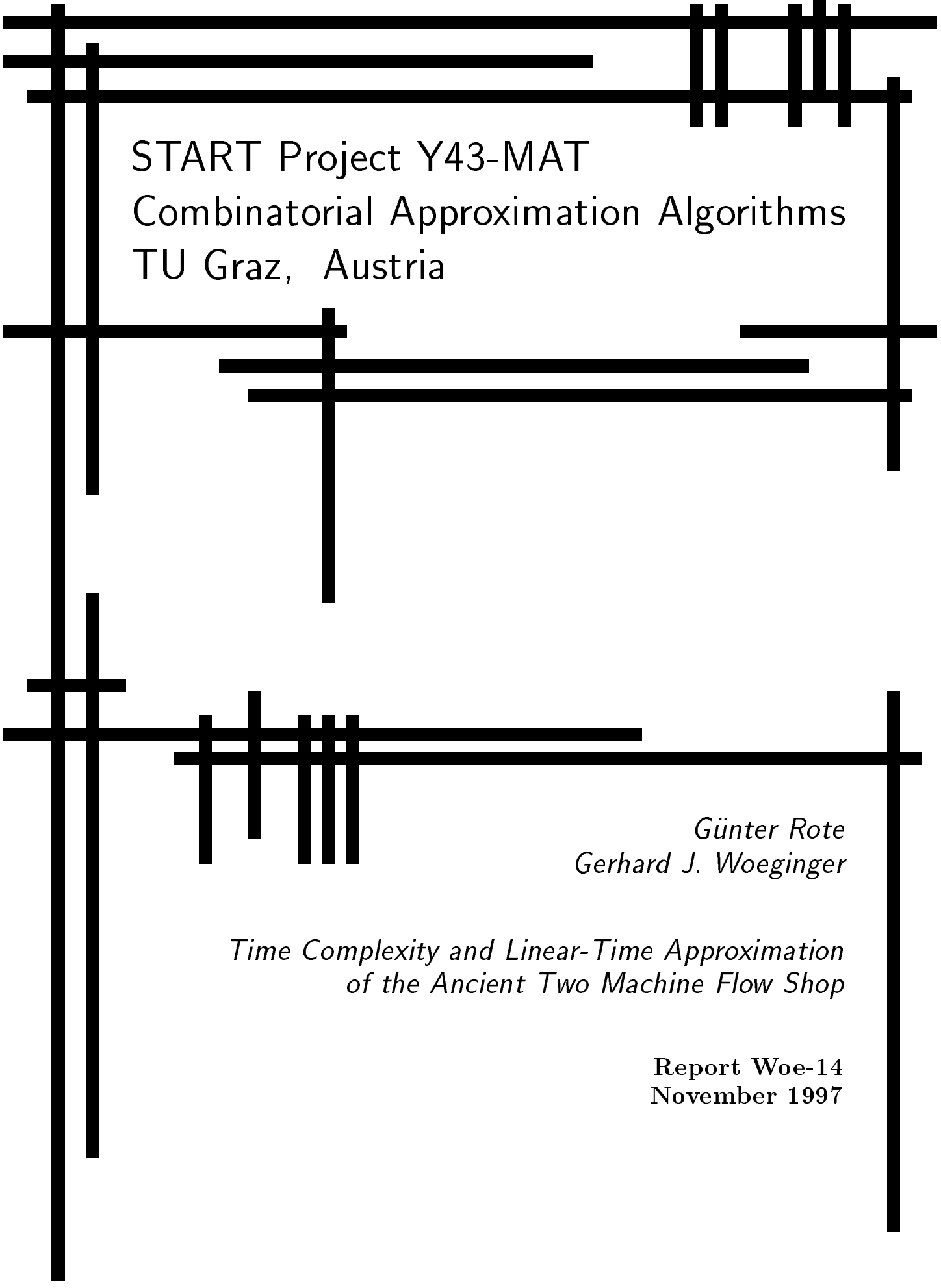

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# *Time Complexity and Linear-Time Approximation of the Ancient Two Machine Flow Shop*

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# Time complexity and linear-time approximation of the ancient two machine flow shop *

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**Abstract**

We consider the scheduling problems $F2\,||\,C_{\max}$ and $F2\,|\,\textit{no-wait}\,|\,C_{\max}$, i.e. makespan minimization in a two machine flow shop, with and without no wait in process. For both problems solution algorithms based on sorting with $O(n \log n)$ running time are known, where $n$ denotes the number of jobs [Johnson 1954, Gilmore & Gomory 1964].

We prove that no asymptotically faster algorithms can solve these problems. This is done by establishing $\Omega(n \log n)$ lower bounds in the algebraic computation tree model of computation. Moreover, we develop for every $\varepsilon > 0$ approximation algorithms with linear running time $O(n \log \frac{1}{\varepsilon})$ that deliver feasible schedules whose makespan is at most $1 + \varepsilon$ times the optimum makespan.

**Keywords.** Flow shop, no wait in progress, computational complexity, algebraic computation tree, lower bounds.

## 1 Introduction

In the two machine flow shop, there are two machines $M_A$ and $M_B$, and $n$ jobs $J_1, \ldots, J_n$. Every job $J_j$ $(j = 1, \ldots, n)$ needs to be processed first for $a_j$ time units on machine $M_A$ and afterwards for $b_j$ time units on machine $M_B$. The goal is to minimize the makespan, i.e. the point in time at which the last job is completed. In the standard scheduling notation (cf. Lawler, Lenstra, Rinnooy Kan & Shmoys [7]) this problem is denoted by $F2\,||\,C_{\max}$. In the closely related two machine *no wait flow shop*, $F2\,|\,\textit{no-wait}\,|\,C_{\max}$, each job, once started, has to be processed without interruption until it is completed.

The problems $F2\,||\,C_{\max}$ and $F2\,|\,\textit{no-wait}\,|\,C_{\max}$ both are solvable in $O(n \log n)$ time. For $F2\,||\,C_{\max}$, Johnson [6] shows that the following rule yields an optimum schedule: first process the jobs with $a_j \le b_j$ in order of nondecreasing $a_j$, and then process the remaining jobs in order of nonincreasing $b_j$. $F2\,|\,\textit{no-wait}\,|\,C_{\max}$ can be formulated as a travelling salesman problem whose distance matrix assumes a special combinatorial structure. The results of Gilmore & Gomory [4] then yield an $O(n \log n)$ solution to this travelling salesman problem (cf. Reddi & Ramamoorthy [10], or Gilmore, Lawler & Shmoys [5]). The only expensive step of the

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Gilmore & Gomory algorithm is an $O(n \log n)$ sorting step, whereas the remaining steps take $O(n)$ overall time. Hence, the time complexity of both algorithms mainly comes from a sorting step. Naturally the question arises whether one can do faster, without using a sorting routine.

The main contribution of this paper is that every algorithm for problems $F2 \,||\, C_{\max}$ and $F2 \,|\, \textit{no-wait} \,|\, C_{\max}$ must use $\Omega(n \log n)$ time in the worst case. This is proved by establishing $\Omega(n \log n)$ lower bounds in the algebraic computation tree model of computation, which is a standard model in theoretical computer science for studying the time complexity of computational problems. We also investigate the strength of *linear-time* algorithms for these problems. We show that for every $\varepsilon > 0$, the value of the optimum makespan in both problems can be approximated in linear time within an multiplicative error of $1+\varepsilon$. These results are essentially based on replacing the sorting steps that cost $O(n \log n)$ time by partitioning steps that cost $O(n)$ time.

The paper is organized as follows. In Section 2 we briefly introduce the algebraic computation tree model, and we establish the $\Omega(n \log n)$ lower bounds on the time complexity of the scheduling problems. Section 3 gives the linear-time approximation algorithms for $F2 \,||\, C_{\max}$ and for $F2 \,|\, \textit{no-wait} \,|\, C_{\max}$. Section 4 contains the conclusion.

## 2 Lower bounds in the algebraic computation tree model

In this section we prove that any algorithm for $F2 \,||\, C_{\max}$ and $F2 \,|\, \textit{no-wait} \,|\, C_{\max}$, respectively, must have an $\Omega(n \log n)$ worst case time complexity in the algebraic computation tree model of computation.

An *algebraic computation tree* (ACT, for short) models a deterministic program in which each step performs either a binary arithmetic operation $z := x \circ y$ with $\circ \in \{+, -, *, /\}$, a unary arithmetic operation $z := \sqrt{x}$, or a branching operation depending on whether $z < 0$, $z = 0$ or $z > 0$. These operations are the *atomic* operations that can be performed within one time-unit; note that computing the floor-function is not an atomic operation. The ACT is one of the standard models of computation in theoretical computer science; cf. e.g. Ben-Or [1], Ramanan [9], Seiferas [11], and Yao [12].

**Definition 2.1** *An instance of the* UNIFORM GAP *problem consists of $n$ real numbers $x_1, \ldots, x_n$ and a real number $\varepsilon > 0$. The problem is to decide whether all gaps between consecutive numbers are equal to $\varepsilon$. (Two numbers $x_i$ and $x_j$ are consecutive if $x_i \leq x_j$ and if there exists no other number $x_k$, $i \neq k \neq j$, such that $x_i \leq x_k \leq x_j$).*

**Proposition 2.2** *(Lee & Wu [8])*
*The* UNIFORM GAP *problem requires $\Omega(n \log n)$ worst case time in the algebraic computation tree model.* ∎

The lower bounds for problems $F2 \,||\, C_{\max}$ and $F2 \,|\, \textit{no-wait} \,|\, C_{\max}$ are proved by reducing the UNIFORM GAP problem in linear time to the scheduling problems. Indeed, let $x_1, \ldots, x_n$ and $\varepsilon$ constitute an instance of UNIFORM GAP. Without loss of generality we may assume that the smallest of the numbers $x_i$ is zero. We determine in linear time the sum $X$ of all values $x_i$. If $X \neq \frac{1}{2}(n-1)n\varepsilon$, the answer to the UNIFORM GAP instance must be NO. Otherwise, we

construct in linear time $O(n)$ an instance of the two machine flow shop: For $j = 1, \ldots, n$ we introduce a job $J_j$ with processing times $a_j = x_j$ on machine $A$ and processing time $b_j = x_j + \varepsilon$ on machine $B$. We claim that this scheduling instance for $F2\,|\,|\,C_{\max}$ and $F2\,|\,\textit{no-wait}\,|\,C_{\max}$ has a feasible schedule with makespan at most $\frac{1}{2}n(n+1)\varepsilon$ if and only if the UNIFORM GAP instance has answer YES.

(If). In case the UNIFORM GAP instance has answer YES, there exists a permutation $\pi \in S_n$ of the jobs such that $a_{\pi(i)} = (i-1)\varepsilon$ and $b_{\pi(i)} = i\varepsilon$ for all $1 \leq i \leq n$. Then the schedule $(J_{\pi(1)}, J_{\pi(2)}, \ldots, J_{\pi(n)})$ where for $i = 2, \ldots, n$ the operations $a_{\pi(i)}$ and $b_{\pi(i-1)}$ always are processed in parallel is a feasible permutation schedule for $F2\,|\,|\,C_{\max}$ and for $F2\,|\,\textit{no-wait}\,|\,C_{\max}$. Moreover, this schedule has makespan $\frac{1}{2}n(n+1)\varepsilon$.

(Only if). Assume that a schedule exists with makespan at most $\frac{1}{2}n(n+1)\varepsilon$. Without loss of generality we may assume that this schedule is a permutation schedule in which both machines process the jobs in the same order $\pi \in S_n$. Then for every $1 \leq k \leq n$, the inequality

$$\sum_{i=1}^{k} a_{\pi(i)} + \sum_{i=k}^{n} b_{\pi(i)} \;\leq\; \frac{1}{2}n(n+1)\varepsilon \tag{1}$$

holds. By rewriting the left-hand side of this inequality, one derives

$$\sum_{i=1}^{k} a_{\pi(i)} + \sum_{i=k}^{n} (a_{\pi(i)} + \varepsilon) \;=\; X + a_{\pi(k)} + (n-k+1)\varepsilon \;\leq\; \frac{1}{2}n(n+1)\varepsilon, \tag{2}$$

which is equivalent to $a_{\pi(k)} \leq (k-1)\varepsilon$. Summing this over all $1 \leq k \leq n$ yields

$$X \;=\; \sum_{k=1}^{n} a_{\pi(k)} \;\leq\; \sum_{k=1}^{n} (k-1)\varepsilon \;=\; \frac{1}{2}n(n-1)\varepsilon \;=\; X. \tag{3}$$

This implies that for all $1 \leq k \leq n$, $a_{\pi(k)} = (k-1)\varepsilon$ must hold. Hence, the considered instance of UNIFORM GAP has answer YES.

Summarizing, any algorithm for $F2\,|\,|\,C_{\max}$ or $F2\,|\,\textit{no-wait}\,|\,C_{\max}$ with worst case running time $O(f(n))$ can be used to construct an $O(n + f(n))$ algorithm for UNIFORM GAP. Now the lower bound result of Lee & Wu [8] (cf. Proposition 2.2) yields that $f(n)$ must be $\Omega(n \log n)$.

**Theorem 2.3** *Any algorithm for $F2\,|\,|\,C_{\max}$ and any algorithm for $F2\,|\,\textit{no-wait}\,|\,C_{\max}$ must have a worst case running time $\Omega(n \log n)$ in the algebraic computation tree model of computation.* ∎

## 3 A linear-time approximation algorithm

In this section, our goal is to approximate the optimum makespan of a two machine flow shop (with or without no wait in process) in linear time. Let $\varepsilon > 0$ be some small fixed real number and define $g = \lceil \log_2(\frac{4}{\varepsilon}) \rceil$. We consider an instance $I$ of $F2\,|\,|\,C_{\max}$ or $F2\,|\,\textit{no-wait}\,|\,C_{\max}$ that is specified by the processing times $a_j$ and $b_j$ $(j = 1, \ldots, n)$ of the $n$ jobs. Let $C^*_{\max}(I)$ denote the optimum makespan of $I$. Hence, our goal is to find in $O(ng) = O(n \log \frac{1}{\varepsilon})$ time a feasible schedule for $I$ with makespan at most $(1+\varepsilon)C^*_{\max}(I)$.

**Observation 3.1** *Without loss of generality we may assume that the number $n$ of jobs is of the form $n = 2^g(2^m - 1)$ for some integer $m \geq 1$.*

**Proof.** If $n < 2^g$ holds, then we can solve the instance to optimality in $O(n \log n) = O(ng)$ time. If $n \geq 2^g$ holds and $n$ is not of the form $2^g(2^m - 1)$, then we add an appropriate set of dummy jobs of length 0. By at most tripling the number of jobs, we get a number of jobs of the desired form. ∎

Consider the $n$ processing times $a_1, \ldots, a_n$ on the first machine and define the following partition $A_{i,k}$, $0 \leq i \leq m-1$ and $1 \leq k \leq 2^g$, of the numbers $a_j$:

- For $0 \leq i \leq m-1$ and $1 \leq k \leq 2^g$, the set $A_{i,k}$ contains $2^i$ elements.
- For $0 \leq i \leq m-2$ and $1 \leq k, \ell \leq 2^g$, all elements in $A_{i+1,k}$ are less or equal to all elements in $A_{i,\ell}$.
- For $0 \leq i \leq m-1$ and $1 \leq k < \ell \leq 2^g - 1$, all elements in $A_{i,k+1}$ are less or equal to all elements in $A_{i,k}$.

Moreover, we define a similar partition $B_{i,k}$, $0 \leq i \leq m-1$ and $1 \leq k \leq 2^g$, of the numbers $b_1, \ldots, b_n$. For $0 \leq i \leq m-1$ and $1 \leq k \leq 2^g$, denote by $\alpha_{i,k}$ the maximum value in $A_{i,k}$ and denote by $\beta_{i,k}$ the maximum value in $B_{i,k}$. Finally, we define a new instance $I'$ of $F2\,|\,|\,C_{\max}$, respectively $F2\,|\,\textit{no-wait}\,|\,C_{\max}$, as follows: For every job $J_j$ in $I$ with processing times $a_j \in A_{i,k}$ and $b_j \in B_{r,s}$ the instance $I'$ contains a corresponding job $J'_j$ with processing times $a'_j = \alpha_{i,k}$ and $b'_j = \beta_{r,s}$.

Since for all $j$ $(j = 1, \ldots, n)$ $a_j \leq a'_j$ and $b_j \leq b'_j$ holds, the optimum makespan $C^*_{\max}(I')$ of instance $I'$ must be greater or equal to $C^*_{\max}(I)$. Conversely, consider an optimum schedule for $I$ in which each machine processes the jobs in the same order $\pi \in S_n$. If we use $\pi$ to process the corresponding jobs in $I'$, the increase in the makespan is upper bounded by the total increase in the processing times. Summarizing, we have

$$C^*_{\max}(I) \;\leq\; C^*_{\max}(I') \;\leq\; C^*_{\max}(I) + \sum_{j=1}^{n}(a'_j - a_j) + \sum_{j=1}^{n}(b'_j - b_j). \tag{4}$$

Observe the following statements for the smallest value in set $A_{i,k}$.

- For $1 \leq k \leq 2^g$, the smallest value in $A_{0,k}$ is equal to its only element $\alpha_{0,k}$.
- For $1 \leq i \leq m-1$ and $1 \leq k \leq 2^g - 1$, the smallest value in $A_{i,k}$ is at least $\alpha_{i,k+1}$.
- If $1 \leq i \leq m-1$, then the smallest value in $A_{i,2^g}$ is at least $\alpha_{i+1,1}$. (Define $\alpha_{m,1} = 0$).
- In $A_{m-1,2^g}$, the smallest value is at least 0.

By using these bounds on $a_j$ and by summing over all sets $A_{i,k}$, one gets that

$$\sum_{j=1}^{n} a_j \;\geq\; \sum_{k=1}^{2^g} \alpha_{0,k} + \sum_{i=1}^{m-1}\sum_{k=1}^{2^g-1} 2^i \alpha_{i,k+1} + \sum_{i=1}^{m-1} 2^i \alpha_{i+1,1}. \tag{5}$$

By using $\alpha_{0,k} \geq \alpha_{1,1}$ together with $\alpha_{i,k+1} \geq \alpha_{i+1,1}$ for $1 \leq i \leq m-2$ and $1 \leq k \leq 2^g - 1$, and together with $\alpha_{m-1,k+1} \geq 0$ for $1 \leq k \leq 2^g - 1$, inequality (5) leads to

$$\sum_{j=1}^{n} a_j \;\geq\; 2^g \alpha_{1,1} + \sum_{i=1}^{m-2} \sum_{k=1}^{2^g-1} 2^i \alpha_{i+1,1} + \sum_{i=1}^{m-2} 2^i \alpha_{i+1,1} \;=\; 2^g \sum_{i=1}^{m-1} 2^{i-1} \alpha_{i,1}. \tag{6}$$

Moreover,

$$\sum_{j=1}^{n} a'_j \;=\; \sum_{k=1}^{2^g} \alpha_{0,k} + \sum_{i=1}^{m-1} \sum_{k=1}^{2^g-1} 2^i \alpha_{i,k} + \sum_{i=1}^{m-1} 2^i \alpha_{i,2^g}. \tag{7}$$

By subtracting (5) from (7) and by telescoping several sums, we get

$$\begin{aligned}
\sum_{j=1}^{n} (a'_j - a_j) \;&\leq\; \sum_{i=1}^{m-1} \sum_{k=1}^{2^g-1} 2^i (\alpha_{i,k} - \alpha_{i,k+1}) + \sum_{i=1}^{m-1} 2^i (\alpha_{i,2^g} - \alpha_{i+1,1}) \\
&=\; \sum_{i=1}^{m-1} 2^i (\alpha_{i,1} - \alpha_{i,2^g}) + \sum_{i=1}^{m-1} 2^i (\alpha_{i,2^g} - \alpha_{i+1,1}) \\
&=\; \sum_{i=1}^{m-1} 2^i (\alpha_{i,1} - \alpha_{i+1,1}) \\
&=\; \alpha_{1,1} + \sum_{i=1}^{m-1} 2^{i-1} \alpha_{i,1}.
\end{aligned} \tag{8}$$

Finally, since the optimum makespan is greater or equal to the total processing time on the first machine, we get from (6) and (8) that

$$\begin{aligned}
C^*_{\max}(I) \;&\geq\; \sum_{j=1}^{n} a_j \;\geq\; 2^g \sum_{i=1}^{m-1} 2^{i-1} \alpha_{i,1} \\
&\geq\; 2^{g-1} (\alpha_{1,1} + \sum_{i=1}^{m-1} 2^{i-1} \alpha_{i,1}) \;\geq\; 2^{g-1} \sum_{j=1}^{n} (a'_j - a_j)
\end{aligned} \tag{9}$$

Now (9) together with the definition of $g$ yields that

$$\sum_{j=1}^{n} (a'_j - a_j) \;\leq\; \frac{\varepsilon}{2} \cdot C^*_{\max}(I). \tag{10}$$

Analogous arguments yield

$$\sum_{j=1}^{n} (b'_j - b_j) \;\leq\; \frac{\varepsilon}{2} \cdot C^*_{\max}(I). \tag{11}$$

Finally, we combine (10) and (11) with (4), and thus get

$$C^*_{\max}(I') \;\leq\; (1+\varepsilon) C^*_{\max}(I). \tag{12}$$

**Lemma 3.2** *From instance $I$, one can compute instance $I'$ in $O(ng)$ time.*

**Proof.** Blum, Floyd, Pratt, Rivest & Tarjan [2] show that the $r$-largest value among $s$ numbers can be determined in $O(s)$ time. We use this algorithm to compute all sets $A_{i,k}$ and all sets $B_{i,k}$ in $O(ng)$ time.

In a first step, we compute $m$ auxiliary sets $A_{m-1}, A_{m-2}, \ldots, A_0$. For $0 \leq i \leq m-1$, $|A_i| = 2^{g+i}$ holds; all elements in $A_{i+1}$ are less or equal to all elements in $A_i$. Set $A_{m-1}$ is computed as follows: Determine the median (i.e. the $\frac{n}{2}$-largest element) of the values $a_1, \ldots, a_n$. Remove $\frac{n}{2} = 2^{g+m-1}$ elements that are less or equal to the median, and put them into $A_{m-1}$. From the remaining elements, successively compute $A_{m-2}$, $A_{m-3}$, and so on. Since the computation of $A_i$ costs $O(2^{g+i})$ time, this yields an overall time complexity of $O(n)$ for the first step.

In the second phase, every set $A_i$ is split into the sets $A_{i,k}$ with $1 \leq k \leq 2^g$. We can split $A_i$ by repeatedly halving it into smaller and smaller pieces. In the $\ell$-th iteration, we call the algorithm of Blum & al. for $2^{\ell-1}$ sets with $2^{g+i-\ell+1}$ elements each, which takes $O(|A_i|)$ time. After $g$ iterations, we are done. Hence, the overall time complexity for splitting $A_i$ is $O(g|A_i|)$, and the overall time complexity for splitting all sets is $O(ng)$.

Analogously, we compute all sets $B_{i,k}$ in $O(ng)$ time. It is straightforward to determine the values $\alpha_{i,k}$ and $\beta_{i,k}$, and the values $a'_j$ and $b'_j$ in linear time. Hence, instance $I'$ can be computed in $O(ng)$ time from instance $I$. ∎

**Lemma 3.3** *For any instance $I$ of $F2\,||\,C_{\max}$ or $F2\,|\,\textit{no-wait}\,|\,C_{\max}$, one can compute an optimum solution to the corresponding instance $I'$ in $O(n)$ time.*

**Proof.** First consider the case where $I$ is an instance of $F2\,||\,C_{\max}$. Johnson's algorithm [6] mainly relies on sorting the jobs according to their processing times. Since we already know the partitions $A_{i,k}$ and $B_{i,k}$, the sorting of the jobs in $I'$ can easily be done in $O(n)$ time.

For the case where $I$ is an instance of $F2\,|\,\textit{no-wait}\,|\,C_{\max}$, we refer the reader to the work of Gilmore & Gomory [4] (see also Gilmore, Lawler & Shmoys [5]). The first step of the Gilmore & Gomory algorithm consists in sorting the values $a'_j$ and in sorting the values $b'_j$. Similarly as above, we observe that this sorting can be done in $O(n)$ time. The remaining steps of the Gilmore & Gomory algorithm – i.e. computing an optimum assignment, finding the subtours, computing a patching tree of minimum cost for the subtours, and performing the subtour patching – all can be done in $O(n)$ time. We stress the fact that in the Gilmore & Gomory case *any* spanning tree for the subtours is an optimum patching tree (cf. Section 5.3 in Burkard, Deineko, van Dal, van der Veen & Woeginger [3]). ∎

**Theorem 3.4** *For each real $\varepsilon > 0$ and for each $n$ job instance $I$ of $F2\,||\,C_{\max}$, respectively $F2\,|\,\textit{no-wait}\,|\,C_{\max}$, one can compute in $O(n \log \frac{1}{\varepsilon})$ time a feasible schedule whose makespan is at most $1+\varepsilon$ times the optimum makespan.*

**Proof.** Compute instance $I'$ as described in Lemma 3.2, and solve it as described in Lemma 3.3. Let $J'_{\pi(1)}, \ldots, J'_{\pi(n)}$ denote the job ordering in the optimum schedule for $I'$. Then by (12), the job ordering $J_{\pi(1)}, \ldots, J_{\pi(n)}$ gives a schedule for instance $I$ with makespan at most $(1+\varepsilon)C^*_{\max}(I)$. ∎

## 4 Conclusion

In this paper we have shown that any algorithm for the two machine flow shop problems $F2\,||\,C_{\max}$ and $F2\,|\,\textit{no-wait}\,|\,C_{\max}$ needs $\Omega(n \log n)$ time in the algebraic computation tree model of computation. We remark that this does not exclude the existence of linear-time algorithms which use the floor-function, indirect addressing, or similar tricks (In fact, the UNIFORM GAP problem can be solved in linear time by bucket sort, using the floor-function). Moreover, we have shown that the optimum solution of these problems can be approximated in linear time with arbitrarily good precision.

Finally, we remark that also the problem $1\,||\,L_{\max}$ of minimizing the maximum lateness, the problem $1\,||\,T_{\max}$ of minimizing the maximum tardiness, and the problem $1\,||\,\sum U_j$ of minimizing the number of tardy jobs have $\Omega(n \log n)$ lower bounds in the algebraic computation tree model of computation. These three problems possess well-known $O(n \log n)$ algorithms (cf. Lawler, Lenstra, Rinnooy Kan & Shmoys [7]).

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