but under a certain *continuity* hypothesis we are able to precisely describe when is  $\lambda(s) = \lambda(t)$ , for a given pair of elements  $s, t \in \mathcal{S}$ .

The issue boils down to the following situation: let  $e \leq f$  be idempotents in  $E(\mathcal{S})$  and suppose that there is no nonzero idempotent  $d \leq f$  such that  $d \perp e$  (meaning that  $de = 0$ ). Very roughly speaking this means that the space between  $e$  and  $f$  is empty, in which case we say that  $e$  is *dense* in  $f$ . Notice however that when  $e \neq f$ , this will never happen in a Boolean inverse semigroup, since  $d := f \wedge \neg e \neq 0$ .

It turns out that when  $e$  is dense in  $f$  one has that  $\lambda(e) = \lambda(f)$ , even when  $e \neq f$ . In case  $e$  is not necessarily less than  $f$ , but  $ef$  is dense in both  $e$  and  $f$ , we will consequently also have that  $\lambda(e) = \lambda(ef) = \lambda(f)$ .

The impossibility of distinguishing between idempotents clearly has consequences for other elements. Suppose for example that  $s, t \in \mathcal{S}$  are such that  $\lambda(s^{-1}s) = \lambda(t^{-1}t)$ . Suppose moreover that<sup>1</sup>  $st^{-1}t = ts^{-1}s$ . Then a simple computation (see (7.5)) shows that  $\lambda(s) = \lambda(t)$ , so we get another instance on non-faithfulness.

Fortunately we are able to prove in Theorem (7.5) that, under the continuity hypothesis already alluded to, these well understood situations are the only ones allowing for  $\lambda(s) = \lambda(t)$ . Another consequence is that when the regular tight representation is unable to separate between two elements of  $\mathcal{S}$ , then no tight representation can possibly do it.

To explain what do we mean by continuity let us say that two elements  $s, t \in \mathcal{S}$  *essentially coincide with each other*, in symbols  $s \equiv t$ , if  $s^{-1}s = t^{-1}t$ , and for every nonzero idempotent  $f \leq s^{-1}s$ , there exists a nonzero idempotent  $e \leq f$ , such that  $se = te$ . Very roughly this means that  $s$  and  $t$  coincide on a dense set, an idea which may be

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<sup>1</sup> In case  $\mathcal{S}$  is contained in some  $\mathcal{I}(X)$ , this means that  $s$  and  $t$  coincide on the intersection of their domains.

made quite precise when we are speaking of *localizations* in the sense of Kumjian [6]. See Proposition (7.2).

Recalling that when two continuous functions agree on a dense set of their common domain they must coincide everywhere, we say that  $\mathcal{S}$  is *continuous* if  $s \equiv t$  implies that  $s = t$ . Localizations are continuous by Proposition (7.2), and so are  $E^*$ -unitary inverse semigroups, as proved in (7.3).

The use of the continuity hypothesis in Theorem (7.5) naturally raises the question of whether or not this hypothesis is really needed. To resolve this issue, in the final section of this work we describe a general construction which leads to a non-continuous Boolean inverse semigroup  $\mathcal{S}$  for which (7.5) does fail.

The concept of a tight representation of an inverse semigroup  $\mathcal{S}$  strongly depends on its idempotent semilattice  $E(\mathcal{S})$ , so we begin by considering representations of general semilattices in Boolean algebras.

The special case of representations of *lattices* has been considered by many authors, perhaps starting with Stone's Theorem [10] on the characterization of Boolean algebras, and giving way to what is now called *Stone duality*, where complete distributive lattices, also known as *frames*, *locales* or *complete Heyting algebras*, are put in correspondence with sober topological spaces. See [9] and the references given there for more information.

There is also a large literature on *continuous lattices* [4], and some attention has been given to pseudocomplemented semilattices [3], or even general semilattices [5], but our notion of tight representations does not seem to have been considered before.

## 2. Representations of semilattices.

Although we are mainly interested in inverse semigroups, their semilattice of idempotents play a particularly important role in the ideas we shall develop. For this reason we will set this section apart focusing exclusively on semilattices.

### 2.1. Definition.

- (i) By a *partially ordered set* we shall mean a set  $X$  equipped with an order relation (i.e. a reflexive, antisymmetric, and transitive relation) " $\leq$ ", such that  $X$  contains a smallest element, denoted 0.
- (ii) A *semilattice* is a partially ordered set  $X$  such that for every  $x, y \in X$ , the set  $\{z \in X : z \leq x, y\}$  contains a maximum element, denoted  $x \wedge y$ .

It is perhaps not usual to require partially ordered sets or semilattices to contain a zero element. However if a partially ordered set  $X$  does not contain zero one can easily embed it in  $X \dot{\cup} \{0\}$ , with the order extended from  $X$  in such a way that  $0 \leq x$ , for all  $x$ . If  $X$  is a semilattice, it is obvious that  $X \dot{\cup} \{0\}$  is also a semilattice.

**2.2. Definition.** If  $X$  is a partially ordered set we shall say that two elements  $x, y \in X$  are *disjoint*, in symbols  $x \perp y$ , if there is no nonzero  $z \in X$  such that  $z \leq x, y$ . Otherwise we shall say that  $x$  and  $y$  *intersect*. We shall express the fact that  $x$  and  $y$  intersect by writing  $x \pitchfork y$ .

If  $E$  is a semilattice it is easy to see that two elements  $x, y \in E$  intersect if and only if  $x \wedge y \neq 0$ .

**2.3. Definition.** Let  $E$  be a semilattice and let  $\mathcal{B} = (\mathcal{B}, 0, 1, \wedge, \vee, \neg)$  be a Boolean algebra. By a *representation* of  $E$  in  $\mathcal{B}$  we shall mean a map  $\beta : E \rightarrow \mathcal{B}$ , such that

- (i)  $\beta(0) = 0$ , and
- (ii)  $\beta(x \wedge y) = \beta(x) \wedge \beta(y)$ , for every  $x, y \in E$ .

Fix for the time being a representation  $\beta$  of a semilattice  $E$  in a Boolean algebra  $\mathcal{B}$ . If  $x, y \in E$  are such that  $x \leq y$ , it is clear that  $\beta(x) \leq \beta(y)$ . On the other hand, if  $x \perp y$ , one has that  $\beta(x) \perp \beta(y)$ , which may also be expressed in  $\mathcal{B}$  as

$$\beta(x) \leq \neg \beta(y).$$

More generally, if  $X$  and  $Y$  are finite subsets of  $E$ , and one is given an element  $z \in E$  such that  $z \leq x$  for every  $x \in X$ , and  $z \perp y$  for every  $y \in Y$ , it follows that

$$\beta(z) \leq \bigwedge_{x \in X} \beta(x) \wedge \bigwedge_{y \in Y} \neg \beta(y). \quad (2.4)$$

The set of all such  $z$ 's will acquire an increasing importance, so we make the following:

**2.5. Definition.** Given finite subsets  $X, Y \subseteq E$ , we shall denote by  $E^{X,Y}$  the subset of  $E$  given by

$$E^{X,Y} = \{z \in E : z \leq x, \forall x \in X, \text{ and } z \perp y, \forall y \in Y\}.$$

Notice that if  $x_{min} = \bigwedge_{x \in X} x$ , one may replace  $X$  in (2.5) by the singleton  $\{x_{min}\}$ , without altering  $E^{X,Y}$ . However there does not seem to be a similar way to replace  $Y$  by a smaller set.

**2.6. Definition.** Given any subset  $F \subseteq E$ , we shall say that a subset  $Z \subseteq F$  is a *cover* for  $F$ , if for every nonzero  $x \in F$ , there exists  $z \in Z$  such that  $z \pitchfork x$ .

The notion of covers is relevant to the introduction of the following central concept:

**2.7. Definition.** Let  $\beta : E \rightarrow \mathcal{B}$  be a representation of the semilattice  $E$  in the Boolean algebra  $\mathcal{B}$ . We shall say that  $\beta$  is *tight* if for all finite subsets  $X, Y \subseteq E$ , and for every finite cover  $Z$  for  $E^{X,Y}$ , one has that

$$\bigvee_{z \in Z} \beta(z) \geq \bigwedge_{x \in X} \beta(x) \wedge \bigwedge_{y \in Y} \neg \beta(y).$$

Notice that the reverse inequality “ $\leq$ ” always holds by (2.4). Thus, when  $\beta$  is tight, we actually get an equality above. We should also remark that in the absence of any finite cover  $Z$ , as above, every representation is considered to be tight by default.

In certain cases the verification of tightness may be greatly simplified:

**2.8. Proposition.** *Let  $\beta$  be a representation of the semilattice  $E$  in the Boolean algebra  $\mathcal{B}$ , such that either*

- (i)  *$E$  contains a finite set  $X$  such that  $\bigvee_{x \in X} \beta(x) = 1$ , or*
- (ii)  *$E$  does not admit any finite cover.*

*Then  $\beta$  is tight if and only if for every nonzero  $x \in E$  and for every finite cover  $Z$  for the interval*

$$[0, x] := \{z \in E : z \leq x\},$$

*one has that  $\bigvee_{z \in Z} \beta(z) \geq \beta(x)$ .*

*Proof.* See [2: 10.8]. □

Whenever  $z \in [0, x]$ , notice that  $\beta(z) \leq \beta(x)$ , so the last inequality in the statement of the result above is in fact equivalent to  $\bigvee_{z \in Z} \beta(z) = \beta(x)$ .

The representation of  $E$  in  $\mathcal{P}(E^*)$  described above is not necessarily tight. In fact, if  $E$  consists of three distinct elements, say  $E = \{0, y, 1\}$ , with the order relation such that  $0 \leq y \leq 1$ , set  $X = \{1\}$  and  $Y = \{y\}$ . Then  $E^{X, Y} = \{0\}$ , so the empty set  $Z$  is a cover for  $E^{X, Y}$ . However

$$\bigvee_{z \in Z} \beta(z) = \emptyset \neq \{1\} = \bigwedge_{x \in X} \beta(x) \wedge \bigwedge_{y \in Y} \neg \beta(y).$$

Not all semilattices admit tight injective representations. In order to study this issue in detail it is convenient to introduce the following:

**2.9. Definition.** Let  $E$  be a semilattice and let  $x, y \in E$  be such that  $y \leq x$ . We shall say that  $y$  is *dense* in  $x$  if there is no nonzero  $z \in E$  such that  $z \leq x$  and  $z \perp y$ . Equivalently, if  $E^{\{x\}, \{y\}} = \{0\}$ .

Obviously each  $x \in E$  is dense in itself but it is conceivable that some  $y \neq x$  is dense in  $x$ . For a concrete example notice that in the semilattice  $E = \{0, y, 1\}$  above one has that  $y$  is dense in 1.

In the general case, whenever  $y$  is dense in  $x$  we have that  $E^{\{x\}, \{y\}} = \{0\}$ , and hence the empty set is a cover for  $E^{\{x\}, \{y\}}$ . Therefore for every tight representation  $\beta$  of  $E$  one has that

$$0 = \beta(x) \wedge \neg \beta(y),$$

which means that  $\beta(x) \leq \beta(y)$ . Since the opposite inequality also holds, we have that  $\beta(x) = \beta(y)$ . Thus no tight representation of  $E$  can possibly separate  $x$  and  $y$ . For future reference we record this conclusion in the next:

**2.10. Proposition.** *If  $y \leq x$  are elements in the semilattice  $E$ , such that  $y$  is dense in  $x$ , then  $\beta(y) = \beta(x)$  for every tight representation  $\beta$  of  $E$ .*

When  $E$  happens to be a Boolean algebra there is a very elementary characterization of tight representations:

**2.11. Proposition.** *Suppose that  $E$  is a semilattice admitting the structure of a Boolean algebra which induces the same order relation as that of  $E$ , and let  $\beta : E \rightarrow \mathcal{B}$  be a representation of  $E$  in some Boolean algebra  $\mathcal{B}$ . Then  $\beta$  is tight if and only if it is a Boolean algebra homomorphism.*

*Proof.* Supposing that  $\beta$  is tight, notice that  $\{1\}$  is a cover for  $E^{\emptyset, \{0\}}$ , so

$$\beta(1) = \neg\beta(0) = \neg 0 = 1.$$

Given  $x \in E$  notice that  $\{\neg x\}$  is a cover for  $E^{\emptyset, \{x\}}$ , therefore

$$\beta(\neg x) = \neg\beta(x).$$

Since  $x \vee y = \neg(\neg x \wedge \neg y)$ , for all  $x, y \in E$ , we may easily prove that  $\beta(x \vee y) = \beta(x) \vee \beta(y)$ . Thus  $\beta$  is a Boolean algebra homomorphism, as required.

In order to prove the converse implication let  $X, Y \subseteq E$  be finite sets and let  $Z$  be a finite cover for  $E^{X, Y}$ . Let

$$z_0 = \bigvee_{z \in Z} z, \quad x_0 = \bigwedge_{x \in X} x, \quad \text{and} \quad \bar{y}_0 = \bigwedge_{y \in Y} \neg y.$$

It is obvious that  $z_0 \leq x_0 \wedge \bar{y}_0$ , and we claim that in fact  $z_0 = x_0 \wedge \bar{y}_0$ . We will prove it by checking that

$$\neg z_0 \wedge x_0 \wedge \bar{y}_0 = 0.$$

Let  $u = \neg z_0 \wedge x_0 \wedge \bar{y}_0$ , and notice that the fact that  $u \leq x_0 \wedge \bar{y}_0$  implies that  $u \in E^{X, Y}$ . Arguing by contradiction, and hence supposing that  $u$  is nonzero, we deduce that  $u \wedge z$ , for some  $z \in Z$ , but this contradicts the fact that  $u \leq \neg z_0$ . This proves our claim so, assuming that  $\beta$  is a Boolean algebra homomorphism, we have

$$\bigvee_{z \in Z} \beta(z) = \beta\left(\bigvee_{z \in Z} z\right) = \beta(z_0) = \beta(x_0 \wedge \bar{y}_0) = \bigwedge_{x \in X} \beta(x) \wedge \bigwedge_{y \in Y} \neg\beta(y),$$

showing that  $\beta$  is tight. □

We shall have a lot more to say about tight representations in the following sections.

### 3. Filters.

A fundamental tool for the study of tight representations of semilattices is the well-known notion of filters, which we briefly describe in this section.

**3.1. Definition.** Let  $X$  be any partially ordered set with minimum element  $0$ . A *filter* in  $X$  is a nonempty subset  $\xi \subseteq X$ , such that

- (i)  $0 \notin \xi$ ,
- (ii) if  $x \in \xi$  and  $y \geq x$ , then  $y \in \xi$ ,
- (iii) if  $x, y \in \xi$ , there exists  $z \in \xi$ , such that  $x, y \geq z$ .

An *ultrafilter* is a filter which is not properly contained in any filter.

Given a partially ordered set  $X$  and any nonzero element  $x \in X$  it is elementary to prove that

$$\xi = \{y \in X : y \geq x\}$$

is a filter containing  $x$ . By Zorn's Lemma there exists an ultrafilter containing  $\xi$ , thus every nonzero element in  $X$  belongs to some ultrafilter.

When  $E$  is a semilattice, given the existence of  $x \wedge y$  for every  $x, y \in E$ , condition (3.1.iii) may be replaced by

$$x, y \in \xi \Rightarrow x \wedge y \in \xi. \quad (3.2)$$

The following is an important fact about filters in semilattices.

**3.3. Lemma.** *Let  $E$  be a semilattice and let  $\xi$  be a filter in  $E$ . Then  $\xi$  is an ultrafilter if and only if  $\xi$  contains every element  $y \in E$  such that  $y \pitchfork x$  for every  $x \in \xi$ .*

*Proof.* In order to prove the “if” part let  $\eta$  be a filter such that  $\xi \subseteq \eta$ . Given  $y \in \eta$  one has that for every  $x \in \xi$ , both  $y$  and  $x$  lie in  $\eta$ , and hence (3.2) implies that  $y \wedge x \in \eta$ , so  $y \wedge x \neq 0$ , and hence  $y \pitchfork x$ . By hypothesis  $y \in \xi$ , proving that  $\eta = \xi$ , and hence that  $\xi$  is an ultrafilter.

Conversely let  $\xi$  be an ultrafilter and suppose that  $y \in E$  is such that  $y \pitchfork x$ , for every  $x \in \xi$ . Defining

$$\eta = \{u \in E : u \geq y \wedge x, \text{ for some } x \in \xi\},$$

we claim that  $\eta$  is a filter. By hypothesis  $0 \notin \eta$ . Also if  $u_1, u_2 \in \eta$ , choose for every  $i = 1, 2$  some  $x_i \in \xi$  such that  $u_i \geq y \wedge x_i$ . Then

$$u_1 \wedge u_2 \geq (y \wedge x_1) \wedge (y \wedge x_2) = y \wedge (x_1 \wedge x_2),$$

so  $u \in \eta$ . Given that (3.1.ii) is obvious we see that  $\eta$  is indeed a filter, as claimed. Noticing that  $\xi \subseteq \eta$  we have that  $\eta = \xi$ , because  $\xi$  is an ultrafilter. Since  $y \in \eta$ , we deduce that  $y \in \xi$ .  $\square$

#### 4. Characters.

We fix, throughout this section, a semilattice  $E$ , always assumed to have a smallest element 0. The study of representations of  $E$  in the most elementary Boolean algebra of all, namely  $\{0, 1\}$ , leads us to the following well known important concept.

**4.1. Definition.** A *character* of  $E$  is a nonzero representation of  $E$  in the Boolean algebra  $\{0, 1\}$ . The set of all characters will be denoted by  $\widehat{E}$ .

We observe that it is implicit in our definition that a character vanishes at zero, a condition which is not required by some authors.

Temporarily denoting by  $\widetilde{E}$  the set of *all* representations of  $E$  in  $\{0, 1\}$ , including the identically zero representation, it is easy to see that  $\widetilde{E}$  is a closed subspace of the compact product space  $\{0, 1\}^E$ , hence  $\widetilde{E}$  is compact. Since  $\widehat{E}$  is obtained by removing the identically zero representation from  $\widetilde{E}$ , we have that  $\widehat{E}$  is locally compact.

Given a character  $\phi$ , observe that

$$\xi_\phi = \{x \in E : \phi(x) = 1\}, \quad (4.2)$$

is a filter in  $E$  (it is nonempty because  $\phi$  is assumed not to be identically zero). Conversely, given a filter  $\xi$ , define for every  $x \in E$ ,

$$\phi_\xi(x) = \begin{cases} 1, & \text{if } x \in \xi, \\ 0, & \text{otherwise.} \end{cases}$$

It is then easy to see that  $\phi_\xi$  is a character. Therefore we see that (4.2) gives a one-to-one correspondence between  $\widehat{E}$  and the set of all filters.

**4.3. Proposition.** *If  $\xi$  is an ultrafilter then  $\phi = \phi_\xi$  is a tight representation of  $E$  in  $\{0, 1\}$ .*

*Proof.* Let  $X, Y \subset E$  be finite subsets and let  $Z$  be a cover for  $E^{X, Y}$ . In order to prove that

$$\bigvee_{z \in Z} \phi(z) \geq \prod_{x \in X} \phi(x) \prod_{y \in Y} (1 - \phi(y)),$$

it is enough to show that if the right-hand side equals 1, then so does the left-hand side. This is to say that if  $x \in \xi$  for every  $x \in X$ , and  $y \notin \xi$  for every  $y \in Y$ , then there is some  $z \in Z$ , such that  $z \in \xi$ .

By (3.3), for each  $y \in Y$  there exists some  $x_y \in \xi$  such that  $y \perp x_y$ . Supposing by contradiction that  $Z \cap \xi = \emptyset$ , then for every  $z \in Z$  there exists, again by (3.3), some  $x_z \in \xi$ , such that  $z \perp x_z$ . Set

$$w = \bigwedge_{x \in X} x \wedge \bigwedge_{y \in Y} x_y \wedge \bigwedge_{z \in Z} x_z.$$

Since  $w \in \xi$  we have that  $w \neq 0$ . Obviously  $w \leq x$  for every  $x \in X$ , and  $w \perp y$  for every  $y \in Y$ , and hence  $w \in E^{X, Y}$ . Since  $Z$  is a cover there exists some  $z_1 \in Z$  such that  $w \pitchfork z_1$ . However, since  $w \leq x_{z_1} \perp z_1$ , we have that  $w \perp z_1$ , a contradiction.  $\square$



**4.4. Definition.** We shall denote by  $\widehat{E}_\infty$  the set of all characters  $\phi \in \widehat{E}$  such that  $\xi_\phi$  is an ultrafilter. Also we will denote by  $\widehat{E}_{tight}$  the set of all tight characters.

Employing the terminology just introduced we may rephrase (4.3) by saying that  $\widehat{E}_\infty \subseteq \widehat{E}_{tight}$ . The following main result further describes the relationship between  $\widehat{E}_\infty$  and  $\widehat{E}_{tight}$ .

**4.5. Theorem.** *Let  $E$  be a semilattice with smallest element  $0$ , and let  $\widehat{E}_\infty$  and  $\widehat{E}_{tight}$  be as defined in (4.4). Then the closure of  $\widehat{E}_\infty$  in  $\widehat{E}$  coincides with  $\widehat{E}_{tight}$ .*

*Proof.* Since the condition for any given  $\phi$  in  $\widehat{E}$  to belong to  $\widehat{E}_{tight}$  is given by equations it is easy to prove that  $\widehat{E}_{tight}$  is closed within  $\widehat{E}$ , and since  $\widehat{E}_\infty \subseteq \widehat{E}_{tight}$  by (4.3), we deduce that  $\overline{\widehat{E}_\infty} \subseteq \widehat{E}_{tight}$ . To prove the reverse inclusion let us be given  $\phi \in \widehat{E}_{tight}$ . We must therefore show that  $\phi$  can be arbitrarily approximated by elements from  $\widehat{E}_\infty$ . Let  $U$  be a neighborhood of  $\phi$  within  $\widehat{E}$ . By definition of the product topology,  $U$  contains a neighborhood of  $\phi$  of the form

$$V = V_{X,Y} = \{\psi \in \widehat{E} : \psi(x) = 1, \text{ for all } x \in X, \text{ and } \psi(y) = 0, \text{ for all } y \in Y\},$$

where  $X$  and  $Y$  are finite subsets of  $E$ . We next claim that  $E^{X,Y} \neq \{0\}$ . In order to prove this suppose the contrary, and hence  $Z = \emptyset$  is a cover for  $E^{X,Y}$ . Since  $\phi$  is tight we conclude that

$$0 = \bigvee_{z \in Z} \phi(z) = \prod_{x \in X} \phi(x) \prod_{y \in Y} (1 - \phi(y)).$$

However, since  $\phi$  is supposed to be in  $V$ , we have that  $\phi(x) = 1$  for all  $x \in X$ , and  $\phi(y) = 0$  for all  $y \in Y$ , which means that the right-hand side of the expression displayed above equals 1. This is a contradiction and hence our claim is proved.

We are therefore allowed to choose a nonzero  $z \in E^{X,Y}$ , and further to pick an ultrafilter  $\xi$  such that  $z \in \xi$ . Observe that  $\phi_\xi \in \widehat{E}_\infty$ , and the proof will be concluded once we show that  $\phi_\xi \in U$ .

For every  $x \in X$  and  $y \in Y$ , we have that  $z \leq x$  and  $z \perp y$ , hence  $x \in \xi$  and  $y \notin \xi$ . This entails  $\phi_\xi(x) = 1$  and  $\phi_\xi(y) = 0$ , so  $\phi_\xi \in V \subseteq U$ , as required.  $\square$

In the correspondence between  $\widehat{E}$  and the set of all filters given by (4.2), we know that elements of  $\widehat{E}_\infty$  correspond to ultrafilters (by definition of the former). Given the importance of the notion of tight characters, highlighted by (4.5), it is sensible to make the following:

**4.6. Definition.** A filter  $\xi$  in  $E$  is said to be *tight* if  $\phi_\xi$  is a tight character, that is, if  $\phi_\xi \in \widehat{E}_{tight}$ .

By (4.3) we see that every ultrafilter is tight.

## 5. Boolean inverse semigroups.

In this section we shall fix an inverse semigroup  $S$ , with idempotent semilattice denoted by  $E(S)$ . We shall always assume that  $S$  contains a zero element  $0$ , meaning that

$$0s = s0 = 0, \quad \forall s \in S. \quad (5.1)$$

Any semigroup  $S$  can be readily embedded in a semigroup with zero by simply adding an extra element, denoted  $0$ , and extending the multiplication operation of  $S$  by means of (5.1).

Suppose that  $S$  is an inverse semigroup whose semilattice of idempotents is a Boolean algebra in the sense that its order relation is induced by the structure of a Boolean algebra on  $E(S)$ . In particular  $E(S)$  must contain a smallest element  $0$ , and a biggest element  $1$ . For every  $e \in E(S)$  one then has that  $1e = e$ . It follows that for every  $s \in S$ ,

$$1s = 1ss^{-1}s = ss^{-1}s = s,$$

and similarly  $s1 = s$ . So we see that  $1$  is a multiplicative unit for  $S$ .

On the other hand  $0e = e0 = 0$ , for every idempotent  $e$ , but this does not necessarily imply (5.1), a counter-example being that of any group  $G$  with more than one element. In this case the smallest element of  $E(G) = \{1\}$  is  $1$ , so  $0 = 1$ . However it is definitely not true that  $0s = s0 = 0$ , for every  $s \in G$ .

**5.2. Definition.** A *Boolean inverse semigroup* is an inverse semigroup  $\mathcal{B}$  whose lattice of idempotents  $E(\mathcal{B})$  is a Boolean algebra, and such that

$$0s = s0 = 0, \quad \forall s \in \mathcal{B},$$

where  $0$  denotes the smallest element of  $E(\mathcal{B})$ .

If  $\mathcal{B}$  is a Boolean inverse semigroup we will freely use Boolean algebra language when referring to  $E(\mathcal{B})$  with the understanding that it relates to the unique Boolean algebra structure compatible with the order structure of  $E(\mathcal{B})$ . However we will refrain from using the *meet* operator “ $\wedge$ ” since it is conveniently substituted by the multiplication operation on  $E(\mathcal{B})$ .

Examples of Boolean inverse semigroups are quite common. If  $X$  is any set and  $\mathcal{I}(X)$  is the set of all partially defined bijections between subsets of  $X$  then it is well known that  $\mathcal{I}(X)$  is an inverse semigroup under composition, called the *symmetric inverse semigroup on  $X$* . The semilattice of idempotents of  $\mathcal{I}(X)$  is identical to  $\mathcal{P}(X)$ , the Boolean algebra of all subsets of  $X$ , and hence  $\mathcal{I}(X)$  is a Boolean inverse semigroup.

## 6. Representations of inverse semigroups.

The representation theory of inverse semigroups (see for example [1]) is a well established field of research investigating semigroup homomorphisms from a given inverse semigroup  $\mathcal{S}$  into some symmetric inverse semigroup. In this section we will study representations in the slightly more general setting of Boolean inverse semigroups, but which are *tight* as defined below. This rather natural concept seems not to have been discussed in the literature so far.

Throughout this section we fix an inverse semigroup  $\mathcal{S}$  with zero. If  $\mathcal{B}$  is a Boolean inverse semigroup and

$$\beta : \mathcal{S} \rightarrow \mathcal{B}$$

is a semigroup homomorphism, observe that  $\beta(E(\mathcal{S})) \subseteq E(\mathcal{B})$ , so the restriction of  $\beta$  to  $E(\mathcal{S})$  is a map from a semilattice into a Boolean algebra. It therefore makes sense to ask whether or not it is a tight representation.

**6.1. Definition.** Let  $\mathcal{B}$  be a Boolean inverse semigroup. A semigroup homomorphism  $\beta : \mathcal{S} \rightarrow \mathcal{B}$  is said to be a *tight representation* if the restriction of  $\beta$  to  $E(\mathcal{S})$  is a tight representation of  $E(\mathcal{S})$  in  $E(\mathcal{B})$ , in the sense of (2.7).

Notice that if  $\beta$  is a tight representation in the above sense then (2.3.i) applies so it is understood that  $\beta(0) = 0$ . The following is an obvious consequence of (2.11).

**6.2. Proposition.** *Let  $\mathcal{S}$  and  $\mathcal{B}$  be Boolean inverse semigroups and let*

$$\beta : \mathcal{S} \rightarrow \mathcal{B}$$

*be a semigroup homomorphism. Then  $\beta$  is a tight representation if and only if the restriction of  $\beta$  to  $E(\mathcal{S})$  is a homomorphism in the category of Boolean algebras.*

Among the better known examples of a semigroup homomorphism from an inverse semigroup  $\mathcal{S}$  to a Boolean inverse semigroup is the Vagner–Preston representation [7], so it is interesting to ask whether or not it is a tight representation. In order to fix notation let us briefly describe it. For every idempotent  $e \in E(\mathcal{S})$ , let

$$D_e = \{t \in \mathcal{S} : tt^{-1} \leq e\},$$

and for  $s \in \mathcal{S}$  consider the map  $\gamma(s) : D_{s^{-1}s} \rightarrow D_{ss^{-1}}$ , given by  $\gamma(s)(t) = st$ . Then each  $\gamma(s)$  is a bijective map and hence  $\gamma$  gives a map

$$\gamma : \mathcal{S} \rightarrow \mathcal{I}(\mathcal{S}),$$

which is well known to be a semigroup monomorphism. The Vagner–Preston Theorem asserts that every inverse semigroup may be realized inside some  $\mathcal{I}(X)$  and  $\gamma$  provides just that realization.

Supposing, as we are, that  $\mathcal{S}$  contains a zero element, notice that  $D_0$  is the singleton  $\{0\}$ , while  $\gamma(0)$  is the identity map on  $D_0$ , so that  $\gamma(0)$  is *not* the zero element of  $\mathcal{I}(\mathcal{S})$ ,

the latter being the empty function. This violates (2.3.i) and hence  $\gamma$  is *not* a tight representation.

One could attempt to remedy this by working in the category of pointed spaces, replacing  $\mathcal{I}(\mathcal{S})$  by the subsemigroup  $\mathcal{I}_0(\mathcal{S})$  of all partially defined maps on  $\mathcal{S}$  which contain zero in its domain, and which send zero to itself. But this will still not give us a tight representation. To see why this is so, consider, for example, the following Boolean algebra viewed as a semilattice, and hence as an inverse semigroup:

$$\mathcal{S} = \{0, 1\} \times \{0, 1\}.$$

Since all elements of  $\mathcal{S}$  are idempotent, the range of  $\gamma$  is contained in  $E(\mathcal{I}(\mathcal{S})) = \mathcal{P}(\mathcal{S})$ .

Observe that if  $X = \{(1, 1)\}$ , and  $Y = \emptyset$ , then  $E(\mathcal{S})^{X,Y} = E(\mathcal{S}) = \mathcal{S}$ , and hence  $Z = \{(1, 0), (0, 1)\}$  is a cover for  $E(\mathcal{S})^{X,Y}$ . However

$$\bigvee_{z \in Z} \gamma(z) = \gamma(1, 0) \cup \gamma(0, 1) = \{(0, 0), (1, 0), (0, 1)\},$$

while

$$\bigwedge_{x \in X} \gamma(x) \wedge \bigwedge_{y \in Y} \neg \gamma(y) = \gamma(1, 1) = \mathcal{S}.$$

The purpose of this section is to exhibit a canonical tight representation of  $\mathcal{S}$ . Filters will again be crucial in achieving this.

While we have so far only considered filters in semilattices, from now on we will also consider filters in  $\mathcal{S}$ , relative to its standard order relation.

If  $\xi \subseteq \mathcal{S}$  is a filter and  $e \in E(\mathcal{S})$ , let

$$e\xi = \{et : t \in \xi\}.$$

We will now turn our attention to filters  $\xi$  such that  $e\xi \subseteq \xi$ .

**6.3. Lemma.** *Given a filter  $\xi \subseteq \mathcal{S}$ , suppose that  $es \in \xi$ , for some  $s \in \mathcal{S}$  and  $e \in E(\mathcal{S})$ . Then  $e\xi \subseteq \xi$ .*

*Proof.* Given  $t \in \xi$  observe that by (3.1.iii) there exists  $r \in \xi$  such that  $es, t \geq r$ . Therefore  $r = esr^{-1}r$ , so

$$et \geq er = e(esr^{-1}r) = esr^{-1}r = r \in \xi,$$

so  $et \in \xi$ . □

**6.4. Corollary.** *If  $\xi$  and  $\eta$  are filters in  $\mathcal{S}$  such that  $\xi \subseteq \eta$ , and  $e$  is an idempotent such that  $e\xi \subseteq \xi$ , then  $e\eta \subseteq \eta$ .*

*Proof.* Given any  $s \in \xi$  we have that  $es \in e\xi \subseteq \xi \subseteq \eta$ , and hence  $e\eta \subseteq \eta$  by (6.3). □

**6.5. Corollary.** *If  $\xi \subseteq \mathcal{S}$  is a filter and  $s \in \xi$ , then  $ss^{-1}\xi \subseteq \xi$ .*

*Proof.* Since  $ss^{-1}s = s \in \xi$ , the result follows from (6.3).  $\square$

For ultrafilters there is another important condition which implies the same conclusion as (6.3):

**6.6. Lemma.** *Let  $\xi \subseteq \mathcal{S}$  be an ultrafilter and let  $e \in E(\mathcal{S})$  be such that  $et \neq 0$ , for all  $t \in \xi$ . Then  $e\xi \subseteq \xi$ .*

*Proof.* Let

$$\eta = \{u \in \mathcal{S} : u \geq et, \text{ for some } t \in \xi\}.$$

Observe that  $\eta$  is a filter, since  $0 \notin \eta$ , by hypothesis, and (3.1.ii-iii) are of easy verification. For every  $t \in \xi$  one has that  $t \geq et$ , and hence  $t \in \eta$ . Thus  $\xi \subseteq \eta$ , and since  $\xi$  is an ultrafilter, we deduce that  $\xi = \eta$ . For  $t \in \xi$ , it is obvious that  $et \in \eta$ . This says that  $e\xi \subseteq \eta = \xi$ .  $\square$

**6.7. Definition.** We will denote by  $\Omega$  the set of all ultrafilters in  $\mathcal{S}$ , and for each idempotent  $e \in E(\mathcal{S})$  we will denote by  $\Omega_e$  the set of all ultrafilters  $\xi$  such that  $e\xi \subseteq \xi$ .

The following result describes how the  $\Omega_e$  behave under intersections.

**6.8. Lemma.** *Let  $e$  and  $f$  be idempotents in  $E(\mathcal{S})$ . Then  $\Omega_e \cap \Omega_f = \Omega_{ef}$ .*

*Proof.* If  $\xi \in \Omega_e \cap \Omega_f$  then

$$ef\xi = e(f\xi) \subseteq e\xi \subseteq \xi,$$

so  $\xi \in \Omega_{ef}$ . Conversely, if  $\xi$  is in  $\Omega_{ef}$ , pick any  $t$  in  $\xi$ . Then  $eft \in ef\xi \subseteq \xi$ , so we deduce from (6.3) that  $e\xi \subseteq \xi$ , and hence  $\xi \in \Omega_e$ . Similarly  $fet \in \xi$ , so  $\xi \in \Omega_f$ , as desired.  $\square$

Notice that if  $e = 0$ , then  $\Omega_e = \emptyset$ . Thus the above result implies that  $\Omega_e$  and  $\Omega_f$  are disjoint when  $ef = 0$ .

Let us now take some time to discuss when is  $\Omega_e = \Omega_f$ , for idempotents  $e$  and  $f$ .

**6.9. Proposition.** *Let  $e$  and  $f$  be idempotents in  $E(\mathcal{S})$ . Then  $\Omega_e = \Omega_f$  if and only if  $ef$  is dense (Definition 2.9) in both  $e$  and  $f$ . In this case, for every tight representation  $\beta$  of  $\mathcal{S}$ , one has that  $\beta(e) = \beta(f)$ .*

*Proof.* Let us first prove the only if part. We begin by treating the special case in which  $e \leq f$ . Thus, assuming that  $\Omega_e = \Omega_f$ , we must prove that  $e$  is dense in  $f$ .

Arguing by contradiction, let  $d$  be a nonzero idempotent such that  $d \perp e$ , and  $d \leq f$ . Choose an ultrafilter  $\xi$  such that  $d \in \xi$  and observe that

$$fd = d \in \xi,$$

so that  $\xi \in \Omega_f$ , by (6.3). By assumption we have that  $\xi \in \Omega_e$  and hence  $e\xi \subseteq \xi$ . In particular

$$0 = ed \in e\xi \subseteq \xi,$$

which is a contradiction. This proves that  $e$  is dense in  $f$ .

Without the assumption that  $e \leq f$ , but still supposing that  $\Omega_e = \Omega_f$ , observe that by (6.8) we have

$$\Omega_e = \Omega_{ef} = \Omega_f.$$

By the first part of the proof we then deduce that  $ef$  is dense in both  $e$  and  $f$ , as required.

Conversely, suppose that  $ef$  is dense in  $e$  and  $f$ . In order to conclude the proof it is obviously enough to prove that  $\Omega_e = \Omega_{ef}$ , and  $\Omega_f = \Omega_{ef}$ , while by symmetry it suffices to prove only the first assertion. Observing that since  $\Omega_{ef} \subseteq \Omega_e$  by (6.8), we must only prove that  $\Omega_e \subseteq \Omega_{ef}$ .

For this let  $\xi \in \Omega_e$ . Given any  $t \in \xi$  we claim that  $eft \neq 0$ . To prove it suppose otherwise so that  $eftt^{-1} = 0$ , for some  $t \in \xi$ . This says that  $ett^{-1} \perp ef$ , and clearly  $ett^{-1} \leq e$ . Since  $ef$  is dense in  $e$  we deduce that  $ett^{-1} = 0$ , hence

$$0 = ett^{-1}t = et \in e\xi \subseteq \xi,$$

which is a contradiction. This shows that  $eft \neq 0$ , for every  $t \in \xi$ . By (6.6) it then follows that  $\xi \in \Omega_{ef}$ , as desired.

Finally, given a tight representation of  $\mathcal{S}$  we have that the restriction of  $\beta$  to  $E(\mathcal{S})$  is a tight representation of  $E(\mathcal{S})$  in the sense of (2.7). Hence we have by (2.10) that

$$\beta(e) = \beta(ef) = \beta(f),$$

proving the last part of the statement.  $\square$

With the next result we shall start to study certain functions on the set of filters, in preparation for introducing the regular tight representation.

**6.10. Proposition.** *Given  $s \in \mathcal{S}$  and a filter  $\xi$  such that  $s^{-1}s\xi \subseteq \xi$ , let*

$$\lambda_s(\xi) = \{u \in \mathcal{S} : u \geq st, \text{ for some } t \in \xi\}.$$

*Then*

- (i)  $\lambda_s(\xi)$  is a filter,
- (ii)  $ss^{-1}\lambda_s(\xi) \subseteq \lambda_s(\xi)$ ,
- (iii)  $s\xi \subseteq \lambda_s(\xi)$ .

*Proof.* With respect to the last assertion let  $t \in \xi$ , and put  $u = st$ . Then obviously  $u \geq st$ , so  $u \in \lambda_s(\xi)$ . In order to prove (i) assume by contradiction that  $0 \in \lambda_s(\xi)$ . Then  $st = 0$ , for some  $t \in \xi$ , and hence

$$0 = s^{-1}st \in s^{-1}s\xi \subseteq \xi,$$

a contradiction, proving that  $0 \notin \lambda_s(\xi)$ . If  $u_1, u_2 \in \lambda_s(\xi)$ , choose for  $i = 1, 2$ , some  $t_i \in \xi$  such that  $u_i \geq st_i$ . Pick  $t \in \xi$  such that  $t_1, t_2 \geq t$ , and set  $u = st$ . By (iii) one has that  $u \in \lambda_s(\xi)$  and we have

$$u_i \geq st_i \geq st = u,$$

proving (3.1.iii). Since (3.1.ii) is obvious we have concluded the proof that  $\lambda_s(\xi)$  is a filter.

In order to prove (ii) let  $u \in \lambda_s(\xi)$ , and pick  $t \in \xi$  such that  $u \geq st$ . Then

$$ss^{-1}u \geq ss^{-1}st = st,$$

so  $ss^{-1}u \in \lambda_s(\xi)$ .  $\square$

Given a filter  $\xi$  such that  $s^{-1}s\xi \subseteq \xi$ , we have seen above that  $ss^{-1}\lambda_s(\xi) \subseteq \lambda_s(\xi)$ , so it makes sense to speak of  $\lambda_{s^{-1}}(\lambda_s(\xi))$ .

**6.11. Proposition.** *Let  $\xi$  be a filter such that  $s^{-1}s\xi \subseteq \xi$ . Then  $\lambda_{s^{-1}}(\lambda_s(\xi)) = \xi$ .*

*Proof.* If  $v \in \lambda_{s^{-1}}(\lambda_s(\xi))$ , there exists  $u \in \lambda_s(\xi)$  such that  $v \geq s^{-1}u$ . In turn there exists  $t \in \xi$  such that  $u \geq st$ , so

$$v \geq s^{-1}u \geq s^{-1}st \in s^{-1}s\xi \subseteq \xi,$$

and hence  $v \in \xi$ . Conversely let  $t \in \xi$ , then  $st \in \lambda_s(\xi)$  by (6.10.iii), and by the same token  $s^{-1}st \in \lambda_{s^{-1}}(\lambda_s(\xi))$ . Since  $t \geq s^{-1}st$ , we have that  $t \in \lambda_{s^{-1}}(\lambda_s(\xi))$ .  $\square$

We shall next prove that  $\lambda_s$  preserves ultrafilters. For this recall that for  $e \in E(\mathcal{S})$ , we denote by  $\Omega_e$  the set of all ultrafilters  $\xi$  such that  $e\xi \subseteq \xi$ .

**6.12. Proposition.** *If  $s \in \mathcal{S}$  and  $\xi \in \Omega_{s^{-1}s}$ , then  $\lambda_s(\xi) \in \Omega_{ss^{-1}}$ .*

*Proof.* In order to prove that  $\lambda_s(\xi)$  is an ultrafilter, suppose that  $\lambda_s(\xi) \subseteq \eta$ , for some filter  $\eta$ . We must show that  $\lambda_s(\xi) = \eta$ . By (6.10.ii) we have that  $ss^{-1}\lambda_s(\xi) \subseteq \lambda_s(\xi)$ , so we may use (6.4) to conclude that  $ss^{-1}\eta \subseteq \eta$ . Thus  $\lambda_{s^{-1}}(\eta)$  is a filter by (6.10.i). Using (6.11) we have

$$\xi = \lambda_{s^{-1}}(\lambda_s(\xi)) \subseteq \lambda_{s^{-1}}(\eta),$$

so  $\xi = \lambda_{s^{-1}}(\eta)$ , by maximality. This implies that

$$\lambda_s(\xi) = \lambda_s(\lambda_{s^{-1}}(\eta)) = \eta.$$

To prove that  $\lambda_s(\xi) \in \Omega_{ss^{-1}}$  it then suffices to show that  $ss^{-1}\lambda_s(\xi) \subseteq \lambda_s(\xi)$ , which is nothing but (6.10.ii).  $\square$

The following is a useful characterization of  $\lambda_s(\xi)$  when  $\xi$  is an ultrafilter.

**6.13. Proposition.** *Let  $s \in \mathcal{S}$  and let  $\xi \in \Omega_{s^{-1}s}$ . Then  $\lambda_s(\xi)$  is the unique filter containing  $s\xi$ .*

*Proof.* By (6.10.iii) we have that  $\lambda_s(\xi)$  does indeed contain  $s\xi$ . So let  $\eta$  be another filter such that  $s\xi \subseteq \eta$ . We must prove that  $\eta = \lambda_s(\xi)$ . Given any  $t \in \xi$  we have that

$$ss^{-1}st = st \in s\xi \subseteq \eta,$$

so  $ss^{-1}\eta \subseteq \eta$ , by (6.3) and hence  $\lambda_{s^{-1}}(\eta)$  is a filter by (6.10.i).

We claim that  $\xi \subseteq \lambda_{s^{-1}}(\eta)$ . In order to prove it let  $t \in \xi$ . Then  $st \in s\xi \subseteq \eta$ , and hence by (6.10.iii) we deduce that  $s^{-1}st \in \lambda_{s^{-1}}(\eta)$ . Since  $t \geq s^{-1}st$ , we conclude that  $t \in \lambda_{s^{-1}}(\eta)$ . This proves our claim and since  $\xi$  is an ultrafilter, we actually get  $\xi = \lambda_{s^{-1}}(\eta)$ . Therefore

$$\lambda_s(\xi) = \lambda_s(\lambda_{s^{-1}}(\eta)) = \eta. \quad \square$$

By (6.12) we have that  $\lambda_s$  defines a map

$$\lambda_s : \Omega_{s^{-1}s} \rightarrow \Omega_{ss^{-1}}$$

which is bijective by (6.11). Obviously  $\lambda_s^{-1} = \lambda_{s^{-1}}$ .

It is our next short term goal to show that  $\lambda$  is a semigroup homomorphism of  $\mathcal{S}$  into  $\mathcal{I}(\Omega)$ . The next result will be useful to help us understand the domain of the composition of these maps.

**6.14. Lemma.** *For every  $s \in \mathcal{S}$  and  $e \in E(\mathcal{S})$  one has that  $\lambda_s(\Omega_{es^{-1}s}) = \Omega_{ses^{-1}}$ .*

*Proof.* Let  $\xi \in \Omega_{es^{-1}s} = \Omega_e \cap \Omega_{s^{-1}s}$ . Given  $t \in \xi$  we have that  $et \in \xi$ , and hence  $set \in s\xi \subseteq \lambda_s(\xi)$ . Since

$$set = ss^{-1}set = ses^{-1}st,$$

we deduce from (6.3) that  $ses^{-1}\lambda_s(\xi) \subseteq \lambda_s(\xi)$ , and hence  $\lambda_s(\xi) \in \Omega_{ses^{-1}}$ . This shows that  $\lambda_s(\Omega_{es^{-1}s}) \subseteq \Omega_{ses^{-1}}$ . Since  $ses^{-1} = ses^{-1}ss^{-1}$ , we may apply the part of the result already proved, with  $s^{-1}$  replacing  $s$ , and  $ses^{-1}$  replacing  $e$ , to obtain

$$\lambda_{s^{-1}}(\Omega_{ses^{-1}}) \subseteq \Omega_{s^{-1}ses^{-1}s} = \Omega_{es^{-1}s},$$

therefore

$$\Omega_{ses^{-1}} = \lambda_s(\lambda_{s^{-1}}(\Omega_{ses^{-1}})) \subseteq \lambda_s(\Omega_{es^{-1}s}),$$

concluding the proof.  $\square$

From now on we will regard the  $\lambda_s$  as partially defined bijections on  $\Omega$ . If  $f$  and  $g$  are partial bijections on a set  $X$ , say

$$f : A \rightarrow B, \quad \text{and} \quad g : C \rightarrow D,$$

where  $A, B, C$  and  $D$  are subsets of  $X$ , then the composition  $gf$  is defined on  $f^{-1}(C \cap B)$  by the expression  $fg(x) = f(g(x))$ .

**6.15. Proposition.** *For every  $t, s \in \mathcal{S}$  one has that  $\lambda_t\lambda_s = \lambda_{ts}$ .*

*Proof.* By the above remark the domain of the composition  $\lambda_t\lambda_s$  is

$$\lambda_s^{-1}(\Omega_{t^{-1}t} \cap \Omega_{ss^{-1}}) = \lambda_{s^{-1}}(\Omega_{t^{-1}ts s^{-1}}) = \Omega_{s^{-1}t^{-1}ts} = \Omega_{(ts)^{-1}ts},$$

which coincides with the domain of  $\lambda_{ts}$ . Moreover for every  $\xi \in \Omega_{(ts)^{-1}ts}$  we have by (6.10.iii) that

$$ts\xi = t(s\xi) \subseteq t\lambda_s(\xi) \subseteq \lambda_t(\lambda_s(\xi)),$$

so  $\lambda_t(\lambda_s(\xi)) = \lambda_{ts}(\xi)$ , by (6.13).  $\square$

The following is one of our main results:

**6.16. Theorem.** *Let  $\mathcal{S}$  be an inverse semigroup with zero. Then the correspondence  $s \mapsto \lambda_s$  is a tight representation of  $\mathcal{S}$  in the Boolean inverse semigroup  $\mathcal{I}(\Omega)$ .*

*Proof.* That  $\lambda$  is a semigroup homomorphism follows from (6.15), so it suffices to prove that the restriction of  $\lambda$  to  $E(\mathcal{S})$  is a tight representation of the latter in  $E(\mathcal{I}(\mathcal{B})) = \mathcal{P}(\mathcal{B})$ .

If  $s = 0$ , then  $\Omega_{s^{-1}s} = \Omega_{ss^{-1}} = \emptyset$ , and hence  $\lambda_s$  is the empty function, namely the zero element of  $\mathcal{I}(\Omega)$ , proving (2.3.i). As for (2.3.ii) it immediately follows from the fact that  $\lambda$  is multiplicative.



In order to prove tightness we would like to use (2.8), so we first need to check the validity of either (i) or (ii) in (2.8). Thus, suppose that (2.8.ii) fails, meaning that  $E(\mathcal{S})$  admits a finite cover, say  $Z$ . We will prove that

$$\Omega = \bigcup_{z \in Z} \Omega_z.$$

By way of contradiction assume that  $\xi$  is an ultrafilter which is not in any  $\Omega_z$ . By (6.6) for each  $z \in Z$  there exists some  $t_z \in \xi$  such that  $zt_z = 0$ . Using (3.1.iii) pick some  $t \in \xi$  such that  $t_z \geq t$ , for all  $z \in Z$ , and notice that

$$zt \leq zt_z = 0,$$

so  $zt = 0$ , and hence  $ztt^{-1} = 0$ , which means that  $z \perp tt^{-1}$ . But since  $Z$  is a cover for  $E(\mathcal{S})$  this implies that  $tt^{-1} = 0$ , and hence that  $t = 0$ , contradicting the fact that  $t \in \xi$ . This proves (2.8.i), so we may use the simplified test given there to prove that  $\lambda$  is tight.

We therefore let  $x \in E(\mathcal{S})$  be a nonzero element and  $Z$  be a cover for  $[0, x]$ . We must prove that

$$\bigcup_{z \in Z} \Omega_z \supseteq \Omega_x.$$

So let  $\xi$  be an ultrafilter in  $\Omega_x$  and suppose by contradiction that  $\xi \notin \Omega_z$ , for any  $z \in Z$ . Then, by (6.6), for each  $z$  in  $Z$  there exists some  $t_z \in \xi$  such that  $zt_z = 0$ , and by (3.1.iii) we may pick  $t \in \xi$  such that  $t_z \geq t$ , for all  $z \in Z$ . As above this gives  $zt \leq zt_z = 0$ , so

$$zt = 0, \quad \forall z \in Z.$$

Given that  $\xi \in \Omega_x$ , we have that  $x\xi \subseteq \xi$ , so in particular  $xt \in \xi$ . Now let  $e = tt^{-1}x$ , and observe that  $e \leq x$ , so  $e \in [0, x]$ . Moreover  $e \neq 0$ , because

$$et = tt^{-1}xt = xtt^{-1}t = xt \in \xi.$$

By hypothesis we deduce that there exists some  $z \in Z$  such that  $z \frown e$ , whence

$$0 \neq ze = ztt^{-1}x = 0,$$

a contradiction. This shows that  $\xi \in \Omega_z$ , for some  $z$  in  $Z$ , hence concluding the proof that  $\lambda$  is tight.  $\square$

**6.17. Definition.** We shall call the above representation  $\lambda$  the *regular tight representation* of  $\mathcal{S}$ .

## 7. Faithfulness of tight representation.

As in the previous section we fix an inverse semigroup  $\mathcal{S}$  with zero. In this section we would like to study conditions under which the regular tight representation of  $\mathcal{S}$  is injective. As we shall see, injectivity does not always hold and in fact it is often the case that different elements  $s$  and  $t$  in  $\mathcal{S}$  are not separated by any tight representation of  $\mathcal{S}$  whatsoever. Among our goals in this section we will characterize precisely when does this happen.

To ease our task we will make an important assumption about the inverse semigroup involved, which fortunately does not rule out some important classes of inverse semigroups, such as the  $E^*$ -unitary ones.

### 7.1. Definition.

- (i) Let  $s, t \in \mathcal{S}$ . We shall say that  $s$  *essentially coincides with*  $t$ , in symbols  $s \equiv t$ , if  $s^{-1}s = t^{-1}t$ , and for every nonzero idempotent  $f \leq s^{-1}s$ , there exists a nonzero idempotent  $e \leq f$ , such that  $se = te$ .
- (ii) We shall say that  $\mathcal{S}$  is *continuous* if  $s \equiv t$  implies that  $s = t$ .

The fact that  $s \equiv t$  is to be interpreted somewhat in the same way as when two functions agree on a dense subset of their common domain. So much so that we have:

**7.2. Proposition.** *Let  $\mathcal{S}$  be a localization in the sense of Kumjian [6], that is,  $\mathcal{S}$  is an inverse subsemigroup of  $\mathcal{I}(X)$ , where  $X$  is a Hausdorff topological space, and  $\mathcal{S}$  consists of homeomorphisms between open subsets of  $X$ , the domains of which form a basis for the topology of  $X$ . We suppose in addition that  $\mathcal{S}$  contains the empty function  $\emptyset$ , and hence  $\mathcal{S}$  is an inverse semigroup with zero. Then  $\mathcal{S}$  is continuous in the sense of (7.1).*

*Proof.* Let  $s, t \in \mathcal{S}$  be such that  $s \equiv t$ . Identifying idempotents with their domains, as usual, let  $U = s^{-1}s = t^{-1}t$ , and put

$$D = \{x \in U : s(x) = t(x)\}.$$

We claim that  $D$  is dense in  $U$ . To prove it let  $A \subseteq U$  be a nonempty open set. By hypothesis there exists a nonzero idempotent (i.e. an open set)  $f$  such that  $f \subseteq A$ , and consequently  $f \leq s^{-1}s$ . Since  $s \equiv t$ , we may find a nonzero idempotent  $e \leq f$ , such that  $se = te$ . Picking any  $x \in e$ , we then have that  $x \in f \subseteq A$ , and  $s(x) = se(x) = te(x) = t(x)$ , so  $x \in A \cap D$ , proving that  $D$  is dense. Since  $s$  and  $t$  are continuous we deduce that  $s = t$ .  $\square$

Not all inverse semigroups are continuous. Suppose for example that  $\mathcal{S}$  is an inverse semigroup with zero and consider  $\mathcal{S}' = \mathcal{S} \dot{\cup} \{z\}$ , where  $z \notin \mathcal{S}$ . Define a multiplication operation on  $\mathcal{S}'$  extending that of  $\mathcal{S}$  and such that

$$zs = sz = z, \quad \forall s \in \mathcal{S}'.$$

It turns out that  $\mathcal{S}'$  is an inverse semigroup with zero, except that the *zero* of  $\mathcal{S}'$  is  $z$ , rather than the original zero of  $\mathcal{S}$  (which we denote by 0).

Given any  $s, t \in \mathcal{S}$  with  $s^{-1}s = t^{-1}t$ , and any nonzero (i.e. different from  $z$ ) idempotent  $f \leq s^{-1}s$ , notice that  $0$  is a nonzero (sic) idempotent with  $0 \leq f$ , and  $s0 = 0 = t0$ . Thus  $s \equiv t$  even though  $s$  and  $t$  might not coincide.

The following additional counter-example is due to Szendrei (personal communication). Let  $X = \{1, 2, 3\}$ , and let  $\mathcal{S}$  be the inverse subsemigroup of  $\mathcal{I}(X)$  consisting of the following four elements:

- $1$  – identity permutation
- $0$  – empty mapping (the zero element)
- $i$  – the partial identity sending  $1$  to  $1$  and undefined otherwise
- $s$  – the transposition interchanging  $2$  and  $3$  (and sending  $1$  to  $1$ ).

Clearly  $1$ ,  $0$ , and  $i$  are idempotents of  $\mathcal{S}$  forming a three-element chain. We have  $s^{-1}s = 1$ , and for both nonzero idempotents  $e$  with  $e \leq s^{-1}s$  (that is, for both  $e = 1$  and  $e = i$ ), the relations  $i \leq e$  and  $si = 1i$  hold, whence  $s \equiv 1$ .

Recall that an inverse semigroup with zero is said to be  $E^*$ -unitary [11], [7: Section 9], if whenever  $s \in \mathcal{S}$ , and  $se = e$ , for some nonzero idempotent  $e$ , then  $s$  is necessarily also idempotent.

**7.3. Proposition.** *Every  $E^*$ -unitary inverse semigroup with zero is continuous.*

*Proof.* Let  $\mathcal{S}$  be an  $E^*$ -unitary inverse semigroup with zero and let  $s, t \in \mathcal{S}$  be such that  $s \equiv t$ . Plugging  $f = s^{-1}s$  in the definition there exists a nonzero idempotent  $e \leq s^{-1}s$ , such that  $se = te$ . Then [2: 5.3] applies giving  $s = t$ .  $\square$

We now return to studying the general case.

**7.4. Proposition.** *Let  $\mathcal{S}$  be an inverse semigroup with zero. If  $s, t \in \mathcal{S}$  are such that  $s^{-1}s = t^{-1}t$ , and  $\lambda_s = \lambda_t$ , where  $\lambda$  is the regular tight representation of  $\mathcal{S}$ , then  $s \equiv t$ .*

*Proof.* Given a nonzero idempotent  $f \leq s^{-1}s$ , choose an ultrafilter  $\xi$  such that  $f \in \xi$ . Since  $s^{-1}sf = f \in \xi$ , we have by (6.3) that  $\xi \in \Omega_{s^{-1}s}$ . Thus  $\lambda_s(\xi) = \lambda_t(\xi)$ . In addition we have by (6.10.iii) that

$$sf \in \lambda_s(\xi) = \lambda_t(\xi) \ni tf,$$

so by (3.1.iii) there exists  $u \in \lambda_s(\xi)$  such that  $sf, tf \geq u$ . Therefore

$$sfu^{-1}u = u = tfu^{-1}u.$$

Thus  $e := fu^{-1}u$  is a nonzero idempotent (because  $u \neq 0$ ) such that  $e \leq f$ , and  $se = te$ . This proves that  $s \equiv t$ .  $\square$

Although it is not crucial for our purposes it would be interesting to decide if the converse of the above result holds.

In the following main result we characterize precisely the extent to which tight representations do not separate points of a continuous inverse semigroup  $\mathcal{S}$ .

**7.5. Theorem.** *Let  $\mathcal{S}$  be a continuous inverse semigroup with zero and let  $s, t \in \mathcal{S}$ . Then the following are equivalent:*

- (i)  $\beta(s) = \beta(t)$  for every tight representation  $\beta$  of  $\mathcal{S}$ ,
- (ii)  $\lambda(s) = \lambda(t)$ ,
- (iii)  $st^{-1}t = ts^{-1}s$ , and  $s^{-1}st^{-1}t$  is dense in both  $s^{-1}s$  and  $t^{-1}t$ ,
- (iv)  $tt^{-1}s = ss^{-1}t$ , and  $ss^{-1}tt^{-1}$  is dense in both  $ss^{-1}$  and  $tt^{-1}$ .

*Proof.* (i)  $\Rightarrow$  (ii): obvious.

(ii)  $\Rightarrow$  (iii): If  $\lambda(s) = \lambda(t)$  then in particular the domains of  $\lambda(s)$  and  $\lambda(t)$  must coincide, and hence  $\Omega_{s^{-1}s} = \Omega_{t^{-1}t}$ . The last assertion in (iii) then follows from (6.9). Next let  $\hat{s} = st^{-1}t$ , and  $\hat{t} = ts^{-1}s$ , and observe that

$$\hat{t}^{-1}\hat{t} = s^{-1}st^{-1}t = \hat{s}^{-1}\hat{s}.$$

Moreover we have

$$\lambda(\hat{s}) = \lambda(st^{-1}t) = \lambda(s)\lambda(t)^{-1}\lambda(t) = \lambda(t)\lambda(s)^{-1}\lambda(s) = \lambda(ts^{-1}s) = \lambda(\hat{t}).$$

Invoking (7.4) we conclude that  $\hat{s} \equiv \hat{t}$ , and hence that  $\hat{s} = \hat{t}$ , because  $\mathcal{S}$  is continuous.

(iii)  $\Rightarrow$  (i): Let  $\beta$  be a tight representation of  $\mathcal{S}$ . Since  $s^{-1}st^{-1}t$  is dense in both  $s^{-1}s$  and  $t^{-1}t$ , we have by the last part of (6.9) that  $\beta(s^{-1}s) = \beta(t^{-1}t)$ . Therefore

$$\begin{aligned} \beta(s) &= \beta(ss^{-1}s) = \beta(s)\beta(s^{-1}s) = \beta(s)\beta(t^{-1}t) = \beta(st^{-1}t) = \\ &= \beta(ts^{-1}s) = \beta(t)\beta(s^{-1}s) = \beta(t)\beta(t^{-1}t) = \beta(tt^{-1}t) = \beta(t). \end{aligned}$$

(ii)  $\Leftrightarrow$  (iv): Since  $\lambda(s^{-1}) = \lambda(s)^{-1}$ , and similarly for  $t$ , we have that (ii) is equivalent to saying that  $\lambda(s^{-1}) = \lambda(t^{-1})$ . Exchanging  $s$  and  $t$ , respectively by  $s^{-1}$  and  $t^{-1}$ , and applying the already proved equivalence between (ii) and (iii), we then see that (ii) is equivalent to saying that  $s^{-1}tt^{-1} = t^{-1}ss^{-1}$ , and that  $ss^{-1}tt^{-1}$  is dense in both  $ss^{-1}$  and  $tt^{-1}$ , which is tantamount to (iv).  $\square$

### 8. A counter-example.

Given the use of the continuity hypothesis in the proof of the implication (ii)  $\Rightarrow$  (iii) of (7.5) it is interesting to decide whether or not that result survives in the absence of such a hypothesis. In this section we present an example to show that it does not.

We begin by outlining a general construction of inverse semigroups related to Clifford semigroups, which has been used by several authors. Let  $E$  be a semilattice with smallest element 0, and let  $G$  be a group. Suppose that for each  $x \in E$  we are given a normal subgroup  $N_x \trianglelefteq G$ , such that  $N_0 = G$ , and whenever  $x \leq y$  in  $E$  one has that  $N_x \supseteq N_y$ . Viewing both  $E$  and  $G$  as inverse semigroups, consider their cartesian product  $\mathcal{T} = E \times G$ , with coordinatewise operations. Clearly  $\mathcal{T}$  is an inverse semigroup as well. We define a congruence on  $\mathcal{T}$  by saying that

$$(x, g) \sim (y, h),$$

if and only if  $x = y$  and  $h^{-1}g \in N_x$ .

If  $\pi_x$  denotes the quotient map from  $G$  to  $G/N_x$ , then the last condition above is perhaps more conveniently stated by saying that  $\pi_x(g) = \pi_x(h)$ . By our assumptions about the  $N_x$  it is evident that, whenever  $x \leq y$ , one has that

$$\pi_y(g) = \pi_y(h) \Rightarrow \pi_x(g) = \pi_x(h), \quad \forall g, h \in G. \quad (8.1)$$

It is easy to show that “ $\sim$ ” is a congruence, and we shall let  $\mathcal{S} = \mathcal{S}(E, G, \{N_x\}_x)$  be the inverse semigroup obtained by taking the quotient of  $\mathcal{T}$  by “ $\sim$ ”. Given  $(x, g) \in \mathcal{T}$ , we will henceforth refer to its equivalence class by  $[x, g]$ . Notice that for every  $(x, g) \in \mathcal{T}$  one has that

$$[0, 1][x, g] = [0 \wedge x, g] = [0, g] = [0, 1],$$

the last equality following from the assumption that  $N_0 = G$ . One may similarly prove that  $[x, g][0, 1] = [0, 1]$ , which means that  $[0, 1]$  is a zero element for  $\mathcal{S}$ . Moreover

$$[x, g][x, g]^{-1} = [x, g][x, g^{-1}] = [x, 1],$$

so  $E(\mathcal{S})$  consists of the set of all equivalence classes  $[x, 1]$ , for  $x \in E$ . Since  $(x, 1) \sim (y, 1)$  if and only if  $x = y$ , we deduce that  $E(\mathcal{S})$  is isomorphic to  $E$ .

We will now consider a more concrete application of these ideas. Let  $\{0, 1\}$  have the obvious Boolean algebra structure and put  $\mathcal{B} = \{0, 1\} \times \{0, 1\}$ . The unit of  $\mathcal{B}$  is clearly  $1 = (1, 1)$ , and its zero element is  $0 = (0, 0)$ . We shall denote the remaining elements by

$$e_1 = (1, 0), \quad \text{and} \quad e_2 = (0, 1),$$

so  $\mathcal{B} = \{0, e_1, e_2, 1\}$ . We will temporarily view  $\mathcal{B}$  simply as a semilattice. Given any two normal subgroups  $N_1, N_2 \trianglelefteq G$ , and setting

$$\begin{aligned} N_0 &= G, \\ N_{e_1} &= N_1, \\ N_{e_2} &= N_2, \\ N_1 &= \{1\}, \end{aligned}$$

one may easily check that the collection  $\{N_x\}_{x \in \mathcal{B}}$  satisfies the conditions above so that we may construct the associated inverse semigroup  $\mathcal{S} = \mathcal{S}(\mathcal{B}, G, \{N_x\}_x)$  as above.

As already noticed  $E(\mathcal{S}) = \mathcal{B}$ , so  $\mathcal{S}$  is a Boolean inverse semigroup. By (2.11) one sees that the identity mapping  $\iota : \mathcal{S} \rightarrow \mathcal{S}$  is a tight representation of  $\mathcal{S}$  in itself. It follows that (7.5.i) only holds when  $s = t$ . However, we will show that (7.5.ii) might hold for  $s \neq t$ .

Before we begin let us agree on a particularly useful notation for elements of  $\mathcal{S}$ . Noticing that  $N_1 = \{1\}$  observe that  $(1, g) \sim (1, h)$  if and only if  $g = h$ . Thus the mapping

$$g \in G \mapsto [1, g] \in \mathcal{S}$$

is a semigroup monomorphism, and hence we may identify  $G$  with its copy within  $\mathcal{S}$ . We have already observed that

$$x \in \mathcal{B} \mapsto [x, 1] \in \mathcal{S}$$

is also a semigroup monomorphism and hence we are allowed to think of  $\mathcal{B}$  as a subsemigroup of  $\mathcal{S}$ . Given any  $(x, g) \in \mathcal{T}$  we have that

$$[x, g] = [x, 1][1, g] = xg,$$

where in the last term we are fully enforcing our identifications. Therefore  $\mathcal{S} = \mathcal{B}G$ , and thanks to the product structure of  $\mathcal{T}$  notice moreover that  $\mathcal{B}$  and  $G$  commute.

With the purpose of understanding the order structure of  $\mathcal{S}$  let  $[x, h]$  and  $[y, g]$  be elements in  $\mathcal{S}$  with  $[x, h] \leq [y, g]$ , that is,

$$[x, h] = [y, g][x, h][x, h]^{-1} = [y, g][x, 1] = [y \wedge x, g].$$

This is the same as saying that  $x \leq y$ , and  $\pi_x(h) = \pi_x(g)$ . This implies in particular that  $[x, h] = [x, g]$ , so every inequality in  $\mathcal{S}$  is of the form

$$[x, g] \leq [y, g],$$

with  $x \leq y$ . The only nontrivial inequalities (i.e. not involving zero nor an equality) are therefore

$$ge_i \leq g,$$

for  $i = 1, 2$ , and  $g \in G$ . It follows that the minimal nonzero elements of  $\mathcal{S}$  are precisely those of the form  $ge_i$ , as above. This said it is easy to see that the most general ultrafilter in  $\mathcal{S}$  is

$$\xi_{ge_i} = \{s \in \mathcal{S} : s \geq ge_i\} = \{ge_i, g\},$$

for  $i = 1, 2$ , and  $g \in G$ .

For the purpose of giving our counter-example we will suppose in addition that  $N_{e_1} \cap N_{e_2} \neq \{1\}$ . Given  $s \in G$ , choose a nontrivial element  $n \in N_{e_1} \cap N_{e_2}$  and put  $t = sn$ . Clearly  $s \neq t$ , but

$$\pi_{e_i}(s) = \pi_{e_i}(t), \quad \forall i = 1, 2.$$

It is our intention to prove that  $\lambda_s = \lambda_t$ . In order to do so notice that  $s^{-1}s = t^{-1}t = 1$ , and that  $\Omega_1 = \Omega$ , so the domain of both  $\lambda_s$  and  $\lambda_t$  is the set of all ultrafilters.

Given any ultrafilter  $\xi$ , write  $\xi = \xi_{ge_i}$ , for some  $g \in G$ , and  $i = 1, 2$ . Since  $ge_i \in \xi$ , we have that  $sge_i \in \lambda_s(\xi)$ , by (6.10.iii). Recalling that  $sge_i$  is a minimal element, we necessarily have  $\lambda_s(\xi) = \xi_{sge_i}$ , and similarly  $\lambda_t(\xi) = \xi_{tge_i}$ . Moreover

$$(e_i, sg) \sim (e_i, tg),$$

because  $\pi_{e_i}(sg) = \pi_{e_i}(tg)$ , so  $sge_i = tge_i$ , and hence

$$\lambda_s(\xi) = \xi_{sge_i} = \xi_{tge_i} = \lambda_t(\xi).$$

Since  $\xi$  is arbitrary we deduce that  $\lambda_s = \lambda_t$ . The final conclusion is that (7.5.ii) holds for  $s$  and  $t$ , but (7.5.i) does not. The trouble is of course that  $\mathcal{S}$  is not continuous, and this concludes our goal of showing that (7.5) cannot be proved without the continuity hypothesis.

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