# TILES WITH NO SPECTRA 

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#### Abstract

We exhibit a subset of a finite Abelian group, which tiles the group by translation, and such that its tiling complements do not have a common spectrum (orthogonal basis for their $L^{2}$ space consisting of group characters). This disproves the Universal Spectrum Conjecture of Lagarias and Wang [7]. Further, we construct a set in some finite Abelian group, which tiles the group but has no spectrum. We extend this last example to the groups $\mathbb{Z}^{d}$ and $\mathbb{R}^{d}$ (for $d \geq 5$ ) thus disproving one direction of the Spectral Set Conjecture of Fuglede [1]. The other direction was recently disproved by Tao [12].


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## 1. Introduction

Let $G$ be a locally compact Abelian group and $\Omega \subseteq G$ be a bounded open set. We call $\Omega$ spectral if there is a set $\Lambda$ of continuous characters of $G$ which forms an orthogonal basis for $L^{2}(\Omega)$. Such a set $\Lambda$ is called a spectrum of $\Omega$. This paper concerns a conjecture of Fuglede [1] (the Spectral Set Conjecture), which states that a domain $\Omega$ in $\mathbb{R}^{d}$ is spectral if and only if it can tile $\mathbb{R}^{d}$ by translation. A set $\Omega$ tiles $\mathbb{R}^{d}$ by translation if there exists a set $T \subseteq \mathbb{R}^{d}$ (called a tiling complement of $\Omega$ ) of translates such that $\sum_{t \in T} \chi_{\Omega}(x-t)=1$, for almost all $x \in \mathbb{R}^{d}$. Here $\chi_{\Omega}$ denotes the indicator function of $\Omega$.

Tao [12] has recently proved that the direction "spectral $\Rightarrow$ tiling" does not hold (in dimension 5 and higher - Matolcsi [9] has reduced this dimension to 4). Here we prove that the direction "tiling $\Rightarrow$ spectral" is also false in dimension 5 and higher.

The Spectral Set Conjecture has attracted considerable attention in the last decade, revealing a wealth of connections to functional analysis, combinatorics, commutative algebra, number theory and Fourier analysis (the papers $[1,2,3,4,5,6,7,8,10,12]$ and references therein give a more or less complete account of results related to Fuglede's conjecture). Until Tao's example [12] there had been many results for special cases of domains, tiling complements or spectra, all of them supporting the conjecture. (Already in [1] Fuglede showed that the conjecture is true if either the tiling complement or the spectrum is assumed to be a lattice.) Despite the failure of the conjecture in general, it may still be true for some rather large natural class of domains, such as the convex domains [3].

The counterexample of Tao [12] to the "spectral $\Rightarrow$ tiling" direction was based, originally, on the existence of (real) Hadamard matrices whose size is not a power of 2. Such matrices immediately lead to counterexamples in appropriate finite groups, due to divisibility reasons. The main difficulty in disproving the "tiling $\Rightarrow$ spectral" direction is the lack of natural necessary conditions (which would play the role of divisibility) for a set in order to be spectral.

[^0]In order to produce a counterexample our strategy is as follows. The Spectral Set Conjecture makes sense in finite groups as well, and we first disprove the direction "tiling $\Rightarrow$ spectral" in an appropriate finite group in $\S 3$. This we do by first finding a counterexample to the Universal Spectrum Conjecture of Lagarias and Wang [7]. (This conjecture states, in a finite group, that if a set $T$ can tile the group with tiling complements $T_{1}, \ldots, T_{n}$ then these sets are all spectral and share a common spectrum. Note that this conjecture is stronger than the original Spectral Set Conjecture.) In §4, using the example found in the finite group setting, we produce a counterexample in the group $\mathbb{Z}^{d}$ and finally in $\mathbb{R}^{d}$, where the Spectral Set Conjecture was originally stated.

In $\S 2$ we give necessary background material and describe notation.

## 2. Preliminaries

Suppose $\Omega$ is a bounded open set in a locally compact Abelian group $G$. We will only be interested in finite groups, $\mathbb{Z}^{d}$ and $\mathbb{R}^{d}$ and the forhtcoming considerations apply to them.

We call $\Omega$ spectral if $L^{2}(\Omega)$ has an orthogonal basis

$$
\Lambda \subseteq \widehat{G}
$$

of characters ( $\widehat{G}$ denotes the dual group of $G[11]$ ). The set $\Lambda$ is then called a spectrum for $\Omega$, and $(\Omega, \Lambda)$ is called a spectral pair in $G$. In the groups we are dealing with the characters are functions of the type $x \rightarrow \exp (2 \pi i\langle\nu, x\rangle)$, where $\nu$ takes values in an appropriate subgroup of the torus $\mathbb{T}^{d}$ (if $G$ is discrete) or in Euclidean space.

The inner product and norm on $L^{2}(\Omega)$ are

$$
\langle f, g\rangle_{\Omega}=\int_{\Omega} f \bar{g}, \quad \text { and } \quad\|f\|_{\Omega}^{2}=\int_{\Omega}|f|^{2}
$$

If $\lambda, \nu \in \widehat{G}$ we have

$$
\langle\lambda, \nu\rangle_{\Omega}=\widehat{\chi_{\Omega}}(\nu-\lambda) .
$$

which gives

$$
\Lambda \text { is an orthogonal set } \Leftrightarrow \forall \lambda, \mu \in \Lambda, \lambda \neq \mu: \widehat{\chi_{\Omega}}(\lambda-\mu)=0
$$

For $\Lambda$ to be complete as well we must in addition have (Parseval)

$$
\begin{equation*}
\forall f \in L^{2}(\Omega): \quad\|f\|_{2}^{2}=\frac{1}{|\Omega|} \sum_{\lambda \in \Lambda}|\langle f, \lambda\rangle|^{2} \tag{1}
\end{equation*}
$$

For the groups we care about (finite groups, $\mathbb{Z}^{d}$ and $\mathbb{R}^{d}$ ) in order for $\Lambda$ to be complete it is sufficient to have (1) for any character $f \in \widehat{G}$, since then we have it in the closed linear span of these functions, which is all of $L^{2}(\Omega)$. An equivalent reformulation for $\Lambda$ to be a spectrum of $\Omega$ is therefore that

$$
\begin{equation*}
\sum_{\lambda \in \Lambda}|\widehat{\chi \Omega}|^{2}(x-\lambda)=|\Omega|^{2}, \tag{2}
\end{equation*}
$$

for almost every $x \in \widehat{G}$. For finite sets $\Omega$ (the group is finite or $\mathbb{Z}^{d}$ ) for a set $\Lambda \subseteq \widehat{G}$ to be a spectrum it is necessary and sufficient that $\Lambda$ satisfy the two conditions:
(a) $\Lambda-\Lambda \subseteq\{\widehat{\chi \Omega}=0\} \cup\{0\}$ (orthogonality), and
(b) $\# \Lambda=\# \Omega$ (maximal dimension).

For subsets $\Omega \subseteq \mathbb{R}^{d}$, when the spectra are infinite, we fall back on (2).
If $f \geq 0$ is in $L^{1}(G)$ and $T \subseteq G$ we say that $f$ tiles with $T$ at level $\ell$ if $\sum_{t \in T} f(x-t)=\ell$ for almost all $x \in G$. We denote this by " $f+T=\ell G$ " and we call $T$ a tiling complement of $f$. If $f=\chi_{\Omega}$ is the indicator function of some set then we just write $\Omega+T=\ell G$ instead of $\chi_{\Omega}+T=\ell G$, and, in this case, if $\ell$ is not specified it assumed to be 1 .

In the finite group case it is immediate to show that $f+T$ is a tiling of $G$ if and only if

$$
\begin{equation*}
\{\widehat{f}=0\} \cup\left\{\widehat{\chi_{T}}=0\right\} \cup\{0\}=\widehat{G} . \tag{3}
\end{equation*}
$$

There are analogs of this relationship that hold in the infinite case as well but we will not need these here (see [5]).

If $f$ is a continuous function we write $Z(f)$ for its zero set. For a set $A$ we write $Z_{A}$ for the zero set of the Fourier Transform of its indicator function $Z\left(\widehat{\chi_{A}}\right)$.

The starting point of our considerations is a generalization of the composition construction appearing in [9], Proposition 2.1.

Proposition 2.1. Let $G$ be a finite Abelian group, and $H \leq G$ a subgroup. Let $T_{1}, T_{2}, \ldots T_{k} \subset$ $H$ be subsets of $H$ such that they share a common tiling set in $H$; i.e. there exists a set $T^{\prime} \subset H$ such that $T_{j}+T^{\prime}=H$ is a tiling for all $1 \leq j \leq k$. Consider any tiling decomposition $S+S^{\prime}=G / H$ of the factor group $G / H$, with $\# S=k$, and take arbitrary representatives $s_{1}, s_{2}, \ldots s_{k}$ from the cosets of $H$ corresponding to the set $S$. Then the set $\Gamma:=\cup_{j=1}^{k}\left(s_{j}+T_{j}\right)$ is a tile in the group $G$.
Proof. The proof is simply the observation that for any system of representatives $\tilde{S}^{\prime}:=$ $\left\{s_{1}^{\prime}, s_{2}^{\prime}, \ldots\right\}$ of $S^{\prime}$ the set $T^{\prime}+\tilde{S}^{\prime}$ is a tiling set for $\Gamma$ in $G$.

Despite the proof being obvious, this construction seems to include a large class of tilings and it leads to some interesting examples.

When taking $T_{1}=T_{2}=\cdots=T_{k}=T$ we (essentially) get back the 'tiling part' of the statement of Proposition 2.1 in [9]. The drawback of that statement, in producing a counterexample to the Spectral Set Conjecture, is that the same construction applies to spectral sets as well (see the 'spectral part' of Proposition 2.1 in [9]). The essence of the generalization here comes from allowing differrent sets $T_{1}, \ldots T_{k}$ to be used.

Let us see the analogous construction for spectral sets.
Proposition 2.2. Let $G$ be a finite Abelian group, and $H \leq G$ a subgroup. Let $T_{1}, T_{2}, \ldots T_{k} \subset$ $H$ be subsets of $H$ such that they share a common spectrum in $\widehat{H}$; i.e. there exists a set $L \subset \widehat{H}$ such that $L$ is a spectrum of $T_{j}$ for all $1 \leq j \leq k$. Consider any spectral pair ( $Q, Q^{\prime}$ ) in the factor group $G / H$, with $\# Q=k$, and take arbitrary representatives $q_{1}, q_{2}, \ldots q_{k}$ from the cosets of $H$ corresponding to the set $Q$. Then the set $\Gamma:=\cup_{j=1}^{k}\left(q_{j}+T_{j}\right)$ is spectral in the group $G$.
Proof. We do not give a detailed proof of this statement, as we will not directly use it in the forthcoming arguments. Let us give an outline of the proof only. For any $l_{j} \in L$ there exists a $g_{j} \in \widehat{G}$ such that $\left.g_{j}\right|_{H}=l_{j}$. Take such characters $g_{1}, \ldots, g_{r}$ (where $r=\# T_{j}$ ). Also, the characters of $G$ which take constant values in cosets of $H$ can be identified with elements of $\widehat{G / H}$. Take such characters $v_{1}, v_{2}, \ldots, v_{k}$ corresponding to the elements of $Q^{\prime}$. Then the spectrum of $\Gamma$ is the set $\Lambda=g_{j} v_{l}(1 \leq j \leq r, 1 \leq l \leq k)$. The calculations proving orthogonality proceed along the same line as in [9], Proposition 2.1. Completeness then follows from the cardinality of $\Lambda$.

The main point of the two preceding constructions is that they are not entirely "compatible". That is, one can hope to find sets $T_{1}, \ldots T_{k} \subset H$ sharing a common tiling complement $T^{\prime}$ but not sharing a common spectrum $L$. This would be a counterexample to the Universal Spectrum Conjecture. Then the construction of Proposition 2.1 will lead to a set $\Gamma$ which tiles $G$, but there is nothing to guarantee that $\Gamma$ is spectral in $G$ (in fact, we will find a way to guarantee that $\Gamma$ is not spectral). This is exactly the route we will follow in $\S 3$.

## 3. Counterexamples in finite groups

Here we follow the path outlined in $\S 2$ in order to produce an example of a set $\Gamma$ in a finite group $G$, such that $\Gamma$ is a tile but is not spectral in $G$.

The first step is to find a counterexample to the Universal Spectrum Conjecture. We are looking for a finite group $G$ and a tile $T^{\prime}$ in $G$ such that the tiling complements $T_{1}, \ldots T_{k}$ of $T^{\prime}$ do not posess a common spectrum $L$.

For a given $T^{\prime} \subset G$, one sufficient condition for the existence of a universal spectrum $L$, as pointed out in [8], is to ensure that

$$
\begin{equation*}
\# L \cdot \# T^{\prime}=\# G \text { and } L-L \subset Z_{T^{\prime}}^{c} \tag{4}
\end{equation*}
$$

Indeed, any tiling complement $T_{j}$ of $T^{\prime}$ must satisfy $Z_{T_{j}} \supset Z_{T^{\prime}}^{c} \backslash\{0\}$, therefore condition (4) ensures that $L$ is a spectrum of $T_{j}$. (We do not know whether condition (4) is also necessary for the existence of a universal spectrum, as suggested in [8] in the remarks following Theorem 3.1.) Therefore, when trying to construct a set $T^{\prime}$ having no universal spectrum, one must exclude the existence of a set $L$ satisfying (4).

Notice, that if $L$ satisfies (4) then $L$ is not only a universal spectrum for all tiling complements of $T^{\prime}$, but also a universal tiling set of all spectra of $T^{\prime}$. Indeed, for any spectrum $Q$ of $T^{\prime}$ we have $Q-Q \subset Z_{T^{\prime}} \cup\{0\}$ therefore $(L-L) \cap(Q-Q)=\{0\}$ and $\# L \cdot \# Q=\# \widehat{G}$, which ensures that $L+Q=\widehat{G}$ is a tiling.

Having this observation in mind, one way to exclude the possibility of (4) is to choose a set $T^{\prime}$ which posseses a particular spectrum $Q$ which does not tile $\widehat{G}$ (but recall that $T^{\prime}$ itself must tile $G$ otherwise the notion of universal spectrum is meaningless). In other words, in some group $\widehat{G}$ take a spectral set $Q$ which does not tile $\widehat{G}$ (such examples already exist, cf. [12], [9]) and choose any spectrum of $Q$ as a candidate for $T^{\prime} \subset G$. However, the examples in [12] and [9] are such that $\# Q$ does not divide $\# \widehat{G}$, therefore any choice for $T^{\prime}$ is also doomed by divisibility reasons, because $T^{\prime}$ cannot tile $G$ either. We circumvent this problem by increasing the size of the group $G$.

The ideas above are summarized in the following theorem, which disproves the Universal Spectrum Conjecture. (In what follows, the notation $\mathbb{Z}_{n}$ refers to the cyclic group $\mathbb{Z} / n \mathbb{Z}$.)

Theorem 3.1. Consider $G=\mathbb{Z}_{6}^{5}$ and $E=\left\{\mathbf{0}, \boldsymbol{e}_{\mathbf{1}}, \boldsymbol{e}_{\mathbf{2}}, \ldots \boldsymbol{e}_{\mathbf{5}}\right\}$ where $\boldsymbol{e}_{\boldsymbol{j}}=(0, \ldots 1, \ldots, 0)^{\top}$. The set $E$ tiles $G$ but has no universal spectrum in $\widehat{G}$.

Proof. We identify the elements of $G$ and $\widehat{G}$ with column and row vectors, respectively.
The existence of a universal spectrum $L$ is equivalent to the conditions $\# L=6^{4}$ and $L-L \subset\left(\bigcap_{j} Z_{T_{j}}\right) \cup\{0\}$, where $T_{j}$ are all the tiling complements of $E$.

Consider the set $K \subset \widehat{G}$ (cf. [9, Theorem 3.1]) consisting of the rows of the following matrix

$$
K:=\left(\begin{array}{ccccc}
0 & 0 & 0 & 0 & 0  \tag{5}\\
0 & 2 & 2 & 4 & 4 \\
2 & 0 & 4 & 4 & 2 \\
2 & 4 & 0 & 2 & 4 \\
4 & 4 & 2 & 0 & 2 \\
4 & 2 & 4 & 2 & 0
\end{array}\right)
$$

(We remark, that $K$ is a spectrum of $E$. In fact, this follows from the fact that the matrix

$$
K^{\prime}:=\frac{1}{3}\left(\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0  \tag{6}\\
0 & 0 & 1 & 1 & 2 & 2 \\
0 & 1 & 0 & 2 & 2 & 1 \\
0 & 1 & 2 & 0 & 1 & 2 \\
0 & 2 & 2 & 1 & 0 & 1 \\
0 & 2 & 1 & 2 & 1 & 0
\end{array}\right)
$$

is $\log$-Hadamard (i.e. the matrix $U_{j k}=\exp \left(2 \pi i K_{j k}^{\prime}\right)$ is orthogonal.). We will not use the fact that $K$ is a spectrum, but it reflects the considerations preceding the theorem.)

Observe that $K$ is contained in the subgroup $H \leq \widehat{G}$ of row-vectors having even coordinates only. However, $\# H=3^{5}$ and $\# K=6$, therefore $K$ cannot tile $H$ and, consequently, it cannot tile $\widehat{G}$ either. (It is easy to see that if a set tiles a group then it tiles the subgroup it generates.) It is also easy to check that the set $K-K$ consists of $\mathbf{0}$ and only coordinate permutations of the vector $(0,2,2,4,4)$. (In fact $K-K$ contains all coordinate permutations of $(0,2,2,4,4)$, but we do not need this.)

Next we show that $E$ admits some tiling complements $T_{0}, \ldots T_{14}$, which have no common spectrum.

Take the vector $\boldsymbol{v}_{\mathbf{0}}=(1,2,3,4,5)^{\top}$ and define a group homomorphism $\phi: G \rightarrow \mathbb{Z}_{6}$ by

$$
\phi(\boldsymbol{x}):=\boldsymbol{v}_{\mathbf{0}}^{\top} \boldsymbol{x}(\bmod 6)
$$

Then $\phi$ is one-to-one on $E$, and the image of $E$ is the whole group $\mathbb{Z}_{6}$. Therefore $T_{0}=\operatorname{ker} \phi$ is a tiling complement for $E$. Notice that $Z_{T_{0}}^{c}$ contains all multiples of $\boldsymbol{v}_{\mathbf{0}}{ }^{\top}$, and, in particular, it contains $2 \boldsymbol{v}_{\mathbf{0}}{ }^{\top}=(2,4,0,2,4)$, and $4 \boldsymbol{v}_{\mathbf{0}}{ }^{\top}=(4,2,0,4,2)$. By appropriate permutations of the coordinates of $\boldsymbol{v}_{\mathbf{0}}$ we can define vectors $\boldsymbol{v}_{\mathbf{1}}, \ldots, \boldsymbol{v}_{\mathbf{1 4}}$ and corresponding tiling sets $T_{0}, \ldots, \mathcal{T}_{14}$ in such a way that $\left(\bigcap_{j=0}^{14} Z_{T_{j}}\right) \cap(K-K)=\{0\}$. Therefore, a set $L$ satisfying $\# L=6^{4}$ and $L-L \subset\left(\bigcap_{j=0}^{14} Z_{T_{j}}\right) \cup\{0\}$ cannot exist because in that case $L+K$ would be a tiling of $\widehat{G}$, and we already know that $K$ is not a tile.

Having found a counterexample to the Universal Spectrum Conjecture, we use the construction of $\S 2$ to exhibit the failure of the Spectral Set Conjecture in finite groups.
Theorem 3.2. Consider $G_{2}=\mathbb{Z}_{6}^{5} \times \mathbb{Z}_{15}$ and $\Gamma=\bigcup_{j=0}^{14}\left(f_{j}+\tilde{T}_{j}\right)$, where $f_{j}=(0,0,0,0,0, j)^{\top}$ and $\tilde{T}_{j}$ are the sets appearing in Theorem 3.1 extended by 0 as the last coordinate. Then $\Gamma$ is a tile in $G_{2}$ but it is not spectral.
Proof. In this proof the notation $\tilde{A}$ always refers to a set $A \subset G=\mathbb{Z}_{6}^{5}$ (or, $A \subset \widehat{G}=\mathbb{Z}_{6}^{5}$ as row vectors) extended by 0 as the last coordinate.

The fact that $\Gamma$ is a tile follows from Proposition 2.1 or can easily be seen directly: the tiling complement of $\Gamma$ is $\tilde{E}$.

To see that $\Gamma$ is not spectral, note first that the set $\tilde{K}$ is contained in the subgroup $\tilde{H}$ ( $K$ and $H$ are defined in the proof of Theorem 3.1), therefore it cannot tile $\widehat{G_{2}}$ because of divisibility reasons.

Any spectrum $Q$ of $\Gamma$ must satisfy $\# Q=\# \Gamma=6^{4} \cdot 15$ and $Q-Q \subset Z_{\Gamma} \cup\{0\}$. Consider the vector $\tilde{\boldsymbol{k}_{\mathbf{1}}}=(0,2,2,4,4,0) \in \tilde{K}-\tilde{K}$. We show that $\tilde{\boldsymbol{k}_{\mathbf{1}}} \notin Z_{\Gamma}$. Indeed,

$$
\widehat{\chi}_{\Gamma}\left(\tilde{\boldsymbol{k}_{1}}\right)=\sum_{j=0}^{14} \widehat{\chi}_{T_{j}}\left(\boldsymbol{k}_{\mathbf{1}}\right)>0
$$

because each term is nonnegative (each $T_{j}$ being a subgroup in $G$ ), and at least one term is non-zero by the construction of Theorem 3.1. The same argument shows that $\tilde{\boldsymbol{k}_{\boldsymbol{j}}} \notin Z_{\Gamma}$ for all $\boldsymbol{k}_{\boldsymbol{j}} \in K-K$.

Therefore, any spectrum $Q$ of $\Gamma$ must satisfy $(Q-Q) \cap(\tilde{K}-\tilde{K})=\{0\}$, which, since $\# Q \cdot \# \tilde{K}=\# \widehat{G_{2}}$, implies $Q+\tilde{K}=\widehat{G_{2}}$. This is a contradiction since $\tilde{K}$ is not a tile in $\widehat{G_{2}}$

## 4. Transition to $\mathbb{Z}^{d}$ and $\mathbb{R}^{d}$

We now describe a general transition scheme from the finite group setting to $\mathbb{Z}^{d}$ and $\mathbb{R}^{d}$. As a result we find a set in $\mathbb{R}^{6}$, which is a finite union of unit cubes (placed at points with integer coordinates), which tiles $\mathbb{R}^{6}$ by translations but is not spectral.

First we prove this in the group $\mathbb{Z}^{d}$.
Theorem 4.1. Let $\boldsymbol{n}=\left(n_{1}, \ldots, n_{d}\right) \in \mathbb{Z}^{d}$ and consider a set $A \subset G=\mathbb{Z}_{n_{1}} \times \cdots \times \mathbb{Z}_{n_{d}}$. Write

$$
\begin{equation*}
T=T(\boldsymbol{n}, k)=\left\{0, n_{1}, 2 n_{1}, \ldots,(k-1) n_{1}\right\} \times \cdots \times\left\{0, n_{d}, 2 n_{d}, \ldots,(k-1) n_{d}\right\} \tag{7}
\end{equation*}
$$

and define $A(k)=A+T$. Then, for large enough values of $k$, the set $A(k) \subset \mathbb{Z}^{d}$ is spectral in $\mathbb{Z}^{d}$ if and only if $A$ is spectral in $G$.

Proof. The 'if' part of the theorem is essentially contained in [9, Proposition 2.1] (but we will not need this direction here).

To see the 'only if' part, observe first that $\chi_{A(k)}=\chi_{A} * \chi_{T}$, hence, writing $Z(f)=\{f=0\}$, we obtain

$$
Z\left(\widehat{\chi_{A(k)}}\right)=Z\left(\widehat{\chi_{A}}\right) \cup Z\left(\widehat{\chi_{T}}\right) .
$$

Elementary calculation of $\widehat{\chi_{T}}$ (it is a cartesian product) shows that it is a union of "hyperplanes"

$$
\begin{equation*}
Z\left(\widehat{\chi_{T}}\right)=\left\{\boldsymbol{\xi} \in \mathbb{T}^{d}: \exists j \exists \nu \in \mathbb{Z}, k \text { does not divide } \nu, \text { such that } \xi_{j}=\frac{\nu}{k n_{j}}\right\} \tag{8}
\end{equation*}
$$

Define the group

$$
H=\left\{\boldsymbol{\xi} \in \mathbb{T}^{d}: \forall j \exists \nu \in \mathbb{Z} \text { such that } \xi_{j}=\frac{\nu}{n_{j}}\right\}
$$

which is the group of characters of the group $G$ and does not depend on $k$. Observe that $H+(Q-Q)$ does not intersect $Z\left(\widehat{\chi_{T}}\right)$, where

$$
Q=\left[0, \frac{1}{k n_{1}}\right) \times \cdots \times\left[0, \frac{1}{k n_{d}}\right)
$$

Assume now that $S \subseteq \mathbb{T}^{d}$ is a spectrum of $A(k)$, so that $\# S=\# A(k)=r k^{d}$, if $r=\# A$. Define, for $\boldsymbol{\nu} \in\{0, \ldots, k-1\}^{d}$, the sets

$$
S_{\boldsymbol{\nu}}=\left\{\boldsymbol{\xi} \in S: \boldsymbol{\xi} \in\left(\frac{\nu_{1}}{k n_{1}}, \ldots, \frac{\nu_{d}}{k n_{d}}\right)+Q+\left(\frac{m_{1}}{n_{1}}, \ldots, \frac{m_{d}}{n_{d}}\right), \text { for some } \boldsymbol{m} \in \mathbb{Z}^{d}\right\} .
$$

Since the number of the $S_{\boldsymbol{\nu}}$ is $k^{d}$ and they partition $S$, it follows that there exists some $\boldsymbol{\mu}$ for which $\# S \boldsymbol{\mu} \geq r$.

We also note that, if $k$ is sufficiently large, then any translate of $Q$ may contain at most one point of the spectrum. The reason is that $Q-Q$ contains no point of $Z\left(\widehat{\chi_{T}}\right)$ (for any $k$ ) and no point of $Z\left(\widehat{\chi_{A}}\right)$ for all large $k$ (as $\left.\widehat{\chi_{A}}(\mathbf{0})>0\right)$.

Observe next that if $\boldsymbol{x}, \boldsymbol{y} \in S \boldsymbol{\mu}$ then

$$
\begin{aligned}
\boldsymbol{x}-\boldsymbol{y} & \in H+(Q-Q) \\
& =H+\left(-\frac{1}{k n_{1}}, \frac{1}{k n_{1}}\right) \times \cdots \times\left(-\frac{1}{k n_{d}}, \frac{1}{k n_{d}}\right)
\end{aligned}
$$

and that this set does not intersect $Z\left(\widehat{\chi_{T}}\right)$ (from (8)). It follows that for all $\boldsymbol{x}, \boldsymbol{y} \in S_{\boldsymbol{\mu}}$ we have $\boldsymbol{x}-\boldsymbol{y} \in Z\left(\widehat{\chi_{A}}\right)$.

Let $k$ be sufficiently large so that for all points $\boldsymbol{h} \in H$ for which $\widehat{\chi_{A}}(\boldsymbol{h}) \neq 0$ the rectangle $\boldsymbol{h}+Q-Q$ does not intersect $Z\left(\widehat{\chi_{A}}\right)$. It follows that if $\boldsymbol{x}, \boldsymbol{y} \in S \boldsymbol{\mu}$ then $\boldsymbol{x}-\boldsymbol{y} \in \boldsymbol{h}+(Q-Q)$, where $\boldsymbol{h} \in Z\left(\widehat{\chi_{A}}\right)$.

For each $\boldsymbol{x} \in \mathbb{T}^{d}$ define $\lambda(\boldsymbol{x})$ to be the unique point $\boldsymbol{z}$ whose $j$-th coordinate is an integer multiple of $\frac{1}{k n_{j}}$ for which $\boldsymbol{x} \in \boldsymbol{z}+Q$. If $\boldsymbol{x}, \boldsymbol{y} \in S \boldsymbol{\mu}$ it follows that $\lambda(\boldsymbol{x})-\lambda(\boldsymbol{y}) \in H \cap Z\left(\widehat{\chi_{A}}\right)$. Define now $\Lambda=\{\lambda(\boldsymbol{x}): \boldsymbol{x} \in S \boldsymbol{\mu}\}$ (and shift $\Lambda$ to contain 0 , so that $\Lambda \subset H$ ). It is obvious that $\# \Lambda \geq r$ and $\Lambda-\Lambda \subseteq Z\left(\widehat{\chi_{A}}\right) \cup\{\mathbf{0}\}$, therefore $\Lambda$ is a spectrum of $A$.

Non-spectral tiles can be pulled from $\mathbb{Z}^{d}$ to $\mathbb{R}^{d}$ using the following.
Theorem 4.2. Suppose $A \subseteq \mathbb{Z}^{d}$ is a finite set and $Q=[0,1)^{d}$. Then $A$ is a spectral set in $\mathbb{Z}^{d}$ if and only if $A+Q$ is a spectral set in $\mathbb{R}^{d}$.
Proof. Write $E=A+Q$. Then $\widehat{\chi_{E}}=\widehat{\chi_{A} \widehat{\chi_{Q}}}$ and $Z\left(\widehat{\chi_{E}}\right)=Z\left(\widehat{\chi_{A}}\right) \cup Z\left(\widehat{\chi_{Q}}\right)$. By calculation we have

$$
Z\left(\widehat{\chi_{Q}}\right)=\left\{\boldsymbol{\xi} \in \mathbb{R}^{d}: \exists j \text { such that } \xi_{j} \in \mathbb{Z} \backslash\{\mathbf{0}\}\right\} .
$$

Now suppose $\Lambda \subset \mathbb{T}^{d}$ is a spectrum of $A$ as a subset of $\mathbb{Z}^{d}$. Viewing $\mathbb{T}^{d}$ as $Q$ we observe that the set $Z\left(\widehat{\chi_{A}}\right)$ is periodic with $\mathbb{Z}^{d}$ as a period lattice. Define now $S=\Lambda+\mathbb{Z}^{d}$. The differences of $S$ are either points which are on $Z\left(\widehat{\chi_{A}}\right)\left(\bmod \mathbb{Z}^{d}\right)$ or points with all integer coordinates. In any case these differences fall in $Z\left(\widehat{\chi_{E}}\right)$, hence $\sum_{s \in S}\left|\widehat{\chi_{E}}(x-s)\right|^{2} \leq(\# A)^{2}$. Furthermore, the density of $S$ is \#A which, along with the periodicity of $S$, implies that $\left|\widehat{\chi_{E}}\right|^{2}+S$ is a tiling of $\mathbb{R}^{d}$ at level $(\# A)^{2}$. That is, $S$ is a spectrum for $E$.

Conversely, assume $S$ is a spectrum for $E$ as a subset of $\mathbb{R}^{d}$. It follows that the density of $S$ is equal to $|E|=\# A$, hence there exists $\boldsymbol{k} \in \mathbb{Z}^{d}$ such that $\boldsymbol{k}+Q$ contains at least \#A points of $S$. Call the set of these points $S_{1}$, and observe that the differences of points of $S_{1}$ are contained in $Q-Q=(-1,1)^{d}$, and that $Q-Q$ does not intersect $Z\left(\widehat{\chi_{Q}}\right)$. It follows that the differences of the points of $S_{1}$ are all in $Z\left(\widehat{\chi_{A}}\right)$, and, since their number is $\# A$, they form a spectrum of $A$ as a subset of $\mathbb{Z}^{d}$.

In conclusion we have proved the following.

Theorem 4.3. In each of the groups $\mathbb{Z}^{d}$ and $\mathbb{R}^{d}$, $d \geq 5$, there exists a set which tiles the group by translation but is not spectral
Proof. It is easy to see that if $A$ is our non-spectral tile in the finite group $\mathbb{Z}_{6}^{5} \times \mathbb{Z}_{15}$ then the set $A(k) \subseteq \mathbb{Z}^{6}$ which appears in Theorem 4.1 is a tile as well, and by that Theorem it is not spectral. Using Theorem 4.2 we can construct a set with these properties in $\mathbb{R}^{6}$ by adding a unit cube at each point.

To get down to dimension 5, notice that the construction in Theorem 3.2 can be repeated verbatim in the group $G_{3}=\mathbb{Z}_{6}^{5} \times \mathbb{Z}_{17}$ instead of the group $\mathbb{Z}_{6}^{5} \times \mathbb{Z}_{15}$. (Just repeat the set $T_{15}$ two more times.) But now we view $G_{3}$ as the group $\mathbb{Z}_{6}^{4} \times \mathbb{Z}_{6.17}$ (as 6 and 17 are coprime). Theorems 4.1 and 4.2 now give examples in dimension 5 .

If $d \geq 6$ then the set of $\mathbb{Z}^{6}$ which is tiling but not spectral will still be such in $\mathbb{Z}^{d}$ when viewed as a subset of that in the obvious way. The preservation of tiling property is obvious, and one can easily show that the existence of any spectrum in $\mathbb{T}^{d}$ implies the existence of a spectrum in $\mathbb{T}^{6}$. We go to $\mathbb{R}^{d}$ again using Theorem 4.2.

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