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TIKHONOV REGULARIZATION FOR DUNKL MULTIPLIER OPERATORS

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Abstract

We study some class of Dunkl multiplier operators $T_{k,m}$; and we give for them an application of the theory of reproducing kernels to the Tikhonov regularization, which gives the best approximation of the operators $T_{k,m}$ on the Dunkl-type Paley-Wiener spaces H_h.

1. Introduction

In this paper, we consider $\mathbf{R}^d$ with the Euclidean inner product $\langle .,. \rangle$ and norm $|y| := \sqrt{\langle y, y \rangle}$. For $\alpha \in \mathbf{R}^d \backslash \{0\}$, let σ_α be the reflection in the hyperplane $H_\alpha \subset \mathbf{R}^d$ orthogonal to α:

$$\sigma_\alpha x := x - \frac{2\langle \alpha, x \rangle}{|\alpha|^2} \alpha.$$

A finite set $\Re \subset \mathbf{R}^d \backslash \{0\}$ is called a root system, if $\Re \cap \mathbf{R}.\alpha = \{-\alpha, \alpha\}$ and $\sigma_\alpha \Re = \Re$ for all $\alpha \in \Re$. We assume that it is normalized by $|\alpha|^2 = 2$ for all $\alpha \in \Re$. For a root system $\Re$, the reflections σ_α, $\alpha \in \Re$, generate a finite group G. The Coxeter group G is a subgroup of the orthogonal group $O(d)$. All reflections in G, correspond to suitable pairs of roots. For a given $\beta \in \mathbf{R}^d \backslash \bigcup_{\alpha \in \Re} H_\alpha$, we fix the positive subsystem $\Re_+ := \{\alpha \in \Re : \langle \alpha, \beta \rangle > 0\}$. Then for each $\alpha \in \Re$ either $\alpha \in \Re_+$ or $-\alpha \in \Re_+$.

Let $k : \Re \to \mathbf{C}$ be a multiplicity function on $\Re$ (a function which are constant on the orbits under the action of G). As an abbreviation, we introduce the index $\gamma_k := \sum_{\alpha \in \Re_+} k(\alpha)$.

Throughout this paper, we will assume that $k(\alpha) \geq 0$ for all $\alpha \in \Re$. Moreover, let w_k denote the weight function $w_k(x) := \prod_{\alpha \in \Re_+} |\langle \alpha, x \rangle|^{2k(\alpha)}$, for all $x \in \mathbf{R}^d$, which is G-invariant and homogeneous of degree $2\gamma_k$.

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Let c_k be the Mehta-type constant given by

$$c_k := \left(\int_{\mathbf{R}^d} e^{-|x|^2/2} w_k(x)\,\mathrm{d}x\right)^{-1}.$$

We denote by μ_k the measure on $\mathbf{R}^d$ given by $\mathrm{d}\mu_k(x) := c_k w_k(x)\,\mathrm{d}x$; and by $L^p(\mu_k)$, $1 \le p \le \infty$, the space of measurable functions f on $\mathbf{R}^d$, such that

$$\|f\|_{L^p(\mu_k)} := \left(\int_{\mathbf{R}^d} |f(x)|^p\,\mathrm{d}\mu_k(x)\right)^{1/p} < \infty, \quad 1 \le p < \infty,$$

$$\|f\|_{L^\infty(\mu_k)} := \operatorname*{ess\,sup}_{x \in \mathbf{R}^d} |f(x)| < \infty.$$

For $f \in L^1(\mu_k)$ the Dunkl transform is defined (see [3]) by

$$\mathscr{F}_k(f)(y) := \int_{\mathbf{R}^d} E_k(-ix, y) f(x)\,\mathrm{d}\mu_k(x), \quad y \in \mathbf{R}^d,$$

where $E_k(-ix, y)$ denotes the Dunkl kernel (for more details, see the next section).

Let m be a function in $L^\infty(\mu_k)$. The Dunkl multiplier operators $T_{k,m}$, are defined for $f \in L^2(\mu_k)$ by

$$T_{k,m} f(x) := \mathscr{F}_k^{-1}(m\mathscr{F}_k(f))(x), \quad x \in \mathbf{R}^d.$$

These operators are studied in [15, 16] where the author established some applications (Calderón's reproducing formulas, best approximation formulas, extremal functions....).

Building on the ideas of Matsuura et al. [6], Saitoh [11, 13] and Yamada et al. [19], and using the theory of reproducing kernels [1, 10], we give best approximation of the operator $T_{k,m}$ on the Dunkl-type Paley-Wiener space H_h. More precisely, for all $\eta > 0$, $g \in L^2(\mu_k)$, the infimum

$$\inf_{f \in H_h} \{\eta \|f\|_{H_h}^2 + \|g - T_{k,m} f\|_{L^2(\mu_k)}^2\},$$

is attained at one function $F_{\eta,g}^*$, called the extremal function, and given by

$$F_{\eta,g}^*(y) = \int_{\mathbf{R}^d} E_k(iy, z) \frac{\chi_h(z)\overline{m(z)}\mathscr{F}_k(g)(z)}{\eta + |m(z)|^2}\,\mathrm{d}\mu_k(z).$$

Next we show for $F_{\eta,g}^*$ the following properties.

(i) $\|F_{\eta,g}^*\|_{H_h} \le \dfrac{1}{2\sqrt{\eta}} \|g\|_{L^2(\mu_k)}$.

(ii) $\lim_{\eta \to 0^+} \|T_{k,m} F_{\eta,g}^* - g\|_{L^2(\mu_k)} = 0$.

(iii) $\lim_{\eta \to 0^+} \|F_{\eta, T_{k,m} f}^* - f\|_{H_h} = 0$.

In the Dunkl setting, the extremal functions are studied in several directions [14, 15, 16, 17].

This paper is organized as follows. In Section 2 we define and study the Dunkl multiplier operators $T_{k,m}$ on the Dunkl-type Paley-Wiener spaces H_h. The last section of this paper is devoted to give an application of the theory of reproducing kernels to the Tikhonov regularization, which gives the best approximation of the operators $T_{k,m}$ on H_h.

2. The Dunkl-type Paley-Wiener spaces

The Dunkl operators $\mathcal{D}_j$; $j = 1, \dots, d$, on $\mathbf{R}^d$ associated with the finite reflection group G and multiplicity function k are given, for a function f of class C^1 on $\mathbf{R}^d$, by

$$\mathcal{D}_j f(x) := \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha \in \mathfrak{R}_+} k(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x\rangle}.$$

For $y \in \mathbf{R}^d$, the initial problem $\mathcal{D}_j u(., y)(x) = y_j u(x, y)$, $j = 1, \dots, d$, with $u(0, y) = 1$ admits a unique analytic solution on $\mathbf{R}^d$, which will be denoted by $E_k(x, y)$ and called Dunkl kernel [2, 4]. This kernel has a unique analytic extension to $\mathbf{C}^d \times \mathbf{C}^d$ (see [8]). In our case (see [2, 3]),

$$|E_k(ix, y)| \le 1, \quad x, y \in \mathbf{R}^d. \tag{2.1}$$

The Dunkl kernel gives rise to an integral transform, which is called Dunkl transform on $\mathbf{R}^d$, and was introduced by Dunkl in [3], where already many basic properties were established. Dunkl's results were completed and extended later by De Jeu [4]. The Dunkl transform of a function f in $L^1(\mu_k)$, is defined by

$$\mathscr{F}_k(f)(y) := \int_{\mathbf{R}^d} E_k(-ix, y) f(x)\, \mathrm{d}\mu_k(x), \quad y \in \mathbf{R}^d.$$

We notice that $\mathscr{F}_0$ agrees with the Fourier transform $\mathscr{F}$ that is given by

$$\mathscr{F}(f)(y) := (2\pi)^{-d/2} \int_{\mathbf{R}^d} e^{-i\langle x, y\rangle} f(x)\, \mathrm{d}x, \quad x \in \mathbf{R}^d.$$

Some of the properties of Dunkl transform $\mathscr{F}_k$ are collected bellow (see [3, 4]).

THEOREM 2.1. (i) $L^1 - L^\infty$-boundedness. *For all* $f \in L^1(\mu_k)$, $\mathscr{F}_k(f) \in L^\infty(\mu_k)$ *and*

$$\|\mathscr{F}_k(f)\|_{L^\infty(\mu_k)} \le \|f\|_{L^1(\mu_k)}.$$

(ii) Inversion theorem. *Let* $f \in L^1(\mu_k)$, *such that* $\mathscr{F}_k(f) \in L^1(\mu_k)$. *Then*

$$f(x) = \mathscr{F}_k(\mathscr{F}_k(f))(-x), \quad \text{a.e. } x \in \mathbf{R}^d.$$

(iii) Plancherel theorem. *The Dunkl transform $\mathscr{F}_k$ extends uniquely to an isometric isomorphism of $L^2(\mu_k)$ onto itself. In particular,*

$$\|\mathscr{F}_k(f)\|_{L^2(\mu_k)} = \|f\|_{L^2(\mu_k)}.$$

Let $h > 0$ and χ_h the function defined by

$$\chi_h(z) := \prod_{i=1}^{d} \chi_{(-1/h,1/h)}(z_i), \quad z = (z_1, \dots, z_d) \in \mathbf{R}^d,$$

where $\chi_{(-1/h,1/h)}$ is the characteristic function of the interval $(-1/h, 1/h)$.

We define the Paley-Wiener space H_h, as

$$H_h := \mathscr{F}_k^{-1}(\chi_h L^2(\mu_k)).$$

The space H_h satisfies

$$H_h \subset L^2(\mu_k), \quad \mathscr{F}_k(H_h) \subset L^1 \cap L^2(\mu_k). \tag{2.2}$$

We see that any element $f \in H_h$ is represented uniquely by a function $F \in L^2(\mu_k)$ in the form

$$f = \mathscr{F}_k^{-1}(\chi_h F).$$

The space H_h provided with the norm

$$\|f\|_{H_h} = \|F\|_{L^2(\mu_k)}.$$

Let m be a function in $L^\infty(\mu_k)$. The Dunkl multiplier operators $T_{k,m}$, are defined for $f \in L^2(\mu_k)$ by

$$T_{k,m}f := \mathscr{F}_k^{-1}(m\mathscr{F}_k(f)). \tag{2.3}$$

THEOREM 2.2. *Let $m \in L^\infty(\mu_k)$. The operators $T_{k,m}$ are bounded linear operators from H_h into $L^2(\mu_k)$, and*

$$\|T_{k,m}f\|_{L^2(\mu_k)} \le \|m\|_{L^\infty(\mu_k)} \|f\|_{H_h}.$$

Proof. Let $m \in L^\infty(\mu_k)$. From Theorem 2.1 (iii) and (2.3) we obtain

$$\begin{aligned}\|T_{k,m}f\|_{L^2(\mu_k)} &= \left(\int_{\mathbf{R}^d} |m(z)|^2 |\mathscr{F}_k(f)(z)|^2 \, \mathrm{d}\mu_k(z)\right)^{1/2} \\ &\le \|m\|_{L^\infty(\mu_k)} \left(\int_{\mathbf{R}^d} \chi_h(z) |F(z)|^2 \, \mathrm{d}\mu_k(z)\right)^{1/2} \\ &\le \|m\|_{L^\infty(\mu_k)} \|f\|_{H_h}.\end{aligned}$$

This gives the result. □

As application, we give the following example.

Example 2.3. Let m be the function defined for $t > 0$ by

$$m(z) := e^{-t|z|^2}, \quad z \in \mathbf{R}^d.$$

Then $T_{k,m}f = W_{k,t}(f)$, where $W_{k,t}$ is the Dunkl-type Weierstrass transform [9, 14].

Let $\eta > 0$. We denote by $\langle .\,, . \rangle_{\eta, H_h}$ the inner product defined on the space H_h by

$$\langle f, g \rangle_{\eta, H_h} := \eta \langle f, g \rangle_{H_h} + \langle T_{k,m} f, T_{k,m} g \rangle_{L^2(\mu_k)}, \tag{2.4}$$

and the norm $\|f\|_{\eta, H_h} := \sqrt{\langle f, f \rangle_{\eta, H_h}}$.

On H_h the two norms $\|.\|_{H_h}$ and $\|.\|_{\eta, H_h}$ are equivalent. This $(H_h, \langle .\,, . \rangle_{\eta, H_h})$ is a Hilbert space with reproducing kernel given by the following theorem.

Theorem 2.4. *Let $\eta > 0$ and $m \in L^\infty(\mu_k)$. The space $(H_h, \langle .\,, . \rangle_{\eta, H_h})$ has the reproducing kernel*

$$K_h(x, y) = \int_{\mathbf{R}^d} \frac{\chi_h(z) E_k(ix, z) E_k(-iy, z)}{\eta + |m(z)|^2} \, d\mu_k(z), \tag{2.5}$$

that is

(i) *For all $y \in \mathbf{R}^d$, the function $x \to K_h(x, y)$ belongs to H_h.*

(ii) *The reproducing property: for all $f \in H_h$ and $y \in \mathbf{R}^d$,*

$$\langle f, K_h(., y) \rangle_{\eta, H_h} = f(y).$$

Proof. (i) Let $y \in \mathbf{R}^d$. From (2.1), the function $z \to \dfrac{\chi_h(z) E_k(-iy, z)}{\eta + |m(z)|^2}$ belongs to $L^1 \cap L^2(\mu_k)$. Then, the function K_h is well defined and by Theorem 2.1 (ii), we have

$$K_h(x, y) = \mathscr{F}_k^{-1}\left(\frac{\chi_h(z) E_k(-iy, z)}{\eta + |m(z)|^2} \right)(x), \quad x \in \mathbf{R}^d. \tag{2.6}$$

Then by Theorem 2.1 (iii) and (2.1), we obtain

$$|\mathscr{F}_k(K_h(., y))(z)| \leq \frac{\chi_h(z)}{\eta} \quad \text{and} \quad \|K_h(., y)\|^2_{H_h} \leq \frac{c_k 2^{\gamma_k + d} d^{\gamma_k}}{\eta^2 h^{2\gamma_k + d}}.$$

This proves that for all $y \in \mathbf{R}^d$ the function $K_h(., y)$ belongs to H_h.

(ii) Let $f \in H_h$ and $y \in \mathbf{R}^d$. From (2.4) and (2.6), we have

$$\langle f, K_h(., y) \rangle_{\eta, H_h} = \int_{\mathbf{R}^d} E_k(iy, z) \mathscr{F}_k(f)(z) \, d\mu_k(z),$$

and from (2.2), we obtain the reproducing property:

$$\langle f, K_h(., y) \rangle_{\eta, H_h} = f(y).$$

This completes the proof of the theorem. □

3. Extremal functions for the operators $T_{k,m}$

In this section, by using the theory of extremal function and reproducing kernel of Hilbert space [10, 11, 12, 13] we study the extremal function associated to the Dunkl multiplier operators $T_{k,m}$. In the particular case when $k = 0$ this function is studied in [7, 18]. The main result of this section can be stated as follows.

THEOREM 3.1. *Let $m \in L^\infty(\mu_k)$. For any $g \in L^2(\mu_k)$ and for any $\eta > 0$, there exists a unique function $F^*_{\eta,g}$, where the infimum*

$$\inf_{f \in H_h} \{\eta \|f\|^2_{H_h} + \|g - T_{k,m} f\|^2_{L^2(\mu_k)}\} \tag{3.1}$$

is attained. Moreover, the extremal function $F^*_{\eta,g}$ is given by

$$F^*_{\eta,g}(y) = \int_{\mathbf{R}^d} g(x) Q_h(x, y)\, \mathrm{d}\mu_k(x),$$

where

$$Q_h(x, y) = \int_{\mathbf{R}^d} \frac{\chi_h(z)\overline{m(z)} E_k(-ix, z) E_k(iy, z)}{\eta + |m(z)|^2}\, \mathrm{d}\mu_k(z).$$

Proof. The existence and unicity of the extremal function $F^*_{\eta,g}$ satisfying (3.1) is obtained in [5, 6, 12]. Especially, $F^*_{\eta,g}$ is given by the reproducing kernel of H_h with $\|.\|_{\eta, H_h}$ norm as

$$F^*_{\eta,g}(y) = \langle g, T_{k,m}(K_h(., y))\rangle_{L^2(\mu_k)}, \tag{3.2}$$

where K_h is the kernel given by (2.5).

But by Theorem 2.1 (ii) and (2.6), we have

$$\begin{aligned} T_{k,m}(K_h(., y))(x) &= \int_{\mathbf{R}^d} m(z) \mathscr{F}_k(K_h(., y))(z) E_k(ix, z)\, \mathrm{d}\mu_k(z) \\ &= \int_{\mathbf{R}^d} \frac{\chi_h(z) m(z) E_k(ix, z) E_k(-iy, z)}{\eta + |m(z)|^2}\, \mathrm{d}\mu_k(z). \end{aligned}$$

This clearly yields the result. □

As application, we give the following example.

Example 3.2. Let $\eta > 0$ and $g \in L^2(\mu_k)$. If $m(z) := e^{-t|z|^2}$, $t > 0$, then

$$F^*_{\eta,g}(y) = \int_{\mathbf{R}^d} g(x) Q_h(x, y)\, \mathrm{d}\mu_k(x),$$

where

$$Q_h(x, y) = \int_{\mathbf{R}^d} \frac{\chi_h(z) E_k(-ix, z) E_k(iy, z)}{\eta e^{t|z|^2} + e^{-t|z|^2}}\, \mathrm{d}\mu_k(z).$$

COROLLARY 3.3. *Let $\eta > 0$ and $g \in L^2(\mu_k)$. The extremal function $F^*_{\eta,g}$ satisfies*:

(i) $|F^*_{\eta,g}(y)| \leq \left(\frac{c_k 2^{\gamma_k+d} d^{\gamma_k}}{4\eta h^{2\gamma_k+d}}\right)^{1/2} \|g\|_{L^2(\mu_k)}$.

(ii) $\|F^*_{\eta,g}\|_{L^2(\mu_k)} \leq \left(\frac{c_k 2^{\gamma_k+d} d^{\gamma_k}}{4\eta h^{2\gamma_k+d}}\right)^{1/2} \left(\int_{\mathbf{R}^d} |g(x)|^2 e^{|x|^2/2} \, \mathrm{d}\mu_k(x)\right)^{1/2}$.

Proof. (i) From (3.2) and Theorem 2.1 (iii), we have

$$\begin{aligned} |F^*_{\eta,g}(y)| &\leq \|g\|_{L^2(\mu_k)} \|T_{k,m}(K_h(.,y))\|_{L^2(\mu_k)} \\ &\leq \|g\|_{L^2(\mu_k)} \|m\mathscr{F}_k(K_h(.,y))\|_{L^2(\mu_k)}. \end{aligned}$$

Then, by (2.6) we deduce

$$|F^*_{\eta,g}(y)| \leq \|g\|_{L^2(\mu_k)} \left(\int_{\mathbf{R}^d} \frac{\chi_h(z)|m(z)|^2 \, \mathrm{d}\mu_k(z)}{[\eta + |m(z)|^2]^2}\right)^{1/2}. \tag{3.3}$$

Using the fact that

$$[\eta + |m(z)|^2]^2 \geq 4\eta |m(z)|^2, \tag{3.4}$$

we obtain the result.

(ii) We write

$$F^*_{\eta,g}(y) = \int_{\mathbf{R}^d} e^{-|x|^2/4} e^{|x|^2/4} g(x) Q_h(x,y) \, \mathrm{d}\mu_k(x).$$

Applying Hölder's inequality, we obtain

$$|F^*_{\eta,g}(y)|^2 \leq \int_{\mathbf{R}^d} |g(x)|^2 e^{|x|^2/2} |Q_h(x,y)|^2 \, \mathrm{d}\mu_k(x).$$

Thus and from Fubini-Tonnelli's theorem, we get

$$\|F^*_{\eta,g}\|^2_{L^2(\mu_k)} \leq \int_{\mathbf{R}^d} |g(x)|^2 e^{|x|^2/2} \|Q_h(x,.)\|^2_{L^2(\mu_k)} \, \mathrm{d}\mu_k(x). \tag{3.5}$$

The function $z \to \frac{\chi_h(z)\overline{m(z)}E_k(-ix,z)}{\eta + |m(z)|^2}$ belongs to $L^1 \cap L^2(\mu_k)$, then by Theorem 2.1 (ii),

$$Q_h(x,y) = \mathscr{F}_k^{-1}\left(\frac{\chi_h(z)\overline{m(z)}E_k(-ix,z)}{\eta + |m(z)|^2}\right)(y).$$

Thus, by Theorem 2.1 (iii) we deduce that

$$\|Q_h(x,.)\|^2_{L^2(\mu_k)} = \int_{\mathbf{R}^d} |\mathscr{F}_k(Q_h(x,.))(z)|^2 \, \mathrm{d}\mu_k(z) \leq \int_{\mathbf{R}^d} \frac{\chi_h(z)|m(z)|^2 \, \mathrm{d}\mu_k(z)}{[\eta + |m(z)|^2]^2}. \tag{3.6}$$

Then using the inequality (3.4), we obtain

$$\|Q_h(x,.)\|_{L^2(\mu_k)} \le \frac{1}{2\sqrt{\eta}}\left(\int_{\mathbf{R}^d}\chi_h(z)\,\mathrm{d}\mu_k(z)\right)^{1/2}.$$

From this inequality and (3.5) we deduce the result. □

COROLLARY 3.4. *Let* $\eta > 0$. *For every* $g \in L^2(\mu_k)$, *we have*

(i) $F^*_{\eta,g}(y) = \int_{\mathbf{R}^d} E_k(iy,z)\dfrac{\chi_h(z)\overline{m(z)}\mathscr{F}_k(g)(z)}{\eta+|m(z)|^2}\,\mathrm{d}\mu_k(z)$.

(ii) $\mathscr{F}_k(F^*_{\eta,g})(z) = \dfrac{\chi_h(z)\overline{m(z)}\mathscr{F}_k(g)(z)}{\eta+|m(z)|^2}$.

(iii) $\|F^*_{\eta,g}\|_{H_h} \le \dfrac{1}{2\sqrt{\eta}}\|g\|_{L^2(\mu_k)}$.

Proof. (i) follows from (3.2) by using Theorem 2.1 (iii) and (2.6).

(ii) The function $z \to \dfrac{\chi_h(z)\overline{m(z)}\mathscr{F}_k(g)(z)}{\eta+|m(z)|^2}$ belongs to $L^1 \cap L^2(\mu_k)$. Then by Theorem 2.1 (ii), we have

$$F^*_{\eta,g}(y) = \mathscr{F}_k^{-1}\left(\frac{\chi_h(z)\overline{m(z)}\mathscr{F}_k(g)(z)}{\eta+|m(z)|^2}\right)(y).$$

Thus, by Theorem 2.1 (iii), we obtain (ii).

(iii) By relation (ii) we have

$$\|F^*_{\eta,g}\|^2_{H_h} = \int_{\mathbf{R}^d}\frac{|\mathscr{F}_k(f^*_{\eta,g})(z)|^2}{\chi_h(z)}\,\mathrm{d}\mu_k(z) = \int_{\mathbf{R}^d}\frac{|m(z)|^2|\mathscr{F}_k(g)(z)|^2}{[\eta+|m(z)|^2]^2}\,\mathrm{d}\mu_k(z).$$

Using the inequality (3.4), we obtain

$$\|F^*_{\eta,g}\|^2_{H_h} \le \frac{1}{4\eta}\int_{\mathbf{R}^d}|\mathscr{F}_k(g)(z)|^2\,\mathrm{d}\mu_k(z) = \frac{1}{4\eta}\|g\|^2_{L^2(\mu_k)},$$

which ends the proof. □

THEOREM 3.5. *Let* $\eta > 0$. *For every* $g \in L^2(\mu_k)$, *we have*

(i) $T_{k,m}F^*_{\eta,g}(y) = \int_{\mathbf{R}^d} E_k(iy,z)\dfrac{\chi_h(z)|m(z)|^2\mathscr{F}_k(g)(z)}{\eta+|m(z)|^2}\,\mathrm{d}\mu_k(z)$.

(ii) $\mathscr{F}_k(T_{k,m}F^*_{\eta,g})(z) = \dfrac{\chi_h(z)|m(z)|^2\mathscr{F}_k(g)(z)}{\eta+|m(z)|^2}$.

(iii) $T_{k,m}F^*_{\eta,g}(y) = F^*_{\eta,T_{k,m}g}(y)$.

(iv) $\lim_{\eta\to 0^+}\|T_{k,m}F^*_{\eta,g} - g\|_{L^2(\mu_k)} = 0$.

Proof. From (2.3) and Corollary 3.4 (ii), we have

$$T_{k,m}F^*_{\eta,g}(y) = \mathscr{F}_k^{-1}\left(\frac{\chi_h(z)|m(z)|^2\mathscr{F}_k(g)(z)}{\eta+|m(z)|^2}\right)(y).$$

The function $z \to \dfrac{\chi_h(z)|m(z)|^2\mathscr{F}_k(g)(z)}{\eta + |m(z)|^2}$ belongs to $L^1 \cap L^2(\mu_k)$. Then by Theorem 2.1 (ii), we obtain (i), and by Theorem 2.1 (iii) we obtain (ii).

(iii) follows from (i) and Corollary 3.4 (i).

(iv) From (ii) we have

$$\mathscr{F}_k(T_{k,m}F^*_{\eta,g} - g)(z) = \frac{-\eta\mathscr{F}_k(g)(z)}{\eta + \chi_h(z)|m(z)|^2}.$$

Thus,

$$\|T_{k,m}F^*_{\eta,g} - g\|^2_{L^2(\mu_k)} = \int_{\mathbf{R}^d} \frac{\eta^2|\mathscr{F}_k(g)(z)|^2}{[\eta + \chi_h(z)|m(z)|^2]^2}\, \mathrm{d}\mu_k(z).$$

Using the dominated convergence theorem and the fact that

$$\frac{\eta^2|\mathscr{F}_k(g)(z)|^2}{[\eta + \chi_h(z)|m(z)|^2]^2} \le |\mathscr{F}_k(g)(z)|^2,$$

we deduce (iv). □

THEOREM 3.6. *Let $\eta > 0$. For every $f \in H_h$, we have*

(i) $\lim_{\eta\to 0^+}\|F^*_{\eta,T_{k,m}f} - f\|_{L^\infty(\mu_k)} = 0$.

(ii) $\lim_{\eta\to 0^+}\|F^*_{\eta,T_{k,m}f} - f\|_{H_h} = 0$.

Proof. (i) From (2.2), the function $\mathscr{F}_k(f) \in L^1 \cap L^2(\mu_k)$. Then by Corollary 3.4 (i) and Theorem 2.1 (ii),

$$F^*_{\eta,T_{k,m}f}(y) - f(y) = \int_{\mathbf{R}^d} \frac{-\eta\mathscr{F}_k(f)(z)}{\eta + \chi_h(z)|m(z)|^2} E_k(iy,z)\, \mathrm{d}\mu_k(z). \tag{3.7}$$

So

$$\|F^*_{\eta,T_{k,m}f} - f\|_{L^\infty(\mu_k)} \le \int_{\mathbf{R}^d} \frac{\eta|\mathscr{F}_k(f)(z)|}{\eta + \chi_h(z)|m(z)|^2}\, \mathrm{d}\mu_k(z).$$

Again, by dominated convergence theorem and the fact that

$$\frac{\eta|\mathscr{F}_k(f)(z)|}{\eta + \chi_h(z)|m(z)|^2} \le |\mathscr{F}_k(f)(z)|,$$

we deduce (i).

(ii) From (3.7) we have

$$\mathscr{F}_k(F^*_{\eta,T_{k,m}f} - f)(z) = \frac{-\eta\mathscr{F}_k(f)(z)}{\eta + \chi_h(z)|m(z)|^2}.$$

Consequently,

$$\|F^*_{\eta,T_{k,m}f} - f\|^2_{H_h} = \int_{\mathbf{R}^d} \frac{\eta^2|\mathscr{F}_k(f)(z)|^2}{\chi_h(z)[\eta + \chi_h(z)|m(z)|^2]^2}\, \mathrm{d}\mu_k(z).$$

Using the dominated convergence theorem and the fact that

$$\frac{\eta^2|\mathscr{F}_k(f)(z)|^2}{\chi_h(z)[\eta+\chi_h(z)|m(z)|^2]^2} \le \frac{|\mathscr{F}_k(f)(z)|^2}{\chi_h(z)},$$

we deduce (ii). □

Remark 3.7. $(\eta = 0)$. Let $m \in L^\infty(\mu_k)$ with $m \neq 0$; and let $g \in L^2(\mu_k)$.
(i) From (3.3) we have

$$|F^*_{0,g}(y)| \le \left(\int_{\mathbf{R}^d} \frac{\chi_h(z)}{|m(z)|^2}\, \mathrm{d}\mu_k(z)\right)^{1/2} \|g\|_{L^2(\mu_k)}.$$

If we take $g = T_{k,m}f$ when $f \in H_h$, then by Theorem 3.6 (i), we get

$$|f(y)| \le \left(\int_{\mathbf{R}^d} \frac{\chi_h(z)}{|m(z)|^2}\, \mathrm{d}\mu_k(z)\right)^{1/2} \|T_{k,m}f\|_{L^2(\mu_k)}.$$

(ii) From (3.6) we have

$$\|Q_h(x,.)\|^2_{L^2(\mu_k)} \le \int_{\mathbf{R}^d} \frac{\chi_h(z)}{|m(z)|^2}\, \mathrm{d}\mu_k(z).$$

Then by (3.5) we obtain

$$\|F^*_{0,g}\|_{L^2(\mu_k)} \le \left(\int_{\mathbf{R}^d} \frac{\chi_h(z)}{|m(z)|^2}\, \mathrm{d}\mu_k(z)\right)^{1/2} \left(\int_{\mathbf{R}^d} |g(x)|^2 e^{|x|^2/2}\, \mathrm{d}\mu_k(x)\right)^{1/2}.$$

By Theorem 3.5 (iii) and (iv), we deduce that

$$\|g\|_{L^2(\mu_k)} \le \left(\int_{\mathbf{R}^d} \frac{\chi_h(z)}{|m(z)|^2}\, \mathrm{d}\mu_k(z)\right)^{1/2} \left(\int_{\mathbf{R}^d} |T_{k,m}g(x)|^2 e^{|x|^2/2}\, \mathrm{d}\mu_k(x)\right)^{1/2}.$$

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