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Tiling with Polyominoes and Combinatorial Group Theory

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When can a given finite region consisting of cells in a regular lattice (triangular, square, or hexagonal) in $\mathbb{R}^2$ be perfectly tiled by tiles drawn from a finite set of tile shapes? This paper gives necessary conditions for the existence of such tilings using *boundary invariants*, which are combinatorial group-theoretic invariants associated to the boundaries of the tile shapes and the regions to be tiled. Boundary invariants are used to solve problems concerning the tiling of triangular-shaped regions of hexagons in the hexagonal lattice with certain tiles consisting of three hexagons. Boundary invariants give stronger conditions for nonexistence of tilings than those obtainable by weighting or coloring arguments. This is shown by considering whether or not a region has a *signed tiling*, which is a placement of tiles assigned weights 1 or -1, such that all cells in the region are covered with total weight 1 and all cells outside with total weight 0. Any coloring (or weighting) argument that proves nonexistence of a tiling of a region also proves nonexistence of any signed tiling of the region as well. A partial converse holds: if a simply connected region has no signed tiling by simply connected tiles, then there is a generalized coloring argument proving that no signed tiling exists. There exist regions possessing a signed tiling which can be shown to have no perfect tiling using boundary invariants.

1. INTRODUCTION

Packing, covering, and tiling problems are among the most basic combinatorial problems. Here we consider problems concerning the possibility or impossibility of tiling finite regions of a regular lattice tiling of $\mathbb{R}^2$ by translations of a finite set of (lattice) tiles.

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There are three regular lattice tilings of $\mathbb{R}^2$, which are the triangular lattice, square lattice, and hexagonal lattice, pictured in Fig. 1.1. Each of these tilings divides $\mathbb{R}^2$ into *cells*, and any cell can be obtained from any other cell by a translation. A *lattice figure* or *region*, is a finite union of (closed) cells that is connected. Lattice figures for the three types of lattices are called *polyiamonds*, *polyominoes*, and *polyhexes*, respectively. Two lattice figures are *equivalent* if one can be obtained from the other by a translation. They are *congruent* if one can be obtained from the other by a Euclidean motion, which includes rotations and reflections. A (lattice) *tile* is a simply connected lattice figure. A set Σ of lattice figures *tiles* a region R if R can be covered with translates of figures in Σ such that each cell in R is covered by exactly one lattice figure.

Tiling problems on lattices are in general computationally difficult problems. Consider the following two problems:

PLANE TILING PROBLEM

Instance. A finite set Σ of tiles.

Question. Does Σ tile the whole lattice?

FINITE TILING PROBLEM

Instance. A region R and a finite set Σ of tiles.

Question. Does Σ tile R?

The Plane Tiling Problem is undecidable, as can be shown by a suitable encoding of the undecidable Wang Tiling Problem (also called the Domino Problem, see [2; 24; 14, Chap. 11]), in which each colored edge of a colored square (Wang tile) is replaced with an appropriately serrated edge following the lattice edges. The Finite Tiling Problem is clearly decidable by exhaustive enumeration and is in the computational complexity class NP because if a tiling exists it can be nondeterministically "guessed." However, it is NP-complete, as may be shown using an encoding of Square Tiling (see [8, p. 257]) again obtained using tiles with serrated edges. Consequently it is unlikely that there exists a polynomial time algorithm to solve

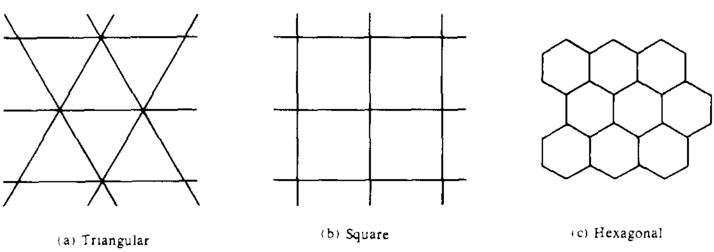


FIG. 1.1. Regular lattice tilings of $\mathbb{R}^2$.

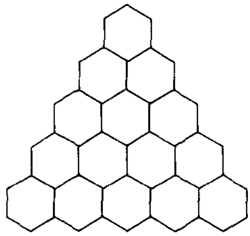


FIG. 1.2. Triangular region T_5.

the Finite Tiling Problem. Special methods do exist which can often be used to prove nonexistence of tilings of regions with a single tile. These include coloring and weighting arguments among others [3–6; 8–13; 16–20; 26].

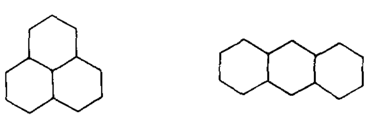
In view of the difficulty of the general Finite Tiling Problem, it is not too surprising that even apparently simple-looking tiling problems can prove difficult to solve. This paper arose from considering the following sets of tiling problems on the hexagonal lattice. Let T_N denote the triangular array of cells in the hexagonal lattice having $\binom{N+1}{2}$ cells pictured in Fig. 1.2. The *triangle tiling by triangles problem* is to decide: for which values of N can T_N be tiled by congruent copies of the triangular tile T_2 pictured in Fig. 1.3a? The *triangle tiling by lines problem* is to decide: for which values of N can T_N be tiled by congruent copies of the three-in-line tile L_3 pictured in Fig. 1.3b? In these problems one permits tiles to be rotated or reflected. In terms of equivalence classes of tiles the first problem above allows tiling by two inequivalent tiles and the second problem allows tiling by three inequivalent tiles, as pictured in Fig. 1.4.

These two tiling problems have the following answers.

THEOREM 1.1. *The triangular region T_N in the hexagonal lattice can be tiled by congruent copies of the triangular tile T_2 if and only if*

$$N \equiv 0,\ 2,\ 9,\ \textit{or}\ 11 \pmod{12}.$$

THEOREM 1.2. *It is impossible to tile the triangular region T_N in the hexagonal lattice with congruent copies of the three-in-line tile L_3.*



(a) Triangular tile T_2 (b) Three-in-line tile L_3

FIG. 1.3. Tiles for triangle tiling problems.

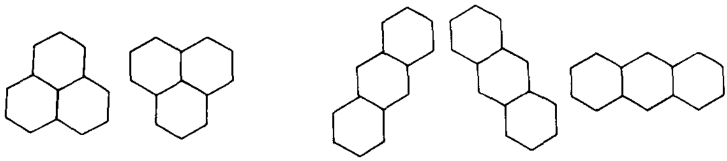


FIG. 1.4. Tile sets of translation-inequivalent tiles.

To solve these problems, we introduce combinatorial group-theoretic invariants associated to the boundaries of the tiles and the region to be tiled; we call these *boundary invariants.* Section 2 defines these invariants and shows that for a simply connected region R a necessary condition for a tiling by tiles in a set Σ to exist is that the combinatorial boundary of the region R be contained in a group $T(\Sigma)$ generated by the boundaries of the tiles in Σ (Theorem 2.1). This group-theoretic criterion seems in general no easier to verify than to solve the original problem. It can, however, be successfully applied to the case of the two triangle tiling problems, using group-theoretic properties specific to these problems. This is done in Section 3.

These solutions to the two triangle tiling problems are somewhat complicated, and it is reasonable to ask if simpler solutions exist. We investigate the relation between boundary invariants and other known necessary conditions for a tiling to exist. A region R has a *signed tiling* using tiles from a set Σ if there exists a placement of a finite number of such tiles, possibly overlapping, with each such tile assigned a weight of $+1$ and -1, such that for each cell in R the sum of the weights of tiles covering that cell add up to $+1$, while for each cell outside R the sum of the weights covering that cell is 0. Clearly a necessary condition for a tiling to exist is that a signed tiling exist. It is easy to determine when signed tilings exist for the triangle tiling problems.

THEOREM 1.3. *The triangular region T_N in the hexagonal lattice has a signed tiling by congruent copies of the triangular tile T_2 if and only if*

$$N \equiv 0 \textit{ or } 2 \pmod 3.$$

THEOREM 1.4. *The triangular region T_N in the hexagonal lattice has a signed tiling by congruent copies of the three-in-line tile L_3 if and only if*

$$N \equiv 0 \textit{ or } 8 \pmod 9.$$

These results are proved in Section 4.

Section 5 studies a notion of *generalized coloring argument* which includes known coloring and weighting arguments as special cases. Any generalized coloring argument that proves nonexistence of a tiling also proves nonexistence of a signed tiling (see Theorem 5.2). In view of the theorems above we immediately obtain the following consequence.

THEOREM 1.5. *It is impossible to solve the triangle tiling problems by a generalized coloring argument.*

This result gives a sense in which the two triangle tiling problems above do not have a simple solution.

Another interesting example is provided by a result of Walkup [26] showing that an $r \times s$ rectangle can be perfectly tiled by T-tetrominoes if and only if $r \equiv s \equiv 0 \pmod 4$. It can be checked that such rectangles have signed tilings by T-tetrominoes if and only if $rs \equiv 0 \pmod 8$. Hence this problem also cannot be solved by a generalized coloring argument. Walkup's ingenious argument is special to the T-tetromino; its relation to the combinatorial group theory approach of this paper is not obvious.

The boundary invariants defined in Section 2 can in principle be defined for tilings on finite subregions of any periodic tiling of $\mathbb{R}^2$ or of hyperbolic space $\mathbb{H}^n$.

We are indebted to Peter Doyle, Roger Lyndon, and Hugh Montgomery for helpful comments.

2. BOUNDARY INVARIANTS: THE TILE GROUP

Boundary invariants can be defined for any regular lattice; for simplicity we treat only the case of the square lattice. The triangle tiling problems described in Section 1 for the hexagonal lattice can be translated into mathematically equivalent tiling problems on the square lattice, see Section 3.

The square lattice in $\mathbb{R}^2$ consists of lattice points, edges, and cells. A *lattice point* is a member of $\mathbb{Z}^2$. Two lattice points are *neighbors* if they are at distance one from each other, so each lattice point has exactly four neighbors. An *edge* is a line segment connecting two neighboring lattice points; it is either *horizontal* or *vertical*. A *cell* is the set of all points making up the interior and boundary of a square of area one having its four vertices at lattice points.

A (*directed*) *path* P in the square lattice consists of a sequence of directed edges specified by a sequence of lattice points $\{(x_i, y_i): 0 \leqslant i \leqslant n\}$, where the ith directed edge connects (x_{i-1}, y_{i-1}) to (x_i, y_i). It is *closed* if $(x_0, y_0) = (x_n, y_n)$. A directed path is *simple* if no edge appears twice and if it does not cross itself, where we say a path *crosses itself* if there is

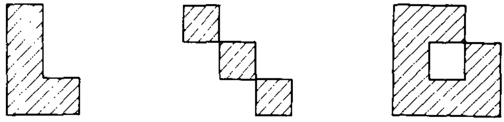


FIG. 2.1. Arrangements of cells, (a) and (b) are simply connected, (c) is not.

$0<i<n$ and $j \neq i$ with $(x_i, y_i) = (x_j, y_j)$ and the two edges from (x_{i-1}, y_{i-1}) to (x_{i+1}, y_{i+1}) consist of either two horizontal or two vertical edges.

A *region* R is a finite connected set of closed cells. The *topological boundary* ∂R of R is an (unordered) set of directed edges found as follows. The topological boundary ∂C of a cell C consists of its four edges, oriented counterclockwise. The boundary of ∂R is formed by taking the set of all edges in ∂C for all cells C in R, and discarding any edges that occur twice with opposite orientations. A region R is *simply connected* if its complement $\bar{R} = \mathbb{R}^2 - R$ is connected and if its boundary edges can be ordered to form a simple closed path. (This definition coincides with R being simply connected in the usual topological sense [23, p. 144].) Some examples illustrating these definitions are pictured in Fig. 2.1.

A simple closed path bounding a simply connected region R is uniquely specified by its first edge $\mathbf{e}$; we call such a path an *oriented boundary of R with leading edge* $\mathbf{e}$ and denote it by $\partial R(\mathbf{e})$. The first vertex in $\partial R(\mathbf{e})$ is called the *base point* of $\partial R(\mathbf{e})$. Some examples are shown in Fig. 2.2.

An *n-tile* is a simply connected region consisting of n cells. The notion of n-tile differs slightly from *n-omino* in that an n-tile may possibly be disconnected by removal of a single point while an n-omino may not, and n-ominos are required to be connected but not necessarily simply connected.

A *tile type* consists of the set of all translations of a tile.

A *tiling problem* consists of a region R and a set Σ of tile types. A region R can be *covered* or *tiled* by Σ if there exists a set of tiles in Σ that cover each cell of R exactly once.

We describe directed paths in the square lattice by words in the free group $\mathbf{F} = \langle A, U \rangle$ on two generators (where $A =$ "across," $U =$ "up"). To

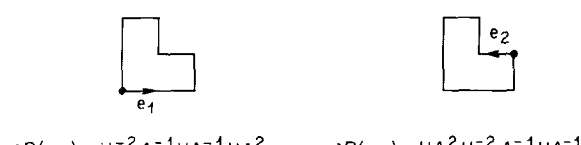


FIG. 2.2. Oriented boundaries and associated words in free group.

the path $P=\{(x_i, y_i): 0 \leqslant i \leqslant n\}$ we assign the word $W=W(P)$ in $\mathbf{F}$ given by

$$W=G_n G_{n-1} \cdots G_1,$$

where

$$G_i=\begin{cases} A & \text{if } (x_i, y_i)=(x_{i-1}+1, y_{i-1}), \\ A^{-1} & \text{if } (x_i, y_i)=(x_{i-1}-1, y_{i-1}), \\ U & \text{if } (x_i, y_i)=(x_{i-1}, y_{i-1}+1), \\ U^{-1} & \text{if } (x_i, y_i)=(x_{i-1}, y_{i-1}-1). \end{cases}$$

Figure 2.2 gives the words associated to the oriented boundaries with specified base points for the regions pictured.

There is an obvious mapping in the reverse direction which assigns to each word W in $\mathbf{F}$ the directed path $P(W)$ starting from the fixed base point $(0, 0)$ in $\mathbb{Z}^2$ obtained by reading the word W from right to left, and one clearly has $W(P(W))=W$.

Given an oriented boundary $\partial R(\mathbf{e})$ of a simply connected region R we let $\partial R(\mathbf{e})$ also stand for the word $W(\partial R(\mathbf{e}))$ in $\mathbf{F}$. The words

$$\{\partial R(\mathbf{e}): \mathbf{e} \text{ a counterclockwise oriented edge of } \partial R\}$$

are cyclic permutations of each other, hence are all conjugate in $\mathbf{F}$. For example for the regions in Fig. 2.2,

$$\partial R(\mathbf{e}_2)=(UA^2)\, \partial R(\mathbf{e}_1)(UA^2)^{-1}.$$

The *combinatorial boundary* $[\partial R]$ of a simply connected region R is the conjugacy class in $\mathbf{F}$ containing all the oriented boundaries $\partial R(\mathbf{e})$ of R, i.e.,

$$[\partial R]=\{W\, \partial R(\mathbf{e})\, W^{-1}: W \in \mathbf{F}\}.$$

In what follows we use standard terminology in combinatorial group theory: $\langle W_1, W_2, \ldots\rangle$ denotes the subgroup of a free group $\mathbf{F}$ generated by the words W_i, for any subgroup $\mathbf{G}$ of $\mathbf{F}$ let $N(\mathbf{G})$ denote the smallest normal subgroup in $\mathbf{F}$ containing $\mathbf{G}$, and let $[\mathbf{G}:\mathbf{G}]$ denote the commutator subgroup of $\mathbf{G}$, i.e., the group generated by the commutators $W_1 W_2 W_1^{-1} W_2^{-1}$ for all $W_1, W_2 \in \mathbf{G}$.

The *cycle group* $\mathbf{C}$ is the subgroup of the free group $\mathbf{F}$ consisting of all words associated to closed directed paths in the square lattice. The combinatorial boundary of any simply connected region is contained in the cycle group $\mathbf{C}$. In Section 5 we show that the cycle group is the commutator subgroup $[\mathbf{F}:\mathbf{F}]$ of $\mathbf{F}$, hence is a normal subgroup of $\mathbf{F}$, and in fact it can be shown that $\mathbf{C}=N(\langle AUA^{-1}U^{-1}\rangle)$.

We assign to a set of tiles $\Sigma = \{R_i\}$ a subgroup of $\mathbf{F}$ that contains all the boundaries of the tiles. The *tile group* $\mathbf{T}(\Sigma)$ is the smallest normal subgroup of $\mathbf{F}$ containing the combinatorial boundaries $[\partial R_i]$ of all tiles in Σ, i.e.,

$$\mathbf{T}(\Sigma) = N(\langle \partial R_i(\mathbf{e}_i) \colon 1 \leqslant i \leqslant m \rangle) = \langle W \, \partial R_i(\mathbf{e}_i) \, W^{-1} \colon W \in \mathbf{F},\ 1 \leqslant i \leqslant m \rangle.$$

Here $\partial R_i(\mathbf{e}_i)$ is an oriented boundary of R_i.

The tile group $\mathbf{T}(\Sigma)$ is contained in the cycle group $\mathbf{C}$ and is certainly a normal subgroup of $\mathbf{C}$. We call the quotient group

$$\mathbf{h}(\Sigma) = \mathbf{C}/\mathbf{T}(\Sigma)$$

the *tile homotopy group*. This name is suggested by analogy with the first homotopy group, based on the observation that $\mathbf{C}$ consists of the set of (allowable) closed paths in the lattice, while (roughly speaking) $\mathbf{T}(\Sigma)$ represents the paths that can be deformed to the empty path by picking up or laying down tiles.

The basic invariant that we assign to a region R to be tiled with a set of tiles Σ is its combinatorial boundary $[\partial R]$ viewed as a conjugacy class in the tile homotopy group $\mathbf{C}/\mathbf{T}(\Sigma)$.

THEOREM 2.1. *A necessary condition that a simply connected region R have a tiling by tiles in a set Σ is that the combinatorial boundary $[\partial R]$ of R be contained in the tile group $\mathbf{T}(\Sigma)$.*

It requires some care to give a completely rigorous proof of this result. Here we sketch a proof indicating the essential ideas, omitting proofs of some facts about 2-dimensional topology that can be proved along the lines of [23, Chaps. 5, 6].

Proof (Sketch). We must show that if R has a tiling in Σ then $[\partial R] \subseteq \mathbf{T}(\Sigma)$. Since $\mathbf{T}(\Sigma)$ is a normal subgroup of $\mathbf{F}$ it suffices to show that some oriented boundary $\partial R(\mathbf{e})$ of R is in $\mathbf{T}(\Sigma)$.

The proof is by induction on the number of tiles in a tiling by Σ. The result is clear when R is tiled by a single tile in Σ. So suppose $\mathscr{T}$ is a tiling of R with $k \geqslant 2$ tiles.

CLAIM. *There exists a decomposition $R = R^* \cup R^{**}$ such that R^*, R^{**} are both nonempty simply connected regions which can be tiled by Σ, and there are directed edges $\mathbf{e}_1$ of ∂R^*, $\mathbf{e}_2$ of ∂R^{**} so that*

$$\partial R(\mathbf{e}_1) = \partial R^{**}(\mathbf{e}_2) \, \partial R^*(\mathbf{e}_1).$$

The claim immediately completes the induction step, because $\partial R^*(\mathbf{e}_1)$, $\partial R^{**}(\mathbf{e}_2) \in \mathbf{T}(\Sigma)$ by the induction hypothesis, hence $\partial R(\mathbf{e}_1) \in \mathbf{T}(\Sigma)$.

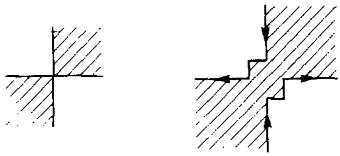


FIG. 2.3. Thickening.

Proof of Claim. First observe that the simple connectivity of R means essentially that it is topologically a disk with a simple closed curve as topological boundary. This is not literally true because R may have separating vertices, but becomes true if R is enlarged by adding two extra small squares of size ε around each separating vertex and deforming ∂R appropriately, see Fig. 2.3. (This process is called *thickening* in [23, p. 142].)

In the following argument we describe simply connected regions as though they were disks with Jordan curve boundaries, and the argument carries over to the general case by thickening.

Pick any tile S in $\mathscr{T}$ such that ∂R and ∂S have an edge in common. Then, since ∂R and ∂S are Jordan curves and $S \subseteq R$, one has joint partitions of ∂R and ∂S as

$$\partial R = \partial R_1 \cup \cdots \cup \partial R_{2j},$$
$$\partial S = \partial S_1 \cup \cdots \cup \partial S_{2j},$$

in which all ∂R_i and ∂S_i are simple paths, each $\partial R_i \neq \varnothing$ is a set of consecutive edges of ∂R, $\partial R_{2i+1} = \partial S_{2i+1}$ and $\partial S_{2i} \cap \partial R = \varnothing$. Figure 2.4 illustrates such a decomposition—the first edge of ∂R_1 is a common edge of ∂R and ∂S. (In this definition ∂S_i and ∂R are treated as sets of edges. In fact ∂S_{2i} and ∂R treated as point sets may have isolated vertices in common, see Fig. 2.4.) Note that this definition allows $\partial S_{2i} = \varnothing$, and this may sometimes occur, see Fig. 2.5b.

Now let ∂R^* denote ∂R_2 together with the reversal of all edges in ∂S_2. Then ∂R^* is a simple closed path and encloses a nonempty region R^* that

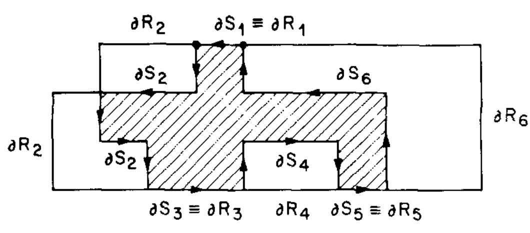
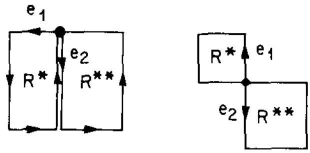


FIG. 2.4. Boundary of region R containing the tile S.



Case (a) Cases (b) and (c)

FIG. 2.5. Combining tile boundaries.

is simply connected. Let $R^{**} = R - R^{*}$. Then R^{**} has the simple closed path

$$\partial R^{**} = \partial S_1 \cup \partial S_2 \cup \partial S_3 \cup \partial R_4 \cup \partial S_5 \cup \cdots \cup \partial R_{2j}$$

as boundary, hence is simply connected. Now the tile S separates all the cells in R^{*} from the cells in $R^{**} - S$, hence all tiles in the tiling $\mathscr{T} - \{S\}$ of $R - S$ lie either in R^{*} or R^{**}, so $\mathscr{T}$ gives tilings $\mathscr{T}^{*}$ of R^{*} and $\mathscr{T}^{**}$ of R^{**}.

Finally, we observe that

$$\partial R(\mathbf{e}_1) = \partial R^{**}(\mathbf{e}_2)\, \partial R^{*}(\mathbf{e}_1),$$

where $\mathbf{e}_1$ is the first edge in ∂R_2, provided $\mathbf{e}_2$ is chosen suitably. The choice is: $\mathbf{e}_2$ is the first edge in ∂S_2 if $\partial S_2 \neq \varnothing$ (case (a)). Otherwise if $\partial S_3 \neq \varnothing$ then $\mathbf{e}_2$ is the first edge in ∂S_3 (case (b)), while if S_3 does not exist then $\mathbf{e}_2$ is the first edge in ∂S_1 (case (c)). These cases are illustrated in Fig. 2.5; case (c) occurs when $R^{**} = S$. This proves the claim. ∎

Theorem 2.1 provides a necessary condition for a perfect tiling to exist, hence serves as a criterion for proving nonexistence of perfect tilings. In general this theorem trades one hard problem for another. However in the special circumstances of the triangle tiling problems of Section 1, this criterion can be successfully applied.

3. Triangle Tiling Problems

The triangle tiling problems of Section 1 are easily converted to equivalent tiling problems on the square lattice. The region to be tiled becomes a "staircase" pictured in Fig. 3.1. The tile sets Σ_1 and Σ_2 for the two tiling problems are pictured in Fig. 3.2. Figure 3.2 gives a representative word for the combinatorial boundary $[\partial R]$ of each of the tiles pictured. A representative word for the boundary of the "staircase" region T_N is

$$\partial T_N = A^N U^{-N} (A^{-1} U)^N. \tag{3.1}$$

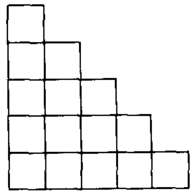
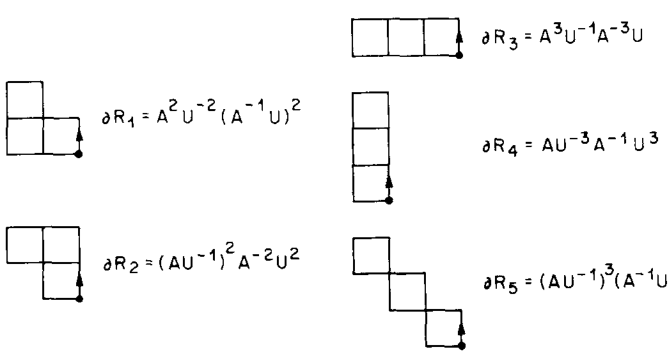


FIG. 3.1. Staircase region T_5.

The nonexistence parts of the proofs of Theorems 1.1 and 1.2 apply the criterion of Theorem 2.1: in these cases the boundary $[\partial T_N]$ is not contained in the appropriate tile group $\mathbf{T}(\Sigma_i)$. The proofs use a group-theoretic argument exploiting the special character of the tile group involved, due to the first author. One computes invariants associated to a special subgroup $\mathbf{H}$ of the free group $\mathbf{F} = \mathbf{F}_2$, defined below. The group $\mathbf{H}$ contains $[\partial T_N]$ and the tile groups $\mathbf{T}(\Sigma_i)$ of the two problems and has easily computable invariants, which are a consequence of the fact that the quotient group $\mathbf{F}/\mathbf{H}$ has a planar Cayley diagram.

Recall that the *Cayley diagram* $\mathscr{G}(\mathbf{F}_g/\mathbf{K})$ (also called the *group diagram*, *graph*, or *color diagram*) is a graph with directed labelled edges associated to a presentation of a quotient group $\mathbf{G} = \mathbf{F}_g/\mathbf{K}$ of the free group $\mathbf{F}_g$ on g generators, where $\mathbf{K}$ is a normal subgroup of relations. In the Cayley diagram of $\mathbf{G}$ each vertex corresponds to an element W of $\mathbf{G}$, and for each generator S_i of $\mathbf{F}_g$ there is a directed edge labelled i from vertex W to vertex $S_i W$. In particular every vertex in a Cayley diagram has $2g$ edges incident on it, with g edges directed inwards and g edges directed outwards.

The subgroup $\mathbf{K}$ of relations defining $\mathbf{G} = \mathbf{F}_g/\mathbf{K}$ has a simple characterization in terms of its Cayley diagram. Let $\bar{\mathscr{G}}(\mathbf{F}_g/\mathbf{K})$ denote the *undirected* labelled graph obtained from $\mathscr{G}(\mathbf{F}_g/\mathbf{K})$ by ignoring the directions on the



(a) Triangle tile set Σ_1 (b) Three-in-line tile set Σ_2

FIG. 3.2. Tile sets for triangle tiling problems.

edges. Associate to any word $W = G_k^{\varepsilon_k} G_{k-1}^{\varepsilon_{k-1}} \cdots G_1^{\varepsilon_1}$ in the free group $\mathbf{F}_g$ (where the G_i are generators and each $\varepsilon_i = \pm 1$) a directed path on the edges of the undirected graph $\mathscr{G}(\mathbf{F}_g/\mathbf{K})$ starting from the identity vertex I which at the ith step follows a directed edge from the vertex labelled $W_i = G_i^{\varepsilon_i} G_{i-1}^{\varepsilon_{i-1}} \cdots G_1^{\varepsilon_1}$ to $W_{i+1} = G_{i+1}^{\varepsilon_{i+1}} W_i$ along the unique edge labelled i between W_i and W_{i+1}. Then a word W is in $\mathbf{K}$ if and only if it corresponds to a closed path in $\mathscr{G}(\mathbf{F}_g/\mathbf{K})$ starting from I.

The special subgroup $\mathbf{H}$ of $\mathbf{F}_2$ is defined by the property that it has associated quotient group $\mathbf{G} = \mathbf{F}_2/\mathbf{H}$ whose (undirected) Cayley diagram $\mathscr{G}(\mathbf{F}_2/\mathbf{H})$ is the infinite planar graph that tiles the plane with hexagons and triangles as pictured in Fig. 3.3. The shaded vertex denotes the identity element, and if $\mathbf{F}_2 = \langle A, U \rangle$ then A-generator edges border triangles labelled A and similarly for indicate U-generator edges. The graph $\mathscr{G}(\mathbf{F}_2/\mathbf{H})$ is the boundary of a lattice tesselation of the plane by equilateral triangles and hexagons. The group $\mathbf{G}$ is isomorphic to one of the 17 plane crystallographic groups (the one labelled **p3** in [6, p. 49]), and the subgroup of relations $\mathbf{H}$ is given by

$$\mathbf{H} = N(\langle A^3, U^3, (U^{-1}A)^3 \rangle). \tag{3.2}$$

In the sequel we take as the definition of $\mathbf{H}$ that its elements correspond to closed paths in the undirected Cayley diagram $\mathscr{G}(\mathbf{F}_2/\mathbf{H})$; the explicit characterization (3.2) of $\mathbf{H}$ is never used.

The relevance of $\mathbf{H}$ to the triangle tiling problems is established by the following claim.

CLAIM. *The tile groups* $\mathbf{T}(\Sigma_1)$, $\mathbf{T}(\Sigma_2)$ *and the combinatorial boundaries* $[\partial T_N]$ *for* $N \equiv 0$ *or* 2 (mod 3) *are all contained in* $\mathbf{H}$.

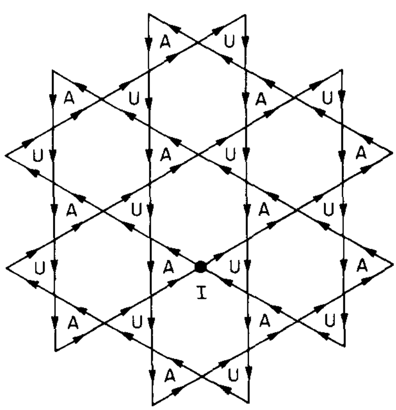


FIG. 3.3. Cayley diagram $\mathscr{G}(\mathbf{F}_2/\mathbf{H})$.

Proof of Claim. Since $\mathbf{H}$ is a normal subgroup of $\mathbf{F}_2$, it suffices to check that representative generators of $\{[\partial R]: R \in \Sigma_i\}$ and of $[\partial T_N]$ are in $\mathbf{H}$. To do this, one checks that such generators give closed paths starting from I in $\mathscr{G}(\mathbf{F}_2/\mathbf{H})$. This is easily done for the boundaries ∂R given in Fig. 3.2. It remains to check ∂T_N. To do this, one observes first that A^3, U^3 and $(A^{-1}U)^3$ are in $\mathbf{H}$. Next, these relations imply that $\partial T_N = A^N U^{-N} (A^{-1}U)^N$ is in $\mathbf{H}$ provided that ∂T_i is in $\mathbf{H}$ for $N \equiv i \pmod 3$, and one easily checks that ∂T_2, ∂T_3 are both in $\mathbf{H}$. ▌

The planar nature of the Cayley diagram $\mathscr{G} = \mathscr{G}(\mathbf{F}_2/\mathbf{H})$ gives rise to a large class of group-theoretic invariants associated to elements of $\mathbf{H}$, which consist of the winding numbers of the paths associated to elements of $\mathbf{H}$ about the hexagonal and triangular cells in the plane of the Cayley diagram. Let s be a cell (either hexagonal or triangular) in this tiling and let x_s be a point in the interior of s. The *winding number* (or *index*) $w(P; s)$ of a closed directed path $\mathbf{P}$ in $\mathscr{G}$ around s counts the number of times $\mathbf{P}$ encloses the cell s in the counterclockwise direction and is given by

$$w(\mathbf{P}; s) = \frac{1}{2\pi i} \oint_P \frac{1}{z - x_s}\, dz. \tag{3.3}$$

This quantity $w(P; s)$ is well defined independent of the choice of point x_s in s, and is *additive* in the sense that for two closed paths P_1 and P_2 starting at the same point W in $\mathscr{G}$ one has

$$w(P_2 P_1; s) = w(P_1; s) + w(P_2; s). \tag{3.4}$$

These facts about winding numbers in $\mathbb{R}^2$ are proved in basic texts on complex analysis, cf. [1, pp. 114–118; 15, pp. 233–241].

The winding number $w(\ ; s)$ induces a map $w(\ ; s)\colon \mathbf{H} \to \mathbb{Z}$ which assigns to a word $V \in \mathbf{H}$ the value $w(P(V); s)$ where $P(V)$ is the closed directed path in $\mathscr{G}$ associated to V, and (3.4) shows that this mapping is a homomorphism. Let S be any finite or infinite set of cells in $\mathscr{G}$ and let

$$w(P; S) = \sum_{s \in S} w(P; s).$$

This is well defined, since any closed path in the Cayley graph encloses a finite number of cells and it is clear that $w(\ ; S)\colon \mathbf{H} \to \mathbb{Z}$ is a homomorphism.

Now we use these invariants to solve the triangle tiling problems.

Proof of Theorem 1.1. Since all tiles in Σ_1 cover three cells, the region T_N cannot be tiled unless the number of cells $N(N+1)/2$ in T_N is a multiple of 3, hence $N \equiv 0$ or $2 \pmod 3$.

We will show that ∂T_N is not in the tile group $\mathbf{T}(\Sigma_1)$ when $N \equiv 3, 5, 6,$ or $8 \pmod{12}$, and hence that no tiling exists in these cases by Theorem 2.1. Consider the homomorphism $\phi\colon \mathbf{H} \to \mathbb{Z}$ with $\phi(V) = w(V; S)$, where S is the set of all hexagons in the Cayley diagram $\mathscr{G} = \mathscr{G}(\mathbf{F}_2/\mathbf{H})$. Using the boundaries in Fig. 3.2a it is easy to calculate that $\phi(\partial R_1) = 1$, $\phi(\partial R_2) = -1$. The translation-invariance of the Cayley graph $\mathscr{G}$ allows one to see that

$$\phi(W\,\partial R_i W^{-1}) = \phi(\partial R_i), \qquad \text{for} \quad i = 1, 2,$$

for any word W in the free group. Hence

$$\phi([\partial R_1]) = 1, \qquad \phi([\partial R_2]) = -1.$$

We know that $\partial T_N \in \mathbf{H}$, and (3.1) yields

$$\phi(\partial T_N) = \left[\frac{N+1}{3}\right]. \tag{3.5}$$

Suppose that ∂T_N is in the tile group $\mathbf{T}(\Sigma_1)$, in which case there exists an integer m and words W_i such that

$$\partial T_N = \prod_{i=1}^{m} W_i (\partial R_{k_i})^{\varepsilon_i} W_i^{-1}, \tag{3.6}$$

where each $k_i = 1$ or 2 and each $\varepsilon_i = 1$ or -1. Then

$$\phi(\partial T_N) = \sum_{i=1}^{m} \phi(W_i (\partial R_{k_i})^{\varepsilon_i} W_i^{-1}) = \sum_{i=1}^{m} \varepsilon_i \phi(\partial R_{k_i}) \equiv m \pmod{2}. \tag{3.7}$$

Next we introduce a second homomorphism $\psi\colon \mathbf{H} \to \mathbb{Z}$ which views a word in $\mathbf{H}$ as defining a closed directed path in the square lattice $\mathbb{Z}^2$ starting at $(0, 0)$ as in the proof of Theorem 2.1 and which associates to each such path the sum of its winding numbers about all cells in the square lattice. That is, for a tile R the mapping $\psi(\partial R)$ counts the number of cells covered by the tile, so that, for example, one has

$$\psi(W\,\partial R_i W^{-1}) = \psi(\partial R_i) = 3 \qquad \text{for} \quad i = 1, 2,$$

for all W in the free group $\mathbf{F}_2$, and one has

$$\psi(\partial T_N) = \binom{N+1}{2}. \tag{3.8}$$

Now the hypothesis (3.6) gives

$$\psi(\partial T_N) = \sum_{i=1}^{m} \psi(W_i (\partial R_{k_i})^{\varepsilon_i} W_i^{-1}) = \sum_{i=1}^{m} \varepsilon_i \psi(\partial R_{k_i}) \equiv m \pmod{2}. \tag{3.9}$$

Combining (3.5), (3.7), (3.8), and (3.9) yields

$$\left[\frac{N+1}{3}\right] \equiv \binom{N+1}{2} \pmod 2,$$

which is a necessary condition for ∂T_N to be in the tile group $T(\Sigma_1)$. Both sides of this congruence are periodic (mod 12), and it is easily checked that this congruence does not hold for $N \equiv 3, 5, 6$, and 8 (mod 12), proving that $\partial T_N \notin \mathbf{T}(\Sigma_1)$ in these cases.

It is easy to construct tilings for $N \equiv 0, 2, 9$, or 11 (mod 12). We leave it to the reader to construct such tilings for T_2, T_9, T_{11}, and T_{12}. One then proceeds by induction on K, constructing tilings for T_{12K+L} for $L = 2, 9$, 11, and 12 from that for T_{12K} using the scheme pictured in Fig. 3.4, noting that since a 2×3 rectangle can be tiled, so can a 5×6 rectangle and an 11×12 rectangle, whence an $L \times 12K$ rectangle can be tiled.

Proof of Theorem 1.2. Since all tiles in Σ_2 cover three cells, one must have $N \equiv 0$ or 2 (mod 3) as above. We will show that ∂T_N is not in the tile group $\mathbf{T}(\Sigma_2)$ in all cases, so that a tiling of T_N is impossible by Theorem 2.1. To do this, consider the homomorphism $\phi\colon \mathbf{H} \to \mathbb{Z}$ which counts the sum of all winding numbers around all triangles labelled U in the Cayley diagram $\mathscr{G}(\mathbf{F}_2/\mathbf{H})$ in Fig. 3.3. One easily calculates using the boundaries in Fig. 3.2b that

$$\phi(\partial R_3) = \phi(\partial R_4) = \phi(\partial R_5) = 0. \tag{3.10}$$

As in the previous proof one has

$$\phi(W\,\partial R_i W^{-1}) = \phi(\partial R_i) \qquad \text{for} \quad i = 3, 4, 5,$$

for all W in the free group. A computation using (3.1) in the Cayley diagram yields

$$\phi(\partial T_N) = \left[\frac{N+1}{3}\right]. \tag{3.11}$$

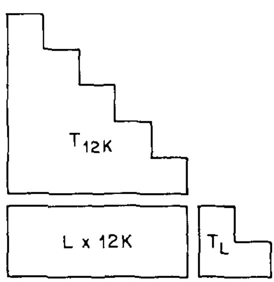


FIG. 3.4. Tiling of T_{12K+L} for $L = 2, 9, 11, 12$.

Suppose that ∂T_N were in the tile group $\mathbf{T}(\Sigma_2)$, so that

$$\partial T_N = \prod_{i=1}^{m} W_i (\partial R_{k_i})^{\varepsilon_i} W_i^{-1},$$

where each $W_i \in \mathbf{F}_2$, each $k_i \in \{3, 4, 5\}$ and each $\varepsilon_i = 1$ or -1. Then

$$\phi(\partial T_N) = \sum_{i=1}^{m} \varepsilon_i \phi(\partial R_{k_i}) = 0,$$

by (3.10). This contradicts (3.11) for $N \geqslant 2$, and this contradiction proves that ∂T_N is not in $\mathbf{T}(\Sigma_2)$ for $N \equiv 0$ or 2 (mod 3). ▌

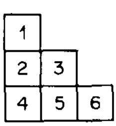
A wide variety of related tiling problems can be solved using invariants associated to groups $\mathbf{H}$ for which $\mathbf{F}_g/\mathbf{H}$ has a planar Cayley diagram. See Thurston [25] for extensions of this approach and more examples.

4. Triangle Tiling Problem: Signed Tilings

Recall that a *signed tiling* of a region R by tiles from a set Σ consists of placements of a finite set of tiles, each assigned a weight of 1 or -1, such that for each cell in R the sum of the weights of the tiles covering this cell is 1 and for each cell not in R the sum of the weights covering this cell is 0.

Proof of Theorem 1.3. Since each tile in a signed tiling covers ± 3 cells (taking weights into account), the number of cells in T_N must be $\equiv 0$ (mod 3). Since T_N has $\binom{N+1}{2}$ cells, this requires $N \equiv 0$ or 2 (mod 3).

It suffices to exhibit signed tilings for $N \equiv 3$, 5, 6, 8 (mod 12). Let $N = 12K + L$ where $L = 3$, 5, 6, or 8. Then the triangular region T_N may be decomposed as pictured in Fig. 3.4 into regions T_{12K}, T_L and a rectangular region $L \times 12K$. The region T_{12K} may be tiled by Theorem 1.1, and the $L \times 12K$ rectangular region can also be tiled by congruent copies of the triangular tile T_3, by observing that a 3×2 rectangle can be so tiled, as can 5×6, 6×6, 8×6 rectangles. Hence to prove the theorem it suffices to find signed tilings of T_3, T_5, T_6, and T_8. Such tilings are easy to find. A signed tiling for T_3 is pictured in Fig. 4.1; signed tilings for T_5, T_6, and T_8 are left as exercises for the reader. ▌



WEIGHT +1 TILES: 123, 245, 356

WEIGHT −1 TILES: 235

FIG. 4.1. Signed tiling of T_3 by triangle tiles.

Proof of Theorem 1.4. We first show that

$$N \equiv 0 \text{ or } 8 \pmod 9$$

is a necessary condition for a signed tiling to exist. Number consecutively the horizontal rows of cells in the staircase region T_N in the square lattice so that row j contains exactly j cells. The tile set Σ_2 consists of three tiles labelled R_3, R_4, R_5 in Fig. 3.2, which we relabel A, B, and C, respectively, for notational convenience. A placement of tile A always has three cells in a single row, while B and C always have one cell in each of three contiguous rows. Suppose that a signed tiling exists for T_N and for this tiling let $n_{BC}^+(j)$ (resp. $n_{BC}^-(j)$) count the number of tiles of type B or C having weight $+1$ (resp. -1) which contain one cell in each of rows j, $j+1$, and $j+2$, and set

$$n_{BC}(j) = n_{BC}^+(j) - n_{BC}^-(j).$$

By counting the number of tiles covered in row j by this signed tiling one finds that

$$n_{BC}(j-2) + n_{BC}(j-1) + n_{BC}(j) \equiv j \pmod 3, \qquad 1 \leqslant j \leqslant N, \tag{4.1a}$$

$$n_{BC}(j-2) + n_{BC}(j-1) + n_{BC}(j) \equiv 0 \pmod 3, \qquad j \leqslant 0 \text{ or } j > N. \tag{4.1b}$$

Since a signed tiling is finite, there is a positive integer k such that all tiles are in rows $-k$ to $+k$. Hence $n_{BC}(-k-1) = n_{BC}(-k-2) = 0$ and applying the congruences for $j = -k, -k+1, ..., 0$ successively one obtains

$$n_{BC}(j) \equiv 0 \pmod 3, \qquad -k \leqslant j \leqslant 0.$$

Similarly starting from $n_{BC}(k+1) = n_{BC}(k+2) = 0$ and working backwards using $j = k+2, k+1, k, ..., N+3$ successively one obtains

$$n_{BC}(j) \equiv 0 \pmod 3, \qquad N+1 \leqslant j \leqslant k. \tag{4.2}$$

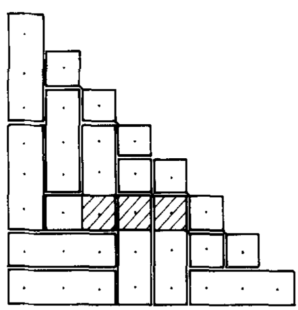


FIG. 4.2. Signed tiling of T_8 by three-in-line tiles. (A weight -1 tile is placed on the shaded squares.)

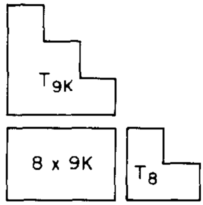


FIG. 4.3. Signed tiling of T_{9K+8} by three-in-line tiles.

Now working forwards using the congruences for $j = 1, 2, ..., N$ one finds for $1 \leqslant j \leqslant N$ that $n_{BC}(j) \pmod 3$ is periodic with period 9 and takes the values $(1, 1, 1, 2, 2, 2, 0, 0, 0)$ for $j = (1, 2, 3, 4, 5, 6, 7, 8, 9)$, respectively. But (4.2) implies that $n_{BC}(N-1) \equiv n_{BC}(N) \equiv 0 \pmod 3$; this is impossible unless $N \equiv 0$ or $8 \pmod 9$.

It remains to construct signed tilings for $N \equiv 0$ or $8 \pmod 9$. A signed tiling for $N = 8$ is easy to find, and one is given in Fig. 4.2. Signed tilings for $N = 9K$, $9K + 8$ can be constructed by induction on K. Given a signed tiling for $9(K-1) + 8$ we obtain one for $9K$ by tiling the last row with tiles of type R_3. Then given a signed tiling of T_{9K} we may subdivide T_{9K+8} as pictured in Fig. 4.3, and use the signed tilings of T_{9K} and T_8 provided by the induction hypothesis together with a tiling of the $8 \times 9K$ rectangular region with R_3 tiles. This completes the construction. ∎

5. Generalized Coloring Arguments and Tile Homology

Many tiling problems have been resolved using arguments involving colorings or weightings of the cells of the underlying lattice. We show that such arguments have a natural interpretation in terms of boundary invariants and that the strongest such arguments are equivalent to detecting the existence of signed tilings.

Consider the square lattice with its associated free group $\mathbf{F} = \langle A, U \rangle$ and cycle group $\mathbf{C} = [\mathbf{F} : \mathbf{F}]$. Coloring or weighting arguments correspond to additive invariants assigned to cells of the square lattice. Part (iii) of the following theorem shows that a natural group encoding such invariants is the maximal abelian quotient group $\mathbf{A}_0 = \mathbf{C}/[\mathbf{C} : \mathbf{C}]$, which we call the *cell group*.

THEOREM 5.1. (i) *The cycle group* $\mathbf{C}$ *consists of all words* W *such that* $P(W)$ *is a closed directed path in* $\mathbb{Z}^2$, *i.e.*, $\mathbf{C} = [\mathbf{F} : \mathbf{F}]$.

(ii) *The group* $[\mathbf{C} : \mathbf{C}]$ *consists of all words* W *such that* $P(W)$ *is a closed directed path in* $\mathbb{Z}^2$ *with winding number* 0 *around every cell in* $\mathbb{Z}^2$. *Consequently*, $[\mathbf{C} : \mathbf{C}]$ *is a normal subgroup of* $\mathbf{F}$.

(iii) *The group* $\mathbf{A}_0 = \mathbf{C}/[\mathbf{C}:\mathbf{C}]$ *is a direct sum of a countable number of copies of* $\mathbb{Z}$, *which are in one-to-one correspondence with the cells* c_{ij} *of the lattice* $\mathbb{Z}^2$. *The projection map* $\pi_{i,j}: \mathbf{C} \to \mathbb{Z}$ *onto the* c_{ij}*th* $\mathbb{Z}$*-summand of* $\mathbf{A}_0$ *is given by the winding number* $w(P(W); c_{ij})$.

We defer the proof of this theorem to the end of this section, in order to proceed directly to the discussion of coloring arguments.

A *generalized coloring map* is any homomorphism $\phi: \mathbf{C} \to \mathbf{A}$, where $\mathbf{A}$ is an abelian group. A *generalized coloring argument* uses a generalized coloring map ϕ to show that a simply connected region R cannot be tiled by tiles in a set Σ by showing that the image of the combinatorial boundary $[\partial R]$ under ϕ is not contained in the image of the tile group $\mathbf{T}(\Sigma)$ under ϕ. Since all such homomorphisms ϕ can be factored as the projection $\pi: \mathbf{C} \to \mathbf{A}_0 = \mathbf{C}/[\mathbf{C}:\mathbf{C}]$ composed with a homomorphism $\tilde{\phi}: \mathbf{A}_0 \to \mathbf{A}$, the strongest generalized coloring map is the projection π onto the cell group $\mathbf{A}_0$.

We justify the name "generalized coloring argument" by showing how the coloring argument in Golomb [9] can be formulated in terms of a generalized coloring map. It is well known that the checkerboard with two opposite corners removed ("mutilated checkerboard") pictured in Fig. 5.1 cannot be tiled with dominoes. To prove this one colors the checkerboard in a checkerboard pattern. Depending on where the mutilated checkerboard is placed on the lattice, it covers either 30 black squares and 32 white squares or 32 black squares and 30 white squares. Since each domino in a tiling covers one square of each color, any perfectly tiled region must contain the same number of squares of each color, hence the mutilated checkerboard cannot be tiled.

To obtain an equivalent generalized coloring argument one colors the cells of the lattice $\mathbb{Z}^2$ in a checkerboard pattern with the cell c_{ij} having lower left corner (i, j) being colored black if $i+j \equiv 0 \pmod 2$ and white if $i+j \equiv 1 \pmod 2$. Now define a map $\phi: \mathbf{C} \to \mathbb{Z} \oplus \mathbb{Z}$ given by $\phi = \phi_1 \oplus \phi_2$, where $\phi_1(W)$ counts the sum of the winding numbers of the closed path $P(W)$ about all cells c_{ij} with $i+j \equiv 0 \pmod 2$ (the "white" cells) and $\phi_2(W)$

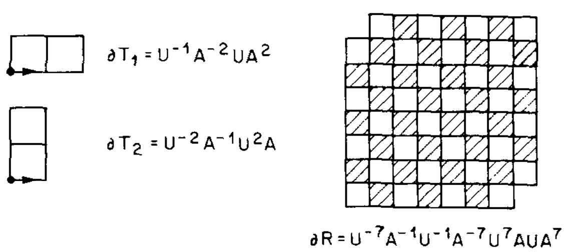


FIG. 5.1. Mutilated checkerboard and dominoes.

denotes the sum of the winding numbers of the closed path $P(W)$ about all cells c_{ij} with $i+j \equiv 1 \pmod 2$ (the "black cells"). The mutilated checkerboard R has boundary $\partial R = U^{-7}(A^{-1}U^{-1})A^{-7}U^{7}(AU)A^{7}$ while

$$\mathbf{T}(\Sigma) = N(\langle U^{-1}A^{-2}UA^{2}, U^{-2}A^{-1}U^{2}A \rangle),$$

see Fig. 5.1. Now $\phi([\partial R]) = \{(30, 32), (32, 30)\}$ while $\phi(\mathbf{T}(\Sigma)) = \{(n, n): n \in \mathbb{Z}\}$, which shows that R cannot be tiled by dominoes.[1]

Other coloring and weighting arguments used in [5, 7, 9, 10, 13, 17] can be framed in terms of generalized coloring maps in similar fashion.

The information about nonexistence of tilings given by any generalized coloring map $\phi: \mathbf{C} \to \mathbf{A}$ is completely expressed in terms of the tile homotopy group, using the quotient map $\bar{\phi}: \mathbf{C}/\mathbf{T}(\Sigma) \to \tilde{\mathbf{A}} = \mathbf{A}/\phi(\mathbf{T}(\Sigma))$ induced by factoring out the tile group $\mathbf{T}(\Sigma)$; indeed $\phi([\partial R])$ is contained in $\phi(\mathbf{T}(\Sigma))$ if and only if $\bar{\phi}([\partial R])$ consists of the identity element in $\tilde{\mathbf{A}}$. Conversely, any homomorphism $\bar{\phi}$ from the tile homotopy group $\mathbf{h}(\Sigma)$ into an abelian group $\tilde{\mathbf{A}}$ arises from the generalized coloring map $\phi: \mathbf{C} \to \tilde{\mathbf{A}}$ given by $\phi = \bar{\phi} \circ \bar{\pi}$, where $\bar{\pi}: \mathbf{C} \to \mathbf{C}/\mathbf{T}(\Sigma)$ is the natural projection. Thus we may equally well consider generalized coloring arguments as specified by homomorphisms $\bar{\phi}$ from the tile homotopy group $\mathbf{h}(\Sigma)$ to abelian groups $\tilde{\mathbf{A}}$.

In this new context the maximal information available about tilings is given by the map $\pi_s: \mathbf{h}(\Sigma) \to \mathbf{H}(\Sigma)$, where $\mathbf{H}(\Sigma)$ is the maximal abelian quotient group of $\mathbf{h}(\Sigma)$.[2] We call $\mathbf{H}(\Sigma)$ the *tile homology group*, by analogy with the well-known fact that the first homology group is the maximal abelian quotient group of the first homotopy group. Using the projection $\bar{\pi}: \mathbf{C} \to \mathbf{C}/\mathbf{T}(\Sigma)$ we have $\mathbf{H}(\Sigma) = \mathbf{C}/\mathbf{B}(\Sigma)$, where $\mathbf{B}(\Sigma)$ is the kernel of $\pi_s \circ \bar{\pi}$. We call $\mathbf{B}(\Sigma)$ the *tile boundary group*. $\mathbf{B}(\Sigma)$ is the smallest normal subgroup of $\mathbf{C}$ containing $\mathbf{T}(\Sigma)$ and $[\mathbf{C}:\mathbf{C}]$. We claim that

$$\mathbf{B}(\Sigma) = \mathbf{T}(\Sigma)[\mathbf{C}:\mathbf{C}], \tag{5.1}$$

and that $\mathbf{B}(\Sigma)$ is a normal subgroup of $\mathbf{F}$. The inclusion $\mathbf{T}(\Sigma)[\mathbf{C}:\mathbf{C}] \subseteq \mathbf{B}(\Sigma)$ is clear. To prove the other inclusion, note that $\mathbf{T}(\Sigma)[\mathbf{C}:\mathbf{C}]$ is a normal subgroup of $\mathbf{F}$ (hence of $\mathbf{C}$) using the general fact that $\mathbf{G}_1\mathbf{G}_2 = \{g_1 g_2: g_1 \in \mathbf{G}_1, g_2 \in \mathbf{G}_2\}$ is a normal subgroup of a group $\mathbf{G}$ whenever $\mathbf{G}_1$ and $\mathbf{G}_2$ are both normal subgroups of $\mathbf{G}$. Since $\mathbf{B}(\Sigma)$ is the smallest normal subgroup of $\mathbf{C}$ containing $\mathbf{T}(\Sigma)$ and $[\mathbf{C}:\mathbf{C}]$, it follows that $\mathbf{B}(\Sigma) \subseteq \mathbf{T}(\Sigma)[\mathbf{C}:\mathbf{C}]$, proving the claim.

[1] Since homomorphisms map conjugacy classes to conjugacy classes, one may ask why the image $\phi([\partial R])$ is not a single element in the *abelian* group $\mathbb{Z} \oplus \mathbb{Z}$. This is because $[\partial R]$ is actually an $\mathbf{F}$-conjugacy class, so its image is actually a conjugacy class in the *nonabelian* group $\mathbf{F}/[\mathbf{C}:\mathbf{C}]$.

[2] The map π_s is exactly the quotient map $\pi_s: \mathbf{C}/\mathbf{T}(\Sigma) - \mathbf{A}_0/\pi(\mathbf{T}(\Sigma))$ induced from the strongest generalized coloring map $\pi: \mathbf{C} \to \mathbf{A}_0 = \mathbf{C}/[\mathbf{C}, \mathbf{C}]$, as is easily checked.

The discussion above shows that the maximal information about non-existence of tilings obtainable by a generalized coloring argument concerns whether or not the combinatorial boundary $[\partial R]$ is in the tile boundary group $B(\Sigma)$. Now we show that this condition is a necessary and sufficient condition for a signed tiling to exist.

THEOREM 5.2. *For a simply connected region R and set of tiles Σ the following conditions are equivalent*:

(i) *R has a signed tiling using tiles in Σ.*

(ii) *The combinatorial boundary $[\partial R]$ is in the tile boundary group $\mathbf{B}(\Sigma)$.*

Proof. (i)$\Rightarrow$(ii). Suppose that R has a signed tiling. Place R on $\mathbb{Z}^2$ so that it has an oriented boundary ∂R with base point $(0, 0)$. Let $\{(T_i, \varepsilon_i): 1 \leqslant i \leqslant k\}$ denote the signed tiling of R, with $\varepsilon_i = 1$ or -1 being the sign of the tile T_i. Let ∂T_i denote an oriented boundary of tile T_i with base point $(0, 0)$, and let W_i be an oriented path from $\mathbf{m}_i$ to $(0, 0)$, where $\mathbf{m}_i$ is the basepoint where the tile T_i is placed. Consider the word

$$W = (\partial R)^{-1} \prod_{i=1}^{k} (W_i (\partial T_i)^{\varepsilon_i} W_i^{-1}).$$

We claim that $P(W)$ is a closed path which has winding number 0 about all cells in $\mathbb{Z}^2$, so that by Theorem 5.1(ii), $W \in [\mathbf{C}:\mathbf{C}]$. To see this, we note that $P((\partial R)^{-1})$ has winding number -1 about all cells in R and winding number 0 elsewhere, while $P(W_i (\partial T_i)^{\varepsilon_i} W_i^{-1})$ has winding number ε_i about all cells in T_i and winding number 0 elsewhere, so the claim follows by definition of a signed tiling. Thus $W \in [\mathbf{C}, \mathbf{C}]$ and

$$\partial R = \left(\prod_{i=1}^{k} (W_i (\partial T_i)^{\varepsilon_i} W_i^{-1}) \right) W^{-1}$$

is expressed as an element of $\mathbf{T}(\Sigma)[\mathbf{C}:\mathbf{C}]$, which is $\mathbf{B}(\Sigma)$. Since $\mathbf{B}(\Sigma)$ is a normal subgroup of $\mathbf{F}$, the conjugacy class $[\partial R] \subseteq \mathbf{B}(\Sigma)$.

(ii)$\Rightarrow$(i). Place R so that it has an oriented boundary ∂R with base point $(0, 0)$. Since $\partial R \in \mathbf{B}(\Sigma)$ and $\mathbf{B}(\Sigma) = \mathbf{T}(\Sigma)[\mathbf{C}:\mathbf{C}]$, one has

$$\partial R = \left(\prod_{i=1}^{k} (W_i (\partial T_i)^{\varepsilon_i} W_i^{-1}) \right) W^{-1},$$

where ∂T_i are oriented boundaries of tiles in Σ with basepoint $(0, 0)$, ε_i takes values ± 1, and $W \in [\mathbf{C}:\mathbf{C}]$. Now we can reverse the previous argument. By Theorem 5.1(ii) the path $P(W)$ associated to the word

$$W = (\partial R)^{-1} \left(\prod_{i=1}^{k} (W_i (\partial T_i)^{\varepsilon_i} W_i^{-1}) \right)$$

has winding number 0 about all cells. Computing the winding numbers of $P((\partial R)^{-1})$ and $P(W_i(\partial T_i)^{\varepsilon_i} W_i^{-1})$ about each cell shows that $\{(T_i, \varepsilon_i)\}$ is a signed tiling of R. ∎

Theorem 1.5 follows easily from this result.

Proof of Theorem 1.5. Theorem 5.3 and the discussion preceding it show that a generalized coloring argument can only prove the nonexistence of a tiling by proving the nonexistence of a signed tiling. Since we have shown that both triangle tiling problems have instances having a signed tiling but no perfect tiling, these problems cannot be solved by any generalized coloring argument. ∎

We have obtained the following hierarchy of successively weaker tiling invariants.

THEOREM 5.3. *Let R be a simply connected region and Σ a set of tiles. Consider the conditions*:

(H1) *R can be tiled using tiles in Σ.*

(H2) *$[\partial R]$ is in the tile group $\mathbf{T}(\Sigma)$.*

(H3) *$[\partial R]$ is in the tile boundary group $\mathbf{B}(\Sigma)$.*

Then (H1) $\Rightarrow$ (H2) $\Rightarrow$ (H3). *These implications are not reversible in general.*

Proof. The assertion (H1) $\Rightarrow$ (H2) is exactly Theorem 2.1. (H2) $\Rightarrow$ (H3) is immediate.

To show (H2) $\not\Rightarrow$ (H1) let the tile set Σ consist of a 2×2 square and a 3×3 square. Consider the L-shaped region R obtained by removing a 2×2 square from the upper right corner of a 3×3 square. It is clear that $[\partial R] \in \mathbf{T}(\Sigma)$ and that R cannot be tiled by translates of these two tiles.

The implication (H3) $\not\Rightarrow$ (H2) follows from the triangle tiling by lines problem. By Theorem 1.4 a signed tiling exists for $N \equiv 0$ or 8 (mod 9), and (H3) then holds by Theorem 5.2. The proof of Theorem 1.2 shows that (H2) does not hold in this case. ∎

By using semigroups instead of groups one can obtain a necessary and sufficient condition for a tiling to exist. The *tile semigroup* $\mathbf{T}^+(\Sigma)$ is defined to be the subsemigroup of the free group $\mathbf{F}$ generated by the conjugacy classes $\{[\partial T]: T \in \Sigma\}$.

THEOREM 5.4. *Let R be a simply connected region and Σ a set of tiles. The following conditions are equivalent*:

(i) *R can be tiled by tiles in Σ.*

(ii) *$[\partial R]$ is contained in the tile semigroup $\mathbf{T}^+(\Sigma)$.*

Proof. (i) $\Rightarrow$ (ii). The proof of Theorem 2.1 actually shows this. (ii) $\Rightarrow$ (i). Using tiles with basepoints, if $[\partial R] \subseteq \mathbf{T}^+(\Sigma)$, then ∂R can be expressed as

$$\partial R = \prod_{i=1}^{k} \dot{W}_i (\partial T_i) W_i^{-1},$$

from which a tiling of R can be directly read off, using winding numbers around cells of R. ∎

Now we give the proof that was deferred.

Proof of Theorem 5.1. (i) Let $\mathbf{C}_0$ consist of all words W such that $P(W)$ is a closed directed path in $\mathbb{Z}^2$. $\mathbf{C}_0$ is clearly a normal subgroup of $\mathbf{F}$.

We first show that $[\mathbf{F}:\mathbf{F}] \subseteq \mathbf{C}_0$. Expressing a word W in the generators A, U, A^{-1}, U^{-1} as a directed path it is easy to see this path is closed iff

$$\# \text{ occurrences}(A) = \# \text{ occurrences}(A^{-1}),$$

$$\# \text{ occurrences}(U) = \# \text{ occurrences}(U^{-1}).$$

All commutators $ABA^{-1}B^{-1}$ have this property, hence $[\mathbf{F}:\mathbf{F}] \subseteq \mathbf{C}_0$.

Now we prove $\mathbf{C}_0 \subseteq [\mathbf{F}:\mathbf{F}]$. Let W be a word representing an element of $\mathbf{C}_0$. We assign to each word W an invariant (n, k, l), where n is the length of the word, k is the maximum value $i^2 + j^2$ of any vertex $(i, j) \in \mathbb{Z}^2$ visited by the path $P(W)$, and l denotes the number of vertices (i, j) with $i^2 + j^2 = k$ (counted with multiplicity) that are visited by the path $P(W)$. Note that k and l are both less than n^2. We proceed by induction on triples (n, k, l) ordered lexicographically. The base case is $(0, 0, 0)$, which is the identity. For the induction step, if W contains any adjacent pairs of generators GG^{-1} we may cancel them and decrease its length. If this is not the case, the path $P(W)$ corresponding to W traverses no edge twice in succession. Let (i, j) be a vertex with $i^2 + j^2 = k$ visited by $P(W)$. If (i, j) is in the first quadrant, then either $W = W_2 A^{-1} U W_1$ with UW_1 visiting vertex (i, j) or $W = W_2 U^{-1} A W_1$ with AW_1 visiting vertex (i, j), as pictured in Fig. 5.2.

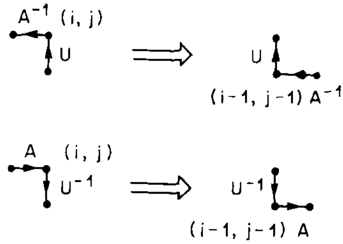


FIG. 5.2. Shortening a word in the first quadrant.

In the first case the word $\tilde{W} = W_2 U A^{-1} W_1$ has a lexicographically smaller value $(n, k, l-1)$ or $(n, k^1, *)$, and in the group $\mathbf{F}$ one has

$$W = (W_2 A^{-1} U A U^{-1} W_2^{-1}) \tilde{W},$$

where $W_2 A^{-1} U A U^{-1} W_2^{-1}$ is a conjugate of a commutator, so it is in $[\mathbf{F}:\mathbf{F}]$. By the induction hypothesis, $\tilde{W}$ is in $[\mathbf{F}:\mathbf{F}]$, hence so is W. In the second case above we use $\tilde{W} = W_2 A U^{-1} W_1$ and the same argument. Similar arguments work for (i, j) in the other three quadrants, completing the induction step, and proving $\mathbf{C}_0 = [\mathbf{F}:\mathbf{F}] = \mathbf{C}$.

(ii) Let $\mathbf{C}_1$ consist of all words W such that $P(W)$ is a closed path with winding number 0 about all cells. $\mathbf{C}_1$ is clearly a normal subgroup of $\mathbf{F}$. We must show $\mathbf{C}_1 = [\mathbf{C}:\mathbf{C}]$.

We show $[\mathbf{C}:\mathbf{C}] \subseteq \mathbf{C}_1$. Since winding numbers are additive, if $W_1, W_2 \in \mathbf{C}$ then both they and their inverses correspond to closed paths, whence

$$\begin{aligned} &w(W_1 W_2 W_1^{-1} W_2^{-1}; c_{ij}) \\ &\quad = w(W_1; c_{ij}) + w(W_2; c_{ij}) + w(W_1^{-1}; c_{ij}) + w(W_2^{-1}; c_{ij}) = 0, \end{aligned}$$

for all cells c_{ij}.

We show $\mathbf{C}_1 \subseteq [\mathbf{C}:\mathbf{C}]$ by induction on the invariant (n, k, l) ordered as in the previous argument. The base case is the empty word, identified with the identity element of $\mathbf{F}$. For the induction step, let W have value (n, k, l). If W contains any adjacent pairs of generators of the form GG^{-1}, we may cancel them and complete the induction step. Otherwise let (i, j) with $i^2 + j^2 = k$ be a vertex visited by the path corresponding to W. For the subsequent argument we relabel the cells so that c_{ij} denotes the cell whose vertex *furthest* from the origin $(0, 0)$ is (i, j). We examine all the visits of the path of W to (i, j). Suppose (i, j) is in the first quadrant. At each visit the path either arrives at this vertex from $(i, j-1)$ and exits to $(i-1, j)$ via $A^{-1}U$, or else arrives from $(i-1, j)$ and exits to $(i, j-1)$ via $U^{-1}A$, as in Fig. 5.2. Now we compute the winding number $w(W; c_{ij})$ using the argument principle, as in [15]. Since the path never crosses the line $i+j=k$, one has

$$w(W; c_{ij}) = \#\ \text{occurrences}(A^{-1}U) - \#\ \text{occurrences}(U^{-1}A),$$

where this sum is over visits to (i, j) only. Since this winding number is zero, there must be at least one visit of each kind, and one has $W = W_3 A^{-1} U W_2 U^{-1} A W_1$ or $W = W_3 U^{-1} A W_2 A^{-1} U W_1$, where W_1, W_2, W_3 are possibly empty words, and the path of W visits (i, j) in the middle of $A^{-1}U$ and of $U^{-1}A$. In the first case, let $\tilde{W} = W_3 U A^{-1} W_2 A U^{-1} W_1$, which has invariant either $(n, k, l-2)$ or $(n, k^1, *)$, and note that as words in F one has

$$W = (W_3 A^{-1} U W_2 U^{-1} A)(U A^{-1} W_2^{-1} A U^{-1} W_3^{-1}) \tilde{W}.$$

Calling the right side of this expression $Z\tilde{W}$, one finds after inserting suitable words of the form DD^{-1} that

$$Z = MNM^{-1}N^{-1},$$

where $M = W_3 A^{-1} U W_2 U^{-1} A W_3^{-1}$ and $N = W_3 U A^{-1} U^{-1} A W_3^{-1}$. Since M and N yield closed paths, it follows that Z is in $[\mathbf{C}:\mathbf{C}]$. By the induction hypothesis $\tilde{W}$ is in $[\mathbf{C}:\mathbf{C}]$, hence so is W. Similar arguments work in the second case $W = W_3 U^{-1} A W_2 A^{-1} U W_1$ and for (i, j) in the other three quadrants. This completes the induction step showing $\mathbf{C}_1 \subseteq [\mathbf{C}:\mathbf{C}]$, and (ii) is proved.

(iii) Define a homomorphism $\pi = \bigoplus_{i,j} \pi_{i,j}$ from $\mathbf{C}$ to $\bigoplus_{(i,j)} \mathbb{Z}$ by $\pi_{i,j} = w(P(W); c_{ij})$. This map is well defined by part (i) and its kernel is $[\mathbf{C}:\mathbf{C}]$ by part (ii). Hence its image is isomorphic to $\mathbf{C}/[\mathbf{C}:\mathbf{C}]$. ∎

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