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TIGHT TOUGHNESS CONDITION FOR FRACTIONAL (g, f, n)-CRITICAL GRAPHS

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Abstract. A graph G is called a fractional (g, f, n)-critical graph if any n vertices are removed from G, then the resulting graph admits a fractional (g, f)-factor. In this paper, we determine the new toughness condition for fractional (g, f, n)-critical graphs. It is proved that G is fractional (g, f, n)-critical if $t(G) \geq \frac{b^2-1+bn}{a}$. This bound is sharp in some sense. Furthermore, the best toughness condition for fractional (a, b, n)-critical graphs is given.

1. Introduction

All graphs considered in this paper are finite, loopless, and without multiple edges. The notation and terminology used but undefined in this paper can be found in [2]. Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. For a vertex $x \in V(G)$, we use $d_G(x)$ and $N_G(x)$ to denote the degree and the neighborhood of x in G, respectively. Let $\delta(G)$ denote the minimum degree of G. For any $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $G[S]$.

Suppose that g and f are two integer-valued functions on $V(G)$ such that $0 \leq g(x) \leq f(x)$ for all $x \in V(G)$. A spanning subgraph F of G is called a *(g, f)-factor* if $g(x) \leq d_F(x) \leq f(x)$ for each $x \in V(G)$. A *fractional (g, f)-factor* is a function h that assigns to each edge of a graph G a number in [0,1] so that for each vertex x we have $g(x) \leq \sum_{e\in E(x)} h(e) \leq f(x)$. If $g(x) = a$, $f(x) = b$ for all $x \in V(G)$, then a fractional (g, f)-factor is a fractional $[a, b]$-factor. Moreover, if $g(x) = f(x) = k$ ($k \geq 1$ is an integer throughout this paper, and we will not reiterate it again) for all $x \in V(G)$, then a fractional (g, f)-factor is just a fractional k-factor.

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Liu and Zhang [4] determined a necessary and sufficient condition for a graph to have a fractional (g, f)-factor.

Theorem 1 (Liu and Zhang [4]). *Suppose that f and g are two integer-valued functions defined on the vertex set of a graph G such that $0 \le g(x) \le f(x)$ for each $x \in V(G)$. Then G has a fractional (g, f)-factor if and only if for any subset S of $V(G)$,*

$$f(S) - g(T) + d_{G-S}(T) \ge 0,$$

where $T = \{x \in V(G) \setminus S : d_{G-S}(x) \le g(x)\}$.

Liu and Zhang [4, 5] showed some characters on fractional (g, f)-factor. A graph G is called a *fractional (g, f, n)-critical graph* if after deleting any n vertices from G, the resulting graph still has a fractional (g, f)-factor. Similarly, a graph G is called a *(g, f, n)-critical graph* if after removing any n vertices from G, the resulting graph admits a (g, f)-factor. Several sufficient conditions for (a, b, n)-critical graphs can refer [9] and [10].

Liu [7] investigated the necessary and sufficient condition for a graph G to be a fractional (g, f, n)-critical graph.

Lemma 2 (Liu [7]). *Let G be a graph and let g, f be two non-negative integer-valued functions defined on $V(G)$ satisfying $g(x) \le f(x)$ for all $x \in V(G)$. Let n be a positive integer. Then G is a fractional (g, f, n)-critical graph if and only if for any subset S of $V(G)$ with $|S| \ge n$*

$$f(S) - g(T) + d_{G-S}(T) \ge \max\{f(U) : U \subseteq S, |U| = n\}, \tag{1}$$

where $T = \{x \in V(G) \setminus S : d_{G-S}(x) \le g(x)\}$.

The proof of our main result relies heavily on the following lemma, which can be regarded as an equal version of Lemma 2.

Lemma 3. *Let G be a graph and let g, f be two non-negative integer-valued functions defined on $V(G)$ satisfying $g(x) \le f(x)$ for all $x \in V(G)$. Let n be a non-negative integer. Then G is a fractional (g, f, n)-critical graph if and only if*

$$f(S) - g(T) + d_{G-S}(T) \ge \max\{f(U) : U \subseteq S, |U| = n\} \tag{2}$$

for any disjoint subsets S and T of $V(G)$ with $|S| \ge n$.

The notion of *toughness* was first introduced by Chvátal in [3]: if G is a complete graph, $t(G) = \infty$; if G is not complete,

$$t(G) = \min\{\frac{|S|}{\omega(G-S)} : \omega(G-S) \ge 2\}$$

and where $\omega(G - S)$ is the number of connected components of $G - S$.

Some toughness conditions for a graph to have a fractional factor were given in [1, 8] by Bian, Liu and Cai. Liu [7] studied the relationship between toughness and fractional (g, f, n)-critical graphs and proved the following result.

Theorem 4 (Liu [7])**.** *Let G be a graph and let g, f be two non-negative integer-valued functions defined on $V(G)$ satisfying $a \le g(x) \le f(x) \le b$ with $1 \le a \le b$ and $b \ge 2$ for all $x \in V(G)$, where a, b are positive integers. If $t(G) \ge \frac{(b^2-1)(n+1)}{a}$, then G is a fractional (g, f, n)-critical graph, where n is a nonnegative integer with $|V(G)| \ge n + 1$.*

However, the author of [7] only verified that this bound of $t(G)$ is sharp when $n = 0$, but didn't know whether the condition is best or not when $n \ge 1$. Thus, the problem of tight $t(G)$ for fractional (g, f, n)-critical graphs is still open. It inspires us to think about the best $t(G)$ for fractional (g, f, n)-critical graphs. In this paper, we determine sharp bound of $t(G)$. Our main result to be proved in next section can be stated as follows.

Theorem 5. *Let G be a graph and let g, f be two integer-valued functions defined on $V(G)$ satisfying $a \le g(x) \le f(x) \le b$ with $1 \le a \le b$ and $b \ge 2$ for all $x \in V(G)$, where a, b are positive integers. Let n be a non-negative integer. $|V(G)| \ge n + b + 1$ if G is complete. If $t(G) \ge \frac{b^2-1+bn}{a}$, then G is a fractional (g, f, n)-critical graph.*

Let m be a positive integer. To see the sharpness of Theorem 5, we construct the following graph G:

$$V(G) = A \cup B \cup C,$$

where A, B and C are disjoint with $|A| = (mb + 1)(b - 1 + n)$, $|B| = (ma + 1)(b - 1)$, and $|C| = m(b - 1)$ with $a = b$. Both A and C are cliques in G, while B is isomorphic to $(ma + 1)K_{b-1}$. Other edges in G are $\{uv : u \in B, v \in C\}$ and $\{u_1v_1, u_2v_2, \ldots, u_{(ma+1)(b-1)}v_{(ma+1)(b-1)}\}$, where $V(B) = \{u_1, u_2, \ldots, u_{(ma+1)(b-1)}\}$ and $\{v_1, v_2, \ldots, v_{(ma+1)(b-1)}\} \subset A$. If $b = 2$, let $S = (A - \{u\}) \cup C$, where $u \in A$, then $|S| = 3m + n(2m + 1)$ and $\omega(G - S) = ma + 1$; if $b \ge 3$, let $S = (A - \{u\}) \cup \{v\} \cup C$, where $u \in A$ and $v \in B$ is matched to u in G. Then $|S| = (mb + m + 1)(b - 1) + n(mb + 1)$ and $\omega(G - S) = ma + 2$. This follows that

$$t(G) = \begin{cases} \frac{(mb+m+1)(b-1)+n(mb+1)}{ma+2}, & b \ge 3, \\ \frac{3m+n(2m+1)}{ma+1}, & b = 2. \end{cases}$$

Thus, $t(G)$ can be made arbitrarily close to $\frac{b^2-1+bn}{a}$ when m is large enough.

Let $V_0 \subset V(A) \setminus \{v_1, v_2, \ldots, v_{(ma+1)(b-1)}\}$ with $|V_0| = n$, $S = C \cup V_0$ and $T = B$. Let $g(x) = f(x) = a$ if $x \in S$ and $g(x) = f(x) = b$ if $x \in T$. We have $f(S) - f(U) = a|S|$ for any $U \subset S$ with $|U| = n$ and $d_{G-S}(x) = b - 1$ for each $x \in T$. Thus,

$$\begin{aligned} &f(S) - g(T) + d_{G-S}(T) - \max\{f(U) : U \subseteq S, |U| = n\} \\ &= am(b - 1) - (ma + 1)(b - 1) < 0. \end{aligned}$$

By Lemma 3, G is not a fractional (g, f, n)-critical graph. In this sense, the toughness bound in Theorem 5 is best possible.

Next, the restriction on $|V(G)|$ for complete graph is necessary and can not be weakened by $|V(G)| \geq n+b$. If $|V(G)| = n+b$. Deleting n vertices from G, the resulting graph G' satisfies $d_{G'}(v) = b-1$ for each $v \in V(G')$. Let $f(x) = g(x) = b$ for every $x \in V(G')$. Then G' has no fractional (g, f)-factor. Thus, G is not a fractional (g, f, n)-critical graph.

To prove Theorem 5, we need the following lemmas.

Lemma 6 (Chvátal [3])**.** *If a graph G is not complete, then $t(G) \leq \frac{1}{2}\delta(G)$.*

Lemma 7 (Liu and Zhang [6])**.** *Let G be a graph and let $H = G[T]$ such that $\delta(H) \geq 1$ and $1 \leq d_G(x) \leq k-1$ for every $x \in V(H)$, where $T \subseteq V(G)$ and $k \geq 2$. Let $T_1, \ldots, T_{k-1}$ be a partition of the vertices of H satisfying $d_G(x) = j$ for each $x \in T_j$ where we allow some T_j to be empty. If each component of H has a vertex of degree at most $k-2$ in G, then H has a maximal independent set I and a covering set $C = V(H) - I$ such that*

$$\sum_{j=1}^{k-1}(k-j)c_j \leq \sum_{j=1}^{k-1}(k-2)(k-j)i_j,$$

where $c_j = |C \cap T_j|$ and $i_j = |I \cap T_j|$ for every $j = 1, \ldots, k-1$.

By analyzing proving process of Lemma 2.2 in [6]: "for each vertex $x \in I_n$ and $d_{H_n}(x) = k-1$, there exists a vertex $y \in I_n$ such that $N_{H_n}(x) \cap N_{H_n}(y) \neq \emptyset$", we infer the following equal version.

Lemma 8 (Liu and Zhang [6])**.** *Let G be a graph and let $H = G[T]$ such that $d_G(x) = k-1$ for every $x \in V(H)$ and no component of H is isomorphic to K_k, where $T \subseteq V(G)$ and $k \geq 2$. Then there exist an independent set I and the covering set $C = V(H) - I$ of H satisfying*

$$|V(H)| \leq \sum_{i=1}^{k}(k-i+1)|I^{(i)}| - \frac{|I^{(1)}|}{2}$$

and

$$|C| \leq \sum_{i=1}^{k}(k-i)|I^{(i)}| - \frac{|I^{(1)}|}{2},$$

where $I^{(i)} = \{x \in I, d_H(x) = k-i\}$, $1 \leq i \leq k$ and $\sum_{i=1}^{k}|I^{(i)}| = |I|$.

2. Proof of Theorem 5

If G is complete, due to $|V(G)| \geq n+b+1$, clearly, G has a fractional (g, f)-factor after deleting any n vertices. In the following, we assume that G is not complete.

Suppose that G satisfies the conditions of Theorem 5, but is not a fractional (g, f, n)-critical graph. According to Lemma 3 there exist disjoint subsets S

and T of $V(G)$ such that

$$a|S| + \sum_{x\in T} d_{G-S}(x) - b|T| \le f(S) - g(T) + d_{G-S}(T) < bn. \tag{3}$$

We choose subsets S and T such that $|T|$ is minimum. Obviously, $T \neq \emptyset$.

Claim 1. *$d_{G-S}(x) \le g(x) - 1 \le b - 1$ for any $x \in T$.*

Proof. If $d_{G-S}(x) \ge g(x)$ for some $x \in T$, then the subsets S and $T \setminus \{x\}$ satisfy (3). This contradicts the choice of S and T. □

Let l be the number of the components of $H' = G[T]$ which are isomorphic to K_b and let $T_0 = \{x \in V(H') : d_{G-S}(x) = 0\}$. Let H be the subgraph obtained from $H' - T_0$ by deleting those l components isomorphic to K_b.

If $|V(H)| = 0$, then by (3), we deduce

$$a|S| < b|T_0| + bl + bn$$

or

$$|S| < \frac{b(|T_0| + l) + bn}{a}.$$

Clearly, $\omega(G - S) \ge |T_0| + l \ge 1$. If $\omega(G - S) > 1$, then $t(G) \le \frac{|S|}{\omega(G-S)} < \frac{b(|T_0|+l)+bn}{a(|T_0|+l)} \le \frac{b+bn}{a}$, which contradicts $t(G) \ge \frac{b^2-1+bn}{a}$ and $b \ge 2$. If $\omega(G-S) = 1$, then $|T_0| + l = 1$. Hence $d_{G-S}(x) = b - 1$ or $d_{G-S}(x) = 0$ for $x \in V(G) \setminus S$. Since $d_{G-S}(x) + |S| \ge d_G(x) \ge \delta(G) \ge 2t(G)$, we have $2t(G) \le b - 1 + |S| < b - 1 + \frac{b(n+1)}{a}$, which contradicts $t(G) \ge \frac{b^2-1+bn}{a}$.

Now we consider that $|V(H)| > 0$. Let $H = H_1 \cup H_2$, where H_1 is the union of components of H which satisfies that $d_{G-S}(x) = b-1$ for every vertex $x \in V(H_1)$ and $H_2 = H - H_1$. By Lemma 8, H_1 has a maximum independent set I_1 and the covering set $C_1 = V(H_1) - I_1$ such that

$$|V(H_1)| \le \sum_{i=1}^{b} (b - i + 1)|I^{(i)}| - \frac{|I^{(1)}|}{2} \tag{4}$$

and

$$|C_1| \le \sum_{i=1}^{b} (b - i)|I^{(i)}| - \frac{|I^{(1)}|}{2}, \tag{5}$$

where $I^{(i)} = \{x \in I_1 : d_{H_1}(x) = b - i\}$, $1 \le i \le b$ and $\sum_{i=1}^{b} |I^{(i)}| = |I_1|$. Let $T_j = \{x \in V(H_2) : d_{G-S}(x) = j\}$ for $1 \le j \le b - 1$. Each component of H_2 has a vertex of degree at most $b - 2$ in $G - S$ by the definitions of H and H_2. According to Lemma 7, H_2 has a maximal independent set I_2 and the covering set $C_2 = V(H_2) - I_2$ such that

$$\sum_{j=1}^{b-1} (b - j)c_j \le \sum_{j=1}^{b-1} (b - 2)(b - j)i_j, \tag{6}$$

where $c_j = |C_2 \cap T_j|$ and $i_j = |I_2 \cap T_j|$ for every $j = 1, \dots, b-1$. Set $W = V(G) - S - T$ and $U = S \cup C_1 \cup (N_G(I_1) \cap W)) \cup C_2 \cup (N_G(I_2) \cap W)$. We infer

$$|U| \leq |S| + |C_1| + \sum_{j=1}^{b-1} j i_j + \sum_{i=1}^{b} (i-1)|I^{(i)}| \tag{7}$$

and

$$\omega(G-U) \geq t_0 + l + |I_1| + \sum_{j=1}^{b-1} i_j, \tag{8}$$

where $t_0 = |T_0|$. Let $t(G) = t$. Then when $\omega(G-S) > 1$, we have

$$|U| \geq t\omega(G-S), \tag{9}$$

and it also holds when $\omega(G-S) = 1$. In terms of (7), (8) and (9), we get

$$|S| + |C_1| \geq \sum_{j=1}^{b-1} (t-j) i_j + t(t_0 + l) + t|I_1| - \sum_{i=1}^{b} (i-1)|I^{(i)}|. \tag{10}$$

In view of $b|T| - d_{G-S}(T) > a|S| - bn$, we obtain

$$bt_0 + bl + |V(H_1)| + \sum_{j=1}^{b-1} (b-j) i_j + \sum_{j=1}^{b-1} (b-j) c_j > a|S| - bn.$$

Combining with (10), we deduce

$$\begin{aligned} &bt_0 + bl + |V(H_1)| + \sum_{j=1}^{b-1} (b-j) i_j + \sum_{j=1}^{b-1} (b-j) c_j + a|C_1| + bn \\ > &\sum_{j=1}^{b-1} (at - aj) i_j + at(t_0 + l) + at|I_1| - a\sum_{i=1}^{b} (i-1)|I^{(i)}|. \end{aligned}$$

Therefore,

$$\begin{aligned} &|V(H_1)| + \sum_{j=1}^{b-1} (b-j) c_j + a|C_1| \\ > &\sum_{j=1}^{b-1} (at - aj - b + j) i_j + (at-b)(t_0 + l) + at|I_1| - a\sum_{i=1}^{b} (i-1)|I^{(i)}| - bn. \end{aligned} \tag{11}$$

By (4) and (5), we have

$$|V(H_1)| + a|C_1| \leq \sum_{i=1}^{b} (ab - ai + b - i + 1)|I^{(i)}| - \frac{(a+1)|I^{(1)}|}{2}. \tag{12}$$

Using (6), (11) and (12), we get

$$\begin{aligned}(13)\qquad &\sum_{j=1}^{b-1}(b-2)(b-j)i_j+\sum_{i=1}^{b}(ab-ai+b-i+1)|I^{(i)}|\\ &> \sum_{j=1}^{b-1}(at-aj-b+j)i_j+at|I_1|+\frac{(a+1)|I^{(1)}|}{2}\\ &\quad - a\sum_{i=1}^{b}(i-1)|I^{(i)}|+(at-b)(t_0+l)-bn.\end{aligned}$$

The following proof splits into two cases by the value of t_0+l.

Case 1. $t_0+l\geq 1$**.** By $at\geq b^2-1+bn$, we have $(at-b)(t_0+l)-bn\geq b^2-b-1>0$. Thus, (13) becomes

$$\begin{aligned}&\sum_{j=1}^{b-1}(b-2)(b-j)i_j+\sum_{i=1}^{b}(ab-ai+b-i+1)|I^{(i)}|\\ &\sum_{j=1}^{b-1}(at-aj-b+j)i_j+at|I_1|+\frac{(a+1)|I^{(1)}|}{2}-a\sum_{i=1}^{b}(i-1)|I^{(i)}|.\end{aligned}$$

And then, at least one of the following two cases must hold.

Subcase 1.1. There is at least one j such that

$$(b-2)(b-j)>at-aj-b+j,$$

which implies

$$at<(b-2)(b-j)+aj+b-j=b(b-2)+(a-b+1)j+b.$$

If $a=b$, then $at<a(a-2)+j+a\leq a^2-1$, which contradicts $t(G)\geq\frac{b^2-1+bn}{a}$. If $a<b$, then $at<b(b-2)+(a-b+1)+b=b(b-2)+a+1=(b^2-1)+(a-b)+(2-b)\leq b^2-1$, also contradicts $t(G)\geq\frac{b^2-1+bn}{a}$.

Subcase 1.2.

$$\begin{aligned}&\sum_{i=1}^{b}(ab-ai+b-i+1)|I^{(i)}|\\ &> at|I_1|+\frac{(a+1)|I^{(1)}|}{2}-a\sum_{i=1}^{b}(i-1)|I^{(i)}|\\ &\geq (b^2-1+bn)|I_1|+\frac{(a+1)|I^{(1)}|}{2}-a\sum_{i=1}^{b}(i-1)|I^{(i)}|\\ &\geq (b^2-1)|I_1|+\frac{(a+1)|I^{(1)}|}{2}-a\sum_{i=1}^{b}(i-1)|I^{(i)}|.\end{aligned}$$

This implies,

$$\sum_{i=2}^{b}(ab+b-a-i+2-b^2)|I^{(i)}|+(ab+b-\frac{3}{2}a-b^2+\frac{1}{2})|I^{(1)}|>0.$$

Let

$$h_1(b)=-b^2+(a+1)b-\frac{3}{2}a+\frac{1}{2}.$$

From $b\geq a$, we get

$$\max\{h_1(b)\}=f_1(a)=-\frac{a}{2}+\frac{1}{2}\leq 0.$$

On the other hand, $ab+b-a-i+2-b^2\leq -b^2+(a+1)b-a$ due to $i\geq 2$. Let

$$h_2(b)=-b^2+(a+1)b-a.$$

We infer

$$\max\{h_2(b)\}=f_2(a)=0$$

by $b\geq a$. This is a contradiction.

Case 2. $t_0+l=0$**.** In this case, (13) becomes

$$\sum_{j=1}^{b-1}(b-2)(b-j)i_j+\sum_{i=1}^{b}(ab-ai+b-i+1)|I^{(i)}|$$
$$>\sum_{j=1}^{b-1}(at-aj-b+j)i_j+at|I_1|+\frac{(a+1)|I^{(1)}|}{2}-a\sum_{i=1}^{b}(i-1)|I^{(i)}|-bn.$$

From what we have discussed in Subcase 1, we get $\sum_{j=1}^{b-1}(b-2)(b-j)i_j\leq\sum_{j=1}^{b-1}(at-aj-b+j)i_j$. If $|I_1|>0$, we deduce

$$\begin{aligned}&\sum_{i=1}^{b}(ab-ai+b-i+1)|I^{(i)}|\\&>at|I_1|+\frac{(a+1)|I^{(1)}|}{2}-a\sum_{i=1}^{b}(i-1)|I^{(i)}|-bn\\&\geq(b^2-1+bn)|I_1|+\frac{(a+1)|I^{(1)}|}{2}-a\sum_{i=1}^{b}(i-1)|I^{(i)}|-bn\\&\geq(b^2-1)|I_1|+\frac{(a+1)|I^{(1)}|}{2}-a\sum_{i=1}^{b}(i-1)|I^{(i)}|.\end{aligned}$$

The result follows from what we discussed in Subcase 2 above.

The last situation is $|I_1|=0$ and $\sum_{j=1}^{b-1}(b-2)(b-j)i_j>\sum_{j=1}^{b-1}(at-aj-b+j)i_j-bn$. Let $h_3=(b-2)(b-j)-(at-aj-b+j)+bn$. If $a=b$, we have

$$h_3=b^2-b+j-at+bn\leq b^2-b+(b-1)-(b^2-1+bn)+bn=0.$$

If $b \geq a+1$, then

$$\begin{aligned} h_3 &= b(b-2)+(a-b+1)j+b-at+bn \\ &\leq b(b-2)+(a-b+1)+b-(b^2-1+bn)+bn \\ &= -2(b-1)+a<0, \end{aligned}$$

a contradiction.

We complete the proof of the theorem.

3. Tight toughness condition for fractional (a, b, n)-critical graph

Let $g(x) = a$, $f(x) = b$ for each $x \in V(G)$. The necessary and sufficient condition for fractional (a, b, n)-critical graph derives from Lemma 3.

Lemma 9. *Let G be a graph. Let a, b, n be non-negative integers such that $a \leq b$. Then G is a fractional (a, b, n)-critical graph if and only if*

$$b|S| - a|T| + d_{G-S}(T) \geq bn \tag{14}$$

for all disjoint subsets S, T of $V(G)$ with $|S| \geq n$.

Using standard techniques similar to that of Section 2. Suppose that G is not a fractional (a, b, n)-critical graph. We infer $T \neq \emptyset$, and there exist disjoint subsets S and T of $V(G)$ such that

$$b|S| - a|T| + d_{G-S}(T) < bn, \tag{15}$$

where $|S| \geq n$. We choose S and T such that $|T|$ is minimum. We obtain $d_{G-S}(x) \leq a-1$ for each $x \in T$.

Applying Lemma 9, using the tricks used in Section 2, and noticing the minor differences between (3) and (15), and $d_{G-S}(x) \leq a-1$ for each $x \in T$ here correspond to $d_{G-S}(x) \leq b-1$ for each $x \in T$ in Section 2. We finally get the following tight condition for fractional (a, b, n)-critical graphs. We skip the proof.

Theorem 10. *Let G be a graph and let a, b be two nonnegative integers satisfying $2 \leq a \leq b$. Let n be a non-negative integer. $|V(G)| \geq n+a+1$ if G is complete. If $t(G) \geq \frac{ab-b+a-1}{b} + n$, then G is a fractional (a, b, n)-critical graph.*

To see Theorem 10 is sharp, we construct the following graph G: $V(G) = A \cup B \cup C$ where A, B and C are disjoint with $|A| = (mb+1)(a-1+n)$, $|B| = (mb+1)(a-1)$, and $|C| = m(a-1)$ with $a = b$. Both A and C are cliques in G, while B is isomorphic to $(mb+1)K_{a-1}$. Other edges in G are $\{uv; u \in B, v \in C\}$ and $\{u_1v_1, u_2v_2, \ldots, u_{(mb+1)(a-1)}v_{(mb+1)(a-1)}\}$, where $V(B) = \{u_1, u_2, \ldots, u_{(mb+1)(a-1)}\}$ and $\{v_1, v_2, \ldots, v_{(mb+1)(a-1)}\} \subset A$. If $a = 2$, let $S = (A - \{u\}) \cup C$, where $u \in A$, then $|S| = (mb+m+1) + n(mb+1)$ and $\omega(G-S) = mb+1$; if $a \geq 3$, let $S = (A - \{u\}) \cup \{v\} \cup C$, where $u \in A$ and

$v \in B$ is matched to u in G. Then $|S| = (mb+m+1)(a-1)+n(mb+1)$ and $\omega(G-S) = mb+2$. This follows that

$$t(G) = \begin{cases} \frac{(mb+m+1)(a-1)+n(mb+1)}{mb+2}, & a \geq 3 \\ \frac{(mb+m+1)+n(mb+1)}{mb+1}, & a = 2. \end{cases}$$

It is easy to see that $t(G) \to \frac{ab-b+a-1}{b} + n$ when $m \to +\infty$.

Let $V_0 \subset V(A) \setminus \{v_1, v_2, \ldots, v_{(mb+1)(a-1)}\}$ with $|V_0| = n$, $S = C \cup V_0$ and $T = B$. We have $d_{G-S}(x) = a-1$ for each $x \in T$, and

$$b|S| - bn - a|T| + d_{G-S}(T) = bm(a-1) - (mb+1)(a-1) < 0.$$

By Lemma 9, G is not a fractional (a, b, n)-critical graph. In this sense, $t(G)$ in Theorem 10 is best.

Again, the restriction on $|V(G)|$ for complete graph is necessary and can not be replaced by $|V(G)| \geq n + a$.

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