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Tightly integrated fuzzy description logic programs under the answer set semantics for the semantic web — Source link

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TIGHTLY INTEGRATED FUZZY DESCRIPTION LOGIC PROGRAMS UNDER THE ANSWER SET SEMANTICS FOR THE SEMANTIC WEB

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TIGHTLY INTEGRATED FUZZY DESCRIPTION LOGIC PROGRAMS UNDER THE ANSWER SET SEMANTICS FOR THE SEMANTIC WEB

FEBRUARY 28, 2007

Thomas Lukasiewicz [1] Umberto Straccia [2]

Abstract. We present a novel approach to fuzzy dl-programs under the answer set semantics, which is a tight integration of fuzzy disjunctive programs under the answer set semantics with fuzzy description logics. From a different perspective, it is a generalization of tightly integrated disjunctive dl-programs by fuzzy vagueness in both the description logic and the logic program component. We show that the new formalism faithfully extends both fuzzy disjunctive programs and fuzzy description logics, and that under suitable assumptions, reasoning in the new formalism is decidable. Furthermore, we present a polynomial reduction of certain fuzzy dl-programs to tightly integrated disjunctive dl-programs. We also provide a special case of fuzzy dl-programs for which deciding consistency and query processing have both a polynomial data complexity.

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Contents

1 Introduction

The *Semantic Web* [1, 9] aims at an extension of the current World Wide Web by standards and technologies that help machines to understand the information on the Web so that they can support richer discovery, data integration, navigation, and automation of tasks. The main ideas behind it are to add a machine-readable meaning to Web pages, to use ontologies for a precise definition of shared terms in Web resources, to use KR technology for automated reasoning from Web resources, and to apply cooperative agent technology for processing the information of the Web.

The Semantic Web consists of several hierarchical layers, where the *Ontology layer*, in form of the *OWL Web Ontology Language* [34, 15], is currently the highest layer of sufficient maturity. OWL consists of three increasingly expressive sublanguages, namely, *OWL Lite*, *OWL DL*, and *OWL Full*. OWL Lite and OWL DL are essentially very expressive description logics with an RDF syntax [15]. As shown in [13], ontology entailment in OWL Lite (resp., OWL DL) reduces to knowledge base (un)satisfiability in the description logic $\mathcal{SHIF}(\mathbf{D})$ (resp., $\mathcal{SHOIN}(\mathbf{D})$). On top of the Ontology layer, the *Rules*, *Logic*, and *Proof layers* of the Semantic Web will be developed next, which should offer sophisticated representation and reasoning capabilities.

In particular, there is a large body of work on integrating rules and ontologies, which is a key requirement of the layered architecture of the Semantic Web. Significant research efforts focus on hybrid integrations of rules and ontologies, called *description logic programs* (or *dl-programs*), which are of the form $KB = (L, P)$, where L is a description logic knowledge base and P is a finite set of rules involving either queries to L in a loose integration (see especially [7, 8, 5, 6]) or concepts and roles from L as unary resp. binary predicates in a tight integration (see especially [25, 26, 20]).

Other works explore formalisms for *handling uncertainty and vagueness / imprecision* in the Semantic Web. In particular, formalisms for dealing with uncertainty and vagueness in ontologies have been applied in ontology mapping and information retrieval. Vagueness and imprecision also abound in multimedia information processing and retrieval. Moreover, handling vagueness is an important aspect of natural language interfaces to the Web. There are several recent extensions of description logics, ontology languages, and description logic programs for the Semantic Web by probabilistic uncertainty and fuzzy vagueness. In particular, description logic programs under probabilistic uncertainty and fuzzy vagueness have been proposed in [18, 17] resp. [31, 32, 19].

In this paper, we continue this line of research. We present *tightly integrated fuzzy description logic programs* (or simply *fuzzy dl-programs*) *under the answer set semantics*, which are a tight integration of fuzzy disjunctive programs under the answer set semantics with fuzzy generalizations of $\mathcal{SHIF}(\mathbf{D})$ and $\mathcal{SHOIN}(\mathbf{D})$. Even though there has been previous work on fuzzy positive dl-programs [31, 32] and on loosely integrated fuzzy normal dl-programs [19], to our knowledge, this is the first approach to tightly integrated fuzzy disjunctive dl-programs (with default negation in rule bodies). The main contributions of this paper can be summarized as follows:

- We present a novel approach to fuzzy dl-programs, which is a tight integration of fuzzy disjunctive programs under the answer set semantics with fuzzy description logics. It is a generalization of the tightly integrated disjunctive dl-programs in [20] by fuzzy vagueness in both the description logic and the logic program component.

- We show that the new fuzzy dl-programs have nice semantic features. In particular, all their answer sets are also minimal models, and the cautious answer set semantics faithfully extends both fuzzy disjunctive programs and fuzzy description logics. Furthermore, the new approach also does not need the unique name assumption.

- In the large class of fuzzy dl-programs that are defined over a finite number of truth values, the problems of deciding consistency, cautious consequence, and brave consequence are all decidable. We also present a polynomial reduction for certain fuzzy dl-programs to the tightly integrated disjunctive dl-programs in [20].
- Finally, we delineate a special case of fuzzy dl-programs where deciding consistency and query processing have both a polynomial data complexity.

The rest of this paper is organized as follows. Sections 2 and 3 recall combination strategies and fuzzy description logics, respectively. Section 4 introduces the syntax of fuzzy dl-programs and defines their answer set semantics. In Section 5, we analyze some semantic properties of fuzzy dl-programs under the answer set semantics. Section 6 presents a reduction of fuzzy dl-programs to disjunctive dl-programs. In Section 7, we delineate a special case of fuzzy dl-programs with polynomial data complexity. Section 8 summarizes our main results and gives an outlook on future research. Note that detailed proofs of all the results in this paper are given in the extended version.

2 Combination Strategies

Rather than being restricted to an ordinary binary truth value among $\mathbf{false}$ and $\mathbf{true}$, *vague propositions* may also have a truth value strictly between $\mathbf{false}$ and $\mathbf{true}$. In the sequel, we use the unit interval $[0,1]$ as the set of all possible truth values, where 0 and 1 represent the ordinary binary truth values $\mathbf{false}$ and $\mathbf{true}$, respectively. For example, the vague proposition "John is a tall man" may be more or less true, and it is thus associated with a truth value in $[0,1]$, depending on the body height of John.

In order to combine and modify the truth values in $[0,1]$, we assume *combination strategies*, namely, *conjunction*, *disjunction*, *implication*, and *negation strategies*, denoted $\otimes$, $\oplus$, $\rhd$, and $\ominus$, respectively, which are functions $\otimes$, $\oplus$, $\rhd\colon [0,1] \times [0,1] \rightarrow [0,1]$ and $\ominus\colon [0,1] \rightarrow [0,1]$ that generalize the ordinary Boolean operators $\wedge$, $\vee$, $\rightarrow$, and $\neg$, respectively, to the set of truth values $[0,1]$. For $a, b \in [0,1]$, we then call $a \otimes b$ (resp., $a \oplus b$, $a \rhd b$) the *conjunction* (resp., *disjunction*, *implication*) of a and b, and we call $\ominus\, a$ the *negation* of a. As usual, we assume that combination strategies have some natural algebraic properties, namely, the properties shown in Tables 1 and 2. Note that conjunction and disjunction strategies (with the properties in Table 1) are also called *triangular norms* and *triangular co-norms* [11], respectively. We do not assume properties that relate the combination strategies to each other (such as de Morgan's law); even though one may additionally assume such properties, they are not required here.

Example 2.1 The combination strategies of various fuzzy logics are shown in Table 3.

3 Fuzzy Description Logics

In this section, we recall fuzzy $\mathcal{SHIF}(\mathbf{D})$ and fuzzy $\mathcal{SHOIN}(\mathbf{D})$ [29, 30, 21] (see also [27]). Note that there also exists an implementation of fuzzy $\mathcal{SHIF}(\mathbf{D})$ (the *fuzzyDL* system; see `http://gaia.isti.cnr.it/~straccia`). Intuitively, description logics model a domain of interest in terms of concepts and roles, which represent classes of individuals and binary relations between classes of individuals, respectively. A description logic knowledge base encodes in particular subset relationships between classes of individuals,

Table 1: Axioms for conjunction and disjunction strategies.

Axiom Name	Conjunction Strategy	Disjunction Strategy
Tautology / Contradiction	$a \otimes 0 = 0$	$a \oplus 1 = 1$
Identity	$a \otimes 1 = a$	$a \oplus 0 = a$
Commutativity	$a \otimes b = b \otimes a$	$a \oplus b = b \oplus a$
Associativity	$(a \otimes b) \otimes c = a \otimes (b \otimes c)$	$(a \oplus b) \oplus c = a \oplus (b \oplus c)$
Monotonicity	if $b \leqslant c$, then $a \otimes b \leqslant a \otimes c$	if $b \leqslant c$, then $a \oplus b \leqslant a \oplus c$

Table 2: Axioms for implication and negation strategies.

Axiom Name	Implication Strategy	Negation Strategy
Tautology / Contradiction	$0 \rhd b = 1,\ a \rhd 1 = 1,\ 1 \rhd 0 = 0$	$\ominus 0 = 1,\ \ominus 1 = 0$
Antitonicity	if $a \leqslant b$, then $a \rhd c \geqslant b \rhd c$	if $a \leqslant b$, then $\ominus a \geqslant \ominus b$
Monotonicity	if $b \leqslant c$, then $a \rhd b \leqslant a \rhd c$	

Table 3: Combination strategies of various fuzzy logics.

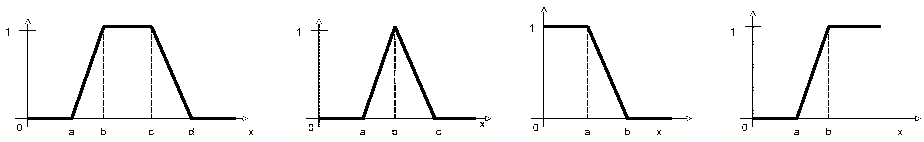
	Łukasiewicz Logic	Gödel Logic	Product Logic	Zadeh Logic
$a \otimes b$	$\max(a+b-1,0)$	$\min(a,b)$	$a \cdot b$	$\min(a,b)$
$a \oplus b$	$\min(a+b,1)$	$\max(a,b)$	$a+b-a\cdot b$	$\max(a,b)$
$a \rhd b$	$\min(1-a+b,1)$	$\begin{cases} 1 & \text{if } a \leqslant b \\ b & \text{otherwise} \end{cases}$	$\min(1,b/a)$	$\max(1-a,b)$
$\ominus a$	$1-a$	$\begin{cases} 1 & \text{if } a = 0 \\ 0 & \text{otherwise} \end{cases}$	$\begin{cases} 1 & \text{if } a = 0 \\ 0 & \text{otherwise} \end{cases}$	$1-a$

subset relationships between binary relations between classes, the membership of individuals to classes, and the membership of pairs of individuals to binary relations between classes. In fuzzy description logics, these relationships and memberships then have a degree of truth in $[0,1]$.

3.1 Syntax

We first describe fuzzy $\mathcal{SHOIN}(\mathbf{D})$, which has the following elementary ingredients. We assume a set of *data values*, a set of *elementary datatypes*, and a set of *datatype predicates* (each with a predefined arity $n \geqslant 1$). A *datatype* is an elementary datatype or a finite set of data values. A *fuzzy datatype theory* $\mathbf{D} = (\Delta^{\mathbf{D}}, \cdot^{\mathbf{D}})$ consists of a datatype domain $\Delta^{\mathbf{D}}$ and a mapping $\cdot^{\mathbf{D}}$ that assigns to each data value an element of $\Delta^{\mathbf{D}}$, to each elementary datatype a subset of $\Delta^{\mathbf{D}}$, and to each datatype predicate of arity n a fuzzy relation over $\Delta^{\mathbf{D}}$ of arity n (that is, a mapping $(\Delta^{\mathbf{D}})^n \to [0,1]$). We extend $\cdot^{\mathbf{D}}$ to all datatypes by $\{v_1, \ldots, v_n\}^{\mathbf{D}} = \{v_1^{\mathbf{D}}, \ldots, v_n^{\mathbf{D}}\}$.

Example 3.1 A crisp unary datatype predicate $\leqslant_{18}$ over the natural numbers denoting the integers of at most 18 may be defined by $\leqslant_{18}(x) = 1$, if $x \leqslant 18$, and $\leqslant_{18}(x) = 0$, otherwise. Then, $Minor = Person \sqcap \exists age.\leqslant_{18}$ defines a person of age at most 18. Non-crisp predicates are usually defined by functions for specifying fuzzy set membership degrees, such as the triangular, the trapezoidal, the L-, and the R-function (see Fig. 1). For example, a fuzzy unary datatype predicate $Young$ over the natural numbers



(a) (b) (c) (d)

Figure 1: (a) Trapezoidal function; (b) Triangular function; (c) L-function; (d) R-function

denoting the degree of youngness of a person's age may be defined by $Young(x) = L(x; 10,\, 30)$. Then, $YoungPerson = Person \sqcap \exists age.\,Young$ denotes a young person.

Let $\mathbf{A}$, $\mathbf{R}_A$, $\mathbf{R}_D$, $\mathbf{I}$, and $\mathbf{M}$ be pairwise disjoint (nonempty) denumerable sets of *atomic concepts*, *abstract roles*, *datatype roles*, *individuals*, and *fuzzy modifiers*, respectively. Here, a *fuzzy modifier* m [12, 33] represents a function f_m on $[0, 1]$, which changes the membership function of a fuzzy set.

Example 3.2 The fuzzy modifiers $very$ resp. $slightly$ may represent the two functions $very(x) = x^2$ resp. $slightly(x) = \sqrt{x}$. Then, the concept of sports cars may be defined as $SportsCar = Car \sqcap \exists speed.very(High)$, where $High$ is a fuzzy datatype predicate over the domain of speed in km/h, which may be defined as $High(x) = R(x; 80, 250)$.

A *role* is any element of $\mathbf{R}_A \cup \mathbf{R}_A^- \cup \mathbf{R}_D$ (where $\mathbf{R}_A^-$ is the set of *inverses* R^- of all $R \in \mathbf{R}_A$). We define *concepts* inductively as follows. Each $A \in \mathbf{A}$ is a concept, $\bot$ and $\top$ are concepts, and if $a_1, \ldots,$ $a_n \in \mathbf{I}$, then $\{a_1, \ldots, a_n\}$ is a concept (called *oneOf*). If C, C_1, C_2 are concepts, $R, S \in \mathbf{R}_A \cup \mathbf{R}_A^-$, and $m \in \mathbf{M}$, then $(C_1 \sqcap C_2)$, $(C_1 \sqcup C_2)$, $\neg C$, and $m(C)$ are concepts (called *conjunction*, *disjunction*, *negation*, and *fuzzy modification*, respectively), as well as $\exists R.C$, $\forall R.C$, $\geqslant nS$, and $\leqslant nS$ (called *exists*, *value*, *atleast*, and *atmost restriction*, respectively) for an integer $n \geqslant 0$. If D is a datatype and $T, T_1, \ldots, T_n \in \mathbf{R}_D$, then $\exists T_1, \ldots, T_n.D$, $\forall T_1, \ldots, T_n.D$, $\geqslant nT$, and $\leqslant nT$ are concepts (called *datatype exists*, *value*, *atleast*, and *atmost restriction*, respectively) for an integer $n \geqslant 0$. We eliminate parentheses as usual.

A *crisp axiom* has one of the following forms: (1) $C \sqsubseteq D$ (called *concept inclusion axiom*), where C and D are concepts; (2) $R \sqsubseteq S$ (called *role inclusion axiom*), where either $R, S \in \mathbf{R}_A$ or $R, S \in \mathbf{R}_D$; (3) $\mathrm{Trans}(R)$ (called *transitivity axiom*), where $R \in \mathbf{R}_A$; (4) $C(a)$ (called *concept assertion axiom*), where C is a concept and $a \in \mathbf{I}$; (5) $R(a, b)$ (resp., $U(a, v)$) (called *role assertion axiom*), where $R \in \mathbf{R}_A$ (resp., $U \in \mathbf{R}_D$) and $a, b \in \mathbf{I}$ (resp., $a \in \mathbf{I}$ and v is a data value); and (6) $a = b$ (resp., $a \neq b$) (*equality* (resp., *inequality*) *axiom*), where $a, b \in \mathbf{I}$. We define *fuzzy axioms* as follows: A *fuzzy concept inclusion* (resp., *fuzzy role inclusion*, *fuzzy concept assertion*, *fuzzy role assertion*) *axiom* is of the form $\alpha\,\theta\,n$, where α is a concept inclusion (resp., role inclusion, concept assertion, role assertion) axiom, $\theta \in \{\leqslant, =, \geqslant\}$, and $n \in [0, 1]$. For example, $C(a) \geqslant 0.1$, $R(a, b) \leqslant 0.3$, $R \sqsubseteq S \geqslant 0.4$, and $C \sqsubseteq D \leqslant 0.6$ are fuzzy axioms. Informally, a fuzzy axiom of the form $\alpha \leqslant n$ (resp., $\alpha = n$, $\alpha \geqslant n$) encodes that the truth degree of α is at most (resp., equal to, at least) n. For example, $TallPerson(jim) \geqslant 0.2$ says that jim is a tall person with a truth degree of at least 0.2, while $C \sqsubseteq D \geqslant n$ says that the subsumption degree between C and D is at least n. We often use $a \colon C$ and α to abbreviate $C(a)$ and $\alpha \geqslant 1$, respectively. A *fuzzy (description logic) knowledge base* L is a finite set of fuzzy axioms, transitivity axioms, and equality and inequality axioms. For decidability, number restrictions in L are restricted to simple abstract roles [16].

Fuzzy $\mathcal{SHIF}(\mathbf{D})$ has the same syntax as fuzzy $\mathcal{SHOIN}(\mathbf{D})$, but without the oneOf constructor and with the atleast and atmost constructors limited to 0 and 1.

Example 3.3 (Shopping Agent) The following axioms are an excerpt of the description logic knowledge base L that conceptualizes a car selling web site:

$$
\begin{aligned}
& \mathit{Cars} \sqcup \mathit{Trucks} \sqcup \mathit{Vans} \sqcup \mathit{SUVs} \sqsubseteq \mathit{Vehicles} && (1)\\
& \mathit{PassengerCars} \sqcup \mathit{LuxuryCars} \sqsubseteq \mathit{Cars} && (2)\\
& \mathit{CompactCars} \sqcup \mathit{MidSizeCars} \sqcup \mathit{SportyCars} \sqsubseteq \mathit{PassengerCars} && (3)\\
& \mathit{Cars} \sqsubseteq (\exists \mathit{hasReview}.\mathit{Integer}) \sqcap (\exists \mathit{hasInvoice}.\mathit{Integer}) && \\
& \qquad \sqcap (\exists \mathit{hasResellValue}.\mathit{Integer}) \sqcap (\exists \mathit{hasMaxSpeed}.\mathit{Integer}) && \\
& \qquad \sqcap (\exists \mathit{hasHorsePower}.\mathit{Integer}) \sqcap \ \ldots && (4)\\
& \mathit{MazdaMX5Miata}\colon \mathit{SportyCar} \sqcap (\exists \mathit{hasInvoice}.18883) && \\
& \qquad \sqcap (\exists \mathit{hasHorsePower}.166) \sqcap \ldots && (5)\\
& \mathit{MitsubishiEclipseSpyder}\colon \mathit{SportyCar} \sqcap (\exists \mathit{hasInvoice}.24029) && \\
& \qquad \sqcap (\exists \mathit{hasHorsePower}.162) \sqcap \ldots && (6)
\end{aligned}
$$

Eqs. 1–3 describe the concept taxonomy of the site, while Eq. 4 describes the datatype attributes of the cars sold in the site. Eqs. 5–6 describe the properties of some sold cars.

We may then encode "costs at most about 22 000 €" and "has a power of around 150 HP" in a buyer's request through the following concepts C and D, respectively:

$$C = \exists \mathit{hasInvoice}.\mathit{LeqAbout22000} \quad \text{and} \quad D = \exists \mathit{hasHorsePower}.\mathit{Around150}\,,$$

where $\mathit{LeqAbout22000} = L(22000, 25000)$ and $\mathit{Around150} = \mathit{Tri}(125, 150, 175)$. The latter two equations define the fuzzy concepts of "at most about 22 000 €" and "around 150 HP". The former is modeled as a left shoulder function stating that if the prize is less than 22 000, then the degree of truth (degree of buyer's satisfaction) is 1, else the truth is linearly decreasing to 0 (reached at the cost of 25 000). In fact, we are modeling a case were the buyer would like to pay less than 22 000, though may still accept a higher price (up to 25 000) to a lesser degree. Similarly, the latter models the fuzzy concept "around 150 HP" as a triangular function with vertice in 150 HP.

3.2 Semantics

Concerning the semantics of fuzzy $\mathcal{SHIF}(\mathbf{D})$ and $\mathcal{SHOIN}(\mathbf{D})$ [30], the main idea is that concepts and roles are interpreted as fuzzy subsets of an interpretation's domain. Therefore, concept inclusion, role inclusion, concept assertion, and role assertion axioms, rather than being satisfied (true) or unsatisfied (false) in an interpretation, have a degree of truth in $[0, 1]$. In the sequel, we assume that $\otimes$, $\oplus$, $\rhd$, and $\ominus$ are some arbitrary but fixed conjunction, disjunction, implication, and negation strategies, respectively. A *fuzzy interpretation* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ relative to a fuzzy datatype theory $\mathbf{D} = (\Delta^{\mathbf{D}}, \cdot^{\mathbf{D}})$ consists of a nonempty set $\Delta^{\mathcal{I}}$ (called the *domain*), disjoint from $\Delta^{\mathbf{D}}$, and a *fuzzy interpretation function* $\cdot^{\mathcal{I}}$, which (i) coincides with $\cdot^{\mathbf{D}}$ on every data value, datatype, and fuzzy datatype predicate, (ii) assigns to each modifier $m \in \mathbf{M}$ its modifier function $f_m\colon [0, 1] \to [0, 1]$, and (iii) assigns

- to each individual $a \in \mathbf{I}$ an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$;
- to each atomic concept $C \in \mathbf{A}$ a function $C^{\mathcal{I}}\colon \Delta^{\mathcal{I}} \to [0, 1]$;
- to each abstract role $R \in \mathbf{R}_A$ a function $R^{\mathcal{I}}\colon \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0, 1]$;

- to each concrete role $T \in \mathbf{R}_D$ a function $T^{\mathcal{I}}\colon \Delta^{\mathcal{I}} \times \Delta^{\mathbf{D}} \to [0,1]$.

The mapping $\cdot^{\mathcal{I}}$ is extended to all roles and concepts as follows (where $x, y \in \Delta^{\mathcal{I}}$):

$$\begin{array}{rcl}
(S^{-})^{\mathcal{I}}(x,y) & = & S^{\mathcal{I}}(y,x) \\
\top^{\mathcal{I}}(x) & = & 1 \\
\bot^{\mathcal{I}}(x) & = & 0 \\
\{a_1,\ldots,a_n\}^{\mathcal{I}}(x) & = & \bigoplus_{i=1}^{n} a_i{}^{\mathcal{I}} = x \\
(C_1 \sqcap C_2)^{\mathcal{I}}(x) & = & C_1{}^{\mathcal{I}}(x) \otimes C_2{}^{\mathcal{I}}(x) \\
(C_1 \sqcup C_2)^{\mathcal{I}}(x) & = & C_1{}^{\mathcal{I}}(x) \oplus C_2{}^{\mathcal{I}}(x) \\
(\neg C)^{\mathcal{I}}(x) & = & \ominus\, C^{\mathcal{I}}(x) \\
(m(C))^{\mathcal{I}}(x) & = & f_m(C^{\mathcal{I}}(x)) \\
(\forall R.C)^{\mathcal{I}}(x) & = & \inf_{y \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(x,y) \rhd C^{\mathcal{I}}(y) \\
(\exists R.C)^{\mathcal{I}}(x) & = & \sup_{y \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(x,y) \otimes C^{\mathcal{I}}(y) \\
(\geqslant n\, S)^{\mathcal{I}}(x) & = & \sup\limits_{\substack{\{y_1,\ldots,y_n\} \subseteq \Delta^{\mathcal{I}} \\ |\{y_1,\ldots,y_n\}| = n}} \bigotimes_{i=1}^{n} S^{\mathcal{I}}(x,y_i) \\
(\leqslant n\, S)^{\mathcal{I}}(x) & = & \ominus\,(\geqslant n+1\; S)^{\mathcal{I}}(x) \\
(\forall T_1,\ldots,T_n.D)^{\mathcal{I}}(x) & = & \inf_{y_1,\ldots,y_n \in \Delta^{\mathbf{D}}} (\bigotimes_{i=1}^{n} T_i{}^{\mathcal{I}}(x,y_i)) \rhd D^{\mathbf{D}}(y_1,\ldots,y_n) \\
(\exists T_1,\ldots,T_n.D)^{\mathcal{I}}(x) & = & \sup_{y_1,\ldots,y_n \in \Delta^{\mathbf{D}}} (\bigotimes_{i=1}^{n} T_i{}^{\mathcal{I}}(x,y_i)) \otimes D^{\mathbf{D}}(y_1,\ldots,y_n)\,.
\end{array}$$

The mapping $\cdot^{\mathcal{I}}$ is extended to concept inclusion, role inclusion, concept assertion, and role assertion axioms as follows (where $a, b \in \mathbf{I}$):

$$\begin{array}{rcl}
(C_1 \sqsubseteq C_2)^{\mathcal{I}} & = & \inf_{x \in \Delta^{\mathcal{I}}} C_1{}^{\mathcal{I}}(x) \rhd C_2{}^{\mathcal{I}}(x) \\
(R_1 \sqsubseteq R_2)^{\mathcal{I}} & = & \inf_{x,y \in \Delta^{\mathcal{I}}} R_1{}^{\mathcal{I}}(x,y) \rhd R_2{}^{\mathcal{I}}(x,y) \\
(C(a))^{\mathcal{I}} & = & C^{\mathcal{I}}(a^{\mathcal{I}}) \\
(R(a,b))^{\mathcal{I}} & = & R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}})\,.
\end{array}$$

The notion of a fuzzy interpretation $\mathcal{I}$ *satisfying* a transitivity, equality, inequality, or fuzzy axiom E, or $\mathcal{I}$ being a *model* of E, denoted $\mathcal{I} \models E$, is defined as follows: (i) $\mathcal{I} \models \mathit{trans}(R)$ iff $R^{\mathcal{I}}(x,y) \geqslant \sup_{z \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(x,z) \otimes R^{\mathcal{I}}(z,y)$ for all $x, y \in \Delta^{\mathcal{I}}$; (ii) $\mathcal{I} \models a = b$ iff $a^{\mathcal{I}} = b^{\mathcal{I}}$, and $\mathcal{I} \models a \neq b$ iff $a^{\mathcal{I}} \neq b^{\mathcal{I}}$; and (iii) $\mathcal{I} \models \alpha\,\theta\, n$ iff $\alpha^{\mathcal{I}}\,\theta\, n$. A concept C is *satisfiable* iff there is an interpretation $\mathcal{I}$ and some $x \in \Delta^{\mathcal{I}}$ such that $C^{\mathcal{I}}(x) > 0$. We say $\mathcal{I}$ *satisfies* a fuzzy knowledge base L, or $\mathcal{I}$ is a *model* of L, denoted $\mathcal{I} \models L$, iff $\mathcal{I}$ is a model of all $E \in L$. We say L is *satisfiable* iff L has a model. A fuzzy axiom E is a *logical consequence* of L, denoted $L \models E$, iff every model of L satisfies E. A fuzzy axiom $\alpha \geqslant n$ is a *tight logical consequence* of L, denoted $L \models_{\mathit{tight}} \alpha \geqslant n$, iff n is the supremum of $m \in [0,1]$ subject to $L \models \alpha \geqslant m$.

Example 3.4 (Shopping Agent cont'd) The following fuzzy axioms are (tight) logical consequences of L in Example 3.3 (under the Zadeh semantics of the connectives):

$$\begin{array}{ll}
C(\mathit{MazdaMX5Miata}) \geqslant 1.0 & C(\mathit{MitsubishiEclipseSpyder}) \geqslant 0.32 \\
D(\mathit{MazdaMX5Miata}) \geqslant 0.36 & D(\mathit{MitsubishiEclipseSpyder}) \geqslant 0.56\,.
\end{array}$$

4 Fuzzy Description Logic Programs

In this section, we present a tightly integrated approach to *fuzzy disjunctive description logic programs* (or simply *fuzzy dl-programs*) under the answer set semantics. Observe that differently from [19] (in addition to being a tightly integrated approach to fuzzy dl-programs), the fuzzy dl-programs here are based on fuzzy description logics as in [30]. Furthermore, they additionally allow for disjunctions in rule heads. We first introduce the syntax of fuzzy dl-programs and then their answer set semantics.

The basic idea behind the tightly integrated approach in this section is as follows. Suppose that we have a fuzzy disjunctive program P. Under the answer set semantics, P is equivalent to its grounding $ground(P)$. Suppose now that some of the ground atoms in $ground(P)$ are additionally related to each other by a fuzzy description logic knowledge base L. That is, some of the ground atoms in $ground(P)$ actually represent concept and role memberships relative to L. Thus, when processing $ground(P)$, we also have to consider L. However, we only want to do it to the extent that we actually need it for processing $ground(P)$. Hence, when taking a fuzzy Herbrand interpretation $I \subseteq HB_\Phi$, we have to ensure that I represents a valid truth value assignment relative to L. In other words, the main idea behind the semantics is to interpret P relative to Herbrand interpretations that also satisfy L, while L is interpreted relative to general interpretations over a first-order domain. Thus, we modularly combine the standard semantics of fuzzy disjunctive programs and of fuzzy description logics as in [19], which allows for building on the standard techniques and the results of both areas. However, our new approach here allows for a much tighter integration of L and P.

4.1 Syntax

We assume a function-free first-order vocabulary Φ with nonempty finite sets of constant and predicate symbols. We use Φ_c to denote the set of all constant symbols in Φ. We also assume pairwise disjoint (nonempty) denumerable sets $\mathbf{A}$, $\mathbf{R}_A$, $\mathbf{R}_D$, $\mathbf{I}$, and $\mathbf{M}$ of atomic concepts, abstract roles, datatype roles, individuals, and fuzzy modifiers, respectively, as in Section 3. We assume that Φ_c is a subset of $\mathbf{I}$. This assumption guarantees that every ground atom constructed from atomic concepts, abstract roles, datatype roles, and constants in Φ_c can be interpreted in the description logic component. We do not assume any other restriction on the vocabularies, that is, Φ and $\mathbf{A}$ (resp., $\mathbf{R}_A \cup \mathbf{R}_D$) may have unary (resp., binary) predicate symbols in common.

Let $\mathcal{X}$ be a set of variables. A *term* is either a variable from $\mathcal{X}$ or a constant symbol from Φ. An *atom* is of the form $p(t_1, \ldots, t_n)$, where p is a predicate symbol of arity $n \geqslant 0$ from Φ, and $t_1, \ldots, t_n$ are terms. A *literal* l is an atom p or a negated atom $not\, p$. A *disjunctive fuzzy rule* (or simply *fuzzy rule*) r is of the form

$$\begin{aligned} a_1 \vee_{\oplus_1} \cdots \vee_{\oplus_{l-1}} a_l \leftarrow_{\otimes_0} b_1 \wedge_{\otimes_1} b_2 \wedge_{\otimes_2} \cdots \wedge_{\otimes_{k-1}} b_k \wedge_{\otimes_k} & \\ not_{\ominus_{k+1}}\, b_{k+1} \wedge_{\otimes_{k+1}} \cdots \wedge_{\otimes_{m-1}} not_{\ominus_m}\, b_m \geqslant v, & \end{aligned} \tag{7}$$

where $l \geqslant 1$, $m \geqslant k \geqslant 0$, $a_1, \ldots, a_l, b_{k+1}, \ldots, b_m$ are atoms, $b_1, \ldots, b_k$ are either atoms or truth values from $[0, 1]$, $\oplus_1, \ldots, \oplus_{l-1}$ are disjunction strategies, $\otimes_0, \ldots, \otimes_{m-1}$ are conjunction strategies, $\ominus_{k+1}, \ldots, \ominus_m$ are negation strategies, and $v \in [0, 1]$. We refer to $a_1 \vee_{\oplus_1} \cdots \vee_{\oplus_{l-1}} a_l$ as the *head* of r, while the conjunction $b_1 \wedge_{\otimes_1} \ldots \wedge_{\otimes_{m-1}} not_{\ominus_m}\, b_m$ is the *body* of r. We define $H(r) = \{a_1, \ldots, a_l\}$ and $B(r) = B^+(r) \cup B^-(r)$, where $B^+(r) = \{b_1, \ldots, b_k\}$ and $B^-(r) = \{b_{k+1}, \ldots, b_m\}$. A *disjunctive fuzzy program* (or simply *fuzzy program* P is a finite set of fuzzy rules of the form (7). We say P is a *normal fuzzy program* iff $l = 1$ for all fuzzy rules (7) in P. We say P is a *positive fuzzy program* iff $l = 1$ and $m = k$ for all fuzzy rules (7) in P.

A *disjunctive fuzzy description logic program* (or simply *fuzzy dl-program*) $KB = (L, P)$ consists of a description logic knowledge base L and a disjunctive fuzzy program P. It is called a *normal fuzzy dl-program* iff P is a normal fuzzy program. It is called a *positive fuzzy dl-program* iff P is a positive fuzzy program.

Example 4.1 (Shopping Agent cont'd) A fuzzy dl-program $KB = (L, P)$ is given by the fuzzy description logic knowledge base L in Example 3.3 and the set of fuzzy dl-rules P, which contains only the following fuzzy dl-rule (where $x \otimes y = \min(x, y)$):

$$\begin{aligned} \mathit{query}(x) \quad \leftarrow_\otimes \quad & \mathit{SportyCar}(x) \wedge_\otimes \mathit{hasInvoice}(x, y_1) \wedge_\otimes \mathit{hasHorsePower}(x, y_2) \wedge_\otimes \\ & \mathit{LeqAbout22000}(y_1) \wedge_\otimes \mathit{Around150}(y_2) \geqslant 1\,. \end{aligned}$$

Informally, the predicate query collects all sporty cars, and ranks them according to whether they cost at most around 22 000 € and have around 150 HP (such a car may be requested by a car buyer with economic needs). Another fuzzy dl-rule is given as follows (where $\ominus\, x = 1 - x$ and $\mathit{Around300} = \mathit{Tri}(250, 300, 350)$):

$$\begin{aligned} \mathit{query}'(x) \quad \leftarrow_\otimes \quad & \mathit{SportyCar}(x) \wedge_\otimes \mathit{hasInvoice}(x, y_1) \wedge_\otimes \mathit{hasMaxSpeed}(x, y_2) \wedge_\otimes \\ & \mathit{not}_\ominus \mathit{LeqAbout22000}(y_1) \wedge_\otimes \mathit{Around300}(y_2) \geqslant 1\,. \end{aligned}$$

Informally, this rule collects all sporty cars, and ranks them according to whether they cost at least around 22 000 € and have a maximum speed of around 300 km/h (such a car may be requested by a car buyer with luxurious needs). Another fuzzy dl-rule involving also a disjunction in its head is given as follows (where $x \oplus y = \max(x, y)$):

$$\mathit{Small}(x) \vee_\oplus \mathit{Old}(x) \quad \leftarrow_\otimes \quad \mathit{Car}(x) \wedge_\otimes \mathit{hasInvoice}(x, y) \wedge_\otimes \mathit{not}_\ominus \mathit{GeqAbout15000}(y) \geqslant 0.7\,.$$

This rule says that a car costing at most around 15 000 € is either small or old. Observe here that Small and Old may be two concepts in the fuzzy description logic knowledge base L. That is, the tightly integrated approach to fuzzy dl-programs under the answer set semantics also allows for using the rules in L to express relationships between the concepts and roles in P. This is not possible in the loosely integrated approach to fuzzy dl-programs under the answer set semantics in [19], since the dl-queries of that framework can only occur in rule bodies, but not in rule heads.

4.2 Semantics

We now define the answer set semantics of fuzzy dl-programs via a generalization of the standard Gelfond-Lifschitz transformation [10].

In the sequel, let $KB = (L, P)$ be a fuzzy dl-program. A *ground instance* of a rule $r \in P$ is obtained from r by replacing every variable that occurs in r by a constant symbol from Φ_c. We denote by $\mathit{ground}(P)$ the set of all ground instances of rules in P. The *Herbrand base* relative to Φ, denoted HB_Φ, is the set of all ground atoms constructed with constant and predicate symbols from Φ. Observe that we define the Herbrand base relative to Φ and not relative to P. This allows for reasoning about ground atoms from the description logic component that do not necessarily occur in P. Observe, however, that the extension from P to Φ is only a notational simplification, since we can always make constant and predicate symbols from Φ occur in P by "dummy" rules such as $\mathit{constant}(c) \leftarrow$ and $p(\boldsymbol{c}) \leftarrow p(\boldsymbol{c})$, respectively. We denote by DL_Φ the set of all ground atoms in HB_Φ that are constructed from atomic concepts in $\mathbf{A}$, abstract roles in $\mathbf{R}_A$, concrete roles in $\mathbf{R}_D$, and constant symbols in Φ_c.

We define Herbrand interpretations and the truth of fuzzy dl-programs in them as follows. An *interpretation* I is a mapping $I\colon HB_\Phi \rightarrow [0, 1]$. We write $\boldsymbol{HB}_\Phi$ to denote the interpretation I such that $I(a) = 1$ for all $a \in HB_\Phi$. For interpretations I and J, we write $I \subseteq J$ iff $I(a) \leqslant J(a)$ for all $a \in HB_\Phi$, and we define the *intersection* of I and J, denoted $I \cap J$, by $(I \cap J)(a) = \min(I(a), J(a))$ for all $a \in HB_\Phi$. Observe that $I \subseteq \boldsymbol{HB}_\Phi$ for all interpretations I. We say that I is a *model* of a ground fuzzy rule r of the form (7), denoted $I \models r$, iff

$$\begin{aligned} I(a_1) \oplus_1 \cdots \oplus_l I(a_l) \geqslant I(b_1) \otimes_1 \cdots \otimes_{k-1} I(b_k) \otimes_k \\ \ominus_{k+1} I(b_{k+1}) \otimes_{k+1} \cdots \otimes_{m-1} \ominus_m I(b_m) \otimes_0 v\,. \end{aligned} \tag{8}$$

Here, we implicitly assume that the disjunction strategies $\oplus_1, \ldots, \oplus_l$ and the conjunction strategies $\otimes_1, \ldots, \otimes_{m-1}, \otimes_0$ are evaluated from left to right. Notice also that the above definition implicitly assumes an implication strategy $\rhd$ that is defined by $a \rhd b = \sup\{c \in [0,1] \mid a \otimes_0 c \leqslant b\}$ for all $a, b \in [0,1]$ (and thus for $n, m \in [0,1]$ and $a = n$, it holds that $a \rhd b \geqslant m$ iff $b \geqslant n \otimes_0 m$, if we assume that the conjunction strategy $\otimes_0$ is continuous). Observe that such a relationship between the implication strategy $\rhd$ and the conjunction strategy $\otimes$ (including also the continuity of $\otimes$) holds in Łukasiewicz, Gödel, and Product Logic (see Table 3). We say that I is a *model* of a fuzzy program P, denoted $I \models P$, iff $I \models r$ for all $r \in \mathit{ground}(P)$. We say I is a *model* of a description logic knowledge base L, denoted $I \models L$, iff $L \cup \{a = I(a) \mid a \in HB_\Phi\}$ is satisfiable. An interpretation $I \subseteq HB_\Phi$ is a *model* of a fuzzy dl-program $KB = (L, P)$, denoted $I \models KB$, iff $I \models L$ and $I \models P$. We say KB is *satisfiable* iff it has a model.

The *Gelfond-Lifschitz transform* of a fuzzy dl-program $KB = (L, P)$ relative to an interpretation $I \subseteq \boldsymbol{HB}_\Phi$, denoted KB^I, is defined as the fuzzy dl-program (L, P^I), where P^I is the set of all fuzzy rules obtained from $\mathit{ground}(P)$ by replacing all default-negated atoms $\mathit{not}_{\ominus_j} b_j$ by the truth value $\ominus_j I(b_j)$. We are now ready to define the answer set semantics of fuzzy dl-programs as follows.

Definition 4.2 Let $KB = (L, P)$ be a fuzzy dl-program. An interpretation $I \subseteq \boldsymbol{HB}_\Phi$ is an *answer set* of KB iff I is a minimal model of KB^I. We say that KB is *consistent* (resp., *inconsistent*) iff KB has an (resp., no) answer set.

We finally define the notions of *cautious* (resp., *brave*) *reasoning* from fuzzy dl-programs under the answer set semantics as follows.

Definition 4.3 Let $KB = (L, P)$ be a fuzzy dl-program. Let $a \in HB_\Phi$ and $n \in [0,1]$. Then, $a \geqslant n$ is a *cautious* (resp., *brave*) *consequence* of a fuzzy dl-program KB under the answer set semantics iff $I(a) \geqslant n$ for every (resp., some) answer set I of KB.

Example 4.4 (Shopping Agent cont'd) Consider again the fuzzy dl-program $KB = (L, P)$ of Example 4.1. The following holds for the answer set M of KB:

$$M(q(\mathit{MazdaMX5Miata})) = 0.36 \qquad M(q(\mathit{MitsubishiEclipseSpyder})) = 0.32\,.$$

5 Semantic Properties

In this section, we summarize some semantic properties (especially those relevant for the Semantic Web) of fuzzy dl-programs under the above answer set semantics.

5.1 Minimal Models

The following theorem shows that, like for ordinary disjunctive programs, every answer set of a fuzzy dl-program KB is also a minimal model of KB, and the answer sets of a positive fuzzy dl-program KB are the minimal models of KB.

Theorem 5.1 *Let $KB = (L, P)$ be a fuzzy dl-program. Then, (a) every answer set of KB is a minimal model of KB, and (b) if KB is positive, then the set of all answer sets of KB is given by the set of all minimal models of KB.*

5.2 Faithfulness

An important property of integrations of rules and ontologies is that they are a faithful [22, 23] extension of both rules and ontologies.

The following theorem shows that the answer set semantics of fuzzy dl-programs faithfully extends its counterpart for fuzzy programs. That is, the answer set semantics of a fuzzy dl-program $KB = (L, P)$ with empty fuzzy description logic knowledge base L coincides with the answer set semantics of its fuzzy program P.

Theorem 5.2 *Let $KB = (L, P)$ be a fuzzy dl-program such that $L = \emptyset$. Then, the set of all answer sets of KB coincides with the set of all answer sets of the fuzzy program P.*

The next theorem shows that the answer set semantics of fuzzy dl-programs also faithfully extends the first-order semantics of fuzzy description logic knowledge bases. That is, for $a \in HB_\Phi$ and $n \in [0, 1]$, it holds that $a \geqslant n$ is true in all answer sets of a positive fuzzy dl-program $KB = (L, P)$ iff $a \geqslant n$ is true in all fuzzy first-order models of $L \cup ground(P)$. The theorem holds also when a is a ground formula constructed from HB_Φ using $\wedge$ and $\vee$, along with conjunction and disjunction strategies $\otimes$ resp. $\oplus$.

Theorem 5.3 *Let $KB = (L, P)$ be a positive fuzzy dl-program, and let $a \in HB_\Phi$ and $n \in [0, 1]$. Then, $a \geqslant n$ is true in all answer sets of KB iff $a \geqslant n$ is true in all fuzzy first-order models of $L \cup ground(P)$.*

As an immediate corollary, we obtain that $a \geqslant n$ is true in all answer sets of a fuzzy dl-program $KB = (L, \emptyset)$ iff $a \geqslant n$ is true in all fuzzy first-order models of L.

Corollary 5.4 *Let $KB = (L, P)$ be a fuzzy dl-program with $P = \emptyset$, and let $a \in HB_\Phi$ and $n \in [0, 1]$. Then, $a \geqslant n$ is true in all answer sets of KB iff $a \geqslant n$ is true in all fuzzy first-order models of L.*

5.3 Unique Name Assumption

Another aspect that may not be very desirable in the Semantic Web [14] is the *unique name assumption* (which says that any two distinct constant symbols in Φ_c represent two distinct domain objects). It turns out that we actually do not have to make this assumption, since the fuzzy description logic knowledge base of a fuzzy dl-program may very well contain or imply equalities between individuals.

This result is included in the following theorem, which shows an alternative characterization of the satisfaction of L in $I \subseteq \boldsymbol{HB}_\Phi$: Rather than being enlarged by a set of axioms of exponential size, L is enlarged by a set of axioms of polynomial size. This characterization essentially shows that the satisfaction of L in I corresponds to checking that (i) I restricted to DL_Φ satisfies L, and (ii) I restricted to $HB_\Phi - DL_\Phi$

does not violate any equality axioms that follow from L. In the theorem, an equivalence relation $\sim$ on Φ_c is *admissible* with an interpretation $I \subseteq \boldsymbol{HB}_\Phi$ iff $I(p(c_1, \ldots, c_n)) = I(p(c'_1, \ldots, c'_n))$ for all n-ary predicate symbols p, where $n > 0$, and constant symbols $c_1, \ldots, c_n, c'_1, \ldots, c'_n \in \Phi_c$ such that $c_i \sim c'_i$ for all $i \in \{1, \ldots, n\}$.

Theorem 5.5 *Let L be a fuzzy description logic knowledge base, and let $I \subseteq \boldsymbol{HB}_\Phi$. Then, $L \cup \{a = I(a) \mid a \in HB_\Phi\}$ is satisfiable iff $L \cup \{a = I(a) \mid a \in DL_\Phi\} \cup \{c \neq c' \mid c \not\sim c'\}$ is satisfiable for some equivalence relation $\sim$ on Φ_c admissible with I.*

6 Reduction of Fuzzy DL-Programs to DL-Programs

In this section, we present a polynomial reduction of fuzzy dl-programs to the tightly integrated dl-programs in [20]. Hence, reasoning in fuzzy dl-programs under the answer set semantics can be reduced to (a) reasoning in tightly integrated dl-programs under the answer set semantics and (b) reasoning in fuzzy description logics. Note that reasoning in fuzzy description logics may additionally be reduced to reasoning in crisp description logics along the lines presented in [28, 2] for fuzzy $\mathcal{ALCH}$ and fuzzy $\mathcal{SHOIN}$.

The reduction applies to all fuzzy dl-programs KB that (i) are closed under $TV_n = \{0, \frac{1}{n}, \ldots, \frac{n}{n}\}$ for some $n > 0$ and (ii) contain only combination strategies from Zadeh Logic. Here, KB is *closed* under TV_n iff (a) every datatype predicate in KB is interpreted by a mapping to TV_n, (b) every fuzzy modifier m in KB is interpreted by a mapping $f_m\colon TV_n \rightarrow TV_n$, (c) every truth value in KB is from TV_n, and (d) every combination strategy in KB is closed under TV_n (which holds, e.g., for the combination strategies of Łukasiewicz, Gödel, and Zadeh Logic). Note that for fuzzy dl-programs KB that are closed under TV_n, the problems of deciding consistency, cautious consequences, and brave consequences are all decidable, since we only have to consider the finite number of interpretations $I \subseteq \boldsymbol{HB}_\Phi$ that map to TV_n.

We denote by Φ^n the alphabet that is obtained from the alphabet Φ by replacing every predicate symbol p by the new predicate symbols p^α with $\alpha \in TV_n^+ = TV_n \setminus \{0\}$. For atoms $a = p(t_1, \ldots, t_k)$ and $\alpha \in TV_n^+$, the atom a^α over Φ^n is defined by $a^\alpha = p^\alpha(t_1, \ldots, t_k)$. Every fuzzy interpretation $I \subseteq \boldsymbol{HB}_\Phi$ is associated with the binary interpretation $t(I) = \{a^\alpha \mid a \in HB_\Phi,\ \alpha \in TV_n^+,\ I(a) \geqslant \alpha\}$.

The *crisp transform* of a fuzzy dl-program $KB = (L, P)$ is the dl-program $t(KB) = (L, t(P))$, where $t(P)$ is the set (i) of all rules $p^\beta(x_1, \ldots, x_k) \leftarrow p^\alpha(x_1, \ldots, x_k)$ such that p is a k-ary predicate symbol from Φ, $x_1, \ldots, x_k$ are distinct variables, $\alpha \in TV_n^+ \setminus \{\frac{1}{n}\}$, and $\beta = \alpha - \frac{1}{n}$, and (ii) of all rules $a_1^\alpha \vee \cdots \vee a_l^\alpha \leftarrow b_1^\alpha \wedge \cdots \wedge b_k^\alpha \wedge \mathit{not}\ b_{k+1}^\gamma \wedge \cdots \wedge \mathit{not}\ b_m^\gamma$ such that a rule of the form (7) belongs to P, $\alpha \in TV_n^+$, $\alpha \leqslant v$, and $\gamma = 1 - \alpha + \frac{1}{n}$. Observe here that the generated dl-program $t(P)$ has a polynomial size in P and TV_n^+ (assuming a unary number encoding for the truth values). The following theorem shows that the answer sets of KB correspond to the answer sets of $t(KB)$ as in [20].

Theorem 6.1 *Let $KB = (L, P)$ be a fuzzy dl-program that (i) is closed under $TV_n = \{0, \frac{1}{n}, \ldots, \frac{n}{n}\}$ for some $n > 0$ and (ii) contains only combination strategies from Zadeh Logic. Then, $I \subseteq \boldsymbol{HB}_\Phi$ is an answer set of KB iff $t(I)$ is an answer set of $t(KB)$.*

Example 6.2 (Shopping Agent cont'd) The last fuzzy dl-rule of Example 4.1 is translated into the following dl-rules in the crisp transform (for $TV_{10} = \{0, 0.1, \ldots, 1\}$):

$$\begin{aligned}
\mathit{Small}^{0.1}(x) \vee \mathit{Old}^{0.1}(x) &\leftarrow \mathit{Car}^{0.1}(x) \wedge \mathit{hasInvoice}^{0.1}(x, y) \wedge \mathit{not}\ \mathit{GeqAbout15000}^{1.0}(y),\\
\mathit{Small}^{0.2}(x) \vee \mathit{Old}^{0.2}(x) &\leftarrow \mathit{Car}^{0.2}(x) \wedge \mathit{hasInvoice}^{0.2}(x, y) \wedge \mathit{not}\ \mathit{GeqAbout15000}^{0.9}(y),
\end{aligned}$$

$$
\begin{array}{rcl}
Small^{0.3}(x) \vee Old^{0.3}(x) & \leftarrow & Car^{0.3}(x) \wedge hasInvoice^{0.3}(x,y) \wedge not\ GeqAbout15000^{0.8}(y),\\
Small^{0.4}(x) \vee Old^{0.4}(x) & \leftarrow & Car^{0.4}(x) \wedge hasInvoice^{0.4}(x,y) \wedge not\ GeqAbout15000^{0.7}(y),\\
Small^{0.5}(x) \vee Old^{0.5}(x) & \leftarrow & Car^{0.5}(x) \wedge hasInvoice^{0.5}(x,y) \wedge not\ GeqAbout15000^{0.6}(y),\\
Small^{0.6}(x) \vee Old^{0.6}(x) & \leftarrow & Car^{0.6}(x) \wedge hasInvoice^{0.6}(x,y) \wedge not\ GeqAbout15000^{0.5}(y),\\
Small^{0.7}(x) \vee Old^{0.7}(x) & \leftarrow & Car^{0.7}(x) \wedge hasInvoice^{0.7}(x,y) \wedge not\ GeqAbout15000^{0.4}(y).
\end{array}
$$

7 Tractability Results

In this section, we present a special class of fuzzy dl-programs KB for which the problems of deciding consistency and of query processing have both a polynomial data complexity. These fuzzy dl-programs are defined relative to *fuzzy DL-Lite* [32], which is a fuzzy generalization of the description logic *DL-Lite* [4]. By [32] (resp., [4]), deciding whether a knowledge base in *DL-Lite* (resp., *fuzzy DL-Lite*) is satisfiable can be done in polynomial time, and conjunctive query processing from a knowledge base in *DL-Lite* (resp., *fuzzy DL-Lite*) has a polynomial data complexity.

We first recall *DL-Lite* and *fuzzy DL-Lite*. Let $\mathbf{A}$, $\mathbf{R}_A$, and $\mathbf{I}$ be pairwise disjoint sets of atomic concepts, abstract roles, and individuals, respectively. A *basic concept in fuzzy DL-Lite* is either an atomic concept from $\mathbf{A}$ or an exists restriction on roles $\exists R.\top$ (abbreviated as $\exists R$), where $R \in \mathbf{R}_A \cup \mathbf{R}_A^-$. A *literal in DL-Lite* is either a basic concept b or the negation of a basic concept $\neg b$. *Concepts in DL-Lite* are defined by induction as follows. Every basic concept in *DL-Lite* is a concept in *DL-Lite*. If b is a basic concept in *DL-Lite*, and ϕ_1 and ϕ_2 are concepts in *DL-Lite*, then $\neg b$ and $\phi_1 \sqcap \phi_2$ are also concepts in *DL-Lite*. An *axiom in DL-Lite* is either (1) a concept inclusion axiom $b \sqsubseteq \psi$, where b is a basic concept in *DL-Lite*, and ϕ is a concept in *DL-Lite*, or (2) a *functionality axiom* $(\mathsf{funct}\ R)$, where $R \in \mathbf{R}_A \cup \mathbf{R}_A^-$, or (3) a concept assertion axiom $b(a)$, where b is a basic concept in *DL-Lite* and $a \in \mathbf{I}$, or (4) a role assertion axiom $R(a,c)$, where $R \in \mathbf{R}_A$ and $a, c \in \mathbf{I}$. A *fuzzy concept* (resp., *role*) *assertion axiom* is of the form $b(a) \geqslant n$ (resp., $R(a,c) \geqslant n$), where $b(a)$ (resp., $R(a,c)$) is a concept (resp., role) assertion axiom in *DL-Lite*, and $n \in [0,1]$. A *fuzzy axiom in DL-Lite* is either a fuzzy concept assertion axiom or a fuzzy role assertion axiom. A *fuzzy knowledge base in DL-Lite* L is a finite set of concept inclusion, functionality, fuzzy concept assertion, and fuzzy role assertion axioms in *DL-Lite*.

For the conjunction strategies of Gödel and Zadeh Logic, every knowledge base in *fuzzy DL-Lite* L can be transformed into an equivalent one in *fuzzy DL-Lite* $trans(L)$ in which every concept inclusion axiom is of form $b \sqsubseteq \ell$, where b (resp., ℓ) is a basic concept (resp., literal) in *DL-Lite*. We then define $trans(KB) = (L, trans(P))$ by $trans(P) = P \cup \{b'(X) \leftarrow b(X) \mid b \sqsubseteq b' \in trans(L), b'$ is a basic concept$\} \cup \{\exists R(X) \leftarrow R(X,Y) \mid R \in \mathbf{R}_A \cap \Phi\} \cup \{\exists R^-(Y) \leftarrow R(X,Y) \mid R \in \mathbf{R}_A \cap \Phi\}$.

We are now ready to define fuzzy dl-programs in *DL-Lite* as follows. We say that a fuzzy dl-program $KB = (L, P)$ is defined in *DL-Lite* iff (i) L is in *fuzzy DL-Lite* and interpreted relative to the conjunction strategies of Gödel or Zadeh Logic, (ii) $trans(P)$ is normal and locally stratified, and (iii) KB is closed under $TV_n = \{0, \frac{1}{n}, \ldots, \frac{n}{n}\}$ for some $n > 0$, where we assume a unary encoding of the numbers in TV_n.

It can be shown that fuzzy dl-programs in *DL-Lite* have either no or a unique answer set, which can be computed by a finite sequence of fixpoint iterations, as usual. This implies immediately that for such programs, consistency checking and query processing have both a polynomial data complexity, which is formally expressed as follows.

Theorem 7.1 *Let $KB = (L, P)$ be a fuzzy dl-program in DL-Lite. Then, (a) deciding whether KB has an answer set, and (b) computing the truth value of a ground atom $a \in HB_\Phi$ in the answer set of KB have both a polynomial data complexity.*

8 Summary and Outlook

We have presented an approach to tightly integrated fuzzy dl-programs under the answer set semantics, which generalizes the tightly integrated disjunctive dl-programs in [20] by fuzzy vagueness in both the description logic and the logic program component. We have shown that the new formalism faithfully extends both fuzzy disjunctive programs and fuzzy description logics, and that under suitable assumptions, reasoning in the new formalism is decidable. Furthermore, we have presented a polynomial reduction for certain fuzzy dl-programs to tightly integrated disjunctive dl-programs. Finally, we have also provided a special case of fuzzy dl-programs for which deciding consistency and query processing have both a polynomial data complexity.

An interesting topic for future research is to analyze the computational complexity of the main reasoning problems in fuzzy dl-programs, and to implement the approach. Another interesting issue is to extend fuzzy dl-programs by classical negation.

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