Invent. math. 124, 341–365 (1996)

Inventiones mathematicae

Tiling the line with translates of one tile

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Received January 1995 / Accepted August 10, 1995

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Summary. A region T is a closed subset of the real line of positive finite Lebesgue measure which has a boundary of measure zero. Call a region T a tile if $\mathbb{R}$ can be tiled by measure-disjoint translates of T. For a bounded tile all tilings of $\mathbb{R}$ with its translates are periodic, and there are finitely many translation-equivalence classes of such tilings. The main result of the paper is that for any tiling of $\mathbb{R}$ by a bounded tile, any two tiles in the tiling differ by a rational multiple of the minimal period of the tiling. From it we a structure theorem characterizing such tiles in terms of complementing sets for finite cyclic groups.

1. Introduction

This paper studies tilings of the real line using translations of a single prototile T. We characterize compact sets T of positive measure that tile $\mathbb{R}$ by translation, and the types of tilings they give.

There exist such prototiles T having many connected components. The simplest case concerns regions consisting of a finite number of unit intervals, all of whose endpoints are integers. Such regions are called *clusters* by Stein and Szabó [30]. Tiling questions for clusters can be reformulated in terms of the set A of left endpoints of unit intervals in the cluster, and then concern which finite subsets A of $\mathbb{Z}$ give tilings of $\mathbb{Z}$, i.e. additive factorizations $A + B = \mathbb{Z}$. This problem has been extensively studied, see Tijdeman [32] for references.

Extra subtleties in this problem arise from the existence of prototiles T having infinitely many connected components. A large class of such prototiles arises from self-similar constructions, e.g. the self-affine tiles studied in Bandt [2], Gröchenig and Haas [11], Kenyon [16, 17], Lagarias and Wang [22, 21]. For

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example, given $\gamma \in \mathbb{R}$ there is a unique compact set $T := T_\gamma$ that satisfies the set-valued functional equation

$$3T = T \cup (T+1) \cup (T+\gamma), \tag{1.1}$$

and such a set T_γ tiles $\mathbb{R}$ by translation if and only if its Lebesgue measure $\mu(T_\gamma) > 0$. It is therefore natural to ask: which $\gamma \in \mathbb{R}$ have $\mu(T_\gamma) > 0$? This question was raised in Odlyzko [27] and was answered in Kenyon [19]: $\mu(T_\gamma) > 0$ if and only if γ is rational and $\gamma = p/q$ with $pq \equiv 2 \pmod 3$. The main result of this paper is a generalized rationality result valid for all bounded regions T that tile $\mathbb{R}$ by translation, which implies the result above as a special case.

The results of this paper exclusively concern bounded tiles, but to allow for generalization we use terminology permitting unbounded tiles. A *region* T is a closed subset of $\mathbb{R}$ which is the closure of its interior, has finite positive Lebesgue measure $\mu(T)$, and has a boundary ∂T of measure zero. Regions may have infinitely many connected components, and may be unbounded. We say that a region T *tiles* $\mathbb{R}$ *by translation* if there is a discrete set $\mathscr{T}$ for which

$$\mathbb{R} = \bigcup_{t \in \mathscr{T}} (T + t), \tag{1.2}$$

such that

$$\mu((T+t) \cap (T+t')) = 0 \quad \text{if} \quad t, t' \in \mathscr{T} \text{ are distinct}, \tag{1.3}$$

or, equivalently, such that the interiors of all translates are disjoint. The *translation set* $\mathscr{T}$ defines the tiling and we say two tilings $\mathscr{T}$ and $\tilde{\mathscr{T}}$ are *translation-equivalent* if

$$\tilde{\mathscr{T}} = \mathscr{T} + c \quad \text{for some} \quad c \in \mathbb{R}.$$

We call any tiling (1.2) a *monohedral translation tiling*. This should be distinguished from the notion of *monohedral tiling* in Grünbaum and Shepard [12], which is a tiling using a single prototile T which may be moved by Euclidean motions and reflections. A monohedral tiling of $\mathbb{R}$ is just a translation tiling using the set of two prototiles $\mathscr{S} = \{T, T^R\}$, in which T^R is the reflection of T about 0. Some questions about monohedral tilings of $\mathbb{R}$ are treated in Adler and Holroyd [1].

In studying arbitrary compact sets that tile $\mathbb{R}$ by translation, we can without loss of generality reduce to the case of regions. In an appendix we show that if T is a compact set of positive Lebesgue measure that tiles $\mathbb{R}$ with tiling set $\mathscr{T}$ then there is a region T' that differs from T on a set of measure zero such that T' tiles $\mathbb{R}$ with the same tiling set $\mathscr{T}$. A tiling $\mathscr{T}$ is *periodic* if

$$\mathscr{T} = \mathscr{T} + \lambda \quad \text{for some} \quad \lambda \in \mathbb{R} \setminus \{0\}. \tag{1.4}$$

and any λ satisfying (1.4) is called a *period* of the tiling $\mathscr{T}$. The set of all periods together with 0 forms a lattice $\Lambda(\mathscr{T})$, which is either $\{0\}$ or else is $\{n\lambda : n \in \mathbb{Z}\}$ for a *minimal period* $\lambda = \lambda(\mathscr{T}) > 0$.

We first show the easy result that one-dimensional translation tilings by a bounded region T are extremely rigid: all of them are periodic.

Theorem 1. *Suppose that a bounded region T of measure $\mu(T)$ tiles $\mathbb{R}$ by translation. Then:*

(i) Every tiling by translations of T is a periodic tiling.
(ii) There are only finitely many translation-equivalence classes of tilings by T.
(iii) Each such tiling has a minimal period which is an integral multiple of $\mu(T)$.

The analogue of Theorem 1 is false in higher dimensions, e.g. the unit square T in $\mathbb{R}^2$ gives infinitely many nonperiodic tilings of $\mathbb{R}^2$ which are translation-inequivalent. Theorem 1 also fails in general for regions T admitting a monohedral tiling of $\mathbb{R}$, as shown in Example 1. A final observation is that there are prototiles T that tile $\mathbb{R}$ by translation but have no lattice tilings, e.g. $T = [0,1] \cup [2,3]$.

Theorem 1 asserts periodicity of all tilings, but it does not give any information about the cosets of such a periodic tiling. The main result of the paper is the following rationality result for such cosets.

Theorem 2 (Rationality Theorem). *Suppose that a bounded region T tiles $\mathbb{R}$ by translation, using a periodic tiling set $\mathscr{T}$ given by*

$$\mathscr{T} = \bigcup_{j=1}^{J} (r_j + \lambda\mathbb{Z}). \tag{1.5}$$

Then all differences $r_j - r_k$ are rational multiples of the period λ.

The analogue of Theorem 2 is false in higher dimensions, e.g. there is a tiling of $\mathbb{R}^2$ with unit squares which is $2\mathbb{Z}^2$-periodic with tiling set $\mathscr{T} = \{(0,0),(1,0),(\gamma,1),(1+\gamma,1)\} + 2\mathbb{Z}^2$ where γ is irrational. The conclusion of Theorem 2 also fails to hold in the more general situation of (indecomposable) tilings of the line by compactly supported nonnegative functions, see [20].

The proof of Theorem 2 is Fourier-analytic, and depends on several facts apparently unrelated to tiling questions, including results on the zeros of bandlimited functions, and the use of either Szemeredi's theorem asserting that sets of integers having positive upper asymptotic density contain arbitrarily long arithmetic progressions or of the Skolem-Mahler-Lech theorem characterizing the set of integer zeros of exponential polynomials. The point of the proof is its validity for arbitrary tiles of $\mathbb{R}$; an easier proof exists for the special case of self-affine tiles (defined below), using the arguments of Kenyon [19].

Using Theorem 2 we obtain a structure theorem for bounded regions T that give tilings of $\mathbb{R}$. To state it we need some further definitions. Given two finite sets of integers $\mathscr{A}, \mathscr{B}$ and an integer $L > 1$, we say that the pair $(\mathscr{A}, \mathscr{B})$ is a *complementing pair* (mod L) if $|\mathscr{A}|\ |\mathscr{B}| = L$ and

$$\mathscr{A} + \mathscr{B} \equiv \{0, 1, 2, \ldots, L-1\} \pmod{L}.$$

We also say that $\mathscr{A}$ is a *complementing set* (*mod L*) if there is some $\mathscr{B}$ such that $(\mathscr{A}, \mathscr{B})$ is a complementing pair (mod L), and we call any such $\mathscr{B}$ a *complement* of $\mathscr{A}$.

In view of Theorem 1 we may rescale the tile T so that it tiles periodically with the period lattice $\mathbb{Z}$. We obtain the following structure theorem.

Theorem 3. *Suppose that the bounded region T tiles $\mathbb{R}$ with a periodic tiling whose period lattice contains $\mathbb{Z}$. Then it tiles $\mathbb{R}$ with a set $\mathscr{T}$ of translations of the form:*

$$\mathscr{T} = \bigcup_{j=1}^{J} \left(\frac{a_j}{L} + \mathbb{Z} \right), \tag{1.6}$$

where $0 = a_1 < a_2 < \cdots < a_J \leq L - 1$ are integers, and the set $\mathscr{A} = \{a_j : 1 \leq j \leq J\}$ is a complementing set (mod L). If $\mathscr{B}$ runs over the (countable) set of complements of $\mathscr{A}$ (mod L), then there is a decomposition

$$T = \bigcup_{\mathscr{B}} (T_{\mathscr{B}} + \mathscr{B}) \tag{1.7}$$

in which only finitely many $T_{\mathscr{B}} \neq \emptyset$, which is determined uniquely by the two requirements that:

(i). The sets $T_{\mathscr{B}}$ are all regions and have mutually disjoint interiors.
(ii). The union of all the sets $T_{\mathscr{B}}$ is the interval $[0, \frac{1}{L}]$.

Conversely, any T having such a decomposition tiles $\mathbb{R}$ with a periodic tile set $\mathscr{T}$ of the form (1.6) above.

In particular, a set $\mathscr{T}$ of the form (1.6) can be a tiling set for some prototile T if and only if $\mathscr{A}$ is a complementing set (mod L). If $\mathscr{A}$ is a complementing set then there exists such a prototile T with the property that LT is a cluster. Thus clusters already yield the most general tiling sets possible for translation tilings in one dimension.

Theorem 3 reduces the classification problem to that of determining all complementing pairs $(\mathscr{A}, \mathscr{B})$ (mod L) for all $L \geq 1$. Complementing pairs were first studied in connection with factorizations of abelian groups, see Hajós [13, 14], and de Bruijn [6]; see Tijdeman [32] for a survey and some new results. There remain several outstanding open questions concerning their structure. These include the question of Hajós [13] whether all complementing pairs $(\mathscr{A}, \mathscr{B})$ (mod L) are quasiperiodic. A complementing pair $(\mathscr{A}, \mathscr{B})$ is *quasiperiodic* if one of $\mathscr{A}$ or $\mathscr{B}$, say $\mathscr{B}$, can be partitioned as $\mathscr{B} = \bigcup_{i=1}^{m} \mathscr{B}_i$ such that $\mathscr{A} + \mathscr{B}_i = g_i + \mathscr{A} + \mathscr{B}_1$ where the elements $\{g_i\}$ form an additive subgroup (mod L).

Theorem 2 can also be used to prove a classification theorem for one-dimensional self-affine tiles, which was first established by Kenyon [16, 19]. Given an integer *base* b with $|b| \geq 2$ and a *digit set* $\mathscr{D}$ of $|b|$ real digits the *attractor* $T := T(b, \mathscr{D})$ is the solution of the set-valued functional equation

$$bT = \bigcup_{d \in \mathscr{D}} (T + d),$$

and is explicitly given by

$$T(b, \mathscr{D}) := \left\{ \sum_{i=1}^{\infty} b^{-i} d_i : \quad \text{all} \quad d_i \in \mathscr{D} \right\}. \tag{1.8}$$

We say that $T(b, \mathscr{D})$ is a *self-affine tile* if its Lebesgue measure

$$\mu(T(b, \mathscr{D})) > 0, \tag{1.9}$$

and that it is an *integral self-affine tile* if in addition $\mathscr{D} \subseteq \mathbb{Z}$. Any self-affine tile $T(b, \mathscr{D})$ tiles $\mathbb{R}$ by translation. In studying such tiles, one can always reduce to the case that $0 \in \mathscr{D}$ by translating the digit set, which has the effect of translating the tile.

Theorem 4. *If $T(b, \mathscr{D})$ is a self-affine tile in $\mathbb{R}$ with $0 \in \mathscr{D}$, then there exists $\lambda > 0$ such that $\lambda\mathscr{D} \subseteq \mathbb{Z}$. Consequently every self-affine tile in $\mathbb{R}$ is the affine image of an integral self-affine tile.*

The analogue of this theorem in higher dimensions is false, e.g. there is a two-dimensional self-affine tile $T(\mathsf{A}, \mathscr{D})$ which is not an affine image of any integral self-affine tile, see Example 2.1 of Lagarias and Wang [22].

The results of Kenyon [19] concerning which real digit sets $\mathscr{D}$ give one-dimensional self-affine tiles follow from Theorem 4, see Section 6.

Our motivation for characterizing one-dimensional tilings was to shed light on the one-dimensional case of a conjecture of Fuglede [7], which concerns the structure of spectral sets in $\mathbb{R}^n$. We say that a region T in $\mathbb{R}^n$ is a *spectral set* if there is a set $\mathscr{S}$ of exponentials, say $\mathscr{S} = \{e_\lambda(\mathbf{x}) : \lambda \in \mathscr{T}\}$, where

$$e_\lambda(\mathbf{x}) := \exp(2\pi i(\lambda_1 x_1 + \cdots + \lambda_n x_n)),$$

which when restricted to T forms an orthogonal basis[1] of $L^2(T)$.

Spectral set conjecture. A region T in $\mathbb{R}^n$ is a spectral set if and only if T tiles $\mathbb{R}^n$ by translation.

This conjecture is not settled in either direction, even in the one-dimensional case. For bounded regions T, Theorem 3 allows us to reduce the “if” direction of the one-dimensional case of the conjecture to problems concerning the structure of complementing sets. In particular we show elsewhere that a conjecture of Tijdeman concerning complementing pairs implies that all bounded tiles T are spectral sets.

The Spectral Set Conjecture applies also to unbounded regions, but the methods of this paper apparently do not extend to the unbounded case. For a survey of previous work done on the spectral set conjecture see Jorgensen and Pedersen [15].

Our results also apply to the following conjecture, which is implicitly raised in Grünbaum and Shepard [12, p. 23].

[1] That is, the set $\{e_\lambda(\mathbf{x})\chi_T(\mathbf{x}) : \lambda \in \mathscr{T}\}$ is an orthogonal basis of $L^2(T)$, where $\chi_T(\mathbf{x})$ is the characteristic function of T.

Periodic tiling conjecture. Any region T that tiles $\mathbb{R}^n$ by translation has a periodic tiling.

The one-dimensional case of this conjecture follows from Theorem 1. In Section 2 we also show that any region T that tiles $\mathbb{R}$ with a monohedral tiling also tiles $\mathbb{R}$ with a periodic monohedral tiling.

There are a number of partial results known concerning the Periodic Tiling Conjecture in dimensions $n \geq 2$. Girault-Beauquier and Nivat [10] proved that the Periodic Tiling Conjecture holds in dimension 2 whenever the region T is a topological disk with a sufficiently smooth boundary (piecewise-C^2). Kenyon [18] asserts that his results permit a proof of this result for all regions T in $\mathbb{R}^2$ that are topological disks, with no restrictions on their boundary. Venkov [34] proved that any convex polytope T that tiles $\mathbb{R}^n$ by translation has a lattice tiling. Thus the Periodic Tiling Conjecture holds for convex polytopes. Venkov's result was independently rediscovered by McMullen [26].

The Periodic Tiling Conjecture depends in an essential way on translations being the only allowed motions. There are known examples of (non-convex) polyhedra in $\mathbb{R}^3$ which tile $\mathbb{R}^3$ by Euclidean motions, but only aperiodically (Schmitt [29], unpublished). Recently J. H. Conway and L. Danzer constructed a three dimensional convex polyhedron (with eight faces) which tiles $\mathbb{R}^3$ by Euclidean motions, with all such tilings being aperiodic (L. Danzer [5]).

The contents of this paper are as follows. Theorems 1, 2, 3 and 4 are proved in Sections 2, 4, 5 and 6, respectively. In Section 3 we obtain an upper bound for the density of integer zeros of the Fourier transform of compactly supported nonnegative functions in $L^2(\mathbb{R})$, when the support of f has measure less than one. This result plays an important role in the proof of Theorem 2.

Acknowledgements. We thank Palle Jorgensen for introducing us to the Spectral Set Conjecture, and Henry Landau, Peter McMullen, Andrew Odlyzko, Bjorn Poonen and Boris Solomyak for helpful comments and references. We also thank the anonymous referee for supplying Example 1, which simplified our earlier example.

2. Periodicity and finiteness of tilings

The existence of periodic tilings is a general fact about one-dimensional tilings using an arbitrary finite set $\mathscr{S}$ of bounded prototiles, as we show below. However the finiteness of translation equivalence-classes of tilings for translation tilings using one tile is a special fact that fails to generalize even to monohedral tilings, see Example 1.

Theorem 5. *Let $\mathscr{S} = \{T_j : 1 \leq j \leq m\}$ be a finite set of bounded regions in $\mathbb{R}$. If there is a translation tiling of $\mathbb{R}$ using tiles drawn from $\mathscr{S}$, then there is a periodic tiling of $\mathbb{R}$ using tiles drawn from $\mathscr{S}$.*

Proof. Since all prototiles in $\mathscr{S}$ are bounded, we may suppose all $T_j \subseteq [-N, N]$. For any prototiles T in $\mathscr{S}$, the interior T° of T is a countable union of open intervals, and the boundary $\partial T := T - T^\circ$ is some (possibly complicated) nowhere dense set of measure zero.

A *patch* $\mathscr{P}$ is any finite set of translates of tiles in $\mathscr{S}$, say

$$\mathscr{P} = \{T^{(i)} + t_i : 1 \le i \le k, \quad \text{each} \quad T^{(i)} \in \mathscr{S}\},$$

that are nonoverlapping, i.e.

$$\mu\left((T^{(i)} + t_i) \cap (T^{(j)} + t_j)\right) = 0 \quad \text{if} \quad i \neq j.$$

Let $\Omega(\mathscr{P})$ denote the closed set covered by the patch $\mathscr{P}$, i.e.

$$\Omega(\mathscr{P}) := \bigcup_{i=1}^{k} (T^{(i)} + t_i).$$

A *tiling* of a finite interval J by $\mathscr{S}$ is a patch $\mathscr{P}$ that covers J and also has the property that every tile $T^{(i)} + t_i$ in $\mathscr{P}$ intersects J. A set of prototiles $\mathscr{S}$ has the *local finiteness property* if given any closed interval J, there are only finitely many ways to tile J by translates of prototiles in $\mathscr{S}$, up to translation-equivalence.

The main step in the proof is:

Claim 1. Any finite set $\mathscr{S}$ of bounded prototiles that tiles the line by translation has the local finiteness property.

To prove this claim, suppose that J is a closed interval and that $T + t$ is a tile which intersects J in a set of positive measure. It suffices to show that it extends in at most finitely many ways to a patch $\mathscr{P}$ that covers J. There is some choice of initial tile $T \in \mathscr{S}$ that extends in at least one way to cover J, because by hypothesis $\mathscr{S}$ tiles the line.

The interior of $T + t$ must include at least one open interval (x_1, x_2) that intersects J, and we suppose that this interval is maximal, so that $x_1, x_2 \in \partial T$. Thus for $\mathscr{P}_0 := \{T + t\}$ we have

$$[x_1, x_2] \subseteq \Omega(\mathscr{P}_0). \tag{2.1}$$

We assert that there are only finitely many choices to place a tile $T' + t'$ so that $x_2 \in T' + t'$ and

$$\mu\left((T + t) \cap (T' + t')\right) = 0.$$

This holds because either x_2 is the extreme left endpoint of $T' + t'$, or else it is a point of $T' + t'$ such that $T' + t'$ contains a gap of size $\ge x_2 - x_1$ to the left of this point. Since T' is bounded there can only be finitely many such gaps in T', indeed at most $\lceil N/(x_2 - x_1)\rceil$ gaps, proving the assertion.

Now suppose that x_2 lies in the interior of J. Then any patch $\mathscr{P}$ covering J that includes $T + t$ must include another tile $T' + t'$ that contains x_2. To see this, take a sequence of points $\{y_i\}$ in J lying outside $T' + t_2$, such that $\lim_{i\to\infty} y_i = x_2$. These are covered by $\mathscr{P}$, so some tile $T' + t'$ in $\mathscr{P}$ contains infinitely many of them, so this tile contains also x_2 since it is closed. By the above argument there are only finitely many choices for $T' + t'$. Let $\mathscr{P}'$ denote the finite set of tiles in the patch $\mathscr{P}$ that contain x_2. We assert that there is a value $\delta'' > 0$ such that

$$[x_1, x_2 + \delta''] \subseteq \Omega(\mathscr{P}'). \tag{2.2}$$

For if not, x_2 would still be a boundary point of $\Omega(\mathscr{P}')$, and the argument above shows that $\mathscr{P}$ then contains another tile $T'' + t''$ not in $\mathscr{P}$ which touches x_2, contradicting the definition of $\mathscr{P}'$. We also note that the value of δ'' can be chosen independent of the extension $\mathscr{P}'$, because we can minimize it over the finite set of possible extensions $\mathscr{P}'$.

Thus we have shown that there are only a finite number of ways to extend the tiling at least δ'' to the right. The argument can now be repeated, since (2.2) is the same form as (2.1), taking x_2' to be the right endpoint of the largest interval in $\Omega(\mathscr{P}')$ that contains $[x_1, x_2 + \delta'']$. Continuing this way, at each step we have finitely many choices for the extension, and each step extends the tiling to the right by at least δ''. Thus the whole process halts in at most in $\lceil |J|/\delta'' \rceil$ iterations.

When x_1 lies in the interior of J, the same argument applies on extending the tiling to the left. Finally there remain the two cases where $x_1 \in \partial J$ or $x_2 \in \partial J$. The argument above shows there are only finitely many choices for a tile $T' + t'$ that intersects J only at one or both of its endpoints. Thus Claim 1 is proved.

Now by Claim 1 there are only finitely many translation-inequivalent ways to tile the interval $[-N, N]$. Call this number M_T. Take a translation-tiling $\mathscr{T}$ of $\mathbb{R}$ from $\mathscr{S}$ and look at how it tiles the $M_T + 1$ intervals

$$J_k = [-N, N] + 7kN, \quad 0 \le k \le M_T.$$

It covers each of these intervals with a patch $\mathscr{R}_k$, and the regions the patches cover are disjoint because all tiles $T_j \subseteq [-N, N]$. By the pigeonhole principle, two such patches are translation-equivalent, say

$$\mathscr{R}_{k_1} = \mathscr{R}_{k_2} + \lambda, \ \lambda > 0. \tag{2.3}$$

Form the patch $\mathscr{P}$ of tiles containing $\mathscr{R}_{k_2}$ plus all tiles in $\mathscr{T}$ containing some point larger than $N + 7k_2N$ and smaller than $-N + 7k_1N$. Then the patch $\mathscr{P}$ tiles $\mathbb{R}$ with a periodic tiling with period λ. Indeed condition (2.3) assures that the ends of translates of $\mathscr{P}$ fit together properly. We omit the remaining details.

Applying Theorem 5 with $\mathscr{S} = \{T, T^R\}$ shows that any region T that tiles $\mathbb{R}$ with a monohedral tiling also has a periodic monohedral tiling.

Example 1. The cluster $T = [0, 2] \cup [5, 6]$ gives uncountably many monohedral tilings that are translation-inequivalent. (These include aperiodic tilings.)

Proof. The reflected tile T^R is $[-6, -5] \cup [-2, 0]$. The interval $[0, 9]$ can be monohedrally tiles in two translation-inequivalent ways, namely

$$(T + \{0, 3\}) \cup (T^R + \{8\})$$

and its “reflection”

$$(T + \{1\}) \cup (T^R + \{6, 9\}).$$

Now $\mathbb{R}$ can be tiled using $\mathscr{S} = \{T, T^R\}$ in uncountably many translation-inequivalent ways, by tiling successive intervals of length 9 arbitrarily using either of these two patches.

Proof of Theorem 1. Suppose that $T \subseteq [-N, N]$. By Claim 1 of the proof of Theorem 5, the prototile T has the local finiteness property. We supplement this with:

Claim 2. If a patch $\mathscr{P}$ covers the interval $[-N, N]$ and $\mathscr{P}$ can be extended to a tiling of the line, then this extended tiling is unique.

To prove this claim, consider the point $x^+ \geq N$ which is the infimum of all points $\geq N$ not covered by $\mathscr{P}$. Suppose that the patch $\mathscr{P}$ extends to some patch $\mathscr{P}'$ that covers $[-N, x^+ + \delta]$ for some $\delta > 0$. Then $\mathscr{P}'$ contains a new tile $T + t'$ that includes some points $x^+ + \epsilon$ for every sufficiently small $\epsilon > 0$. Now the patch $\mathscr{P}$ completely covers the closed interval $[-N, x^+]$; hence by measure-disjointness of $\Omega(\mathscr{P})$ and $T + t'$, and the fact that any two points in T are at distance at most $2N$ apart, it follows that

$$T + t' \subseteq [x^+, \infty).$$

Since $T + t'$ is closed and contains points arbitrarily close to x^+, it also contains x^+. But now x^+ is the left endpoint of $T + t'$, so the translation t' is uniquely specified. In particular, any extension of the patch $\mathscr{P}$ to a tiling of $\mathbb{R}$ must include the tile $T + t'$. Furthermore, the new patch $\mathscr{P}'' = \mathscr{P} \cup \{T + t'\}$ must cover some interval $[-2N, x^+ + \delta'']$ with $\delta'' > 0$. To see this, suppose not, so that x^+ is still a boundary point of the set $\Omega(\mathscr{P}'')$ covered by the patch $\mathscr{P}''$. If the patch $\mathscr{P}''$ can be extended to a tiling of $\mathbb{R}$, then a new tile $T + t''$ could be added to it that covers some points arbitrarily near x^+, and by the argument above we must have $t'' = t'$, which gives a contradiction because the tiles $T + t'$ and $T + t''$ overlap in a set of positive measure.

Now we have extended the tiling slightly to the right, by adding a uniquely determined tile $T + t'$. We can now repeat the argument, to conclude that, if the patch $\mathscr{P}$ extends to a tiling of $\mathbb{R}$, it extends in a unique manner to the interval $[x^+, \infty)$. By a similar argument, the tiling extends uniquely to the left, to cover $(-\infty, x']$. Thus Claim 2 is proved.

Parts (i) and (ii) of the theorem follow easily using Claims 1 and 2. By Claim 1 there are only finitely many translation-inequivalent ways to tile the interval $[-N, N]$. By Claim 2 each of these tilings of $[-N, N]$ extends to at most one tiling of $\mathbb{R}$. Thus there are only finitely many translation-inequivalent tilings of $\mathbb{R}$ by T, which is (ii).

The pigeonhole principle argument used in proving Theorem 5 shows that any tiling $\mathscr{T}$ contains some patch $\mathscr{R}$ such that:

(a) $[-N, N] + t \subseteq \mathscr{R}$ for some t.
(b) $\mathscr{R}$ and some disjoint translate $\mathscr{R} + \lambda$ both occur in $\mathscr{T}$.

Consider now the tiling $\mathscr{T} - \lambda$. It contains $\mathscr{R}$, and Claim 2 applies to show that $\mathscr{R}$ determines the tiling $\mathscr{T} - \lambda$ uniquely. Since $\mathscr{T}$ is also a tiling containing $\mathscr{R}$, we have $\mathscr{T} - \lambda = \mathscr{T}$. Thus $\mathscr{T}$ is periodic, which is (i).

Finally, we verify (iii). Let $\mathscr{T}$ be a periodic tiling set for T with period lattice $\lambda\mathbb{Z}$. Set $\mathscr{T} = \bigcup_{i=1}^{J}(r_i + \lambda\mathbb{Z})$, in which case $U := \bigcup_{i=1}^{J}(T + r_i)$ tiles $\mathbb{R}$ with tile set $\lambda\mathbb{Z}$. We count the number of elements t in $\mathscr{T}$ such that $T + t$ intersects the interval $[-M, M]$ in two ways. Counted directly, it is

$$\frac{2M}{\mu(T)} + O(1) \quad \text{as} \quad M \to \infty,$$

while counted in terms of tiles U that intersect $[-M, M]$, it is

$$\frac{2MJ}{\lambda} + O(1) \quad \text{as} \quad M \to \infty.$$

Thus $\lambda = J\mu(T)$, which is (iii).

Remark. Is Theorem 1 true for unbounded regions? The proof above used the boundedness assumption in proving both Claim 1 and Claim 2.

The following example shows that translation-inequivalent tilings do occur.

Example 2. The cluster $T = [0, 1] \cup [4, 5] \cup [8, 9]$ gives several translation-inequivalent tilings of $\mathbb{R}$.

Proof. Two tiling sets with period $\lambda = 12$ are $\mathscr{T}_1 = \{0, 1, 2, 3\} + 12\mathbb{Z}$ and $\mathscr{T}_2 = \{0, 1, 2, 7\} + 12\mathbb{Z}$, and there are others.

3. Density bound for integer Fourier zeros

Given a function $f(t) \in L^1(\mathbb{R})$, its Fourier transform

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(t)e^{2\pi i\lambda t}dt \tag{3.1}$$

is defined for all $\lambda \in \mathbb{R}$ and lies in $L^\infty(\mathbb{R})$. We use the *support* of $f \in L^1(\mathbb{R})$ in the sense of distributions, denoting it Supp (f), and note that it is a closed set, cf. Rudin [28, p. 149]. Without loss of generality we may redefine such an f on a set of measure zero so that it vanishes outside Supp(f). The *Fourier series zero set* of $f \in L^1(\mathbb{R})$ is

$$Z(f) := \{n : n \in \mathbb{Z} \quad \text{and} \quad \hat{f}(n) = 0\}. \tag{3.2}$$

Finally, the *upper asymptotic density* $\overline{d}(V)$ of a (discrete) set Λ of real numbers is

$$\overline{d}(\Lambda) := \limsup_{T\to\infty} \frac{1}{2T}\#\{\lambda : \lambda \in \Lambda \quad \text{and} \quad |\lambda| \le T\}, \tag{3.3}$$

We prove:

Theorem 6. *Let $f(t)$ be a compactly supported nonnegative function in $L^2(\mathbb{R})$, whose support has measure*

$$0 < \mu(\mathrm{Supp}(f)) < 1. \tag{3.4}$$

Then the Fourier series zero set $Z(f)$ of f has upper asymptotic density

$$\overline{d}(Z(f)) < 1. \tag{3.5}$$

Proof. Let $L_c^2(\mathbb{R})$ denote the linear space of compactly supported functions in $L^2(\mathbb{R})$. Note that $L_c^2(\mathbb{R}) \subseteq L^1(\mathbb{R})$. By the Paley-Wiener theorem the Fourier transforms of functions in $L_c^2(\mathbb{R})$ are exactly the entire functions of exponential type whose restrictions to the real axis are in $L^2(\mathbb{R})$.

We will apply two linear operators on $L_c^2(\mathbb{R})$ which change f but do not affect the Fourier series zero set. The simplest of these is translation

$$\mathsf{T}_y f(t) := f(t-y). \tag{3.6}$$

Clearly T_y is a linear operator on $L_c^2(\mathbb{R})$, with

$$\mathrm{Supp}(\mathsf{T}_y f) = \mathrm{Supp}(f) + y.$$

The Fourier series zero set V_f is invariant under T_y, i.e.

$$Z(\mathsf{T}_y f) = Z(f), \quad \text{all} \quad y \in \mathbb{R}, \tag{3.7}$$

since $\widehat{\mathsf{T}_y f}(\lambda) = e^{2\pi i y\lambda}\hat{f}(\lambda)$, all $\lambda \in \mathbb{R}$.

The second operation P, which is a projection onto functions supported on $\left[-\frac{1}{2}, \frac{1}{2}\right]$, takes $f \in L_c^2(\mathbb{R})$ to the compactly supported function

$$\mathsf{P}f(t) = \begin{cases} \displaystyle\sum_{m\in\mathbb{Z}} f(t+m) & -1/2 \le t < 1/2, \\ \\ 0 & \text{otherwise.} \end{cases} \tag{3.8}$$

To see that $\mathsf{P}f \in L_c^2(\mathbb{R})$, we need only verify that $\mathsf{P}f \in L^2(\mathbb{R})$. For this, note that if $f(t)$ has support in $[-M, M]$, then the sum defining $f(t)$ for $-1/2 \le t \le 1/2$ is finite, whence

$$\|\mathsf{P}f(t)\|_{L^2}^2 \le (2M+1)^2 \|f\|_{L^2}^2.$$

The operator P obviously does not change the values of the Fourier transform at $n \in \mathbb{Z}$, i.e.

$$\widehat{\mathsf{P}f}(n) = \hat{f}(n) \quad \text{for all} \quad n \in \mathbb{Z},$$

hence the Fourier series zero set $Z(f)$ is invariant, i.e.

$$Z(\mathsf{P}f) = Z(f). \tag{3.9}$$

Furthermore

$$\mu(\mathrm{Supp}(\mathsf{P}f)) \le \mu(\mathrm{Supp}(f)) < 1, \tag{3.10}$$

for if f is supported in $[-M, M]$, then

$$\mathrm{Supp}(\mathsf{P}f) \subseteq \bigcup_{m=-M}^{M} \left\{ \left(\mathrm{Supp}(f) \cap \left[m - 1/2,\ m + 1/2 \right] \right) - m \right\}, \tag{3.11}$$

from which (3.10) follows.

Our object is to apply the operators P and T_y repeatedly to produce a nonzero function h having support in an interval $[-\frac{1}{2} + \delta, \frac{1}{2} - \delta]$ for some $\delta > 0$. Since $\mathrm{Supp}(\mathsf{P}f)$ is a closed set of measure less than 1 in $[-\frac{1}{2}, \frac{1}{2}]$, its complement in $[-\frac{1}{2}, \frac{1}{2}]$ contains an open interval, call it $(x_0 - \delta, x_0 + \delta)$, with $-\frac{1}{2} + \delta \le x_0 \le \frac{1}{2} - \delta$. Now we apply the translation operator $\mathsf{T}_{1/2-x_0}$ to $\mathsf{P}f$ to get a function $g := \mathsf{T}_{1/2-x_0}\mathsf{P}f$ with

$$\begin{aligned} \mathrm{Supp}(\mathsf{T}_{1/2-x_0}\mathsf{P}f) &= \mathrm{Supp}(\mathsf{P}f) + \left(1/2 - x_0\right) \\ &\subseteq [-x_0, 1 - x_0] \subseteq \left[-1/2 + \delta, 3/2 - \delta\right]. \end{aligned}$$

By construction the support of g lies in $[-\frac{1}{2}, \frac{3}{2}]$ and omits intervals of width 2δ centered about $-\frac{1}{2}, \frac{1}{2}$ and $\frac{3}{2}$. Now apply the operator P again, to get the function

$$h := \mathsf{P}g = \mathsf{P}\mathsf{T}_{1/2-x_0}\mathsf{P}f,$$

which has

$$\mathrm{Supp}(h) = \mathrm{Supp}(\mathsf{P}\mathsf{T}_{1/2-x_0}\mathsf{P}f) \subseteq \left[-1/2 + \delta,\ 1/2 - \delta\right], \tag{3.12}$$

using (3.11) applied to g. The invariance of Fourier series zero sets gives

$$Z(h) = Z(g) = Z(f). \tag{3.13}$$

Certainly $h \in L_c^2(\mathbb{R})$, hence $\hat{h} \in L^2(\mathbb{R})$. We next show that $\hat{h}(\lambda) \not\equiv 0$. To see this, note that both operators T_y and P take nonnegative functions in $L_c^2(\mathbb{R})$ to nonnegative functions in $L_c^2(\mathbb{R})$, and

$$\int_{-\infty}^{\infty} \mathsf{T}_y f(t)dt = \int_{-\infty}^{\infty} \mathsf{P}_y f(t)dt = \int_{-\infty}^{\infty} f(t)dt > 0$$

implies that $\hat{h}(\lambda) \not\equiv 0$.

Since h is compactly supported, the Paley-Wiener theorem applied to h says that its Fourier transform $\hat{h}(\lambda)$ is the restriction to $\mathbb{R}$ of an entire function of exponential type ρ, and (3.12) implies that

$$\rho \le 2\pi \left(1/2 - \delta\right). \tag{3.14}$$

However it is known that entire functions $\phi(\lambda)$ of exponential type having restricted growth on the real axis cannot have too large a density of real zeros. Let $N_\phi(R)$ count the number of real zeros of such a function in the interval $[0, R]$. Then we have:

Proposition 1. *If $\phi(\lambda) \not\equiv 0$ is an entire function of exponential type ρ and if its restriction to the real axis is in $L^2(\mathbb{R})$, then*

$$\limsup_{R\to\infty} \frac{N_\phi(R)}{R} \le \frac{\rho}{\pi}. \tag{3.15}$$

Proof. This appears as Theorem 5.4.1 in Logan [25], with the following proof. The Paley-Wiener theorem states that

$$\phi(t) = \int_{-\rho}^{\rho} h(t)e^{it\lambda}dt$$

where $h(t) \in L^2([-\rho, \rho])$. Now $h(t) \in L^1(\mathbb{R})$ so $\phi(t) \in L^\infty(\mathbb{R})$, whence

$$\int_{-\infty}^{\infty} \frac{\log^+(\phi(\lambda))}{1+\lambda^2} d\lambda < \infty,$$

where $\log^+(|x|) = \max(0, \log|x|)$. The hypotheses of Theorem VIII of Levinson [24] are then satisfied, and its conclusion yields (3.15). (An alternate proof can be derived using Boas [4], Theorem 8.4.16.)

To complete the proof of Theorem 6, we note that the bound (3.15) also applies to zeros of $\phi(\lambda)$ on the negative real axis – just consider $\phi(-\lambda)$. Thus Proposition 1 implies that the upper asymptotic density of all real zeros is at most ρ/π. Now the upper asymptotic density V_h of integer zeros of $\hat{h}(\lambda)$ can be no larger than that of all real zeros of $\hat{h}(\lambda)$, and by Proposition 1 this is at most ρ/π. Since (3.13) gives $\rho/\pi \le 1 - 2\delta$, Theorem 6 follows.

Remarks. (1). Theorem 6 cannot be strengthened to give any quantitative upper bound between the measure of Supp(f) and the density $\overline{d}(Z(f))$. For any $\epsilon > 0$ there are examples where f is the characteristic function χ_T of a tile T, having $\mu(\mathrm{Supp}(f)) < \epsilon$ and nevertheless $\overline{d}(Z(f)) \ge 1 - \epsilon$, see Lemma 1 in Section 4.

(2). The hypothesis that f be nonnegative cannot be removed from Theorem 6. The function

$$f(t) := \begin{cases} 1 & \text{for } +1/2 \le x \le 1/2 + \delta, \\ \\ -1 & \text{for } -1/2 \le x \le -1/2 + \delta, \end{cases}$$

has $\mathsf{P}f(t) \equiv 0$, so that $Z(f) = \mathbb{Z}$, and $\overline{d}(Z(f)) = 1$.

(3). The requirement that $Z(f)$ be the set of *integer* zeros is also crucial to the statement of Theorem 6. If we study instead the set of *half-integer zeros*

$$\tilde{Z}(f) := \left\{ n \in \mathbb{Z} : \hat{f}\left(n + \tfrac{1}{2}\right) = 0 \right\},$$

then the conclusion of Theorem 6 is no longer valid. For any $\delta > 0$, take f to be the characteristic function χ_S for $S = \left[-\frac{1}{2} - \delta, -\frac{1}{2} + \delta\right] \cup \left[\frac{1}{2} - \delta, \frac{1}{2} + \delta\right]$, with $\mu(\mathrm{Supp}(f)) = 4\delta$. Then

$$\hat{f}(\lambda) = \frac{2}{\pi\lambda} \sin(2\pi\lambda\delta)\cos(\pi\lambda),$$

which vanishes on the entire set $\frac{1}{2} + \mathbb{Z}$, so that $\tilde{Z}(f) = \mathbb{Z}$.

4. Rationality of translates

We now prove the rationality of translates in tiling of $\mathbb{R}$ by translates of a bounded region T. Theorem 6 plays an important role in this proof.

Proof of Theorem 2. Without loss of generality we may take the period lattice of $\mathscr{T}$ to be $\mathbb{Z}$, by rescaling T and $\mathscr{T}$ to $\frac{1}{\lambda}T$ and $\frac{1}{\lambda}\mathscr{T}$, respectively. We are now given a bounded region T that tiles $\mathbb{R}$ with a tiling set $\mathscr{T}$ which has $\mathbb{Z}$ as a period, so that

$$\mathscr{T} := \bigcup_{j=1}^{J}(r_j + \mathbb{Z}). \tag{4.1}$$

Our object is to show that all $r_i - r_j \in \mathbb{Q}$. Set

$$\mathscr{R} := \{r_j : 1 \le j \le J\}$$

and define the new region

$$U := \bigcup_{j=1}^{J}(T + r_j) \tag{4.2}$$

The hypotheses show that the region U tiles $\mathbb{R}$ with the lattice tiling $\mathbb{Z}$. Now U is a bounded region, so it must be a fundamental domain for $\mathbb{R}/\mathbb{Z}$ (up to a set of measure 0), hence $\mu(U) = 1$. (The measure-disjointness of the union (4.2) then implies that $\mu(T) = \frac{1}{J}$.)

We use the Fourier transforms of the characteristic function $\chi_T(t)$ of T and of the measure

$$\delta_{\mathscr{R}}(t) := \sum_{r\in\mathscr{R}} \delta_r(t), \tag{4.3}$$

where $\delta_r(t) := \delta(t - r)$ is a δ-function centered at r. These are

$$\hat{\chi}_T(\lambda) = \int_T \exp(2\pi it\lambda)dt,\ \ \lambda \in \mathbb{C}, \tag{4.4}$$

and

$$\hat{\delta}_{\mathscr{R}}(\lambda) = \sum_{r\in\mathscr{R}} \exp(2\pi ir\lambda),\ \ \lambda \in \mathbb{C}, \tag{4.5}$$

respectively. Then the characteristic function χ_U of U has Fourier transform

$$\begin{aligned} \hat{\chi}_U(\lambda) &= \int_{\mathscr{U}} \exp(2\pi it\lambda)dt \\ &= \sum_{i=1}^{m}\int_{T+r_i} \exp(2\pi it\lambda)dt \\ &= \hat{\delta}_{\mathscr{R}}(\lambda)\hat{\chi}_T(\lambda),\ \ \lambda \in \mathbb{C}. \end{aligned} \tag{4.6}$$

Since U tiles $\mathbb{R}$ with tiling set $\mathbb{Z}$, we have

$$\hat{\chi}_U(n) = \begin{cases} 1 & \text{if } n = 0\,, \\ 0 & \text{if } n \in \mathbb{Z} \setminus \{0\}. \end{cases} \tag{4.7}$$

because $U \equiv [0,1]$ (mod 1), aside from a set of measure zero.

In terms of the Fourier series zero sets $Z(\delta_{\mathscr{R}})$ and $Z(\chi_T)$, (4.6) and (4.7) combine to give

$$Z(\delta_{\mathscr{R}}) \cup Z(\chi_T) = \mathbb{Z} \setminus \{0\}. \tag{4.8}$$

By making a translation of $\mathscr{T}$ we may reduce to the case that $r_1 = 0$ without loss of generality. The theorem then reduces to proving that

$$\mathscr{R} \subseteq \mathbb{Q}. \tag{4.9}$$

We begin by partitioning $\mathscr{R}$ into nonempty equivalence classes modulo $\mathbb{Q}$. Call the resulting partition

$$\mathscr{R} = \bigcup_{k=1}^{K} \mathscr{R}_k^*,$$

where $r - r' \in \mathbb{Q}$ if $r, r' \in \mathscr{R}_k^*$, and $r - r' \notin \mathbb{Q}$ if $r \in \mathscr{R}_{k_1}^*$, $r' \in \mathscr{R}_{k_2}^*$ with $k_1 \neq k_2$. We thus have a decomposition

$$\mathscr{R}_k^* = \tilde{r}_k + \mathscr{C}_k^* \quad \text{with} \quad \mathscr{C}_k^* \subseteq \mathbb{Q}, \quad 1 \le k \le K,$$

where each $\tilde{r}_k \in \mathscr{R}$. Define N to be the least common denominator for this decomposition, i.e.

$$N := \min \left\{ M \in \mathbb{Z}^+ : M \left(\bigcup_{k=1}^{K} \mathscr{C}_k^* \right) \subseteq \mathbb{Z} \right\}. \tag{4.10}$$

Next, for each $\mathscr{C}_k^*$, set

$$f_k^*(\lambda) = \sum_{c \in \mathscr{C}_k^*} \exp(2\pi i c \lambda), \quad 1 \le k \le K. \tag{4.11}$$

If $f_k^*(n) = 0$ for $n \in \mathbb{Z}$ then

$$f_k^*(n + Nm) = 0, \quad \text{all} \quad m \in \mathbb{Z}, \tag{4.12}$$

because N is a common denominator for all elements of $\mathscr{C}_k^*$.

We define the *common integer zero set* X of the f_k^* by

$$X := \{n \in \mathbb{Z} : f_k^*(n) = 0 \quad \text{for} \quad 1 \le k \le K\}. \tag{4.13}$$

Since $\hat{\delta}_{\mathscr{R}}(\lambda) = \sum_{k=1}^{K} f_k^*(\lambda)$, we have $X \subseteq Z(\delta_{\mathscr{R}})$. (4.13) shows that X is a union of arithmetic progressions (mod N), and certainly $0 \notin X$ because $0 \notin Z(\delta_{\mathscr{R}})$.

Claim. The Fourier series zero set has a partition

$$Z(\delta_{\mathscr{R}}) = X \cup Y,$$

in which X is the common integer zero set and Y is a set of density zero, i.e.

$$\overline{d}(Y) = 0. \tag{4.14}$$

In fact, Y is a finite set.

Proof of Claim. We define

$$Y := Z(\delta_{\mathscr{R}}) \setminus X, \tag{4.15}$$

so that $\{X, Y\}$ is a partition of $Z(\hat{\delta}_{\mathscr{R}})$. We must show that $\overline{d}(Y) = 0$.

We now prove that Y contains no arithmetic progression of length $J \geq |\mathscr{R}|$. We argue by contradiction. Suppose that it contains one of length $J = |\mathscr{R}|$, call it

$$s, s + d, s + 2d, \dots, s + (J - 1)d.$$

Now

$$\hat{\delta}_{\mathscr{R}}(s + ld) := \sum_{j=1}^{J} \exp(2\pi i r_j(s + ld)) = 0, \quad 0 \leq l \leq J - 1. \tag{4.16}$$

Define an equivalence relation on the elements of the set $\mathscr{R}$ by

$$r \approx r' \Longleftrightarrow \exp(2\pi i r d) = \exp(2\pi i r' d).$$

This relation $\approx$ induces a partition of $\mathscr{R}$ into nonempty equivalence classes, call it

$$\mathscr{R} = \bigcup_{l=1}^{L} \tilde{\mathscr{R}}_l,$$

and set $z_l = \exp(2\pi i r d)$ for some $r \in \tilde{\mathscr{R}}_l$. We have

$$\hat{\delta}_{\tilde{\mathscr{R}}_l}(\lambda) := \sum_{r \in \tilde{\mathscr{R}}_l} \exp(2\pi i r\lambda), \quad 1 \leq l \leq L,$$

and (4.16) yields

$$\hat{\delta}_{\mathscr{R}}(s + md) = \sum_{l=1}^{L} z_l^m \hat{\delta}_{\tilde{\mathscr{R}}_l}(s) = 0, \quad 1 \leq m \leq J - 1.$$

This is a linear system with unknowns $x_l = \hat{\delta}_{\tilde{\mathscr{R}}_l}(s)$. It is coefficients for $1 \leq m \leq J - 1$ form a Vandermonde matrix with distinct z_l, hence

$$\hat{\delta}_{\tilde{\mathscr{R}}_l}(s) = 0 \quad \text{for} \quad 1 \leq l \leq L. \tag{4.17}$$

We next assert that the partition $\{\tilde{\mathscr{R}}_l : 1 \leq l \leq L\}$ *refines* the partition $\{\mathscr{R}_k^* : 1 \leq k \leq K\}$. For $r \approx r'$ implies that $\exp(2\pi i(r - r')d) = 1$, which since $d \in \mathbb{Z} \setminus \{0\}$ gives $r - r' \in \mathbb{Q}$, so r and r' are in the same $\mathbb{Q}$-equivalence class, as asserted. In consequence,

$$\begin{aligned} \hat{\delta}_{\mathscr{R}_k^*}(s) &= \sum_{r \in \mathscr{R}_k^*} \exp(2\pi i r s) \\ &= \sum_{\tilde{\mathscr{R}}_l \subseteq \mathscr{R}_k^*} \hat{\delta}_{\tilde{\mathscr{R}}_l}(s) = 0, \quad 1 \leq k \leq K. \end{aligned}$$

By definition of X this makes $s \in X$ so $s \in X \cap Y \neq \emptyset$, a contradiction.

To complete the proof of the claim, suppose that $\overline{d}(Y) > 0$. We apply Szemeredi's theorem asserting that if $Y \subseteq \mathbb{Z}^+$ has $\overline{d}(Y) > 0$ then Y contains arbitrarily long arithmetic progressions, cf. Szemeredi [31], Furstenberg [8, 9]. This contradicts Y containing no arithmetic progression of length $|\mathscr{R}|$.

An alternative argument uses the Skolem-Mahler-Lech theorem, and yields the stronger result that Y is a finite set. The Skolem-Mahler-Lech theorem states that the integer zero set of an exponential polynomial is a finite union of complete arithmetic progressions plus a finite set, cf. Lech [23], van der Poorten [33]. In particular $Z(\delta_{\mathscr{R}})$ and X both have this structure, from which it follows that Y differs from a finite union of complete arithmetic progressions on a finite set. So if Y were infinite then it would contain arbitrarily long arithmetic progressions, which gives the same contradiction.

To continue the proof of Theorem 2, introduce the regions

$$U_k := \bigcup_{r \in \mathscr{R}_k^*} (T + r), \quad 1 \leq k \leq K. \tag{4.18}$$

A calculation identical to (4.6) gives

$$\hat{\chi}_{U_k}(\lambda) = \hat{\delta}_{\mathscr{R}_k}(\lambda)\hat{\chi}_T(\lambda), \quad \lambda \in \mathbb{C}, \tag{4.19}$$

which implies that

$$Z(\chi_{U_k}) = Z(\delta_{\mathscr{R}_k^*}) \cup Z(\chi_T). \tag{4.20}$$

The definition (4.13) of X guarantees that

$$X \subseteq Z(\delta_{\mathscr{R}_k^*}) \quad \text{for} \quad 1 \leq k \leq K, \tag{4.21}$$

whence

$$\begin{aligned} \mathbb{Z} \setminus \{0\} &= Z(\delta_{\mathscr{R}}) \cup Z(\chi_T) = X \cup Y \cup Z(\chi_T) \\ &\subseteq Y \cup Z(\delta_{\mathscr{R}_k^*}) \cup Z(\chi_T). \end{aligned}$$

The claim states that $\overline{d}(Y) = 0$, so this yields

$$\overline{d}(Z(\chi_{U_k})) = \overline{d}(Z(\chi_{\mathscr{R}_k^*}) \cup Z(\chi_T)) \geq 1. \tag{4.22}$$

If it were true that $\mu(U_k) < 1$, then Theorem 6 would give

$$\overline{d}(Z(\chi_{U_k})) < 1,$$

contradicting (4.22). Thus $\mu(U_k) = 1$, which means that $\mathscr{R}_k = \mathscr{R}$ so $k = K = 1$, and, since $0 \in \mathscr{R}$, we have $\mathscr{R} \subseteq \mathbb{Q}$.

We now show that Theorem 6 cannot be improved, using some particular regions T that tile $\mathbb{R}$.

Lemma 1. *For any* $\epsilon > 0$ *there exists a region* T *in* $[0, 1]$ *which has measure* $\mu(T) < \epsilon$ *and which tiles* $\mathbb{R}$ *with a periodic tiling whose period lattice contains* $\mathbb{Z}$*, yet whose characteristic function* χ_T *has Fourier series zero set satisfying*

$$\overline{d}(Z(\chi_T)) \geq 1 - \epsilon.$$

Proof. For any $N \geq 1$ take

$$T = \left[0, \frac{1}{N^2}\right] + \frac{1}{N^2}\mathscr{A}$$

where $\mathscr{A} = \{0, N, 2N, \ldots, (N-1)N\}$, so that

$$\mu(T) = \frac{1}{N}. \tag{4.23}$$

If $\mathscr{B} = \{0, 1, \ldots, N-1\}$ then $\mathscr{A} + \mathscr{B} + N^2\mathbb{Z} = \mathbb{Z}$, hence T tiles $\mathbb{R}$ with tile set

$$T = \frac{1}{N^2}\mathscr{B} + \mathbb{Z}.$$

Taking $\mathscr{R} = \frac{1}{N^2}\mathscr{B}$, the function $\delta_{\mathscr{R}}$ has Fourier transform

$$\hat{\delta}_{\mathscr{R}}(\lambda) = \sum_{j=0}^{N-1} \exp\left(\frac{2\pi i j\lambda}{N^2}\right) = \frac{1 - \exp\left(\frac{2\pi i\lambda}{N}\right)}{1 - \exp\left(\frac{2\pi i\lambda}{N^2}\right)},$$

hence it has Fourier series zero set

$$Z(\delta_{\mathscr{R}}) := \{N, 2N, \ldots, (N-1)N\} + N^2\mathbb{Z}.$$

Thus

$$\overline{d}(Z(\delta_{\mathscr{R}})) = \frac{N-1}{N^2},$$

and (4.8) now implies that

$$\overline{d}(Z(\chi_T)) \geq \underline{d}(Z(\chi_T)) \geq \frac{N^2 - N + 1}{N^2} \geq 1 - \frac{1}{N}, \tag{4.24}$$

from which the lemma follows on choosing N large enough.

5. Structure theorem for tiles

We classify the structure of bounded regions T that tile $\mathbb{R}$, using Theorem 2. We let $T_1 \simeq T_2$ mean that T_1 and T_2 differ on a set of measure zero.

Proof of Theorem 3. We are given that T tiles $\mathbb{R}$ with a periodic tiling $\mathscr{T}$ whose period lattice contains $\mathbb{Z}$. Without loss of generality we may suppose that $0 \in \mathscr{T}$, by translating the tile set. The $\mathbb{Z}$-periodicity of $\mathscr{T}$ yields

$$\mathscr{T} = \bigcup_{j=1}^{J} (r_j + \mathbb{Z}), \quad 0 \le r_j < 1, \tag{5.1}$$

and we may suppose that $r_1 = 0$. By Theorem 2 all $r_j = r_j - r_1$ are rational. Taking L to be their common denominator, we set $r_j = \frac{a_j}{L}$, and then $\mathscr{T}$ is of the form (1.6).

Now set

$$\mathscr{A} = \{a_i : 1 \le i \le J\}$$

and, for each $t \in [0, \frac{1}{L})$, define the set of integers

$$\mathscr{B}(t) := \left\{ j \in \mathbb{Z} : t + \frac{j}{L} \in T \right\}. \tag{5.2}$$

Since T is bounded, say $T \subseteq [-N, N]$, there are only finitely many possibilities for the set $\mathscr{B}(t)$, i.e. $\mathscr{B}(t)$ lies in $\mathscr{S}_N := \{\text{all subsets of } [-LN, LN] \cap \mathbb{Z}\}$. For each set of integers $\mathscr{B} \in \mathscr{S}_N$, we let

$$T^*_{\mathscr{B}} := \{t : \mathscr{B}(t) = \mathscr{B}\}.$$

By discarding sets $\mathscr{B}$ with $\mu(T^*_{\mathscr{B}}) = 0$, we have

$$T \simeq \bigcup_{\mu(T^*_{\mathscr{B}})>0} (T^*_{\mathscr{B}} + \mathscr{B}). \tag{5.3}$$

Furthermore, we have

$$\left[0, \frac{1}{L}\right] \simeq \bigcup_{\mu(T^*_{\mathscr{B}})>0} T^*_{\mathscr{B}}. \tag{5.4}$$

This will turn out to be the required decomposition (1.7), after replacing each of the sets $T^*_{\mathscr{B}}$ by its closure $\overline{T^*_{\mathscr{B}}}$.

We assert that for each $\mathscr{B}$ with $\mu(T^*_{\mathscr{B}}) > 0$, the pair $(\mathscr{A}, \mathscr{B})$ is a complementing pair (mod L). To show this, look at the tiling restricted to the subset

$$S_{\mathscr{B}} := T^*_{\mathscr{B}} + \frac{1}{L}\mathbb{Z}$$

of $\mathbb{R}$. Now $S_{\mathscr{B}}$ is tiled (up to a measure zero set) by the tiles

$$U_{\mathscr{B}} := T^*_{\mathscr{B}} + \frac{1}{L}\mathscr{B} \simeq T \cap S_{\mathscr{B}},$$

using the tile set $\mathscr{T}$. Thus

$$\begin{aligned} T^*_{\mathscr{B}} + \frac{1}{L}\mathbb{Z} &\simeq \left(T^*_{\mathscr{B}} + \frac{1}{L}\mathscr{B}\right) + \left(\frac{1}{L}\mathscr{A} + \mathbb{Z}\right) \\ &\simeq \left(T^*_{\mathscr{B}} + \frac{1}{L}(\mathscr{A} + \mathscr{B})\right) + \mathbb{Z}. \end{aligned}$$

Since $T^*_{\mathscr{B}} \subseteq \left[0, \frac{1}{L}\right]$ has positive measure, this forces

$$\frac{1}{L}\mathbb{Z} = \frac{1}{L}(\mathscr{A} + \mathscr{B}) + \mathbb{Z} \tag{5.5}$$

viewed as sets *with multiplicity*, which requires that $(\mathscr{A}, \mathscr{B})$ be a complementing pair (mod L), proving the assertion. It follows that $\mathscr{A}$ is a complementing set (mod L), and also that

$$|\mathscr{B}| = \frac{L}{|\mathscr{A}|} \quad \text{when} \quad \mu(T^*_{\mathscr{B}}) > 0. \tag{5.6}$$

Now, for each $\mathscr{B}$ with $\mu(T^*_{\mathscr{B}}) > 0$, we set

$$T_{\mathscr{B}} := \overline{T^*_{\mathscr{B}}},$$

and proceed to show that these sets satisfy (1.7) with properties (i) and (ii).

We first observe that

$$\left[0, \frac{1}{L}\right] = \overline{\bigcup_{\mu(T^*_{\mathscr{B}})>0} T^*_{\mathscr{B}}} = \bigcup_{\mu(T^*_{\mathscr{B}})>0} \overline{T^*_{\mathscr{B}}} \tag{5.7}$$

is a direct consequence of (5.4), so (ii) holds.

To continue the proof, we study the points in $\overline{T^*_{\mathscr{B}}} \setminus T^*_{\mathscr{B}}$.

Claim. The set

$$X := \bigcup_{\mu(T^*_{\mathscr{B}})>0} (\overline{T^*_{\mathscr{B}}} \setminus T^*_{\mathscr{B}}) \tag{5.8}$$

is a closed set of measure zero and is given by

$$X = \bigcup_{\mu(T^*_{\mathscr{B}})=0} T^*_{\mathscr{B}}. \tag{5.9}$$

Proof. We proceed in three steps. First observe that

$$|\mathscr{B}(t)| \geq \frac{L}{|\mathscr{A}|} \quad \text{for all} \quad t \in \left[0, \frac{1}{L}\right). \tag{5.10}$$

Indeed since T covers $\mathbb{R}$ using the tiling set $\mathscr{T}$, it follows that the discrete set $t + \frac{1}{L}\mathbb{Z}$ must be completely covered by the discrete set

$$t + \frac{1}{L}\mathscr{B}(t) + \mathscr{T} = t + \frac{1}{L}(\mathscr{A} + \mathscr{B}(t)) + \mathbb{Z}.$$

This requires that $\mathscr{A} + \mathscr{B}(t) = \mathbb{Z}$ as sets (not counting multiplicity), which implies (5.10).

Second, consider for any set $\mathscr{B}$ a limit point $t^* \in \overline{T^*_{\mathscr{B}}} \setminus T^*_{\mathscr{B}}$. We have

$$\mathscr{B} \subseteq \mathscr{B}(t^*), \tag{5.11}$$

for if we take a sequence $\{t_j\} \subseteq T^*_{\mathscr{B}}$ with $t_j \to t^*$, then $t_j + \mathscr{B} \subseteq T$ and $t_j + \mathscr{B} \to t^* + \mathscr{B}$, hence $t^* + \mathscr{B} \subseteq T$ since T is closed, and (5.11) follows. However $\mathscr{B}(t^*) \neq \mathscr{B}$ since $t^* \notin T^*_{\mathscr{B}}$ so that

$$|\mathscr{B}(t^*)| > |\mathscr{B}|. \tag{5.12}$$

If $\mu(T^*_{\mathscr{B}}) > 0$ then (5.6) implies that $\mu(T^*_{\mathscr{B}(t^*)}) = 0$. This shows that

$$X \subseteq \bigcup_{\mu(T^*_{\mathscr{B}})=0} T^*_{\mathscr{B}}, \tag{5.13}$$

hence $\mu(X) = 0$. Next, every point $t^* \in T^*_{\mathscr{B}}$ with $\mu(T^*_{\mathscr{B}}) = 0$ has $t^* \in \left[0, \frac{1}{L}\right]$, and (5.7) shows that it arises as a member of some $\overline{T^*_{\mathscr{B}}}$, hence the inclusion (5.13) is an equality and (5.9) holds.

Third, we show that X is a closed set. Any limit point t^* of X is a limit point of some $T^*_{\mathscr{B}}$ with $|\mathscr{B}| > \frac{L}{|\mathscr{A}|}$, and (5.11) applies, so that $|\mathscr{B}(t^*)| \geq |\mathscr{B}| > \frac{t}{|\mathscr{A}|}$ hence $\mu(T^*_{\mathscr{B}(t^*)}) = 0$ and $t^* \in X$. The claim follows.

Now form the set

$$\tilde{T}^\circ := \mathrm{Int}(T) \setminus X,$$

which is an open set with $\mu(\tilde{T}^\circ) = \mu(T)$ because T is a region and X is closed and of measure zero. Now the claim gives

$$\tilde{T}^\circ \subseteq \bigcup_{\mu(T^*_{\mathscr{B}})>0} (T^*_{\mathscr{B}} + \mathscr{B}). \tag{5.14}$$

Since T is a region, every point in T is a limit point of $\mathrm{Int}(T)$, hence is still a limit point of $\tilde{T}^\circ$, because X is in the limit set of $\tilde{T}^\circ$.

We next show that if $\mu(T^*_{\mathscr{B}}) > 0$ then each $\tilde{T}^\circ \cap (T^*_{\mathscr{B}} + \mathscr{B})$ is an open set. This follows from (5.14) because the sets $T^*_{\mathscr{B}} + \mathscr{B}$ are disjoint and no point in any one of them is a limit point of any other, by (5.12). Thus

$$\tilde{T}^\circ \cap T^*_{\mathscr{B}} \subseteq \mathrm{Int}(T^*_{\mathscr{B}} + \mathscr{B}). \tag{5.15}$$

We now have

$$\begin{aligned} \mu(T) &\geq \sum_{\mu(T^*_{\mathscr{B}})>0} \mu(\mathrm{Int}(T^*_{\mathscr{B}})) \\ &\geq \sum_{\mu(T^*_{\mathscr{B}})>0} \mu(\tilde{T}^\circ \cap (T^*_{\mathscr{B}} + \mathscr{B})) \\ &= \mu(\tilde{T}^\circ) = \mu(T). \end{aligned}$$

and this gives

$$\mu(\mathrm{Int}(T^*_{\mathscr{B}}) + \mathscr{B}) = \mu(T^*_{\mathscr{B}} + \mathscr{B}). \tag{5.16}$$

Intersecting with $\left(0, \frac{1}{L}\right)$, we get

$$\mu(\mathrm{Int}(T^*_{\mathscr{B}})) = \mu(T^*_{\mathscr{B}}). \tag{5.17}$$

Furthermore (5.14) yields that every point of T is a limit point of some $\mathrm{Int}(T^*_{\mathscr{B}} + \mathscr{B})$, hence

$$T = \bigcup_{\mu(T^*_{\mathscr{B}})>0} (\overline{T^*_{\mathscr{B}}} + \mathscr{B}),$$

which verifies (1.7).

Finally, since limit points in the open interval $\left(0, \frac{1}{L}\right)$ can only arise from $T^*_{\mathscr{B}}$ itself,

$$\overline{T^*_{\mathscr{B}}} = \overline{\mathrm{Int}(T^*_{\mathscr{B}})}.$$

Thus $\overline{T^*_{\mathscr{B}}}$ is a region, and (i) is verified.

We have proved existence of a decomposition (1.7), and it remains to prove uniqueness. So let $\tilde{T}_{\mathscr{B}}$ be another choice. We use the fact that a region U is uniquely determined by its interior $\mathrm{Int}(U)$. The interior disjointness and covering properties (i) and (ii) guarantee that

$$\tilde{T}^\circ \cap \left(0, \frac{1}{L}\right) \cap T^*_{\mathscr{B}} \subseteq \tilde{T}_{\mathscr{B}}.$$

By earlier arguments, the closure of the left side is $\overline{T^*_{\mathscr{B}}}$ so $\overline{T^*_{\mathscr{B}}} \subseteq \tilde{T}_{\mathscr{B}}$, whence

$$\mathrm{Int}(\overline{T^*_{\mathscr{B}}}) \subseteq \mathrm{Int}(\tilde{T}_{\mathscr{B}}).$$

But $\mu(\mathrm{Int}(\tilde{T}_{\mathscr{B}})) \leq \mu(\mathrm{Int}(\overline{T^*_{\mathscr{B}}}))$, for if it were larger it would intersect the interior of some other $\overline{T^*_{\mathscr{B}'}}$, because the sets $\mathrm{Int}(\overline{T^*_{\mathscr{B}}})$ have full measure in $[0, \frac{1}{L}]$ by (5.7) and (5.17), hence it would intersect $\mathrm{Int}(\tilde{T}_{\mathscr{B}'})$, contradicting property (i). Thus

$$\mu(\mathrm{Int}(\tilde{T}_{\mathscr{B}})) = \mu(\mathrm{Int}(\overline{T^*_{\mathscr{B}}})).$$

Now $\mathrm{Int}(\overline{T^*_{\mathscr{B}}}) = \mathrm{Int}(\tilde{T}_{\mathscr{B}})$, so $\overline{T^*_{\mathscr{B}}} = \tilde{T}_{\mathscr{B}}$, verifying uniqueness.

6. One-dimensional self-affine tiles

We show that all one-dimensional self-affine tiles are affine images of integral self-affine tiles. An easier proof of this result can be obtained along the lines of Kenyon [19], Lemma 4.

Proof of Theorem 4. Suppose first that $0 \in \mathscr{D}$. It is well-known that self-affine tiles $T(b, \mathscr{D})$ tile $\mathbb{R}^n$ by a translation tiling $\mathscr{T}$, cf. Theorem 2 of Lagarias and Wang [22]. That proof showed moreover that if $0 \in \mathscr{D}$ and if one sets

$$\mathscr{D}_{b,k} = \left\{ \sum_{j=0}^{k-1} b^j d_j : \quad \text{all} \quad d_j \in \mathscr{D} \right\}$$

then there is one such tiling $\mathscr{T}$ which for some $k \geq 1$ has

$$\mathscr{D}_{b,k} - d^* \subseteq \mathscr{T}, \quad d^* \in \mathscr{D}_{b,k}.$$

In particular for each $d \in \mathscr{D}$ there are two tiles in this tiling $\mathscr{T}$ translated from each other by d. Now Theorem 6 shows that every such $d = d - 0$ is a rational multiple of the minimal period λ of the tiling $\mathscr{T}$. If $m \in \mathbb{Z}$ is the least common denominator of all the rationals $\left\{\frac{d}{\lambda} : d \in \mathscr{D}\right\}$ then

$$\frac{m}{\lambda}\mathscr{D} \subseteq \mathbb{Z}, \tag{6.1}$$

which is the second part of the theorem.

To complete the proof by an affine transformation we reduce the general case to the case that $0 \in \mathscr{D}$. To do this we use

$$T(b, t\mathscr{D}) = tT(b, \mathscr{D}), \tag{6.2}$$

and

$$T(b, \mathscr{D} - t) = T(b, \mathscr{D}) - t^* \tag{6.3}$$

with $t^* = \sum_{j=1}^{\infty} b^{-j} t = \frac{bt}{t-1}$.

Theorem 4 has immediate consequences concerning digit sets for positional number systems, extending those of Kenyon [19].

Theorem 7. *(i). Given an integer base b with $|b| \geq 2$ and a digit set $\mathscr{D} = \{0, 1, x_2, \ldots, x_{|b|-1}\}$ with all $x_i \in \mathbb{R}$ then a necessary condition for $\mu(T(b, \mathscr{D})) > 0$ is that all $x_i \in \mathbb{Q}$.*

(ii). Suppose further that $|b| = p$ is prime. Then $\mu(T(b, \mathscr{D})) > 0$ if and only if there are integers $\{m_i : 1 \leq i \leq p-1\}$ such that

$$x_i = \frac{m_i}{m_1} \quad \text{with} \quad g.c.d.(m_1, m_2, \ldots, m_{p-1}) = 1,$$

and $\{0, m_1, m_2, \ldots, m_{p-1}\}$ is a complete residue system (mod p).

Proof. (i). This follows from Theorem 4.

(ii). Certainly $\mu(T(b, \mathscr{D})) > 0$ if and only if $\mu(T(b, m_1\mathscr{D})) > 0$, where $m_1\mathscr{D} = \{0, m_1, m_2, \ldots, m_{p-1}\}$. Now apply Theorem 4.1 of Lagarias and Wang [21].

Appendix. Tilings of $\mathbb{R}$ by compact sets

We reduce the study of compact sets that tile $\mathbb{R}$ by translation to the case of regions that tile $\mathbb{R}$ by translation.

Lemma A.1. *Let T be a compact set of positive measure that tiles $\mathbb{R}$ with a measure-disjoint tiling using the tile set $\mathscr{T}$. If T' is the closure of the interior of T then $\mu(T \setminus T') = 0$, and T' tiles $\mathbb{R}$ using the same tile set $\mathscr{T}$.*

Proof. Since the set T is measurable with $\mu(T) > 0$, the tile set must be uniformly discrete, i.e. there exists an $\epsilon > 0$ such that $|t - t'| > \epsilon$ for distinct $t, t' \in \mathscr{T}$, for $\mu\big((T+t) \cap (T+t')\big) > 0$ whenever $|t - t'|$ is sufficiently small.

Let $T \subseteq T'$ be the closure of the interior of T. We assert that the set $E = T \setminus T'$ has $\mu(E) = 0$. By translation if necessary, we may assume that $0 \in \mathscr{T}$. Let $T \subseteq [-M, M]$ and consider the finite set $\mathscr{S} := \{t \in \mathscr{T} : |t| \leq 3M,\ t \neq 0\}$. These tiles $T+t$ with $t \in \mathscr{S}$ are the only ones that can possibly intersect the tile T. If $x \in T \setminus T'$ then it can be approximated as $x = \lim_{i \to \infty} x_i$ with all $x_i \notin T$. Hence infinitely many x_i lie in some fixed $T + t^*$ for some $t^* \in \mathscr{S}$. We have $x \in T + t^*$, since T is closed. Thus

$$E \subseteq \bigcup_{t \in \mathscr{S}} (T+t) \cap T,$$

and

$$\mu(E) \leq \sum_{t \in \mathscr{S}} \mu\big((T+t) \cap T\big) = 0,$$

using (1.3).

Now T' is a region, and it has the same measure as T. Since T is discrete, the set $\mathbb{R} \setminus \bigcup_{t \in \mathscr{S}} (T+t)$ is an open set, and it has zero measure; hence it must be empty, proving that T' tiles $\mathbb{R}$ with the tile set $\mathscr{T}$.

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