Tilting Preenvelopes and Cotilting Precovers

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Dedicated to Professor Helmut Lenzing on his 60th birthday

Abstract

We relate the theory of envelopes and covers to tilting and cotilting theory, for (infinitely generated) modules over arbitrary rings. Our main result characterizes tilting torsion classes as the pretorsion classes providing special preenvelopes for all modules. A dual characterization is proved for cotilting torsion-free classes using the new notion of a cofinendo module. We also construct unique representing modules for these classes.

Introduction

Tilting and cotilting modules have first been considered in the context of finitely generated modules over finite dimensional algebras. The category equivalences and dualities induced by them have provided tools for exploring the structure of module categories over their endomorphism algebras, as well as the structure of tilting torsion, and cotilting torsion-free, classes of modules. The pioneering works of Brenner-Butler [4], Happel-Ringel [16], Assem [1] and Smalø [19] have later been extended to the setting of (infinitely generated) modules over arbitrary rings [17], [6], [10], [7], [9] et al.

Similarly, the idea of (minimal) left and right aproximations of modules developed by Auslander, Smalø and Reiten [3], [2] for finitely generated modules over artin algebras turned out to be useful for modules over arbitrary rings. In fact, the corresponding general notions have independently been discovered by Enochs [11] and termed (pre)envelopes and (pre)covers, see also [15] and [20]. Enochs also noticed that many important concepts occur naturally as particular instances of the general notions. This concerns injective and pure injective hulls, or projective, injective and flat covers. Indeed, one of the key open problems of contemporary module theory deals with covers: the Flat Cover Conjecture asks whether each module over an arbitrary ring has a flat cover, [22].

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In the present paper, we investigate relations between the theory of precovers and preenvelopes, and the tilting and cotilting theory for (infinitely generated) modules over arbitrary rings.

Our main results are Theorem 2.1 and its dual, Theorem 2.5. Tilting torsion classes are characterized in Theorem 2.1 as those pretorsion classes which provide special preenvelopes for all modules or, equivalently, for the regular module. The dual result is proved with help of the new notion of a cofinendo module. By Theorem 2.5, cotilting torsion-free classes are characterized as those pretorsion-free classes which provide special precovers for all modules or, equivalently, for an injective cogenerator. The proofs rely on the facts that preenvelope pretorsion (precover pretorsion-free) classes coincide with the classes of modules (co)generated by (co)finendo modules, Corollary 1.3 (Corollary 1.7). We also consider module representations of tilting and cotilting classes: in Section 3, we introduce the notions of maximal tilting (cotilting) modules and envelope (cover) modules, and compute some of these modules explicitely.

Preliminaries

Let R be a ring, Mod-R the category of right R-modules, and $M \in$ Mod-R. Then M is *cotorsion* provided that $\mathrm{Ext}^1_R(F, M) = 0$ for all flat modules $F \in$ Mod-R. For example, any pure-injective module is cotorsion, [22, 3.1]. M is *finendo* if M is finitely generated over its endomorphism ring $\mathrm{End}(M_R)$. The additive groups of all rationals, p-adic integers and the Prüfer p-group are denoted by $\mathbb{Q}$, $\mathbb{J}_p$ and $\mathbb{Z}_{p^\infty}$, respectively. For a group A and a prime p, $T_p(A)$ denotes the p-torsion subgroup of A.

Denote by $\mathrm{Gen}(M)$ the class of all homomorphic images of direct sums of copies of M, and by $\mathrm{Cogen}(M)$ the class of all submodules of products of copies of M. Given a class $\mathcal{M} \subseteq$ Mod-R, we denote by $\mathrm{Add}(\mathcal{M})$ ($\mathrm{add}(\mathcal{M})$) the class consisting of all summands of (finite) direct sums of elements of $\mathcal{M}$. Similarly, $\mathrm{Prod}(\mathcal{M})$ denotes the class of all summands of arbitrary products of elements of $\mathcal{M}$. Further, let $\mathcal{M}^\perp = \{X \in \text{Mod-}R \mid \mathrm{Ext}^1_R(M, X) = 0 \text{ for all } M \in \mathcal{M}\}$ and ${}^\perp\mathcal{M} = \{X \in \text{Mod-}R \mid \mathrm{Ext}^1_R(X, M) = 0 \text{ for all } M \in \mathcal{M}\}$. If $\mathcal{M} = \{M\}$, then we just write $\mathrm{Add}\, M$, $\mathrm{add}\, M$, $M^\perp$ and ${}^\perp M$, respectively.

The classical definition of a tilting module is that of a finitely presented module, T, of projective dimension ≤ 1, with no self-extensions, and such that the regular module R has an $\mathrm{add}\, T$-copresentation $0 \to R \to T' \to T'' \to 0$, cf. [1], [16]. We will use a more category-theoretic definition which is nevertheless equivalent to the classical one when restricted to finitely presented modules (cf. [10]):

$M \in$ Mod-R is a *tilting* module provided that $\mathrm{Gen}(M) = M^\perp$. We will say that M is a *partial tilting* module provided that $\mathrm{Gen}(M) \subseteq M^\perp$

and $M^{\perp}$ is a torsion class. Dually, M is a *cotilting* module provided that $\mathrm{Cogen}(M) = {}^{\perp}M$; M is *partial cotilting* provided that $\mathrm{Cogen}(M) \subseteq {}^{\perp}M$ and ${}^{\perp}M$ is a torsion free class, cf. [10], [7].
Let $\mathcal{C} \subseteq$ Mod-R. Then $\mathcal{C}$ is a *pretorsion* class provided that $\mathcal{C}$ is closed under direct sums and factors. Moreover, $\mathcal{C}$ is a *tilting torsion class* provided that $\mathcal{C} = \mathrm{Gen}(M)$ for a tilting module M. Dually, $\mathcal{C}$ is a *pretorsion-free* class provided that $\mathcal{C}$ is closed under products and submodules. Moreover, $\mathcal{C}$ is a *cotilting torsion-free class* provided that $\mathcal{C} = \mathrm{Cogen}(M)$ for a cotilting module M.

We will also need the following notions concerning (pre)envelopes and (pre)covers:

Let $\mathcal{E} \subseteq$ Mod-R and $M \in$ Mod-R. Then $\phi \in \mathrm{Hom}_R(M, X)$ with $X \in \mathcal{E}$ is an *$\mathcal{E}$-preenvelope* of M provided that the abelian group homomorphism $\mathrm{Hom}_R(\phi, E) : \mathrm{Hom}_R(X, E) \to \mathrm{Hom}_R(M, E)$ is surjective for each $E \in \mathcal{E}$. An $\mathcal{E}$-preenvelope $\phi \in \mathrm{Hom}_R(M, X)$ of M is called

- *special* provided that ϕ is injective and $\mathrm{Coker}\,\phi \in {}^{\perp}\mathcal{E}$.
- an *$\mathcal{E}$-envelope* of M provided that $\xi\phi = \phi$ and $\xi \in \mathrm{End}(X_R)$ imply that ξ is an automorphism of X.

$\mathcal{E} \subseteq$ Mod-R is a *preenvelope class* (*envelope class*) provided that each module has an $\mathcal{E}$-preenvelope ($\mathcal{E}$-envelope).
Let $\mathcal{C} \subseteq$ Mod-R and $M \in$ Mod-R. Then $\phi \in \mathrm{Hom}_R(X, M)$ with $X \in \mathcal{C}$ is a *$\mathcal{C}$-precover* of M provided that the abelian group homomorphism $\mathrm{Hom}_R(C, \phi) : \mathrm{Hom}_R(C, X) \to \mathrm{Hom}_R(C, M)$ is surjective for each $C \in \mathcal{C}$. A $\mathcal{C}$-precover $\phi \in \mathrm{Hom}_R(X, M)$ of M is called

- *special* provided that ϕ is surjective and $\mathrm{Ker}\,\phi \in \mathcal{C}^{\perp}$.
- *$\mathcal{C}$-cover* of M provided that $\phi\xi = \phi$ and $\xi \in \mathrm{End}(X_R)$ implies that ξ is an automorphism of X.

$\mathcal{C} \subseteq$ Mod-R is a *precover class* (*cover class*) provided that each module has a $\mathcal{C}$-precover ($\mathcal{C}$-cover).

Assume that $\mathcal{X}$ is an envelope (cover) class closed under extensions and containing all injective (projective) modules. Then any $\mathcal{X}$-envelope ($\mathcal{X}$-cover) is special, and unique up to isomorphism, [22, §§1.2 and 2.1]. For further properties, we refer to [22].

1 Preenvelope and precover classes

Lemma 1.1 *Consider a class* $\mathcal{M} \subseteq$ Mod-R.

1. *Let* $b : R \to B$ *be an* $\mathcal{M}$*-preenvelope of the regular module* R. *Then* B *is a cyclic* $\mathrm{End}(B)$*-module and* $\mathrm{Prod}(\mathcal{M}) \subseteq \mathrm{Gen}(B)$.

2. *Let W be an injective cogenerator and $a : A \to W$ an $\mathcal{M}$-precover of W. Then* $\mathrm{Hom}_R(A, W)$ *is a cyclic* $\mathrm{End}(A)$*-module and* $\mathrm{Add}(\mathcal{M}) \subseteq \mathrm{Cogen}(A)$.

Proof. 1. The $\mathrm{End}\, B$-epimorphism $\mathrm{Hom}_R(b, B) : \mathrm{End}\, B \to \mathrm{Hom}_R(R, B) \cong B$ shows the first claim. To prove the second, consider a sequence $(M_\alpha \mid \alpha < \kappa)$ in $\mathcal{M}$ and set $M = \prod_{\alpha<\kappa} M_\alpha$. Since every epimorphism $R^{(I)} \to X$ factors through the map $b^{(I)} : R^{(I)} \to B^{(I)}$ for each $X = M_\alpha$, $\alpha < \kappa$, hence also for $X = M$, we have $M \in \mathrm{Gen}\, B$.
2. By a dual argument. ∎

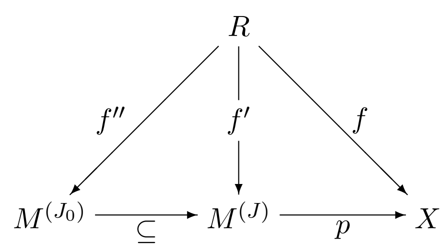
An investigation of tilting and cotilting classes naturally leads to testing when pretorsion and pretorsion-free classes are preenvelope and precover classes, respectively. We will develop criteria for this purpose.
Note that pretorsion (pretorsion-free) classes of modules, $\mathcal{P}$, are characterized by the property that every module has a monomorphic $\mathcal{P}$-cover (an epimorphic $\mathcal{P}$-envelope), [18, 4.10, 4.1].

We start with a characterization of finendo modules:

Proposition 1.2 *The following are equivalent for a module M:*

1. *M is finendo;*
2. *there is an* $\mathrm{add}(M)$*-preenvelope $R \to B$;*
3. $\mathrm{Gen}(M)$ *is a preenvelope class.*

Proof. 1. $\Longrightarrow$ 2. The map $R \to M^n$ induced by a spanning set $f_1, \ldots, f_n$ of the left $\mathrm{End}\, M$-module $\mathrm{Hom}_R(R, M)$ is an $\mathrm{add}\, M$-preenvelope.
2. $\Longrightarrow$ 3. Let $b : R \to B$ be an $\mathrm{add}\, M$-preenvelope. We first show that b is also a $\mathrm{Gen}(M)$-preenvelope. Take a map $f : R \to X \in \mathrm{Gen}(M)$. Then there is an epimorphism $p : M^{(J)} \to X$, and we obtain a commutative diagram



where J_0 is a finite subset of J. Since the map $f'' : R \to M^{(J_0)}$ factors through b, we deduce that f factors through b as well.
Take now an arbitrary module A_R and an epimorphism $\pi : R^{(I)} \to A$. Consider the push-out diagram

$$\begin{array}{ccc} R^{(I)} & \xrightarrow{b^{(I)}} & B^{(I)} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \sigma} \\ A & \xrightarrow[b']{} & B' \end{array}$$

Note that σ is an epimorphism since π is, thus $B' \in \mathrm{Gen}(M)$. Moreover, for any map $f : A \to X \in \mathrm{Gen}(M)$, we have that $f\pi$ factors through $b^{(I)}$, because $b^{(I)}$ is a $\mathrm{Gen}(M)$-preenvelope by [22, 1.2.4]. But then f factors through b' by the push-out property. So, b' is a $\mathrm{Gen}(M)$-preenvelope of A.
3. $\Longrightarrow$ 1. Take a $\mathrm{Gen}(M)$-preenvelope $b : R \to B$. We know from Lemma 1.1 that $\mathrm{Prod}\,\mathrm{Gen}(M) \subset \mathrm{Gen}\,B$. Since $\mathrm{Gen}\,B \subset \mathrm{Gen}(M)$, we conclude that $\mathrm{Gen}(M)$ is closed under products, and this implies (1) by [8, 1.5]. ∎

Alternatively, one can prove the equivalence 1. ⇔ 3. in Proposition 1.2 combining [8, Lemma 1.5] with [18, 3.3].

Corollary 1.3 *Let $\mathcal{T} \subseteq$ Mod-R be a pretorsion class. Then the following are equivalent:*

1. *$\mathcal{T}$ is a preenvelope class;*
2. *R has a $\mathcal{T}$-preenvelope;*
3. *$\mathcal{T} = \mathrm{Gen}(T)$ for a finendo module T.*

Proof. 1. $\Longrightarrow$ 2. is clear. 2. $\Longrightarrow$ 3. follows immediately from Lemma 1.1 and the assumption on $\mathcal{T}$, 3. $\Longrightarrow$ 1. from Proposition 1.2. ∎

We turn to the dual setting. First, notice that a module B is finendo if and only if there exists $f : R \to B^n$, $n \in \mathbb{N}$, such that $\mathrm{Hom}_R(f, B) : \mathrm{Hom}_R(B^n, B) \to \mathrm{Hom}_R(R, B)$ is surjective. The latter is equivalent to the statement "there are a cardinal γ and a map $f : R \to B^{(\gamma)}$ such that for any cardinal α, all maps $R \to B^{(\alpha)}$ factor through f". So we can define the dual notion of a cofinendo module as follows:

Definition 1.4 (i) Let W be an injective cogenerator of Mod-R. A module C is called W-*cofinendo* if there exist a cardinal γ and a map $f : C^\gamma \to W$ such that for any cardinal α, all maps $C^\alpha \to W$ factor through f.

(ii) A module C is *cofinendo* if there is an injective cogenerator W of Mod-R such that C is W-cofinendo.

Lemma 1.5 *Let $C \in$ Mod-R and let W be an injective cogenerator of* Mod-R. *The following are equivalent:*

1. *C is W-cofinendo;*

2. *there is a cardinal β such that for any cardinal α, all maps $C^\alpha \to W$ factor through some coproduct of copies of C^β.*

Proof. 1. $\implies$ 2. is clear. 2. $\implies$ 1. Set $I = \mathrm{Hom}_R(C^\beta, W)$. Denote by $\tilde{f} : [C^\beta]^{(I)} \to W$ the codiagonal map induced by all maps in I. There are a cardinal γ and a map $f : C^\gamma \to W$ such that $[C^\beta]^{(I)} \hookrightarrow C^\gamma$ and $f_{|[C^\beta]^{(I)}} = \tilde{f}$. By hypothesis, for any cardinal α each map $\phi : C^\alpha \to W$ factors through some $\psi : [C^\beta]^{(J)} \to W$. Let ι_j be the embedding of the j-th copy, $j \in J$, of C^β in $[C^\beta]^{(J)}$. For each $j \in J$, $\psi\iota_j$ factors through $\tilde{f}$, hence through f. Let $\xi : [C^\beta]^{(J)} \to C^\gamma$ the codiagonal morphism of all $\psi\iota_j$; then ψ and hence ϕ, factor through ξ.
■

Proposition 1.6 *The following are equivalent for a module C:*

1. *C is cofinendo;*

2. *there is a $\mathrm{Prod}(C)$-precover of an injective cogenerator of $\text{Mod-}R$;*

3. *$\mathrm{Cogen}(C)$ is a precover class.*

Proof. 1. $\implies$ 2. Every morphism $X \to W$, $X \in \mathrm{Prod}\, C$, extends to a morphism $C^\alpha \to W$. Then $f : C^\gamma \to W$ in Definition 1.4 is a $\mathrm{Prod}\, C$-precover.
2. $\implies$ 3. by arguments dual to 2. $\Rightarrow$ 3. in Proposition 1.2.
3. $\implies$ 1. A $\mathrm{Cogen}\, C$-precover of an injective cogenerator W extends to a map $f : C^\gamma \to W$ as in Definition 1.4. ■

Corollary 1.7 *Let $\mathcal{F} \subseteq \text{Mod-}R$ be a pretorsion-free class. Then the following are equivalent*

1. *$\mathcal{F}$ is a precover class;*

2. *there is an $\mathcal{F}$-precover of an injective cogenerator of $\text{Mod-}R$;*

3. *$\mathcal{F} = \mathrm{Cogen}\, C$ for a cofinendo module C.*

Proof. 1. $\implies$ 2. is clear. 2. $\implies$ 3. Let W be an injective cogenerator having an $\mathcal{F}$-precover $a : A \to W$. By Lemma 1.1 and the assumption on $\mathcal{F}$ we know $\mathcal{F} = \mathrm{Cogen}\, A$. Moreover, for any cardinal α all maps $A^\alpha \to W$ factor through a, hence A is cofinendo.
3. $\Rightarrow$ 1. by Proposition 1.6. ■

A corresponding version of Corollary 1.3 and 1.7 in the setting of finitely generated modules over an artin algebra can be found in [3, 4.6, 4.7].

2 Tilting and Cotilting Classes

In the category of finitely generated modules over an artin algebra, torsion classes which are generated by a tilting module can be described in terms of covariant finiteness. Similarly, torsion-free classes which are cogenerated by a cotilting module can be described in terms of contravariant finiteness, [1],[19]. We provide analogous results for arbitrary (co)tilting torsion(-free) classes:

Let $\mathcal{C} \subseteq \text{Mod-}R$. A module $M \in \text{Mod-}R$ is *$\mathcal{C}$-projective* (*$\mathcal{C}$-injective*) provided that the functor $\text{Hom}_R(M, -)$ ($\text{Hom}_R(-, M)$) is exact on short exact sequences of the form $0 \to X \to Y \to Z \to 0$ where $X, Y, Z \in \mathcal{C}$, cf. [10, 2.6].

Theorem 2.1 *Let R be a ring and $\mathcal{T} \subseteq \text{Mod-}R$ be a pretorsion class. The following are equivalent:*

1. *$\mathcal{T}$ is a tilting torsion class;*
2. *every module has a special $\mathcal{T}$-preenvelope;*
3. *there is a special $\mathcal{T}$-preenvelope of R;*
4. *there is a $\mathcal{T}$-preenvelope of R, $b : R \to B$, such that b is injective and B is $\mathcal{T}$-projective.*

Proof. 1. $\implies$ 2. Assume $\mathcal{T}$ is a tilting torsion class, so $\mathcal{T} = \text{Gen}\, T = T^{\perp}$ for a tilting module $T \in \text{Mod-}R$. By [21, 6.8], for any module M, there is a $\mathcal{T}$-torsion resolution of M of the form

$$0 \to M \to T' \to T^{(\lambda)} \to 0,$$

where $T' \in \mathcal{T}$, λ is a cardinal, and $\text{Ext}^1_R(T^{(\lambda)}, N) = 0$ for all $N \in \mathcal{T}$. So the map $M \to T'$ is a special $\mathcal{T}$-preenvelope of M.
2. $\implies$ 3. is clear.
3. $\implies$ 4. Let $b : R \to B$ be a special $\mathcal{T}$-preenvelope. We then have an exact sequence

$$0 \longrightarrow R \xrightarrow{b} B \to L \longrightarrow 0$$

where $B \in \mathcal{T}$, and R and L are in ${}^{\perp}\mathcal{T}$. Hence $B \in {}^{\perp}\mathcal{T}$, which shows that $\text{Hom}_R(B, -)$ is exact on any epimorphism with kernel in $\mathcal{T}$. In particular, B is $\mathcal{T}$-projective.
4. $\implies$ 1. Take $b : R \to B$ as in 4. As in Corollary 1.3, 2.⇒3., we get that $\mathcal{T} = \text{Gen}(B)$ and B is finendo. Since b is injective, B is faithful. Now, [10, 2.7] shows that $\mathcal{T}$ is a tilting torsion class. ∎

In the theorem above, the assumption of $\mathcal{T}$ being a pretorsion class is necessary. This is seen, e.g., for $\mathcal{T}$ equal to the class of all injective modules (which always satisfies conditions 2., 3. and 4., but condition 1. is true iff R is right hereditary and right noetherian).

In view of Theorem 2.1, it is an interesting problem to determine when a tilting torsion class is an envelope class. We will present a sufficient condition.

Given a tilting torsion class $\mathcal{T} \subseteq \text{Mod-}R$, define $\mathfrak{T}_{\mathcal{T}}$ as the family of all classes $\mathcal{H}$ such that $\mathcal{H}$ is closed under extensions, there exists $P_{\mathcal{H}} \in \mathcal{T}$ with $P_{\mathcal{H}}^{\perp} = \mathcal{T}$, and ${}^{\perp}\mathcal{T} \supseteq \mathcal{H} \supseteq \text{Add}(P_{\mathcal{H}})$. For example, ${}^{\perp}\mathcal{T} \in \mathfrak{T}_{\mathcal{T}}$. Also, $\text{Add}(P) \in \mathfrak{T}_{\mathcal{T}}$ for each (partial tilting) $P \in \mathcal{T}$ such that $P^{\perp} = \mathcal{T}$.

Proposition 2.2 *Let R be a ring and $\mathcal{T}$ be a tilting torsion class in* Mod-R. *Assume that there is $\mathcal{G} \in \mathfrak{T}_{\mathcal{T}}$ such that $\mathcal{G}$ is closed under direct limits. Then $\mathcal{T}$ is an envelope class.*

Proof. Let $M \in \text{Mod-}R$. Since $\text{Ext}_R^1(P_{\mathcal{G}}, P_{\mathcal{G}}^{(\kappa)}) = 0$ for all cardinals κ, [21, 6.8] gives an exact sequence

$$(*) \qquad 0 \longrightarrow M \to G \to P_{\mathcal{G}}^{(\lambda)} \longrightarrow 0,$$

where $G \in \mathcal{T}$ and λ is a cardinal. Then $(*)$ is a generator for $\text{Ext}_R^1(\mathcal{G}, M)$ in the sense of [22, 2.2.1]. From=20our assumption and [22, 2.2.6] it follows that M has an $\mathcal{G}^{\perp}$-envelope. Since

$$\mathcal{T} = ({}^{\perp}\mathcal{T})^{\perp} \subseteq \mathcal{G}^{\perp} \subseteq (\text{Add}(P_{\mathcal{G}}))^{\perp} = \mathcal{T},$$

the claim is proved. ∎

For example, $\mathcal{T}$ is a tilting envelope class if $\mathcal{T} = \text{Mod-}R$, or if R is right hereditary and right noetherian and $\mathcal{T}$ is the class of all injective modules. These easy facts are consequences of Proposition 2.2 for $\mathcal{G} = \text{Add}(0)$ and $\mathcal{G} = {}^{\perp}\mathcal{T}$, respectively.

We turn to the dual setting. First, notice the following property of cotilting modules which underlines the duality with the tilting case:

Proposition 2.3 *Let R be a ring, W be an injective module and C be a cotilting module. Then there is an exact sequence*

$$0 \longrightarrow C_1 \to C_0 \to W \longrightarrow 0$$

where $C_0, C_1 \in \text{Prod}(C)$.

Proof. Let $p : R^{(\alpha)} \to W$ be an epimorphism, where α is a cardinal. Since $R^{(\alpha)} \in {}^{\perp}C = \text{Cogen}(C)$, there is a monomorphism $R^{(\alpha)} \to C^{\beta}$. So p extends to an epimorphism $f : C^{\beta} \to W$. Let $K = \text{Ker}(f) \in \text{Cogen}\, C$. By [7, 1.8], there is an exact sequence $0 \longrightarrow K \to C^\gamma \to L \longrightarrow 0$ where $L \in \mathrm{Cogen}(C)$. Consider the push-out diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K & \longrightarrow & C^\beta & \xrightarrow{f} & W & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & C^\gamma & \longrightarrow & E & \longrightarrow & W & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & L & = & L & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

We are going to show that $0 \longrightarrow C^\gamma \to E \to W \longrightarrow 0$ is the desired sequence. Observe that L and C^β are in ${}^\perp C$, thus $E \in {}^\perp C = \mathrm{Cogen}(C)$. Again by [7, 1.8], there is an exact sequence $0 \longrightarrow E \to C^\delta \to Y \longrightarrow 0$ where $Y \in \mathrm{Cogen}(C) = {}^\perp C$. But this sequence is split exact, because $\mathrm{Ext}^1_R(Y, C^\gamma) \cong \mathrm{Ext}^1_R(Y, C)^\gamma = 0 = \mathrm{Ext}^1_R(Y, W)$, so $\mathrm{Ext}^1_R(Y, E) = 0$. This shows that $E \in \mathrm{Prod}(C)$. ∎

The following Lemma is dual to [10, 2.7]:

Lemma 2.4 *Let R be a ring and $\mathcal{F} \subseteq$ Mod-R be a class of modules. Then $\mathcal{F}$ is a cotilting torsion-free class iff $\mathcal{F} = \mathrm{Cogen}(P)$ for a faithful, cofinendo and $\mathcal{F}$-injective module P.*

Proof. For the direct implication, we take C as the cotilting module with $\mathcal{F} = \mathrm{Cogen}(C)$. Clearly, C is faithful and $\mathcal{F}$-injective. Let W be an injective cogenerator of Mod-R. By Proposition 2.3, there is an exact sequence

$$0 \longrightarrow C_1 \to C_0 \to W \longrightarrow 0$$

where $C_0, C_1 \in \mathrm{Prod}(C)$. Since $\mathrm{Ext}^1_R(N, C_1) = 0$ for all $N \in \mathrm{Prod}(C)$, the map $C_0 \to W$ is a $\mathrm{Prod}(C)$-precover of W. By Proposition 1.6, C is cofinendo. For the other implication, let $\mathrm{Cogen}(P) = \mathcal{F}$ with P faithful, cofinendo and $\mathcal{F}$-injective. By Proposition 1.6, we have an $\mathcal{F}$-precover $\pi : P^\gamma \to W$ of an injective cogenerator W of Mod-R. Since P is faithful, $\mathcal{F}$ contains all projective modules. Then π is surjective, hence gives rise to an exact sequence

$$(*) \qquad 0 \to P' \to P^\gamma \xrightarrow{\pi} W \to 0.$$

If $M \in \mathcal{F}$, then any presentation $0 \to K \to F \to M \to 0$ with F free is in $\mathcal{F}$. Since P is $\mathcal{F}$-injective, the induced map $\mathrm{Ext}^1_R(M, P) \to \mathrm{Ext}^1_R(F, P)$ is monic. Since $\mathrm{Ext}^1_R(F,P)=0$, we get $M \in {}^\perp P$, so $\mathrm{Cogen}(P') \subseteq \mathcal{F} \subseteq {}^\perp P$. We prove that $\mathcal{F} = {}^\perp P'$. Let $M \in \text{Mod-}R$. Applying $\mathrm{Hom}_R(M,-)$ to $(*)$, we get

$$(**)\qquad \mathrm{Hom}_R(M,P^\gamma) \xrightarrow{\pi^*} \mathrm{Hom}_R(M,W) \to \mathrm{Ext}^1_R(M,P') \to \mathrm{Ext}^1_R(M,P^\gamma)\,.$$

If $M \in \mathcal{F}$, then the precover property shows that π^* is surjective. Since $\mathcal{F} \subseteq {}^\perp P$, $(**)$ gives $M \in {}^\perp P'$. Conversely, if $M \in {}^\perp P'$, then π^* is surjective. Since W is an injective cogenerator, $\cap\{\mathrm{Ker}\,\varphi \mid \varphi \in \mathrm{Hom}_R(M,W)\} = 0$. By surjectivity of π^*, $\cap\{\mathrm{Ker}\,\psi \mid \psi \in \mathrm{Hom}_R(M,P^\gamma)\} = 0$, so $M \in \mathcal{F}$. It follows that

$$\mathrm{Cogen}(P \oplus P') = \mathrm{Cogen}(P) = \mathcal{F} = {}^\perp P' = {}^\perp(P \oplus P'),$$

so $P \oplus P'$ is a cotilting module cogenerating $\mathcal{F}$. ∎

Theorem 2.5 *Let R be a ring and $\mathcal{F} \subseteq \text{Mod-}R$ be a pretorsion-free class. The following are equivalent:*

1. *$\mathcal{F}$ is a cotilting torsion-free class;*
2. *every module has a special $\mathcal{F}$-precover;*
3. *there exists a special $\mathcal{F}$-precover of an injective cogenerator of* Mod-R*;*
4. *there is an $\mathcal{F}$-precover, $\pi : P \to W$, of an injective cogenerator W of* Mod-R *such that π is surjective and P is faithful and $\mathcal{F}$-injective.*

Proof. 1. $\Longrightarrow$ 2. Assume $\mathcal{F}$ is a cotilting torsion-free class, so $\mathcal{F} = \mathrm{Cogen}(C) = {}^\perp C$ for a cotilting module $C \in \text{Mod-}R$. By [21, 6.9] (or [9, 2.14]), for any module M, there is an $\mathcal{F}$-torsion-free resolution of M of the form

$$0 \to C^\lambda \to C' \to M \to 0,$$

where $C' \in \mathcal{F}$ and λ is a cardinal. Since $\mathrm{Ext}^1_R(N, C^\lambda) = 0$ for all $N \in \mathcal{F}$, the map $C' \to M$ is a special $\mathcal{F}$-precover of M. 2. $\Longrightarrow$ 3. is clear. 3. $\Longrightarrow$ 4. Let $\pi : P \to W$ be a special $\mathcal{F}$-cover of an injective cogenerator W. There is κ such that R embeds into W^κ. Since π is surjective, so is the induced map $\pi^\kappa : P^\kappa \to W^\kappa$. Since R is projective, the embedding factors through π^κ, hence R embeds into P^κ, and P is faithful. The $\mathcal{F}$-injectivity is shown by a dual argument to the one in the proof of Theorem 2.1, 3. $\Rightarrow$ 4. 4. $\Longrightarrow$ 1. Take $\pi : P \to W$ as in 4. As in Corollary 1.7, 2. $\Rightarrow$ 3., we get that $\mathcal{F} = \mathrm{Cogen}(P)$ and P is cofinendo. Now, Lemma 2.4 shows that $\mathcal{F}$ is a cotilting torsion-free class. ∎

In the theorem above, the assumption of $\mathcal{F}$ being a pretorsion-free class is necessary as seen, e.g., for $\mathcal{F}$ equal to the classes of all projective or flat modules.

Corollary 2.6 *Let R be a ring and $\mathcal{F}$ be a torsion-free class closed under direct limits. Then $\mathcal{F}$ is a cover class containing all projective modules iff $\mathcal{F}$ is a cotilting torsion-free class.*
In this case, C is cotorsion whenever C is a module such that $\mathcal{F} \subseteq^{\perp} C$.

Proof. Assume that $\mathcal{F}$ is a cotilting torsion-free class. By [22, 2.2.8] and Theorem 2.5, $\mathcal{F}$ is a cover class. Clearly, $\mathcal{F}$ contains all projective modules. Conversely, assume that $\mathcal{F}$ is a cover class containing all projective modules. Let P be the $\mathcal{F}$-cover of an injective cogenerator W. We have the exact sequence

$$0 \to P' \to P \to W \to 0,$$

where $P' \in \mathcal{F}$, and by [22, Lemma 2.1.1], $P' \in \mathcal{F}^{\perp}$. So, W has a special $\mathcal{F}$-precover, and Theorem 2.5 applies. Finally, by the premise, $\mathcal{F}$ contains all flat modules. So any module $C \in$ Mod-R with ${}^{\perp}C \supseteq \mathcal{F}$ is cotorsion. ∎

As illustrated by the following examples, the cotilting torsion-free classes are quite frequent:

Example 2.7 Let R be a commutative domain and $\mathcal{F}$ be the class of all torsion-free modules. By Enochs' Theorem [22, Theorem 1.3.2], $\mathcal{F}$ is a cover class. Clearly, $\mathcal{F}$ is a torsion-free class, so Theorem 2.5 implies that $\mathcal{F}$ is a cotilting torsion-free class. Moreover, if W is an injective cogenerator for Mod-R and $0 \to F' \to F \to W \to 0$ is an $\mathcal{F}$-cover of W then $F \oplus F'$ is a cotilting module cogenerating $\mathcal{F}$.

This can be generalized in the spirit of [20]: Let R be a ring and $\mathfrak{H} = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory for Mod-R. Assume that $R \in \mathcal{F}$ and that the radical filter corresponding to $\mathfrak{H}$ has a cofinal subset of finitely generated right ideals. Then $\mathcal{F}$ is a cover class by [20, Theorem], hence a cotilting torsion-free class by Theorem 2.5.

The next example deals with the class of all flat modules. In case R is a Prüfer domain, it is just a restatement of the previous one (cf. [13, IV.§1]):

Example 2.8 Let R be a ring and $\mathcal{F}$ the class of all flat modules. Clearly, $\mathcal{F}$ is closed under direct limits. It is well known that $\mathcal{F}$ is a torsion-free class iff R is left coherent and of weak global dimension ≤ 1. In that case, by [22, 3.1.12] and Corollary 2.6, $\mathcal{F}$ is a cotilting torsion-free class.

Observe that an analogous result for the class $\mathcal{P}$ of all projective modules was already proved in [9, 2.13]. Namely, if $\mathcal{P}$ is torsion-free, then R is a right hereditary left coherent and semiprimary ring, and $\mathcal{P}$ is a cotilting torsion-free class.

Example 2.9 Let R be a ring and $P \in$ Mod-R be $*$-module (e.g., let P be a finitely generated tilting module). Then $C = \mathrm{Hom}_R(P, Q)$ is a cotilting right S-module, where $S = \operatorname{End} P_R$ and Q is any injective cogenerator of Mod-R, [8, 1.2]. Since $\operatorname{Ext}^1_S(M, C) \cong \operatorname{Hom}_R(\operatorname{Tor}^1_S(M, P), Q)$ [5, VI.5.1], and Tor commutes with direct limits [5, VI.1.3], Cogen(C) is a cotilting torsion-free class closed under direct limits. Since Q is injective, C is even a pure-injective right S-module.

For other examples where the cotilting modules are not duals of finitely generated tilting modules, we refer to Example 3.6 below.

3 Unique Representing Modules

In the finite dimensional case, each finitely generated tilting and cotilting module is a direct sum of indecomposable modules. So one can represent tilting and cotilting classes by multiplicity free, or basic, generating tilting modules and cogenerating cotilting modules, respectively, [2]. Such representation is not available for classes of modules over arbitrary rings. Nevertheless, there are other module representations:

Let R be a ring and $M \in$ Mod-R. Denote by spn(M) the minimal cardinality of an R-spanning subset of M. Let $\mathcal{T} \subseteq$ Mod-R be a tilting torsion class. Then trank($\mathcal{T}$) is defined as the minimum of spn(T), where T ranges over all tilting modules such that $\mathcal{T} = \operatorname{Gen}(T)$.

As an illustration, recall [10, 2.12] which shows that for each cardinal λ there are a ring R and a tilting torsion class $\mathcal{T}$ such that trank($\mathcal{T}$) $\geq \lambda$.

Dually, let $\mathcal{F} \subseteq$ Mod-R be a cotilting torsion-free class. Then crank($\mathcal{F}$) is defined as the minimum of spn(C), where C ranges over all cotilting modules such that $\mathcal{F} = \operatorname{Cogen}(C)$.

For example, if $R = \mathbb{Z}$, then crank(Mod-R) $= \omega$, and crank($\mathcal{F}$) $= 2^\omega$ for all cotilting torsion-free classes $\mathcal{F} \subsetneq$ Mod-R, [14, 2.2].

Definition 3.1 Let R be a ring and $T \in$ Mod-R. Then T is *maximal tilting* provided that $T = \bigoplus_{\alpha<\kappa} T_\alpha$, where

(1) T_α is a tilting module for each $\alpha < \kappa$;

(2) $T_\alpha \ncong T_\beta$ and $\operatorname{Gen}(T_\alpha) = \operatorname{Gen}(T_\beta)$ for all $\alpha \neq \beta < \kappa$;

(3) $\operatorname{spn}(T_\alpha) = \operatorname{trank}(\operatorname{Gen}(T))$ for all $\alpha < \kappa$;

(4) if T' is a tilting module such that $\operatorname{Gen}(T') = \operatorname{Gen}(T)$ and $\operatorname{spn}(T') = \operatorname{trank}(\operatorname{Gen}(T))$, then there exists $\alpha < \kappa$ such that $T' \cong T_\alpha$.

Let $C \in$ Mod-R. Then C is *maximal cotilting* provided that $C = \prod_{\alpha<\kappa} C_\alpha$, where

(1') C_α is a cotilting module for each $\alpha < \kappa$;

(2') $C_\alpha \ncong C_\beta$ and $\operatorname{Cogen}(C_\alpha) = \operatorname{Cogen}(C_\beta)$ for all $\alpha \neq \beta < \kappa$;

(3') $\mathrm{spn}(C_\alpha) = \mathrm{crank}(\mathrm{Cogen}(T))$ for all $\alpha < \kappa$;

(4') if C' is a cotilting module such that $\mathrm{Cogen}(C') = \mathrm{Cogen}(C)$ and $\mathrm{spn}(C') = \mathrm{crank}(\mathrm{Cogen}(C))$, then there exists $\alpha < \kappa$ such that $C' \cong C_\alpha$.

Note that any maximal tilting module is tilting, and any maximal cotilting module is cotilting. We will show that there is a bijective correspondence between tilting torsion classes (cotilting torsion free classes) and maximal tilting (cotilting) modules.

For a ring R, denote by $\mathfrak{T}$ the family of all tilting torsion classes in Mod-R. Dually, $\mathfrak{C}$ denotes the family of all cotilting torsion free classes in Mod-R.

Proposition 3.2 *Let R be a ring.*

(i) *Denote by $\mathfrak{M}$ the class of all maximal tilting modules (up to isomorphism). Then the map $b : \mathfrak{M} \to \mathfrak{T}$ defined by $b(T) = \mathrm{Gen}(T)$ is bijective.*

(ii) *Denote by $\mathfrak{N}$ the class of all maximal cotilting modules (up to isomorphism). Then the map $c : \mathfrak{N} \to \mathfrak{C}$ defined by $c(C) = \mathrm{Cogen}(C)$ is bijective.*

Proof. (i) By Definition 3.1, b is well-defined and monic.
Let $\mathcal{T}$ be a tilting torsion class. Let A be a representative set of the isomorphism classes of all tilting modules M such that $\mathrm{Gen}(M) = \mathcal{T}$ and $\mathrm{spn}(M) = \mathrm{trank}(\mathcal{T})$. Let $T = \oplus_{M \in A} M$. Then T is maximal tilting and $\mathrm{Gen}(T) = \mathcal{T}$, so b is surjective.
(ii) By a dual argument to (i). ■

If $\mathcal{T} = $ Mod-R, then the corresponding maximal tilting module is just the direct sum of a representative set of all cyclic progenerators.

Let $R = \mathbb{Z}$. If $\mathcal{F} = $ Mod-R, then the corresponding maximal cotilting module is $\mathbb{Q}^\omega \oplus \prod_{p \in \mathbb{P}} \mathbb{Z}_{p^\infty}^\omega$. If $\mathcal{F} \subset$ Mod-R, then there is a non-empty set, P, of primes such that $\mathcal{F} = \{A \in \text{Mod-}\mathbb{Z} \mid T_p(A) = 0 \ \forall p \in P\}$, [14, 2.2]. The corresponding maximal cotilting module is $\mathbb{Q}^{2^\omega} \oplus \prod_{p \in P} \mathbb{Z}_{p^\infty}^{2^\omega} \oplus \prod_{p \notin P} \mathbb{J}_p^\omega$.

In the case when the tilting class $\mathcal{T}$ (cotilting class $\mathcal{C}$) is an envelope (cover) class, there is another module representation available:

Definition 3.3 Let R be a ring and Q be a (fixed) injective cogenerator for Mod-R. Define

$$\mathfrak{T}' = \{\mathcal{T} \in \mathfrak{T} \mid \mathcal{T} \text{ is an envelope class}\},$$

$$\mathfrak{C}' = \{\mathcal{C} \in \mathfrak{C} \mid \mathcal{C} \text{ is a cover class}\}.$$

Further, call $P \in$ Mod-R an *envelope* module provided that there is a submodule $P' \subseteq P$ satisfying

(i) $P' \cong R$,

(ii) $P \oplus P/P'$ is a partial tilting module, and

(iii) there exists no proper summand of P containing P'.

Dually, $C \in \text{Mod-}R$ is called a *cover* module provided that there is a submodule $C' \subseteq C$ satisfying

(i') $C/C' \cong Q$,

(ii') $C \oplus C'$ is a partial cotilting module,

(iii') there is no non-zero submodule of C' which is a summand of C.

By condition (i'), each cover module is faithful.

Proposition 3.4 *Let R be a ring and Q an injective cogenerator for* Mod-R. *Denote by $\mathcal{B}$ the class of all cover modules (up to isomorphism). Define $\rho : \mathfrak{C}' \to \mathcal{B}$ by $\rho(\mathcal{C}) = C$, where $C \to Q$ is the $\mathcal{C}$-cover of Q.*
Then ρ is injective and $\operatorname{Im}\rho$ *contains all cover modules C such that* $\operatorname{Cogen}(C)$ *is closed under direct limits.*

Proof. First, we show that ρ is well-defined. Take $\mathcal{C} \in \mathfrak{C}'$. Let $0 \to C' \hookrightarrow C \to Q \to 0$ be an exact sequence, where $C \to Q$ is the $\mathcal{C}$-cover of Q (the $\mathcal{C}$-cover is surjective as $\mathcal{C}$ contains all projective modules). Then (i') is clear, and (iii') holds by [22, 1.2.8]. Moreover, $\mathcal{C} = \operatorname{Cogen}(C)$ by Lemma 1.1, and as in (the proof of) Lemma 2.4, we get

$$\operatorname{Cogen}(C') \subseteq \operatorname{Cogen}(C) = \operatorname{Cogen}(C \oplus C') = \mathcal{C} = {}^{\perp}C' \subseteq {}^{\perp}C,$$

so ${}^{\perp}(C \oplus C') = {}^{\perp}C'$, and (ii') holds. So ρ is well-defined. Since $\operatorname{Cogen}(\rho(\mathcal{C})) = \mathcal{C}$, ρ is injective.
Let $C \in \mathcal{B}$ be such that $\operatorname{Cogen}(C)$ is closed under direct limits. Put $\mathcal{C} = \operatorname{Cogen}(C)$. By (ii'), $\mathcal{C} \subseteq {}^{\perp}C'$, so $C \to C/C' \cong Q$ is a special $\mathcal{C}$-precover of Q. Again by (ii'), we get that C is $\mathcal{C}$-injective. Since C is faithful, we obtain from Theorem 2.5 that $\mathcal{C} \in \mathfrak{C}$, and Corollary 2.6 gives $\mathcal{C} \in \mathfrak{C}'$. By the uniqueness of covers [22, 1.2.6], we get $\rho(\mathcal{C}) \cong C$. ∎

Proposition 3.5 *Let R be a ring. Denote by $\mathcal{Q}$ the class of all envelope modules (up to isomorphism). Define $\pi : \mathfrak{T}' \to \mathcal{Q}$ by $\rho(\mathcal{T}) = P$, where $R \to P$ is a $\mathcal{Q}$-envelope of R.*
Then π is injective, and $\operatorname{Im}\pi$ *contains all envelope modules P such that* ${}^{\perp}\operatorname{Gen}(P)$ *is closed under direct limits.*

Proof. Dual to the proof of Proposition 3.4, using Corollary 2.2 instead of Corollary 2.6. ∎

Note that if ${}^{\perp}\operatorname{Gen}(P)$ is closed under direct limits in Proposition 3.5, then $\operatorname{Gen}(P)$ consists of cotorsion modules; in particular, P is then cotorsion. In fact, the module $T' = P \oplus P/P'$ in Proposition 3.5 is a tilting module such that $\operatorname{Gen}(T') = \mathcal{T}$. Let T be any tilting module with $\operatorname{Gen}(T) = \mathcal{T}$. Then there is a $\mathcal{T}$-preenvelope of R of the form

$$0 \to R \to T^n \to T'' \to 0.$$

By (the proof of) [22, 1.2.2], T' is isomorphic to a direct summand of $T^n \oplus T''$. In particular, $\operatorname{spn}(T') \leq \operatorname{spn}(T^n \oplus T'')$. Since T'' is a factor of T^n, we infer that $\omega \cdot \operatorname{trank}(\mathcal{T}) = \omega \cdot \operatorname{spn}(T')$. In particular, $\operatorname{trank}(\mathcal{T}) = \operatorname{spn}(T')$ provided that $\operatorname{trank}(\mathcal{T}) \geq \omega$.

Dually, the module $C \oplus C'$ in Proposition 3.4 is a cotilting module. Nevertheless, $\kappa = \operatorname{spn}(C \oplus C')$ may be much bigger than $\operatorname{crank}(\mathcal{F})$, since $\kappa \geq \operatorname{spn}(Q)$ where Q is the (fixed) injective cogenerator for Mod-R. We don't know whether a dual version of the previous remark holds e.g. when Q is the minimal injective cogenerator of Mod-R.

We finish by illustrating the above notions in the category of abelian groups. So assume $R = \mathbb{Z}$. Then the cotilting torsion-free classes are always closed under direct limits, cf. [14, 2.2]. So Proposition 3.4 implies that the classes are uniquely represented by the (cotorsion) cover modules. We will show that in this case all cover modules are injective:

Example 3.6 Let $R = \mathbb{Z}$, $\mathbb{P}$ be the set of all primes in $\mathbb{Z}$, and W an injective cogenerator of Mod-$\mathbb{Z}$. Let $Q = \prod_{p \in \mathbb{P}} \mathbb{Z}_{p^\infty}$. Clearly, there is a cardinal λ such that W is isomorphic to a summand in Q^λ.

Let $\mathcal{F}$ be a cotilting class. By [14, 2.2], there is a subset $P \subseteq \mathbb{P}$ such that $\mathcal{F} = \{A \in \text{Mod-}\mathbb{Z} \mid T_p(A) = 0,\ \forall p \in P\}$. Note that $\mathcal{F}$ is closed under direct limits. If $P = \emptyset$, then $\mathcal{F} = $ Mod-R, so the corresponding cover module is $C = W$ (and $C' = 0$).

Let $p \in P$. Consider the canonical exact sequence

$$(*) \qquad 0 \to \mathbb{J}_p \hookrightarrow \mathbb{D}_p \xrightarrow{\phi} \mathbb{Z}_{p^\infty} \to 0,$$

where $\mathbb{D}_p = \bigcup_{n<\omega} \mathbb{J}_p p^{-n} \cong \mathbb{Q}^{(2^\omega)}$ is injective.

Since $\mathbb{J}_p$ is a cotorsion group, we have $\operatorname{Ext}(F, \mathbb{J}_p) = 0$ for all torsion-free groups F. Let $A \in \mathcal{F}$. We have the exact sequence $0 \to T(A) \to A \to A/T(A) \to 0$. We claim that ϕ is an $\mathcal{F}$-cover of $\mathbb{Z}_{p^\infty}$. Since $\mathbb{J}_p$ is reduced, it suffices to show that $\operatorname{Ext}(T(A), \mathbb{J}_p) = 0$ (and then apply Corollary 2.6 and [22, 1.2.8]). However, there is an exact sequence $0 \to T(A) \to \bigoplus_{q \in \mathbb{P} \setminus P} \mathbb{Z}_{q^\infty}^{(\gamma_q)} \to D \to 0$, where D is divisible. The claim now follows from the fact that $\operatorname{Ext}(\mathbb{Z}_{q^\infty}, \mathbb{J}_p) = 0$ for all $p \in P$ and $q \in \mathbb{P} \setminus P$. (Indeed, we have the exact sequence $(*)$, and $\operatorname{Hom}(\mathbb{Z}_{q^\infty}, \mathbb{Z}_{p^\infty}) = 0$).

Let $q \in \mathbb{P} \setminus P$. Then the identity map $\mathbb{Z}_{q^\infty} \to \mathbb{Z}_{q^\infty}$ is an $\mathcal{F}$-cover of $\mathbb{Z}_{q^\infty}$.

Let $D' = \prod_{p\in P} \mathbb{J}_p^\lambda$ and $D = \prod_{p\in P} \mathbb{D}_p^\lambda \oplus \prod_{q\in\mathbb{P}\setminus P} \mathbb{Z}_{q^\infty}^\lambda$. By [22, 1.2.9],

$$0 \to D' \hookrightarrow D \xrightarrow{\psi} Q^\lambda \to 0$$

is an $\mathcal{F}$-precover of Q^λ. Since D' is reduced, Corollary 2.6 and [22, 1.2.8] show that ψ is an $\mathcal{F}$-cover of Q^λ.

Finally, the composition of ψ with the projection, π, of Q^λ on to the summand W, is an $\mathcal{F}$-precover of W. Let E be the divisible part of $\mathrm{Ker}(\pi\psi)$. Then $D = E \oplus C$ for some (divisible) C. The restriction, f, of $\pi\psi$ to C, maps on to W. Let $C' = \mathrm{Ker}\, f = \mathrm{Ker}(\pi\psi) \cap C$. Then f is an $\mathcal{F}$-precover of W. Since C' is reduced, Corollary 2.6 and [22, 1.2.8] show that f is the $\mathcal{F}$-cover of W. Clearly, C is injective (and C' is reduced cotorsion, hence pure injective).

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