NBER WORKING PAPER SERIES

TIME CONSISTENCY AND THE DURATION OF GOVERNMENT DEBT: A SIGNALLING THEORY OF QUANTITATIVE EASING

Saroj Bhattarai
Gauti B. Eggertsson
Bulat Gafarov

Working Paper 21336
http://www.nber.org/papers/w21336

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
July 2015

We thank Roberto Billi, Jim Bullard, Guillermo Calvo, Oli Coibion, Giuseppe Ferrero, Mark Gertler, Marc Giannoni, Andy Levin, Emi Nakamura, Ricardo Reis, Tao Zha, seminar participants at HEC Montreal, Emory University, University of Texas at Austin, and Brown University, and conference participants at NBER ME spring meeting, HKIMR/New York Fed Conference on Domestic and International Dimensions of Unconventional Monetary Policy, Society of Economic Dynamics Annual meeting, Mid-west Macro spring meeting, CEPR European Summer Symposium in International Macroeconomics, Annual Conference on Computing and Finance, NBER Japan Project Meeting, Annual Research Conference at Swiss National Bank, ECB Workshop on Non-Standard Monetary Policy Measures, Annual Research Conference at De Nederlandsche Bank, Latin American Meetings of Econometric Society, and Columbia University Conference on Macroeconomic Policy and Safe Assets for helpful comments and suggestions. First version: Sept 2013; This version: June 2015. The views expressed herein are those of the authors and do not necessarily reflect the views of the National Bureau of Economic Research.

Time Consistency and the Duration of Government Debt: A Signalling Theory of Quantitative Easing
Saroj Bhattarai, Gauti B. Eggertsson, and Bulat Gafarov
NBER Working Paper No. 21336
July 2015
JEL No. E31,E4,E42,E43,E5,E52,E62,E63

ABSTRACT

We present a signalling theory of Quantitative Easing (QE) at the zero lower bound on the short term nominal interest rate. QE is effective because it generates a credible signal of low future real interest rates in a time consistent equilibrium. We show these results in two models. One has coordinated monetary and fiscal policy. The other an independent central bank with balance sheet concerns. Numerical experiments show that the signalling effect can be substantial in both models.

Saroj Bhattarai
University of Texas at Austin
Department of Economics
2225 Speedway, Stop C3100
Austin, TX 78712-1690
saroj.bhattarai@austin.utexas.edu

Gauti B. Eggertsson
Department of Economics
Brown University
64 Waterman Street
Providence, RI 02912
and NBER
gauti_eggertsson@brown.edu

Bulat Gafarov
Pennsylvania State University
and National Research University
Higher School of Economics
gafarovb@gmail.com

"The problem with Quantitative Easing (QE) is it works in practice, but it doesn't work in theory," Ben Bernanke, Chairman of the Federal Reserve, Jan 16, 2014 just before leaving office.

1 Introduction

Since the onset of the economics crisis in 2008, the Federal Reserve has expanded its balance sheet by large amounts, on the order of 3 trillion mostly under the rubric of Quantitative Easing (QE). To date the accumulated amount of QE corresponds to about 20% percent of annual GDP. The enormous scale of this policy has largely been explained by the fact that the Federal Reserve was unable to cut the Federal Fund rate further, due the zero lower bound on the short term nominal interest rate. Meanwhile, high unemployment, slow growth, and low inflation desperately called for further stimulus measures.

Many commentators argue that QE in the United States prevented a much stronger contraction, and that QE is a key reason for why the US has recovered more rapidly from the Great Recession than some its counterparts. As pointed out by Ben Bernanke, however, one problem is that even if this might be true in practice, a coherent theoretical rationale has been hard to formulate. This paper contributes to filling this gap. We providing an explicit theoretical underpinning for QE: It works because it allows the central bank to credibly commit to expansionary future policy in a zero lower bound situation. We not only explicitly account for QE in theory, but also show some numerical examples in which the effect is non-trivial.

What is QE? Under our interpretation, it is when the central bank buys long-term government debt with money. We interpret this action though the lenses of two models. First, we treat the central bank and the treasury as one agent, i.e. policy is coordinated and budget constraints of the Treasury and the central bank are consolidated. Second, we treat the central bank as "independent" in the sense that it faces its own budget constraint (and thus cares about it own balance sheet) and its objective may be different from social welfare.

Consider first QE under coordinated monetary and fiscal policy, the main benchmark in the paper.[1] Since the nominal interest rate was zero when QE was implemented in the United States, it makes no difference if QE was done by printing money (or more precisely bank reserves) or by issuing short-term government debt: both are government issued papers that yield a zero interest rate. From the perspective of the government as a whole, QE at zero nominal interest rates can then simply be thought of as shortening the maturity of outstanding government debt. The government is simply exchanging long term bonds in the hands of the public with short term ones.

Consider next QE from the perspective of the central bank in isolation. QE creates a "duration mismatch" on the balance sheet of the central bank as it is issuing money/reserves ("short-term debt") in exchange for long term treasuries ("long term assets"). This opens up the possibility of possible future balance sheet losses/gains by the central bank, because the price of its liabilities

[1] Here we will be abstracting from variation in real government spending, a focus of Christiano, Eichenbaum and Rebelo (2011), Eggertsson (2010), Woodford (2011) and Werning (2012).

may fall/rise relative to its assets. We can interpret QE as simply increasing the size of the balance sheet of the central bank, keeping the extent of the duration mismatch on its balance sheet fixed. Alternatively we can interpret it as only increasing the duration on its asset side, keeping the liability side and size of the balance sheet unchanged. Either interpretation is valid, and we will look at the data to sort out which interpretation fits the facts better for a particular QE episode.

Whether we consider QE from the perspective of a consolidated government budget constraint, or an independent central bank, we arrive at the same conclusion. In both settings, QE operates as a "signal" for lower future short term interest rates in a way we make precise.

The main goal of QE in the United States was to reduce long-term interest rates, even when the short-term nominal interest rate could not be reduced further, and thereby, stimulate the economy. Indeed, several empirical studies find evidence of reduction in long-term interest rates following these policy interventions by the Federal Reserve (see e.g. Gagnon et al (2011), Krishnamurthy and Vissing-Jorgensen (2011), Hamilton and Wu (2012), Swanson and Williams (2013) and Bauer and Rudebusch (2013)).[2]

From a theoretical perspective however, the effect of such policy is not obvious since open market operations of this kind are neutral (or irrelevant) in standard macroeconomic models holding the future interest rate reaction function constant. This may have motivated Ben Bernanke's quote cited above. This was pointed out first in a well-known contribution by Wallace (1981) and further extended by Eggertsson and Woodford (2003) to a model with sticky prices and an explicit zero lower bound on nominal interest rates. These papers showed how absent some restrictions in asset trade that prevent arbitrage, a change in the relative supplies of various assets in the hands of the private sector has no effect on equilibrium quantities and asset prices.

For this reason, some papers have recently incorporated frictions such as participation constraints due to "preferred habitat" motives in order to make assets of different maturities imperfect substitutes. This in turn negates the neutrality of open market operations as in such an environment, QE can reduce long-term interest rates because it decreases the risk-premium, see for example Chen, Curdia, and Ferrero (2012). Others, such as Gertler and Karadi (2012), provide a framework in which these operations can have an effect due to limits to arbitrage.[3] Overall, our reading of this literature is that the effect of QE is modest in these models, with the possible exception of QE1 when there were significant disruptions in the financial markets.

As pointed out by Eggertsson and Woodford (2003) (and further illustrated in Woodford (2012) in the context of the crisis) QE need not be effective only because it reduces risk premiums or due to limits on arbitrage. QE can also reduce long-term interest rates if it signals to the private

[2] For example, Gagnon et al (2011) estimate that the $1.75 trillion worth 2009 program reduced long-term interest rates by 58 basis points while Krishnamurthy and Vissing-Jorgensen (2011) estimate that the $600 billion worth 2010 program reduced long-term interest rates by 33 basis points. In addition, Hamilton and Wu (2012), Swanson and Williams (2013), and Bauer and Rudebusch (2013) also find similar effects on long-term interest rates. Note however that empirical studies typically measure nominal interest rates, while theoretically, it is the ability to influence real interest rates that matter.

[3] Del Negro et al (2012) is another example which focuses more on QE1 which has an effect in their model due to imperfect liquidity of private paper. That work, however, is less suitable to think about QE2 and QE3.

sector that the central bank will keep the short-term interest rates low once the zero lower bound is no longer a constraint, i.e. signals a change in the policy rule taken as given in Eggertsson and Woodford's (2003) irrelevant result. In fact, arguably, much of the findings of the empirical literature on reduction of long-term interest rates due to QE can be attributed to expectations of low future short-term interest rates. Indeed, Krishnamurthy and Vissing-Jorgensen (2011) and Bauer and Rudebusch (2013) find evidence in support of this channel in their study of the various QE programs.

Our contribution in this paper is to provide a formal theoretical model of such a "signalling" role of QE in a standard general equilibrium model. To do this we analyze a Markov Perfect Equilibrium (MPE) in a game between the government and the private sector. In this equilibrium, agents will use the "natural" state variables of the game to predict the behavior of future governments. QE will have an effect because it changes the endogenous state variables of the game. In this respect our model of signaling is different from models in a related literature on signalling in which central bank types are fixed (they can either be "doves" or "hawks", see e.g. Barro (1986)). In these models, central banks use nominal interest rates to signal how much they care about inflation. Our signalling mechanism is different from this literature, because the central bank's "type" or preference for inflation is derived endogenously and depends upon the size and composition of the asset holdings of the central bank (moreover, there is full information about the preferences of the government). Thus our interpretation of "signalling" is somewhat different, namely, it has to do with credibly changing the central banks' future policy incentives, or "types" in the language of this earlier literature.

The paper connects more closely to the theoretical literature on how the maturity structure of debt can be manipulated to eliminate the dynamic inconsistency problems in monetary models. Well known examples include Lucas and Stokey (1983), Persson, Persson, and Svensson (1987 and 2006), Calvo and Guidotti (1990 and 1992) and Alvarez, Kehoe, and Neumeyer (2004). While the focus of these papers is generally on policies that eliminates the government's incentive to inflate, our application is the opposite. In our setting, the maturity structure of debt is made shorter to solve the deflation bias (Eggertsson (2006)) that arises when the zero bound is binding. In terms of modelling strategy, a key difference relative to this work is that because we assume sticky prices, the government has an effect not only on inflation, but also on the real interest rate which gives rise to a new margin for policy that will prove to be important. Finally, the part of our paper with an independent central bank that has balance sheet concerns is related to Jeanne and Svensson (2007) and Berriel and Bhattarai (2009). The key difference is that this work focuses on the effects of foreign exchange intervention and does not have long-term assets, while we analyze the size and maturity composition of central bank balance sheet in order to connect to QE.

Below we outline the organization of the paper and preview some of the key findings. We start out by defining a Markov Perfect Equilibrium (MPE) in the standard New Keynesian model (Section 2). The key difference relative to standard treatments is that we allow for long-term government debt. An important simplification is that we assume that the long-term debt is of some fixed duration and we will interpret QE as a one-time reduction in this duration. We defer to Section 5 to define the MPE with time varying and optimally chosen duration of government debt.

We first define the equilibrium in the fully non-linear model, assuming that monetary and fiscal policy are coordinated. In this case a natural objective for the government is utility function of the representative agent. We then (Section 3) show how the model can be approximated via log-linearization of the constraints and quadratic approximation of social welfare. This is helpful because it simplifies the model considerably and allows for a more transparent discussion of the main results. A key proposition (Proposition 1) shows that the MPE of this approximate economy is equivalent to a first order approximation of the MPE of the non-linear model. This is an important step, because linear-quadratic approximation are in general not valid for this class of problems.[4]

An important element of Section 3 is that we define the MPE not only in the context of coordinated monetary and fiscal policy but also if the central bank has its own objective and budget constraint. One conclusion that emerges from Section 3 is that which model one adopts has critical effects on how one interprets the data. If we think of the government as a unified entity then what is important is the term structure of the government debt held in the hands of the private sector. In contrast, for an independent central bank, what is important is the size and duration of the central banks assets and liabilities on its balance sheet. We use the model to organize the data under both approaches. The main findings of the data section will then serve as the basis of the numerical experiments in Section 4.

The baseline New Keynesian model is perhaps too simple for us to take numerical simulations literally as point estimates of the effect of QE. Nevertheless, we think it is useful to explicitly parameterize the model to organize the key results and get some sense for the orders of magnitudes. This is what we do in Section 4. To parameterize the model we ask it to replicate five targets, which we formalize by choosing parameters to minimize the mean squared errors of the model variables relative to the targets. We construct the targets as follows. First, we want the model to generate a substantial recession. To do so, we ask the model to generate a recession at zero interest rate due to drop in the efficient rate of interest as in Eggertsson and Woodford (2003). More specifically, we ask the model to generate a fall in inflation of 2 percent, an output gap of -10 percent, and an expected duration of liquidity trap of 3 years. We then use the numbers about the size of QE we construct in Section 3 and ask the model to generate a response to this policy on future inflation and long-term yields estimated by Krishnamurthy and Vissing-Jorgensen (2011). The approach is then to ask the parameterized model the following question: Given that we match these five targets as best as we can, what does the model predict would have happened to output and inflation in the absence of QE? The answer to this question is that output would have been 30 basis point lower and inflation 14 basis points lower (annualized) in the model with coordinated monetary and fiscal policy for the episode we label QE2. For the central bank with balance sheet

[4] Recent literature on log-linear approximations at the ZLB suggests that these approximations can be surprisingly accurate, even under extreme circumstances such as those that are meant to replicate the Great Depression at the ZLB (Eggertsson and Singh (2015)). We do no consider such extreme examples here, however.

concerns model, the analogous numbers are 45 basis points for output and 14 for inflation. What we label as the Maturity Extension Program or QE3 in contrast had a bigger effect, yielding 90 basis points for output and 40 basis point for inflation under coordinated policy but these effects are smaller for a central bank with balance sheet concerns. These experiments suggest that the signalling effect can in principle be substantial in modern monetary models.

2 Benchmark model

We start by outlining our benchmark model in which case monetary and fiscal policy are coordinated to maximize social welfare under discretion (i.e. the government cannot commit to future policy). The model is a standard general equilibrium sticky-price closed economy set-up with an output cost of taxation, along the lines of Eggertsson (2006). The main difference in the model from the literature is the introduction of long-term government debt. While it may seem like a distraction to write out the fully non-linear model and define the equilibrium in that context, as we will later on analyze a linear quadratic version of this model, this is useful for two reasons. First, we will formally show that the linearized first-order conditions of the government's original non-linear problem are the same as in our linear quadratic model (and this in general need not be the case, see e.g. Eggertsson (2006)). Second, the non-linear version of the problem will be important in section 5 once we allow for fully time-varying duration of government debt, where the linear quadratic approximation is no longer valid. Laying out the model in this way, also, makes transparent the relationship between social welfare and the ad-hoc objectives we will assign to the central bank when it is independent in an alternative variation of the model which we propose in Section 3. There, again, we will be working in a linear quadratic framework.

2.1 Private sector

A representative household maximizes expected discounted utility over the infinite horizon

$$E_t \sum_{t=0}^{\infty} \beta^t U_t = E_t \sum_{t=0}^{\infty} \beta^t \left[u\left(C_t\right) + g\left(G_t\right) - v(h_t) \right] \xi_t \tag{1}$$

where β is the discount factor, C_t is household consumption of the final good, G_t is government consumption of the final good, h_t is labor supplied, and ξ_t is a shock. E_t is the mathematical expectation operator conditional on period-t information, $u\left(.\right)$ is concave and strictly increasing in C_t, $g\left(.\right)$ is concave and strictly increasing in G_t, and $v\left(.\right)$ is increasing and convex in h_t.[5]

The final good is an aggregate of a continuum of varieties indexed by i, $C_t = \int_0^1 \left[c_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}}$, where $\varepsilon > 1$ is the elasticity of substitution among the varieties. The optimal price index for the final good is given by $P_t = \left[\int_0^1 p_t(i)^{1-\varepsilon} di \right]^{\frac{1}{1-\varepsilon}}$, where $p_t(i)$ is the price of the variety i. The demand

[5] We abstract from money in the model and are thus directly considering the "cash-less limit."

for the individual varieties is then given by $\frac{c_t(i)}{C_t} = \left(\frac{p_t(i)}{P_t}\right)^{-\varepsilon}$. Finally, G_t is defined analogously to C_t and so we omit detailed description of government spending.

The household is subject to a sequence of flow budget constraints

$$P_t C_t + B_t^S + S_t B_t + E_t\{Q_{t,t+1} A_{t+1}\} \leq n_t h_t + (1+i_{t-1}) B_{t-1}^S + (1+\rho S_t) B_{t-1} + A_t - P_t T_t + \int_0^1 Z_t(i) di \tag{2}$$

where n_t is nominal wage, $Z_t(i)$ is nominal profit of firm i, B_t^S is the household's holding of one-period risk-less nominal government bond at the beginning of period $t+1$, B_t is a perpetuity bond, S_t its price, and ρ its decay factor (further described below). A_{t+1} is the value of the complete set of state-contingent securities at the beginning of period $t+1$ and $Q_{t,t+1}$ is the stochastic discount factor between periods t and $t+1$ that is used to value random nominal income in period $t+1$ in monetary units at date t.[6] Finally, i_{t-1} is the nominal interest rate on government bonds at the beginning of period t and T_t is government taxes.

The way we introduce long term bonds into the model is to assume that government debt not only takes the form of a one period risk-free debt, B_t^S, but that the government also issues a perpetuity in period t which pays ρ^j dollars $j+1$ periods later, for each $j \geq 0$ and some decay factor $0 \leq \rho < \beta^{-1}$.[7] S_t is the price of the perpetuity nominal bond which depends on the decay factor ρ. The main convenience of introducing long term bond in this way is that we can consider government debt of arbitrary duration. For example, a value of $\rho = 0$ implies that this bond is simply a short-term bond while $\rho = 1$ corresponds to a classic console bond. More generally, in an environment with stable prices, the duration of this bond is $(1-\beta\rho)^{-1}$. Thus, this simple assumption allows us to explore a change in the duration of government debt in a transparent way. The appendix contains details on why the budget constraint takes the form (2). In particular, the modeling of long-term bond in this way admits a simple recursive formulation of the price of old government bonds.

For now, observe that we treat ρ as a constant. We will explore a one-time reduction in this duration as a main "comparative static" of interest. In other words, a reduction in ρ answers the question: What does a permanent reduction in the maturity of government debt do?[8] Toward the end of the paper, however, we will extend the analysis so that ρ becomes a time varying choice variable ρ_t. The main reason for our initial benchmark assumption is simplicity (and the fact that we get a clean comparative static). But perhaps more importantly, we will see later that a one-time reduction in ρ (in a liquidity trap) turns out to be a reasonably good approximation because ρ_t is close to a random walk under optimal policy under discretion at a positive interest rate.

The maximization problem of the household is now entirely standard, with the additional feature of the portfolio choice between long and short term bonds.[9] Let us now turn to the firm

[6] The household is subject to a standard no-Ponzi game condition.

[7] We follow Woodford (2001).

[8] When we move towards an independent central bank, the thought experiment will be somewhat different, as we soon explain.

[9] The problem of the household is thus to choose $\{C_{t+s}, h_{t+s}, B_{t+s}^S, B_{t+s}, A_{t+s}\}$ to maximize (1) subject to a sequence of flow budget constraints given by (2), while taking as exogenously given initial wealth and $\{P_{t+s}, n_{t+s}, i_{t+s}, S_{t+s}(\rho), Q_{t,t+s}, \xi_{t+s}, Z_{t+s}(i), T_{t+s}\}$.

side of the model. There is a continuum of monopolistically competitive firms indexed by i. Each firm produces a variety i according to the production function that is linear in labor $y_t(i) = h_t(i)$. As in Rotemberg (1983), firms face a cost of changing prices given by $d\left(\frac{p(i)}{p_{t-1}(i)}\right)$.[10] The demand function for variety i is given by

$$\frac{y_t(i)}{Y_t} = \left(\frac{p_t(i)}{P_t}\right)^{-\varepsilon} \tag{3}$$

where Y_t is total demand for goods. The firm maximizes expected discounted profits

$$E_t \sum_{s=0}^{\infty} Q_{t,t+s} Z_{t+s}(i) \tag{4}$$

where the period profits $Z_t(i)$ are given by

$$Z_t(i) = \left[(1+s)\, Y_t p_t(i)^{1-\varepsilon} P_t^{\varepsilon} - n_t(i) Y_t p_t(i)^{-\varepsilon} P_t^{\varepsilon} - d\left(\frac{p_t(i)}{p_{t-1}(i)}\right) P_t\right]$$

where s is a production subsidy which we will set to eliminate the steady state distortion of monopolistic competition as is common in the literature.[11]

We can now write down the necessary conditions for equilibrium that arise from the maximization problems of the private sector described above. We focus on a symmetric equilibrium where all firms charge the same price and produce the same amount of output. The households optimality conditions are given by

$$\frac{v_h\left(h_t\right)}{u_C\left(C_t\right)} = \frac{n_t}{P_t} \tag{5}$$

$$\frac{1}{1+i_t} = E_t\left[\beta \frac{u_C(C_{t+1})\xi_{t+1}}{u_C(C_t)\xi_t} \Pi_{t+1}^{-1}\right] \tag{6}$$

$$S_t = E_t\left[\beta \frac{u_C(C_{t+1})\xi_{t+1}}{u_C(C_t)\xi_t} \Pi_{t+1}^{-1}\left(1+\rho S_{t+1}\right)\right] \tag{7}$$

where $\Pi_t = \frac{P_t}{P_{t-1}}$ is gross inflation.[12] The firm's optimality condition from price-setting is given by

$$\varepsilon Y_t\left[u_C\left(C_t\right) - v_y(Y_t)\right]\xi_t + u_C(C_t)\xi_t d'\left(\Pi_t\right)\Pi_t = E_t\left[\beta u_C(C_{t+1})\xi_{t+1} d'\left(\Pi_{t+1}\right)\Pi_{t+1}\right] \tag{8}$$

where with some abuse of notion we have replaced v_h with v_y since in a symmetric equilibrium $h_t(i) = y_t(i) = Y_t$.

[10] Our result are not sensitive to assuming instead the alternative Calvo model of price setting, provided we do not assume there are large resource costs of price changes. This is explained in detail in Eggertsson and Singh (2015). The reason we adapt the Rotemberg specification is simplicity, i.e., it allows us to abstract from price dispersion as a state variable.

[11] The problem of the firm is thus to choose $\{p_{t+s}(i)\}$ to maximize (4), while taking as exogenously given $\{P_{t+s}, Y_{t+s}, n_{t+s}, Q_{t,t+s}, \xi_{t+s}\}$

[12] We may also add a standard transversality condition as a part of these conditions or a natural borrowing limit.

2.2 Government

There is an output cost of taxation (for example, as in Barro (1979)) captured by the function $s(T_t - T)$ where T is the steady-state level of taxes. Thus, in steady-state, there is no tax cost. Total government spending is then given by

$$F_t = G_t + s(T_t - T)$$

where G_t is aggregate government consumption of the composite final good defined before.

It remains to write down the (consolidated) flow budget constraint of the government. Note that the government issues both a one-period bond B_t^S and the perpetuity B_t. We can write the flow budget constraint as

$$B_t^S + S_t B_t = (1 + i_{t-1}) B_{t-1} + (1 + \rho S_t) B_{t-1} + P_t (F_t - T_t).$$

Next, we assume that the one-period bond is in net-zero supply (i.e. $B_t^S = 0$, which makes clear that we only introduce this bond explicitly as the one period risk free short term nominal rate is the key policy instrument of monetary policy), and write the budget constraint in real terms as

$$S_t b_t = (1 + \rho S_t) b_{t-1} \Pi_t^{-1} + (F_t - T_t) \tag{9}$$

where $b_t = \frac{B_t}{P_t}$. We now define fiscal policy as the choice of T_t, F_t , and b_t. For simplicity, we will from now on suppose that total government spending is constant so that $F_t = F$. Conventional monetary policy is the choice of i_t. We simply impose the zero bound constraint on the setting of monetary policy so that[13]

$$i_t \geq 0. \tag{10}$$

2.3 Private sector equilibrium

The goods market clearing condition gives the overall resource constraint as

$$Y_t = C_t + F_t + d(\Pi_t). \tag{11}$$

We can then define the private sector equilibrium, that is the set of possible equilibria that are consistent with household and firm maximization and the technological constraints of the model. A private sector equilibrium is a collection of stochastic processes $\{Y_{t+s}, C_{t+s}, b_{t+s}, S_{t+s}, \Pi_{t+s}, i_{t+s}, Q_{t,t+s}, T_{t+s}, F_{t+s}, G_{t+s}\}$ for $s \geq 0$ that satisfy equations (5)-(10), for each $s \geq 0$, given b_{t-1} and an exogenous stochastic process for $\{\xi_{t+s}\}$. To determine the set of possible equilibria in the model, we now need to be explicit about how policy is determined.

[13] This bound can be explicitly derived in a variety of environments, see e.g. Eggertsson and Woodford (2003) who assume money in the utility function.

2.4 Markov-perfect equilibrium

We characterize a Markov-perfect (time-consistent) Equilibrium in which the government cannot commit and acts with discretion every period.[14] A key assumption in a Markov-perfect Equilibrium is that government policy cannot commit to actions for the future government. Following Lucas and Stokey (1983), however, we suppose that the government is able to commit to paying back the nominal value of its debt.[15] The only way the government can influence future governments, then, is via any endogenous state variables that may enter the private sector equilibrium conditions. Before writing up the problem of the government, it is therefore necessary to write the system in a way that makes clear what are the endogenous state variables of the game we study.

Define the expectation variables f_t^E, g_t^E, and h_t^E. The necessary and sufficient conditions for a private sector equilibrium are now that the variables $\{Y_t,\, C_t,\, b_t,\, S_t,\, \Pi_t,\, i_t,\, T_t\}$ satisfy: (a) the following conditions

$$S_t(\rho)b_t = (1+\rho S_t(\rho))\, b_{t-1}\Pi_t^{-1} + (F - T_t) \tag{12}$$
$$1 + i_t = \frac{u_C\left(C_t\right)\xi_t}{\beta f_t^E},\ i_t \geq 0 \tag{13}$$
$$S_t(\rho) = \frac{1}{u_C\left(C_t\right)\xi_t}\beta g_t^E \tag{14}$$
$$\beta h_t^E = \varepsilon Y_t\left[\frac{\varepsilon-1}{\varepsilon}u_C\left(C_t\right)\xi_t - \tilde{v}_y\left(Y_t\right)\xi_t\right] + u_C\left(C_t\right)\xi_t d'\left(\Pi_t\right)\Pi_t \tag{15}$$
$$Y_t = C_t + F + d\left(\Pi_t\right) \tag{16}$$

given b_{t-1} and the expectations $f_t^E,\ g_t^E,$ and h_t^E; (b) expectations are rational so that

$$f_t^E = E_t\left[u_C\left(C_{t+1}\right)\xi_{t+1}\Pi_{t+1}^{-1}\right] \tag{17}$$
$$g_t^E = E_t\left[u_C\left(C_{t+1}\right)\xi_{t+1}\Pi_{t+1}^{-1}\left(1+\rho S_{t+1}(\rho)\right)\right] \tag{18}$$
$$h_t^E = E_t\left[u_C\left(C_{t+1}\right)\xi_{t+1}d'\left(\Pi_{t+1}\right)\Pi_{t+1}\right]. \tag{19}$$

Note that the possible private sector equilibrium defined above depends only on the endogenous state variable b_{t-1} and the shock ξ_t. Given that the government cannot commit to future policy (apart from through the endogenous state variable), a Markov-perfect Equilibrium then requires that the expectations f_t^E, g_t^E, and h_t^E are only a function of these two state variables, i.e, we can define the expectation functions

$$f_t^E = \bar{f}^E(b_t, \xi_t),\ g_t^E = \bar{g}^E(b_t, \xi_t),\ \text{and}\ h_t^E = \bar{h}^E(b_t, \xi_t). \tag{20}$$

We can now write the discretionary government's optimization problem as a dynamic program-

[14] See Maskin and Tirole (2001) for a formal definition of the Markov-perfect Equilibrium.

[15] One could model this more explicitly by assuming that the cost of outright default is arbitrarily high.

ming problem

$$V(b_{t-1}, \xi_t) = \max_{i_t, T_t} [U(.) + \beta E_t V(b_t, \xi_{t+1})] \tag{21}$$

subject to the private sector equilibrium conditions (12)-(16) and the expectation functions (20). Note that in equilibrium, the expectation functions satisfy the rational expectation restrictions (17)-(19). Here, $U(.)$ is the utility function of the household in (1) and $V(.)$ is the value function.[16] The detailed formulation of this maximization problem and the associated first-order necessary conditions, as well as their linear approximation, are provided in the appendix.[17]

3 Linear-quadratic approach

For most of our analysis we take a linear-quadratic approach to the optimal policy problem, which we will show explicitly is a correct approximation to the original non-linear optimal policy problem of the government that is maximizing social welfare. This characterization will directly apply when we consider the consolidated government. In this section, we also consider an independent central bank. The problem of the independent central bank will be similar to that of the consolidated government, apart from that it has its own budget constraint and an objective that may deviate from social welfare due to political economy constraints. We consider first the coordinated policy, and then move to an independent central bank.

3.1 Coordinated monetary and fiscal policy

We start with the baseline model above of the coordinated government case where the budget constraints are consolidated and government objective is to maximize welfare. We approximate our non-linear model of the previous section around an efficient non-stochastic steady-state with zero inflation.[18] Moreover, there are no tax collection costs in steady-state.[19] Thus, there is a non-zero steady-state level of debt.[20] In steady-state, we assume that there is some fixed total market-value of public debt $Sb = \bar{\Gamma}$. Then, the following relationships hold

$$1 + i = \beta^{-1}, \ S = \frac{\beta}{1 - \rho\beta}, \ b = \frac{1 - \rho\beta}{\beta}\bar{\Gamma} \text{ and } T = F + \frac{1 - \beta}{\beta}\bar{\Gamma}.$$

We log-linearize the private sector equilibrium conditions around the steady state above to obtain

$$\hat{Y}_t = E_t \hat{Y}_{t+1} - \sigma(\hat{\imath}_t - E_t \pi_{t+1} - r_t^e) \tag{22}$$

$$\pi_t = \kappa \hat{Y}_t + \beta E_t \pi_{t+1} \tag{23}$$

[16] Using compact notation, note that we can write the utility function as $[u(C_t) + g(F - s(T_t - T)) - v(Y_t)]\xi_t$.

[17] Note here that we assume that the government and the private-sector move simultaneously.

[18] Variables without a t subscript denote a variable in steady state. Note that output is going to be at the efficient level in steady state because of the assumption of the production subsidy (appropriately chosen) we have made before.

[19] We can think of this as being due to a limited set of lump sum taxation.

[20] The steady-state is efficient even with non-zero steady-state debt because of our assumption that taxes do not entail output loss in steady-state.

$$\hat{b}_t = \beta^{-1}\hat{b}_{t-1} - \beta^{-1}\pi_t - (1-\rho)\hat{S}_t - \psi\hat{T}_t \tag{24}$$

$$\hat{S}_t = -\hat{\imath}_t + \rho\beta E_t\hat{S}_{t+1} \tag{25}$$

where κ and σ are a function of structural model parameters that do not depend upon ρ and r_t^e is the efficient rate of interest that is a function of the shock ξ_t.[21] The coefficient $\psi \equiv \frac{T}{\Gamma}$ is also independent of ρ in our experiment.[22]

Here, (22) is the linearized household Euler equation, (23) is the linearized Phillips curve, (24) is the linearized government budget constraint, and (25) is the linearized forward-looking asset-pricing condition.[23] (22) and (23) are standard relationships depicting how current output depends on expected future output and the current real interest rate gap and how current inflation depends on expected future inflation and the current output respectively.[24]

(24) shows that since debt is nominal, its real value is decreased by inflation. Higher taxes also reduce the debt burden. Moreover, an increase in the price of the perpetuity bond decreases the real value of debt, with the effect depending on the duration of debt: longer the duration, lower is the effect of the bond price on debt. Finally, (25) shows that the price of the perpetuity bond is determined by (the negative of) expected present value of future short-term interest rates. Hence, lower current or future short-term nominal interest rate will increase the price of the perpetuity bond. Note that when $\rho = 0$, all debt is of one-period duration and (24) reduces to the standard linearized government budget constraint while (25) reduces to $\hat{S}_t = -\hat{\imath}_t$.[25]

A second-order approximation of household utility around the efficient non-stochastic steady state gives

$$U_t = -\left[\lambda_\pi \pi_t^2 + \hat{Y}_t^2 + \lambda_T \hat{T}_t^2\right] \tag{26}$$

where λ_π and λ_T are a function of structural model parameters.[26] Compared to the standard loss-function in models with sticky prices that contains inflation and output, (26) features losses that arise from output costs of taxation outside of steady-state.

To analyze optimal policy under discretion in the linear-quadratic framework we once again maximize utility, subject to the now linear private sector equilibrium conditions, taking into account that the expectation are functions of the state variables of the game. In the linear system, the exogenous state is now summarized with r_t^e while the endogenous state variable is once again $\hat{b}_{t-1}$.

[21] The details of the derivation are in the appendix.

[22] Since we are thinking of changes in ρ in our experiment as exchanging short bonds with long bonds – effectively reducing/increasing ρ – this interpretation would imply that total value of debt in steady-state – $\bar{\Gamma}$– remains unchanged.

[23] Variables with hats denote log-deviations from steady state except for the nominal interest rate, which is given as $\hat{\imath}_t = \frac{i_t - i}{1+i}$. Since in the non-stochastic steady state with zero inflation, $1 + i = \frac{1}{\beta}$, this means that the zero lower bound on nominal interest rates imposes the following bound on $\hat{\imath}_t : \hat{\imath}_t \geq -(1-\beta)$.

[24] We write directly in terms of output rather than the output gap since we will not be considering shocks that perturb the efficient level of output in the model.

[25] It is important to point out one technical detail in this case. The interpretation in this case of $\hat{b}_t$ is that it is the real value of the debt inclusive of the interest rate payment to be paid next period, that is, if all debt were one period $b_t = (1 + i_t)\frac{B_t^S}{P_t}$.

[26] The details of the derivation are in the appendix. In particular, $\lambda_\pi = \frac{\varepsilon}{k}$.

Moreover, the expectation variables appearing in the system are now $E_t\hat{Y}_{t+1}$, $E_t\hat{S}_{t+1}$, and $E_t\pi_{t+1}$. Accordingly, we will define the game in terms of the state variables $(r_t^e, \hat{b}_{t-1})$ and the government now takes as given the expectation functions $\bar{Y}^E(\hat{b}_t, r_t^e)$, $\bar{S}^E(\hat{b}_t, r_t^e)$, and $\bar{\pi}^E(\hat{b}_t, r_t^e)$.

The discretionary government's optimization problem can then be written recursively as a linear-quadratic dynamic programming problem

$$V(\hat{b}_{t-1}, r_t^e) = \min[\lambda_\pi \pi_t^2 + \hat{Y}_t^2 + \lambda_T \hat{T}_t^2 + \beta E_t V(\hat{b}_t, r_{t+1}^e)]$$

s.t.

$$\hat{Y}_t = \bar{Y}^E(\hat{b}_t, r_t^e) - \sigma(\hat{\imath}_t - \bar{\pi}^E(\hat{b}_t, r_t^e) - r_t^e)$$

$$\pi_t = \kappa \hat{Y}_t + \beta \bar{\pi}^E(\hat{b}_t, r_t^e)$$

$$\hat{b}_t = \beta^{-1}\hat{b}_{t-1} - \beta^{-1}\pi_t - (1-\rho)\hat{S}_t - \psi \hat{T}_t$$

$$\hat{S}_t = -\hat{\imath}_t + \rho\beta \bar{S}^E(\hat{b}_t, r_t^e).$$

Observe that once again, in equilibrium, the expectation functions need to satisfy the rational expectations restrictions that $E_t\hat{Y}_{t+1} = \bar{Y}^E(\hat{b}_t, r_t^e)$, $E_t S_{t+1} = \bar{S}^E(\hat{b}_t, r_t^e)$, and $E_t\pi_{t+1} = \bar{\pi}^E(\hat{b}_t, r_t^e)$.

We prove in the proposition below that this linear-quadratic approach gives identical linear optimality conditions as the one obtained by linearizing the non-linear optimality conditions of the original non-linear government maximization problem that we described in the previous section. This provides the formal justification for our simplified approach.

Proposition 1 *The linearized dynamic system of the non-linear Markov Perfect Equilibrium is equivalent to the linear dynamic system of the linear-quadratic Markov Perfect Equilibrium.*

Proof. In Appendix. ■

3.1.1 Interpreting data from QE through the lens of the model

Let us now briefly review the data we will use when we do numerical experiments with this benchmark model. We are asking the data to give us some numbers for the following thought experiment: What happens when you reduce the maturity of government debt? In the context of the model, we are interested in getting some values for changes in ρ as representing a particular unconventional monetary policy intervention.

According to the model, under coordinated policy and consolidated budget constraints we should only be considering the debt held by the public (thus, we net out government debt held by the Federal Reserve). Consistent with the model, also, we count reserves issued by the Federal Reserve as short-term government debt. The duration of the consolidated government's debt is given below in Fig. 1.[27] The vertical dashed lines are important events associated with the Federal

[27] In generating this figure, we first use estimates from Chadha, Turner, and Zampoli (2013) on the duration of treasury debt held outside the Federal Reserve, which we then augment with data on reserves issued by the Federal Reserve that is available from public sources (FRED).

Reserve buying long-term treasury bonds: November 2008 and March 2009 (Quantitative Easing 1); November 2010 (Quantitative Easing 2 (QE 2)); September 2011 (Maturity Extension Program (MEP)); and September 2012 and December 2012 (Quantitative Easing 3 (QE3)). Around those dates, the maturity of outstanding government debt declined. The baseline estimation of our model will be based on the November 2010 or the Quantitative Easing 2 (QE2) program, a common focal point in the literature. Given the parameter estimates from the QE2 program, we can also assess the macroeconomic impact of the September 2011 or the Maturity Extension Program (MEP). The reduction in maturity observed in Fig. 1 will be the input in our policy experiments under coordinated policy and consolidated budget constraints.[28]

3.2 An independent central bank

We now present an alternate model where the central bank faces its own budget constraint and minimizes an ad-hoc loss function that captures directly its balance sheet concerns. This model has a political economy related justification to why the central bank might care about transfers to the treasury. This alternative formulation, as will become clear, will also require us to view the data through different lenses than in the last subsection. As we shall see later on, however, the central insights will remain the same, i.e. QE has an effect via signalling.

We are now interested in studying the problem from the perspective of an independent central bank that need not act in concert with the rest of the government. The first step is to explicitly write down its budget constraint, before we move on to its objectives. We consider the case where the central bank holds long-term assets (long term government debt). It buys these assets by issuing one-period liabilities (approximating interest-bearing reserves). We also introduce some "seigniorage net of operations cost" of the central bank that is not time-varying (similar to fiscal spending that is not time-varying in our previous characterization when monetary and fiscal policy are coordinated). Denote central bank holdings of assets by B_t^{CB} and its liabilities by L_t (with prices S_t^γ and Q_t respectively) and the "seigniorage net of operations cost" by K. The assets of the central bank are in the form of a perpetuity bond of the same kind we analyzed before with duration γ. Moreover, let V_t be the transfers to the treasury.

The flow budget constraint of the central bank is then given by

$$S_t^\gamma B_t^{CB} + P_t V_t - Q_t L_t - P_t K = \left(1 + \gamma S_t^\gamma\right) B_{t-1}^{CB} - L_{t-1}$$

where the price of the liabilities, Q_t, is inverse of the (gross) short-term nominal interest rate.[29] This can be written in real terms as

$$S_t^\gamma b_t^{CB} - Q_t l_t = \left(1 + \gamma S_t^\gamma\right) b_{t-1}^{CB} \Pi_t^{-1} - l_{t-1} \Pi_t^{-1} + K - V_t.$$

[28] We will for the rest of the paper take September 2011, September 2012, December 2012 as one policy intervention and with abuse of terminology, refer it as MEP.

[29] For earlier work in this vein, see Jeanne and Svensson (2007) and Berriel and Bhattarai (2009). For recent work exploring the implications of the central bank budget constraint, see Hall and Reis (2013) and Del Negro and Sims (2015), who explore positive issues related to solvency and determinacy.

The debt of the treasury is now either held by the central bank or the general public. The market clearing condition for treasury debt (B_t^T) is thus given by $B_t^T = B_t^{CB} + B_t$ where B_t is debt held by the public. We assume that the treasury follows passive fiscal policy that ensures stable debt dynamics. Thus we completely abstract from fiscal policy considerations. We also assume for simplicity that all central bank reserves are held by the public.

With some algebra outlined in the footnote, and some simplifying assumptions, we can write the linearized budget constraint as

$$\left[\hat{b}_t^{CB} - \hat{l}_t\right] = \beta^{-1}\left[\hat{b}_{t-1}^{CB} - \hat{l}_{t-1}\right] - (1-\gamma)\,\hat{S}_t^{\gamma} + \hat{Q}_t - \psi_V \hat{V}_t \tag{27}$$

where $\psi_V \equiv \frac{V}{S^{\gamma}b^{CB}}$.[30] Moreover, the price of the long term assets, $\hat{S}_t^{\gamma}$, is given by the same asset-pricing condition as before

$$\hat{S}_t^{\gamma} = -\hat{\imath}_t + \gamma\beta E_t \hat{S}_{t+1}^{\gamma}$$

while the price of the short-term asset, which is just the negative of the short-term interest rate, is given by $\hat{Q}_t = -\hat{\imath}_t$.

Before we made the critical assumption that in steady state the market value of public debt was given by some fixed number $Sb = \bar{\Gamma}$ that we linearized around. Here, the most important parameter is the scale of the balance sheet, measured by ψ_V. This number reflects how large the asset side of the balance sheet, $S^{\gamma}b^{CB}$, is relative to the steady state transfers to the treasury, V. Also, note here that the budget constraint is written in terms of the difference between the central banks holding of long term government debt $\hat{b}_t^{CB}$ which has a fixed duration of γ and it own issuance of short term debt (interest bearing reserves) $\hat{l}_t$. It can thus be re-written in terms of the net asset position of the bank, which we define as $\hat{b}_t^{N,CB} = \hat{b}_t^{CB} - \hat{l}_t$. We will use that formulation later below.

There are several noteworthy features in (27). First notice that up to first-order inflation has

[30] The two nonlinear asset pricing conditions will take the form

$$S_t^{\gamma} = E_t\left[\beta\frac{u_C(C_{t+1})\xi_{t+1}}{u_C C_t)\xi_t}\Pi_{t+1}^{-1}\left(1+\gamma S_{t+1}^{\gamma}\right)\right],\ Q_t = \frac{1}{1+i_t} = E_t\left[\beta\frac{u_C(C_{t+1})\xi_{t+1}}{u_C(C_t)\xi_t}\Pi_{t+1}^{-1}\right].$$

In steady-state, like before, we have $Q^{-1} = 1+i = \beta^{-1}, S^{\gamma} = \frac{\beta}{1-\gamma\beta}$. Moreover, define, as before $S^{\gamma}b^T = \bar{\Gamma}$, which from market clearing gives $\left(b^{CB}+b\right) = \bar{\Gamma}$. The central bank budget constraint is then given in steady-state by

$$S^{\gamma}b^{CB} - Ql = (1+\gamma S^{\gamma})\,b^{CB} - l + K - V.$$

We will focus on a steady-state where $S^{\gamma}b^{CB} = Ql$. Since $\frac{S^{\gamma}}{Q} = \frac{1}{1-\gamma\beta}$, we will have $\frac{b^{CB}}{l} = (1-\gamma\beta)$. This then implies that $K = V$. We can now linearize the non-linear asset pricing and central bank budget constraint. First, we have $\hat{S}_t^{\gamma} = \hat{Q}_t + \gamma\beta E_t\hat{S}_{t+1}^{\gamma}$where in terms of our previous notation the following holds $\hat{Q}_t = -\hat{\imath}_t$. Thus, we have exactly like before the asset pricing condition for the long-term asset $\hat{S}_t^{\gamma} = -\hat{\imath}_t + \gamma\beta E_t\hat{S}_{t+1}^{\gamma}$. The linearized budget constraint is now given by

$$\left[\hat{b}_t^{CB} - \frac{Ql}{S^{\gamma}b^{CB}}\hat{l}_t\right] = \beta^{-1}\left[\hat{b}_{t-1}^{CB} - \frac{Ql}{S^{\gamma}b^{CB}}\hat{l}_{t-1}\right] - \beta^{-1}\left[1 - \frac{Ql}{S^{\gamma}b^{CB}}\right]\pi_t - (1-\gamma)\,\hat{S}_t + \frac{Ql}{S^{\gamma}b^{CB}}\hat{Q}_t - \psi_V\hat{V}_t$$

where $\psi_V = \frac{V}{S^{\gamma}b^{CB}}$ is a parameter. We have to make some assumptions on the steady-state ratio of (mkt value) of liabilities to assets of the central bank: $\frac{Ql}{S^{\gamma}b^{CB}}$. We will assume that it is 1. That gives the linearized budget constraint in the text.

no effect on the net asset position of the bank. The reason for this is that inflation depreciates the value of the assets (nominal long term bonds) and liabilities (short term nominal debt) to exactly the same extent. This is a critical difference relative to the consolidated budget constraint. Long term debt does, however, have an important effect on the net asset position of the central bank. This is because while assets are long-term, the liabilities are short term. There is thus generally a "duration mismatch" in the central bank's balance sheet.

To see fully the implications of this duration mismatch, consider first the case in which $\gamma = 0$. Then, both the assets and the liabilities have the same duration and variations in the short term nominal interest rate have no effect on the net worth of the central bank, as $\hat{Q}_t$ and $\hat{S}_t$ cancel out. Consider now the case in which $\gamma > 0$. Now the central bank is holding long dated assets and short dated liabilities. In this case we see that an increase in the short-term nominal interest rate reduces the net worth of the bank. The reason for this is that an increase in the short rate increases the borrowing cost of the bank one-to-one. Meanwhile, on the assets side, the increase in the short term nominal interest rate has a more limited effect , since they are long-dated securities (so the nominal interest rate is multiplied by $1 - \gamma$ in (27)). Thus, increasing the short-term nominal interest rate sharply will lead to balance sheet losses for the central bank.

What does quantitative easing mean in the context of this budget constraint? We can model it in two ways. First, recall that for the consolidated budget constraint we interpreted it as a one time decline in ρ. Here, an analogous experiment is that it corresponds to a one time increase in γ, i.e. the degree of duration mismatch is enhanced by making longer the duration of the assets held by the central bank. This interpretation is about the change in composition of the asset side of the central bank's balance sheet.

While an increase in γ is one way of interpreting quantitative easing, there is another complementary interpretation. An increase in γ means that it is replacing shorter term bonds on its asset side with bonds of longer duration. But the total value of bonds on the asset side is constant. Thus, it is simply a change in the composition of the balance sheet. Now consider the following experiment: Suppose all bonds on the asset side have some fixed γ (say, corresponding to five years). Now imagine the central bank increases the purchases of these these bonds by printing interest-bearing reserves. While this is not affecting the average duration of bonds on the asset side of the balance sheet (or average duration mismatch since γ is fixed), it is increasing the scale of the central bank balance sheet. The scale of the balance sheet in steady state was evaluated by the parameter $\psi_V = \frac{V}{S^{\gamma}b^{CB}}$. A permanent expansion in the balance sheet is therefore measured as a drop in ψ_V. Observe that the increase in scale will simply expand the number of assets and liabilities, but in steady state the remittances, V, remain unchanged. Hence an alternative way of interpreting quantitative easing is an increase in the size of the balance sheet of the Fed, or a drop in $\psi_{V.}$, without changing the composition. We will use both the first and second interpretation to study the implications of different QE episodes.

We can write (27) in terms of one state variable, $\hat{b}_{t-1}^{N,CB} = \hat{b}_{t-1}^{CB} - \hat{l}_{t-1}$, as

$$\hat{b}_t^{N,CB} = \beta^{-1}\hat{b}_{t-1}^{N,CB} - (1-\gamma)\,\hat{S}_t^{\gamma} + \hat{Q}_t - \psi_V \hat{V}_t. \tag{28}$$

Observe that all the private-sector equilibrium conditions remain the same as when we studied the consolidated budget constraint. We can now conduct the experiment of an increase in the central bank's balance sheet via decrease in ψ_V, holding γ fixed. We can also conduct the experiment of larger holding of long-term assets as an increase in γ, holding ψ_V fixed.

In terms of the central bank's objective, we take here an ad-hoc loss-function approach, which has a rich and long history in monetary economics. We posit that the central bank directly cares about transfers to the treasury for political economy reasons. This means that its period loss function now incorporates a term related to target transfers to the Treasury, V_t, in addition to the usual terms related to inflation and output. It is thus given by

$$\left[\lambda_\pi \pi_t^2 + \hat{Y}_t^2 + \lambda_V \hat{V}_t^2\right].$$

Does this objective make sense? There is some evidence that central banks care about the transfers to the treasury.[31] Moreover, as we have seen, to the extent that these transfers affect tax collection, the central bank should care also from a social welfare point of view.

The discretionary central bank's optimization problem can then be written recursively as a linear-quadratic dynamic programming problem

$$V(\hat{b}_{t-1}^{N,CB}, r_t^e) = \min[\lambda_\pi \pi_t^2 + \hat{Y}_t^2 + \lambda_T \hat{T}_t^2 + \beta E_t V(\hat{b}_t^{N,CB}, r_{t+1}^e)]$$

s.t.

$$\hat{Y}_t = \bar{Y}^E(\hat{b}_t^{N,CB}, r_t^e) - \sigma(\hat{\imath}_t - \bar{\pi}^E(\hat{b}_t^{N,CB}, r_t^e) - r_t^e)$$

$$\pi_t = \kappa \hat{Y}_t + \beta \bar{\pi}^E(\hat{b}_t^{N,CB}, r_t^e)$$

$$\hat{b}_t^{N,CB} = \beta^{-1}\hat{b}_{t-1}^{N,CB} - (1-\gamma)\,\hat{S}_t^{\gamma} + \hat{Q}_t - \psi_V \hat{V}_t$$

$$\hat{S}_t^{\gamma} = -\hat{\imath}_t + \gamma\beta \bar{S}^{\gamma E}(\hat{b}_t^{N,CB}, r_t^e).$$

Observe that once again, in equilibrium, the expectation functions need to satisfy the rational expectations restrictions that $E_t\hat{Y}_{t+1} = \bar{Y}^E(\hat{b}_t^{N,CB}, r_t^e)$, $E_t S_{t+1}^{\gamma} = \bar{S}^{\gamma E}(\hat{b}_t^{N,CB}, r_t^e)$, and $E_t\pi_{t+1} = \bar{\pi}^E(\hat{b}_t^{N,CB}, r_t^e)$.

The behavior of this model will critically depend up on the initial value of the net capital of the bank $\hat{b}_{t-1}^{N,CB}$. If this number is negative, then the path for $\hat{V}_t$ will generally be below what the central bank would ideally like it to be during the transition to steady state. Moreover, interest-

[31] Central bank governors that incur large balance sheet losses — e.g. the Central Bank of Iceland in 2008 which lost money corresponding to 30% of GDP – usually find themselves without a job shortly thereafter. Berriel and Bhattarai (2009) contains some other anecdotal evidence of such central bank worries. In related frameworks, Jeanne and Svensson (2007) and Berriel and Bhattarai (2009) include the central bank's net worth directly in the loss function. Our modeling approach here is thus different.

rates will be rising during this transition. We think this provides the most natural interpretation of the political economy objectives of a central bank as we further discuss in Section 4.2 when we calibrate the model.

3.2.1 Interpreting data from QE through the lens of the model

This alternative model of QE provided by modeling an independent central bank leads one to consider different aspects of the data relative to if we were to assume coordinated government policy. Now we are not interested in the composition of the net government debt held by the public, but instead the balance sheet of the central bank directly. Ratio of the assets of the Federal Reserve (holdings of treasuries like our model) to average pre-crisis GDP is given below in Fig. 2. This is the counterpart to the parameter $\frac{1}{\psi_V}$ in the model above.[32] Average maturity of treasury holdings by the Federal Reserve is given below in Fig. 3. This is the counterpart to the parameter γ in the model above.[33] Again, the vertical dashed lines are important events associated with the Federal Reserve buying long-term treasury bonds: November 2008 and March 2009 (Quantitative Easing 1); November 2010 (Quantitative Easing 2 (QE 2)); September 2011 (Maturity Extension Program (MEP)); and September 2012 and December 2012 (Quantitative Easing 3 (QE3)). The baseline parameterization of our model will be based on the November 2010 or the Quantitative Easing 2 (QE2) program. Fig. 2 shows that the defining feature of that program was an increase in the size of the Federal Reserve's balance sheet with the average maturity not changing by much. The figures also help us assess the macroeconomic impact of the September 2011 or the Maturity Extension Program (MEP). Fig. 3 shows that the defining feature of that program was an increase in the maturity of treasury holdings, with the size of the balance sheet not changing by much.

4 Results

4.1 Coordinated monetary and fiscal policy

Let us start with our baseline model where the budget constraints of the treasury and the central bank are consolidated and the government maximizes the welfare of the representative household under discretion. We discuss the calibration of this model next. As we mentioned before, the baseline calibration of this model will be based on the effects of the QE2 program. For the steady-state level of debt-to-taxes, $\frac{bS}{T} = \frac{\bar{\Gamma}}{T}$, we use data from the Federal Reserve Bank of Dallas to get the long-run average of market value of debt over output $(\frac{bS}{Y})$ and NIPA data to estimate the ratio of taxes over output $(\frac{T}{Y})$. This gives us the value $\frac{bS}{T} = \frac{\bar{\Gamma}}{T} = 7.2$. We start with a baseline maturity of 16.87 q., which is the level at the beginning of the QE2 program. Then, as a measure of the effects on the average maturity of outstanding government debt from QE2, we take the difference

[32] In the model, this parameter is the Ratio of assets to remittances to Treasury and that is what we indeed use for calibration. For ease of interpretation, we present in the figure the Ratio of assets to annual GDP, which is a scaled version. These two measures are simply related by a ratio: that of pre-crisis remittances to Treasury to GDP.

[33] We used data from publicly available sources to construct these figures (FRED and the Federal Reserve Board of Governors website).

in Fig. 1 between QE2 and the MEP (the third and fourth dashed vertical lines), which is 0.67 q. That is, according to our measure, the reduction in maturity was from 16.87 q to 16.2 q, with a difference of 0.67q.

To model the case of a liquidity trap, we follow Eggertsson and Woodford (2003) and assume a negative shock to the exogenous efficient rate of interest, r_t^e, which makes the zero lower bound binding.[34] The process for r_t^e follows a two-state Markov process with an absorbing state: From period 0 on then r_t^e takes on a negative value of r_L^e. It remains at this value with probability μ in every period, while with probability $1-\mu$, it reverts back to steady-state and stays there forever after. This means that the economy will exit the liquidity trap with a constant probability of $1-\mu$ every period and that once it exits, it does not get into the trap again. The appendix contains details about the computation algorithm.

To parameterize the model we use a mix of calibration/estimation based on the effects of QE2 on inflation and yields calculated by Krishnamurthy and Vissing-Jorgensen (2011) as well as the expected duration of the zero lower bound episode and its effects on output and inflation. We pick the quarterly discount factor of $\beta = 0.99$ and we fix the elasticity of substitution among varieties of goods at a standard value of 8. We allow for debt while at the liquidity trap to be 30% above its steady-state value, which is in line with the Federal Reserve Bank of Dallas data.[35]

Then, we estimate $(\sigma,\ \lambda_T,\ \kappa,\ r_L^e,\ \mu)$ by matching five targets. Our first two targets are a reduction in 8 quarters ahead yield of $\Delta i^*(8) = -16$ b.p. and an increase in expected cumulative inflation over 10 years of $\Delta\pi^*(40) = 5$ b.p. as a result of the QE2 program when the economy is initially in a ZLB situation. These were the estimates in Krishnamurthy and Vissing-Jorgensen (2011) of the effects of the QE2 program. Our third target is a 3-year average duration of the ZLB period (for an average ZLB duration of about 3 years, we target $\mu* = 0.91$). Finally, our fourth and fifth targets are a drop in output of 10% and a 2% percent drop in inflation during the ZLB episode to make the experiment relevant for the recent "Great Recession" in the United States ($Y^*(1)$ and $\pi^*(1)$ of -0.10 and -0.02 respectively).[36]

Our criterion for estimation is the mean squared weighted relative error given by

$$L = \sqrt{\left(\frac{\Delta\pi(40)}{\Delta\pi^*(40)} - 1\right)^2 + \left(\frac{\Delta i(8)}{\Delta i^*(8)} - 1\right)^2 + \left(\frac{Q(\mu)}{Q(\mu^*)} - 1\right)^2 + \left(\frac{Y(1)}{Y^*(1)} - 1\right)^2 + \left(\frac{\pi(1)}{\pi^*(1)} - 1\right)^2}$$

where $Q(\mu) = \frac{1}{1-\mu}$ is the expected duration in quarters of the ZLB episode. The values of the targets for our best match are $\pi(1) = -0.021$, $Y(1) = -0.091$, $\Delta\pi(40) = 4.72$ b.p., $\Delta i(8) = -5.87$ b.p., and $\mu = 0.89$ (about 2.25 years). Our estimated parameter values are given in Table 1.

[34] One can think of this here as being driven by a preference shock. For an alternate way of generating a liquidity trap in monetary models, based on an exogenous drop in the borrowing limit, see Eggertsson and Krugman (2012).

[35] We also adjust the quantity of debt level after QE2 to keep the market value of the debt fixed during the QE2 intervention (it has to be adjusted to 0.297).

[36] One can alternatively impose some priors on the parameters we estimate using Bayesian methods, but we felt this strategy is more transparent, given that our estimated value for the parameters are relatively reasonable.

4.1.1 Solution at positive interest rates

We start by showing the solution at positive interest rates to show how debt maturity changes the policy incentives of the government. The complication in solving a MPE is that we do not know the unknown expectation functions $\bar{\pi}^E$, $\bar{Y}^E$, and $\bar{S}^E$. To solve this, we use the method of undetermined coefficients. All the details of the derivations are provided in the appendix.

Let us first consider the most basic exercise to clarify the logic of the government's problem. How do the dynamics of the model look like in the absence of shocks when the only difference from steady state is that there is some initial value of debt with some fixed value for debt duration? Fig. 4 shows the dynamics of the endogenous variables in the model for an initial value of debt that is 30 percent above the steady state. We see that if debt is above steady state, it is paid over time back to steady state. For our baseline duration of 16.87 quarters (solid line), the half-life of debt repayment is about 12 quarters. In the transition inflation is about 0.75 percent above steady state and the real interest rate is below its steady state. As a consequence, output is also above its steady state value. This result is in contrast to the classic Barro tax smoothing result whereby debt follows a random walk. The reason is that debt creates an incentive to create inflation for a discretionary government as further described below. By paying down debt back to steady state, the government eliminates this incentive and achieves the first best outcome in the model.

The figure illustrates that for a given maturity of government debt, debt is inflationary and implies a lower future real interest rate until a new steady state is reached. What is the logic for this result? Perhaps the best way to understand the logic is by inspecting the government budget constraint (24). Recall that debt issued in nominal terms, although in the budget constraint we have rewritten it in terms of $\hat{b}_t = \frac{\frac{B_t}{P_t} - \bar{b}}{\bar{b}}$. This implies that for a given outstanding debt $\hat{b}_{t-1}$, any actual inflation will reduce the real value of the outstanding debt. Accordingly we have the term $\beta^{-1}\pi_t$ term in the budget constraint which reflects this inflation incentive. As the literature has stressed in the past (see e.g. Calvo and Guidotti (1990 and 1992)), if prices are flexible then this will reduce actual debt in equilibrium only if the inflation is unanticipated. The reason for this is that otherwise anticipated inflation will be reflected one-to-one in the interest rate paid on the debt.

Apart from the incentive to depreciate the real value of the debt via inflation, there is a second force at work. In our model, the government is not only able to affect the price level, it can also have an effect on the real interest rate. Hence, we see that in Fig. 4 the real interest rate is below steady state during the entire transition path back to steady state. This reduces the real interest rate payments the government needs to pay on debt – in contrast to the classic literature with flexible prices where the (ex-ante) real interest rate is exogenous. We refer to this as the rollover incentive of the government.

Intuitively, it may be most straight forward to see the rollover incentive by simplifying the model down to the case in which $\rho = 0$ and there is only one period debt. In that case, the budget constraint of the government can be written as

$$\hat{b}_t = \beta^{-1}\hat{b}_{t-1} - \beta^{-1}\pi_t + \hat{\imath}_t - \psi\hat{T}_t \tag{29}$$

and now $\hat{b}_t$ is the real value of one period risk-free nominal debt in period t which is *inclusive of interest paid* (to relate to our prevision notation in (2) then when $\rho = 0$ we have $b_t = \frac{B_t}{P_t} = \frac{(1+i_t)B_t^S}{P_t}$ where B_t^S was the one period government debt that did not include interest payment). This expression shows that while π_t has a direct effect by depreciating the real value of government debt, the government has another important margin by which it can influence its debt burden. The term $\hat{\imath}_t$ reflects the rolling-over-cost of the one-period debt. In particular, we see that if the interest rate is low, then the cost of rolling over debt is smaller. This latter mechanism will be critical when considering the effects of varying debt maturity since its force depends on the value of ρ.

How are these dynamics affected by the term structure of government debt? We now consider the importance of variations in ρ. As noted before, our main interest in understanding this effect is that a natural interpretation of QE is that it corresponds to a reduction in ρ as in our model the duration of debt is given by $(1-\beta\rho)^{-1}$. In Fig. 4 we consider a reduction in duration from 16.8 q to 16.2 and 15.6 q. As the figure shows this increases inflation in equilibrium considerably, but also reduces the real rate further. Similarly, we see that the debt is now paid down at a faster clip as higher output gap and inflation causes increased distortions, a point we will return to.

To obtain some intuition for this result, let us again write out the budget constraint, this time substituting out for S_t to obtain

$$\hat{b}_t = \beta^{-1}\hat{b}_{t-1} - \beta^{-1}\pi_t + (1-\rho)[\hat{\imath}_t - \rho\beta\hat{S}^E(\hat{b}_t, r_t^e)] - \psi\hat{T}_t. \tag{30}$$

We observe here that the rollover interest rate is now multiplied by the term $(1-\rho)$. Intuitively, if a larger part of government debt is held with long maturity, the short-term rollover rate matters less, as the terms of the loans are to a greater extent predetermined. Hence, the incentive of the government to lower the short-term interest rate is reduced.

Again, considering special cases here can be useful. We already noted the case in which $\rho = 0$ (only one-period debt) which gave us equation (29). It shows that the short-term interest rate affects debt burden in next period one-to-one, this is the rollover incentive of the government. Consider now the other polar case in which $\rho = 1$ (with classic console only). Then we get

$$\hat{b}_t = \beta^{-1}\hat{b}_{t-1} - \beta^{-1}\pi_t - \psi\hat{T}_t$$

which shows that the short-term nominal interest rate does not affect debt dynamics at all. Thus the rollover incentive is completely eliminated by making debt long-term. This is the key reason why short-term debt creates more inflationary and lower interest rate incentive than long term debt.

Having established intuitively and numerically that at positive interest rates, decreasing the duration of debt increases the incentives of the government to lower short-term real interest rates, we now move on to analyzing the case where the nominal interest rate is at the zero lower bound. At the ZLB, manipulating this incentive can be particularly valuable.

4.1.2 Solution at the ZLB: QE2

Consider the following policy experiment: In the liquidity trap, the level of debt is constant at $\hat{b}_L$. When out of the trap, then $\hat{b}_L$ is optimally determined by the government according to the MPE previously described. Why is this an interesting environment? Now we can ask the following question: What would be the effect of changing the duration of debt once-and-for-all, while the zero lower bound is binding? In other words, we are interested in the comparative static of the model as we vary the duration of debt in the liquidity trap, but at the same time holding aggregate debt, $\hat{b}_L$, constant. We think this is an interesting comparative static, because it corresponds so closely to QE. QE did not involve increasing aggregate government debt, as pointed out in the introduction. Instead it just involved exchanging long-term government debt with short-term government debt (money), which we interpret here as a reduction in ρ. In our experiment the steady-state market value of debt to taxes is always kept fixed. For now, however, a key abstraction is that the value of ρ is fixed so that once you change ρ (QE), it does not revert back to where it was. We will come back to this issue in Section 5.

Before exploring the comparative static at the heart of this section, let us review first how the model behaves in the absence of any intervention, i.e., the evolution of each of the endogenous variables in the face of the shock we chose in the last section. Figs. 5 and 6 show the response of inflation and output to a negative shock to the efficient rate of interest in the benchmark economy, when the duration of government debt is fixed at 16.87 quarters. We will be interested in understanding if QE can improve upon the outcome we see in these figures, which feature an output drop of approximately 10 percent and inflation drop of 2 percent (by construction of our calibration).

Some comments are in order about the baseline economy in the absence of QE. First note that the shock here generates a considerable recession and a drop in inflation. This is driven entirely by the fact that the central bank cannot accommodate the shock via cuts in the nominal interest rate. This creates a gap between the equilibrium real interest rate, r_t, and the efficient rate of interest r_t^e (i.e. the real interest rate needed for output to remain at the first best steady state). This interest rate gap is shown in Fig. 7. It is well known from the existing literature (see e.g. Eggertsson and Woodford (2003)), that this situation can be greatly improved if the central bank could commit to keeping the nominal interest rate low for some time after the shock is over. This is beneficial because aggregate demand depends not only on the current interest rate gap but the entire path of future interest rates.

The optimal commitment analyzed by Eggertsson and Woodford (2003) is not possible in our environment, however. The reason is that we are considering a MPE, so the government cannot commit to future policy that is dynamically inconsistent (this is the so called “deflationary bias” of discretionary policy at the ZLB, see Eggertsson (2006)). The optimal commitment involves promising real interest rate below the efficient rate of interest rate when the shock is over – but at that time the government has little incentive to deliver on this promise.

Another point worth stressing in Fig. 5 is that once the shock is over (and each of the thin lines revert up) then inflation overshoots its long run value in our MPE. This is a feature of our calibration, as we assumed that there is outstanding government debt of 30 percent above steady state. Thus the government does already have some incentive to inflate which is reflected in these numbers. The fact that output drops by about 10 percent in Fig. 6 simply suggests that this incentive is not strong enough for the government to be able to escape the ZLB. One solution to this problem, then, would be simply to issue even more nominal debt (this is a solution analyzed in Eggertsson (2006)), an approach we abstract from here by virtue of $\hat{b}_L$ being constant.[37]

To motivate this abstraction, i.e. fixed $\hat{b}_L$, we can think of some political or economic limits on how much total aggregate government debt can be issued (e.g. a debt limit imposed by Congress or that too high debt gives rise to perception of default, a consideration we have not included in our model). Moreover, when we consider the case of the independent central bank, the option of raising total number of government bonds may not be available. In any case, when the government has two instruments – the stock of nominal debt and its composition in terms of duration – we want to understand how *both* margins work, and our focus is on the latter. This leads us to consider next the central comparative static of this paper: What happens when you permanently reduce the duration of government debt in the MPE? Can manipulating the term structure make expansionary future monetary policy “credible” without further cuts in the current nominal interest rate?

In Figs. 8 and 9 we see what happens if the government reduces the duration of government debt from 16.87 quarters to 16.2 quarters, this is the number we computed on the basis of the data from QE2. The figures shows the change in output and inflation as a result of this policy intervention. The bottom-line is that inflation now increases by 14 basis points (annualized) and output increases by 30 basis points. We can also ask how large intervention would have been needed to fully close the output gap. The answer to this question is that the duration would have had to go down from 16.87 to 8.5 quarters.

What is the key logic? Because government has more short-term debt the central bank keeps the short-term real interest rates lower in future in order to keep the real interest rate low on the debt it is rolling over. Thus, QE provides a “signal” about the future conduct of monetary policy. In particular it generates a credible signal about the future path of short-term interest rates. This then enables it to have effect on macroeconomic prices and quantities at the zero lower bound. The change in the response of the real interest rate is given in Fig. 10, where one can see that the real interest rate is lower throughout the horizon post QE.[38]

[37] Note that we keep the market value of debt constant (before and after change in duration) in our numerical experiments via appropriate adjustments.

[38] This result thus connects our paper with Persson, Persson, and Svensson (1987 and 2006), who show in a flexible price environment that a manipulation of the maturity structure of both nominal and indexed debt can generate an

4.1.3 Capital losses from reneging on optimal policy

We have emphasized so far that the reason why lowering the duration of debt during a liquidity trap situation is beneficial is that it provides incentives for the government to keep the real interest rate low in future as it is now rolling over more short-term debt. We have shown these results by comparing the path of the real interest rate under optimal policy at a baseline and lower duration of debt.

Another way of framing this is that otherwise it would suffer capital losses on its balance sheet. These losses then would have to be accounted for by raising costly taxes. One way to illustrate the mechanism behind this result is to conduct the following thought experiment: suppose that once the liquidity trap is over, the government reneges on the path for inflation and output dictated by optimal policy under discretion and instead perfectly stabilizes them at zero. In such a situation, how large are capital losses, or equivalently, how high do taxes have to rise out of zero lower bound compared to if the government had continued to follow optimal policy? In particular, is this increase in taxes more when debt is of shorter duration ? We show in Fig. 11 the change in taxes (which are scaled as a fraction of output) if the government were to renege on optimal policy at different durations of debt. The increase in taxes out of zero lower bound are higher at a shorter duration of outstanding debt. Thus, *lowering* the duration of government debt provides the government with *more* of an incentive to keep the real interest rate low in future in order to avoid having to raise costly taxes.

4.1.4 Quantitative assessment of MEP/QE3

Given the parameter estimates based on QE2 and its effects on expected inflation and future short-term interest rates given in Table 1, we now conduct an assessment of the macroeconomic effects of MEP/QE3. This policy involved a reduction in the duration of outstanding government debt as well, as can be seen from Fig. 1. Our experiment is based on the difference in duration between September 2011 and September 2013 (we thus use the entire period following MEP for this calibration), which is equal to 1.8 q and bigger than the effect from QE2.[39] Accordingly, the macroeconomic effects, as shown below in Figs. 12-13 are larger as well as the drop in the real interest rate, as shown in Fig. 14 below, is now bigger. The extent of deflation is now reduced by 30 basis points, and output is higher as a consequence on the order of 90 basis points.

4.2 An independent central bank

We now model the effect of QE using the political economy set-up where the central bank directly cares about transfers to the treasury. To parameterize the model we use quarterly average from 2003-2008 of remittances to Treasury from the Federal Reserve to get the steady state value of V. Our baseline value of $\frac{1}{\psi_V} = \frac{S^{\gamma} b^{CB}}{V} = \frac{Ql}{V} = 111.47$ corresponds to the average ratio of holdings of

equivalence between discretion and commitment outcomes.

[39] As noted before, this measure of policy intervention will also include policy changes brought about by QE3.

Treasuries by the Federal Reserve to remittances to Treasury over the period of 2003-2008. When considering QE as an increase in the scale of the balance sheet, we assume V is fixed at the steady state value while the scale of assets and liabilities ($S^{\gamma}b^{CB} = Ql$) increases. In Fig. 2 we show the asset side of the Fed's balance sheet as a ratio of annual GDP, where we use Treasury holdings of the Fed as a proxy for the asset side. This gives a baseline ratio of the Fed's assets over annual GDP pre-crisis as 5.1%. Then, as a measure of the effects on the size of the Federal Reserve's balance sheet from QE2, we take the difference in Fig. 2 between QE2 and the MEP (the third and fourth dashed vertical lines), which is an increase in the Fed's balance sheet (ratio of asset to annual GDP) from 5.1% to 10.6%. Finally, we also pick the value of γ based on the average maturity of Treasury holdings of the FED during 2003-2008, which is equal to 7.57 q. We model the liquidity trap in the same way as before.

For the other structural parameters of the model, like before for the consolidated budget constraint case, we estimate them using a mean squared relative error criterion. Unlike for the consolidated government debt, however, since there is no clear metric on the net asset position of the Federal Reserve during the crisis (in terms of deviation from steady-state), we choose to estimate that parameter $b_L^{N,CB}$, along with five others from before.[40] The targets in the criterion are the same as before and the values of the targets for our best match are $\pi(1) = -0.02$, $Y(1) = -0.10$, $\Delta\pi(40) = 5.1$ b.p., $\Delta i(8) = -16.5$ b.p., and $\mu = 0.85$ (6.7 quarters). Our estimated parameter values are given in Table 2.

4.2.1 Solution at positive interest rates

Let us again first consider the most basic exercise to clarify the logic of the central bank's problem. How do the dynamics of the model look like in the absence of shocks when the only difference from steady state is that there is some initial negative value of net asset of the central bank (thus, net assets is below steady-state) with some fixed value of asset duration? We consider transition dynamics when net asset is below steady-state initially, while in the consolidated budget constraint case we had considered a case where debt is above steady-state initially. These are exactly the same thought experiments because in both cases there is asset accumulation along the transition. Moreover, in both cases, the real interest rate is rising along the transition, which is the main mechanism behind our paper. Our assumption is saying that the Fed would like to have higher net worth at the onset of our experiment, which we believe is a reasonable assumption under QE when the Fed exposed itself to possible balance sheet losses.

We will be looking at comparative statics of those transition dynamics with respect to the size of the central bank balance sheet (holding the duration of the assets fixed) as well as the duration of the assets on the central bank's balance sheet (holding the size of the balance sheet fixed). These, as we have already stressed, is how we interpret QE2 and MEP/QE3.

Fig. 15 shows the dynamics of the endogenous variables in the model for an initial value of net asset of the central bank that is 15 percent below the steady state. Analogous to the consolidated

[40] We also adjust this to keep the market value of the net asset position constant as a result of the QE intervention.

government budget constraint model, we see that if net asset is below steady state, it is accumulated over time back to steady state. Moreover, in this transition, inflation is above steady state and the real interest rate is below its steady state. As a consequence, output is also above its steady state value. What is the logic? Consider the linearized budget constraint of the central bank (28)

$$\hat{b}_t^{N,CB} = \beta^{-1}\hat{b}_{t-1}^{N,CB} - (1-\gamma)\,\hat{S}_t^{\gamma} + \hat{Q}_t - \psi_V \hat{V}_t.$$

The liabilities of the central bank are affected directly by the short-term nominal interest rate whose price is $\hat{Q}_t = -\hat{\imath}_t$. The price of its assets, which is long term, however, is given by $\hat{S}_t^{\gamma} = -\hat{\imath}_t + \gamma\beta E_t \hat{S}_{t+1}^{\gamma}$. If $\gamma = 0$ then its assets are simply equivalent in duration to the liabilities and cutting or raising the nominal interest rate has no effect on the evolution of the net worth of the bank, $\hat{b}_t^{N,CB}$. Once $\gamma > 0$, however, then cutting the nominal interest rate will have a bigger effect on the rollover cost of the liabilities of the bank than on the price of the long-term assets, thus generating capital gains. That is precisely what you want to obtain when $\hat{b}_t^{N,CB} < 0$ and the more so, the higher is γ.

We now consider the effect of increasing the size of the balance sheet. We define the size as the ratio of assets to transfers to the treasury while holding the duration of the assets on the balance sheet fixed. In Fig. 15 we show that bigger the size of the balance sheet (our baseline calibration is 5.1), bigger is the drop in the real interest rate. What is the intuition for this result? Consider the linearized flow budget constraint of the central bank (28) again where recall that ψ_V^{-1} is our measure of the size of the central bank's balance sheet. Now multiply both sides of the equation above with ψ_V^{-1} to get

$$\psi_V^{-1}\hat{b}_t^{N,CB} = \beta^{-1}\psi_V^{-1}\hat{b}_{t-1}^{N,CB} - \psi_V^{-1}\left[(1-\gamma)\,\hat{S}_t^{\gamma} - \hat{Q}_t\right] - \hat{V}_t.$$

This shows that even if we hold γ fixed, an increase in ψ_V^{-1} makes the effect of the duration mismatch between assets and liabilities of the central bank worse. Thus, when assets are long-term and liabilities short-term, for a higher ψ_V^{-1}, the central bank's balance sheet is more exposed to interest-rate risk along the transition where assets are being accumulated and the interest rate is rising, like in Fig. 15. Thus, in order to avoid capital losses on its balance, the central bank has incentives to keep interest rates lower with a higher ψ_V^{-1}.

Fig. 16 considers how increasing the duration of the assets of the central bank, while holding the size of the balance sheet fixed, affects the transition dynamics we described above. Fig. 16 shows that longer is the duration of assets held by the central bank (our baseline calibration is a duration of 7.57 quarters), larger is the drop in the real interest rate along the transition. This is a counterpart of our results for the consolidated government budget constraint model. Let us now move to the case in which the ZLB is binding.

4.2.2 Solution at the ZLB: QE2 and MEP/QE3

At the ZLB we consider a thought experiment that is comparable to what we did with coordinated policy. Suppose $b_L^{N,CB}$ is fixed at some constant number at the ZLB but then optimally determined by the central bank according to the MPE out of ZLB. We are interested in two comparative statics. First, what is the effect of increasing the scale of the balance sheet once and for all holding the duration on the asset side fixed? Second, what is the effect of changing the duration of the bonds on the asset side, holding the scale of the balance sheet fixed? As we saw in Section 3.2.1, QE2 corresponds quite closely to a thought experiment in which the Fed increased the scale of its balance sheet while holding the duration on its asset side fixed. Meanwhile, the MPE/QE3 corresponds more closely to a change in the maturity of the bonds on its asset side, while the scale of the balance sheet remained fixed.

Figs. 17 and 18 show the response of inflation and output to a negative shock to the efficient rate of interest when the size of the central bank's balance sheet is 5.1 and the duration of the central bank's assets is 7.57 quarters. As before, because of the zero lower bound constraint, the economy suffers from deflation and because of the increase in the real interest rate that it creates (that is, a gap between the real interest rate and its efficient counterpart), also from a negative effect on output. The real interest rate gap is shown in Fig. 19. Thus again, optimal policy at the zero lower bound would entail reducing the real interest rate gap at the present and in future, but this is not credible in a MPE.

An increase in the size of the central bank's balance sheet, given by the parameter ψ_V^{-1}, helps solve the problem. Figs. 20 and 21 show the *change* in response of output and inflation to a negative shock to the efficient rate of interest when balance sheet is more than doubled from 5.1 to 10.6 based upon the data from QE2 in Section 3.2.1 (again the duration of the central bank's assets is still fixed at 7.57 quarters). The extent of deflation is reduced by 14 basis points (annualized) as well as the negative effect on output by 45 basis points. Once the shock is over and the zero lower bound is no longer a constraint, the change in response of inflation and output is positive. These effects are in the same order of magnitude as the effects obtained with the consolidated government budget constraint.

The main reason why this is achieved is that because the central bank's balance sheet bigger, it magnifies the effects of interest rate changes on the balance sheet due to the mismatch in duration between its assets (which are long-term) and liabilities (which are short-term). The central bank then keeps the short-term real interest rates lower in future, especially once the zero lower bound is not binding, in order to keep the real interest rate low. Otherwise, it would suffer from capital losses which are costly for the central bank. Thus, quantitative easing indeed provides a signal about the future conduct of monetary policy and in particular, the future path of short-term interest rates. The change in the response of the real interest rate comparing smaller to bigger duration of debt is given in Fig. 22, where one can see that the real interest rate is lower through out the horizon. This, then, is a reaffirmation of the central result of our paper: quantitative easing acts as a commitment device during a liquidity trap situation.

Consider now the effect of increasing the duration of the bonds on the asset side of the Fed, holding the scale of the balance sheet constant. As we pointed out in Section 3.2.1 this corresponds relatively closely to MEP/QE3. For this exercise we compute the average maturity of the treasuries on the Federal Reserve's balance sheet at September 2011 (13.6 q) and September 2013 (18.95 q). The difference between the maturities is 5.4 q. Figs. 24-25 show the associated increase in output and inflation while Fig. 26 shows the resulting decrease in the real interest rate due to this policy intervention that increases maturity of central bank assets by 5.4 q compared to baseline maturity. We see that output increases by 12 bp while inflation by 2.5 bp. For an independent central bank the MEP is therefore less powerful than QE2. This is in contrast to our previous result under coordinated policy which suggested that MEP was more effective than QE2.

4.2.3 Capital losses from reneging on optimal policy

As in the model with a consolidated government, we now provide an alternative illustration of the mechanisms behind the model with an independent central bank. The exercise shows the capital losses on the balance sheet if the central bank were to renege on optimal policy. These balance sheet losses would be costly for the central bank in the model as it would reduce remittances to the treasury too sharply.

We thus conduct the following thought experiment: suppose that once the liquidity trap is over, the central bank reneges on the path for inflation and output dictated by optimal policy under discretion and instead perfectly stabilizes them at zero. In such a situation, how large are capital losses, or equivalently, how much lower are transfers to the treasury out of the zero lower bound compared to if the government had continued to follow optimal policy? In particular, is this decline in transfers more when the central bank's balance sheet is bigger ? We show in Fig. 23 the change in transfers to the treasury if the central bank were to renege on optimal policy at different balance sheet sizes (the transfers here are scaled as a fraction of output). The decrease in transfers out of zero lower bound is higher at a larger balance sheet of the central bank. Thus, increasing the size of its balance sheet provides the central bank with *more* of an incentive to keep the real interest rate low in future in order to avoid having to suffer from costly capital losses.

5 Extensions

In the previous section with coordinated monetary and fiscal policy, we focused on analyzing a situation where the duration of government debt is reduced once-and-for all. That is, we have studied comparative statics experiments with respect to ρ. A natural question that arises in this context is whether there is an incentive for the government to increase the duration of debt once the economy has recovered and if by not considering that, we are overstating our results.[41] Another interesting extension to consider is whether the particular way we have modeled the term-structure,

[41] Similar considerations would apply for the independent central bank model. We focus on the first model only for brevity.

using a perpetuity bond that pays geometrically declining coupons, might be overly restrictive and driving some of our results. While this was a very convenient modeling device, in particular while calibrating the model to various average debt durations using a single parameter, it is natural to explore a model with zero-coupon nominal government bonds with finite maturity.

5.1 Time-varying optimal duration

To address the first question, we now extend our model to allow the government (again, we only consider the consolidated government model here for brevity) to pick the duration of government debt optimally, period by period. That is, now, ρ is time-varying. In particular, the government issues a perpetuity bond in period t (B_t) which pays ρ_t^j dollars $j+1$ periods later. Following very similar manipulations as for the fixed duration case, the flow budget constraint of the government can be written as

$$S_t(\rho_t)b_t = (1 + \rho_{t-1}W_t(\rho_{t-1}))\, b_{t-1}\Pi_t^{-1} + (F - T_t) \tag{31}$$

where $S_t(\rho_t)$ is the period-t price of the government bond that pays ρ_t^j dollars $j+1$ periods later while $W_t(\rho_{t-1})$ is the period-t price of the government bond that pays ρ_{t-1}^j dollars $j+1$ periods later. Moreover, $b_t = \frac{B_t}{P_t}$. Given these types of government bonds, the asset-pricing conditions then take the form

$$S_t(\rho_t) = E_t\left[\beta\frac{u_C\left(C_{t+1},\xi_{t+1}\right)}{u_C\left(C_t,\xi_t\right)}\Pi_{t+1}^{-1}\left(1 + \rho_t S_{t+1}(\rho_t)\right)\right] \tag{32}$$

$$W_t(\rho_{t-1}) = E_t\left[\beta\frac{u_C\left(C_{t+1},\xi_{t+1}\right)}{u_C\left(C_t,\xi_t\right)}\Pi_{t+1}^{-1}\left(1 + \rho_{t-1} W_{t+1}(\rho_{t-1})\right)\right]. \tag{33}$$

The rest of the model is the same as before.

The government's instruments are now i_t, T_t , and ρ_t. Moreover, it follows from the expressions above that in addition to b_{t-1}, now, ρ_{t-1} is also a state variable in the model. Then, we can write the discretionary government's problem recursively as

$$J\left(b_{t-1},\rho_{t-1},\xi_t\right) = \max\left[U\left(.\right) + \beta E_t J\left(b_t,\rho_t,\xi_{t+1}\right)\right]$$

subject to the three new constraints, (31)-(33), as well as the other private sector equilibrium conditions that are common from the model in the previous section. Here, $U\left(.\right)$ is the utility function of the household in (1) and $J(.)$ is the value function.[42] The detailed formulation of this maximization problem and the associated first-order necessary conditions are provided in the appendix. We discuss below why we take this non-linear as opposed to a linear-quadratic approach.

We proceed by computing the non-stochastic steady-state and then taking a first-order approximation of the non-linear government optimality conditions as well as the non-linear private sector equilibrium conditions around the steady-state. Of particular note is that a first-order approxima-

[42] Using compact notation, note that we can write the utility function as $u\left(C_t,\xi_t\right) + g\left(F - s(T_t - T),\xi_t\right) - \tilde{v}\left(Y_t,\xi_t\right)$.

tion of (31)-(33) leads to

$$\hat{v}_t = \beta^{-1}\hat{v}_{t-1} - \beta^{-1}\hat{\pi}_t - (1-\rho)\hat{L}_t - \psi\hat{T}_t \tag{34}$$

where

$$\hat{v}_t = \hat{b}_t + \frac{\rho\beta}{1-\rho\beta}\hat{\rho}_t \tag{35}$$

and $\hat{L}_t = -\hat{\imath}_t + \rho\beta E_t\hat{L}_{t+1}$, $\hat{S}_t - \hat{W}_t = \hat{\rho}_t - \hat{\rho}_{t-1}$. Thus, after undertaking a transformation of variables as given by (35), (34) takes the same form as (24), the linearized government budget constraint when there was no time variation in duration. Thus, time-variation in duration does not play a separate role in government debt dynamics up to first order.[43] Therefore, if we had taken a linear-quadratic approach, like before, then with the quadratic loss-function (26) and the linearized private sector equilibrium conditions including (34), we would not be able to solve for optimal debt duration dynamics at all.

Given this appropriate redefinition of the state variable, we can show that the (bounded) solution of the model at positive interest rates takes the form

$$\begin{bmatrix} \hat{v}_t \\ \hat{\rho}_t \end{bmatrix} = \begin{bmatrix} \rho & 0 \\ \rho_v & 1 \end{bmatrix}\begin{bmatrix} \hat{v}_{t-1} \\ \hat{\rho}_{t-1} \end{bmatrix}, \begin{bmatrix} \hat{Y}_t \\ \pi_t \\ \hat{r}_t \end{bmatrix} = \begin{bmatrix} Y_v & 0 \\ \pi_v & 0 \\ r_v & 0 \end{bmatrix}\begin{bmatrix} \hat{v}_{t-1} \\ \hat{\rho}_{t-1} \end{bmatrix}$$

where, ρ_v, Y_v, π_v, and r_v are functions of the model parameters. Here, for simplicity, we are only focussing on solution of some of the endogenous model variables and do not consider shocks.

Two results stand out: first, $\hat{\rho}_t$ follows a random-walk like behavior; second, $\hat{\rho}_{t-1}$ does not affect directly other variables such as output, inflation, and the real interest rate. This suggests then that even if we allow the government to pick the duration optimally period by period, there is no incentive for it to increase the duration of debt after the economy has recovered following a liquidity trap episode. Therefore, the simple comparative static analysis that we focussed on in the main part of the paper does not appear to be overstating our results. We show the transition dynamics at positive interest rates with debt above steady-state in Fig. 27 below that highlights the solution of this extended model. Given these two properties of the solution, it is now clear that if we consider the sufficient state variable $\hat{v}_{t-1}$ similar to $\hat{b}_{t-1}$, then this model with time-varying duration would lead to similar results as our baseline model with a fixed duration.

5.2 Alternate model of maturity structure

We now consider a model where the government issues zero-coupon nominal bonds with finite maturity (again for brevity, we only consider the baseline model with a consolidated government). Since the goal of this section is only qualitative, we consider a simple environment of one-period (B_t^S) and two-period $\left(B_t^L\right)$ zero-coupon nominal bonds. Total bond supply then is given by $B_t =$

[43] This is also the case in the portfolio choice literature based on approximations, e.g. Berriel and Bhattarai (2013).

$B_t^S + B_t^L$ and we use a notation similar to before, where the ratio of the two-period to one-period bonds is given by $\rho = B_t^L / B_t$. Thus, a reduction in ρ will imply that a larger fraction of government debt is short-term. Again, similar to the notation above, we will denote the two bond prices by $(1+i_t)^{-1}$ and S_t respectively.

We can write down the flow budget constraints of the consumer and the government respectively as follows

$$P_t C_t + (1+i_t)^{-1} B_t^S + S_t B_t^L = B_{t-1}^S + (1+i_t)^{-1} B_{t-1}^L - P_t T_t$$

$$(1+i_t)^{-1} B_t^S + S_t B_t^L = B_{t-1}^S + (1+i_t)^{-1} B_{t-1}^L + P_t (F - T_t).$$

Note here that in writing these flow budget constraints, we already impose some arbitrage restrictions. For future purpose, using the market clearing condition we can write the government budget constraint in real terms as

$$b_t \left(\frac{(1-\rho)}{(1+i_t)} + \rho S_t \right) = b_{t-1} \left(1 - \rho + \frac{\rho}{(1+i_t)} \right) \Pi_t^{-1} + (F - T_t) \tag{36}$$

where $b_t = \frac{B_t}{P_t}$ and $\Pi_t = \frac{P_t}{P_{t-1}}$. The asset pricing conditions then are given by the two relationships

$$(1+i_t)^{-1} = E_t \left[\beta \frac{u_C(C_{t+1})}{u_C(C_t)} \Pi_{t+1}^{-1} \right], \; S_t = E_t \left[\beta \frac{u_C(C_{t+1})}{u_C(C_t)} \Pi_{t+1}^{-1} (1+i_{t+1})^{-1} \right]. \tag{37}$$

Like before, we can now proceed with a log-linearization around a zero-inflation steady-state, where we will use ψ for the steady-state ratio of taxes to market-value of debt $\left(\psi = \frac{T}{b(1-\rho+\rho\beta)\beta} \right)$. Then, (37) log-linearized gives

$$\hat{S}_t = -\left(\hat{\imath}_t + E_t \hat{\imath}_{t+1} \right) \tag{38}$$

which is an illustration of the expectation hypothesis in the model while (36) log-linearized gives

$$\hat{b}_t = \beta^{-1} \hat{b}_{t-1} - \beta^{-1} \pi_t + \left(\frac{1-2\rho}{(1-\rho)+\rho\beta} \right) \hat{\imath}_t - \frac{\rho\beta}{(1-\rho)+\rho\beta} \hat{S}_t - \psi \hat{T}_t. \tag{39}$$

Further, using (38), one can then write (39) as

$$\hat{b}_t = \beta^{-1} \hat{b}_{t-1} - \beta^{-1} \pi_t + \left(\frac{1-2\rho}{(1-\rho)+\rho\beta} \right) \hat{\imath}_t + \frac{\rho\beta}{(1-\rho)+\rho\beta} \left(\hat{\imath}_t + E_t \hat{\imath}_{t+1} \right) - \psi \hat{T}_t.$$

To get some insights on how the maturity composition of debt affects the roll-over incentives of the government in determining the short-term interest rate, denote by $\gamma = \frac{E_t \hat{\imath}_{t1}}{\hat{\imath}_t}$, the equilibrium rate of mean reversion of debt. That is, suppose that b_t is the only state variable (as is the case in this simple model), which in the stationary Markov-perfect equilibrium solution with no shocks, gives $\hat{b}_t = b_b \hat{b}_{t-1}$ and $\hat{\imath}_t = \iota_b b_{t-1}$. Then this rate is constant over time and equal to $\gamma = b_b < 1$. This rate is, in general equilibrium, an endogenous function of the model parameters (including ρ). But for now, we will next take it as given to gain some insights using only the government budget constraint, like we did before to illustrate the intuition of our mechanism. We can then re-write the final log-linearized government budget constraint as

$$\hat{b}_t = \beta^{-1}\hat{b}_{t-1} - \beta^{-1}\pi_t + \Gamma\hat{\imath}_t - \psi\hat{T}_t \tag{40}$$

where $\Gamma(\rho,\beta,\gamma) = \frac{1-\rho+\rho\beta(1+\gamma-\beta^{-1})}{1-\rho+\rho\beta} < 1$ for $\gamma < \beta^{-1}$ and $\rho > 0$. (Note that in a stationary equilibrium, $\gamma < 1$). What we call the"roll-over incentive" in the paper is captured by how much the current, short-term interest rate affects debt/tax dynamics in the flow budget constraint. That is, we are interested in how, in (40), Γ is affected by ρ. In particular, for our mechanism to be at work, we need Γ to decrease as ρ increases. Here, since we hold γ constant in this thought-experiment, we have directly that Γ depends negatively on ρ

$$\frac{\partial\Gamma(\rho,\beta,\gamma)}{\partial\rho} = \frac{\beta\gamma-1}{((\beta-1)\rho+1)^2} < 0.$$

Thus, longer is the maturity composition of government debt, lower is the effect on debt dynamics of the current, short-term interest rate. This is the main mechanism behind the results in our paper and it continues to be in operation in this alternate model of the term structure.

6 Conclusion

We present a theoretical model where open market operations that reduce the duration of outstanding government debt or change the size/composition of the central bank's balance sheet, so called "quantitative easing," are not neutral because they affect the incentive structure of the central bank. In particular, in a Markov-perfect equilibrium of our model, reducing the duration of outstanding government debt or increasing the balance sheet size and duration of assets held by an independent central bank, provides an incentive for the central bank to keep short-term interest rates low in future in order to avoid balance sheet losses. When the economy is in a liquidity trap, such a policy is thus effective at generating inflationary expectations and lowering long-term interest rates, which in turn, helps mitigate the deflation and negative output gap that would ensue otherwise. In other words, quantitative easing is effective because it provides a "signal" to the private sector that the central bank will keep the short-term real interest rates low even when the zero lower bound is no longer a constraint in future.

In future work, it would be of interest to evaluate fully the quantitative importance of our model mechanism in a medium-scale sticky price model, along the lines of Christiano, Eichenbaum, and Evans (2005) and Smets and Wouters (2007). Computation of Markov-perfect Equilibrium under coordinated monetary and fiscal policy at the ZLB appears to not have been investigated for such models in the literature. Moreover, as a methodological extension, it would be fruitful to allow for time-varying duration of government debt to affect real variables. To do so, it will be necessary to take a higher order approximation of the equilibrium conditions and modify the guess-and-verify algorithm to compute the Markov-perfect equilibrium accordingly. Needless to say one could also do such higher order approximations to more accurately characterize the current model. We leave that to future extensions.

References

[1] Adam, Klaus, and Roberto M. Billi (2007), "Discretionary Monetary Policy and the Zero Lower Bound on Nominal Interest Rates," *Journal of Monetary Economics,* 54, 728-752.

[2] Alvarez, Fernando, Patrick J. Kehoe, and Pablo A. Neumeyer (2004), "The Time Consistency of Optimal Monetary and Fiscal Policies," *Econometrica,* 72, 541-567.

[3] Barro, Robert J. (1979), "On the Determination of the Public Debt," *Journal of Political Economy,* 87, 940-971.

[4] Barro, Robert J. (1986), "Reputation in a Model of Monetary Policy with Incomplete Information," *Journal of Monetary Economics,* 17, 3-20.

[5] Bauer, Michael D., and Glenn D. Rudebusch (2013), "The Signaling Channel for Federal Reserve Bond Purchases," working paper.

[6] Berriel, Tiago C., and Saroj Bhattarai (2009), "Monetary Policy and Central Bank Balance Sheet Concerns," *BE Journal of Macroeconomics,* 9.

[7] Berriel, Tiago C., and Saroj Bhattarai (2013), "Hedging Against the Government: A Solution to the Home Asset Bias Puzzle," *American Economic Journal: Macroeconomics,* 5, 102-134.

[8] Calvo, Guillermo A., and Pablo E. Guidotti (1990), "Credibility and Nominal Debt: Exploring the Role of Maturity in Managing Inflation," *IMF Staff Papers*, 37, 612-635.

[9] Calvo, Guillermo A., and Pablo E. Guidotti (1992), "Optimal Maturity of Nominal Government Debt: An Infinite-Horizon Model," *International Economic Review*, 37, 612-635.

[10] Chadha, Jagjit S., Philip Turner, and Fabrizio Zampoli (2013), "The Interest Rate Effects of Government Debt Maturity," working paper.

[11] Chen, Han, Vasco Curdia, and Andrea Ferrero (2012), "The Macroeconomic Effects of Large-scale Asset Purchase Programmes," *Economic Journal,* 122, 289-315.

[12] Christiano, Lawrence J., Martin Eichenbaum, and Charles Evans (2005), "Nominal Rigidities and the Dynamic Effects of a Shock to Monetary Policy," *Journal of Political Economy,* 113, 1-45.

[13] Christiano, Lawrence J., Martin Eichenbaum, and Sergio Rebelo (2011), "When is the Government Spending Multiplier Large?" *Journal of Political Economy, 1, 78-121.*

[14] Del Negro, Marco, and Christopher A. Sims (2015), "When Does a Central Bank Balance Sheet Require Fiscal Support?" working paper.

[15] Eggertsson, Gauti B. (2006), "The Deflation Bias and Committing to Being Irresponsible," *Journal of Money, Credit, and Banking,* 38, 283-321.

[16] Eggertsson, Gauti B. (2010), "What Fiscal Policy is Effective at Zero Interest Rates," *NBER Macroeconomic Annual* 2010.

[17] Eggertsson, Gauti B., and Paul Krugman (2012), "Debt, Deleveraging, and the Liquidity Trap: A Fisher-Minsky-Koo Approach," *Quarterly Journal of Economics,* 127, 1469-1513.

[18] Eggertsson, Gauti B. and Sanjay Singh (2015), "The Non-linear Calvo model at the Zero Bound: Some Analytic Solutions and Numerical Results," working paper.

[19] Eggertsson, Gauti B., and Michael Woodford (2003), "The Zero Bound on Interest Rates and Optimal Monetary Policy," *Brookings Papers on Economic Activity,* 34, 139-235.

[20] Gagnon, Joseph, Matthew Raskin, Julie Ramache, and Brian Sack (2011), "The Financial Market Effects of the Federal Reserve's Large-Scale Asset Purchases," *International Journal of Central Banking,* 7, 3-43.

[21] Gertler, Mark and Peter Karadi (2012), "QE1 vs. 2 vs. 3...: A Framework For Analyzing Large Scale Asset Purchases as a Monetary Policy Tool," working paper.

[22] Hall, Robert E., and Ricardo Reis (2013), "Controlling Inflation and Maintaining Central-Bank Solvency under New-Style Central Banking," working paper.

[23] Hamilton, James D., and Cynthia Wu (2012), "The Effectiveness of Alternative Monetary Policy Tools in a Zero Lower Bound Environment," *Journal of Money, Credit, and Banking,* 44, 3-46.

[24] Jeanne, Olivier, and Lars E. O. Svensson (2007), "Credible Commitment to Optimal Escape from a Liquidity Trap: The Role of the Balance Sheet of an Independent Central Bank," *American Economic Review,* 97, 474-490.

[25] Krishnamurthy, Arvind, and Annette Vissing-Jorgensen (2011), "The Effects of Quantitative Easing on Interest Rates: Channels and Implications for Policy," *Brookings Papers on Economic Activity,* 43, 215-287.

[26] Krugman, Paul (1998), "It's Baaack: Japan's Slump and the Return of the Liquidity Trap," *Brookings Papers on Economic Activity,* 29, 137-206.

[27] Lucas, Robert, and Nancy Stokey (1983), "Optimal Fiscal and Monetary Policy in an Economy without Capital," *Journal of Monetary Economics,* 12, 55-93.

[28] Maskin, Eric, and Jean Tirole (2001), "Markov Perfect Equilibrium I: Observable Actions," *Journal of Economic Theory,* 100, 191-219.

[29] Mussa, Michael (1981), "The Role of Official Intervention," *Group of Thirty Occasional Paper.*

[30] Persson, Mats, Torsten Persson, and Lars E. O. Svensson (1987), "Time Consistency of Fiscal and Monetary Policy," *Econometrica,* 55, 1419-1431.

[31] Persson, Mats, Torsten Persson, and Lars E. O. Svensson (2006), "Time Consistency of Fiscal and Monetary Policy: A Solution," *Econometrica,* 74, 193-212.

[32] Rotemberg, Julio (1982), "Sticky Prices in the United States," *Journal of Political Economy,* 90, 1187-1211.

[33] Sims, Christopher A. (2011), "Stepping on a Rake: The Role of Fiscal Policy in the Inflation of the 1970s," *European Economic Review,* 55, 48-56.

[34] Sims, Christopher A. (2013), "Paper Money," *American Economic Review,* 103, 563-584.

[35] Smets, Frank, and Raf Wouters (2007), "Shocks and Frictions in US Business Cycles: A Bayesian DSGE Approach," *American Economic Review,* 97, 586-606.

[36] Swanson, Eric T, and John C. Williams (2013), "Measuring the Effect of the Zero Lower Bound on Medium-and Longer-Term Interest Rates," working paper.

[37] Wallace, Neil (1981), "A Modigliani-Miller Theorem for Open-Market Operations," *American Economic Review,* 71, 267-274.

[38] Werning, Ivan (2012), "Managing a Liquidity Trap: Monetary and Fiscal Policy," working paper.

[39] Woodford, Michael (2001), "Fiscal Requirements for Price Stability," *Journal of Money, Credit, and Banking,* 33, 669-728.

[40] Woodford, Michael (2003), *Interest and Prices.* Princeton, NJ: Princeton University Press.

[41] Woodford, Michael (2012), "Methods of Policy Accommodation at the Interest-Rate Lower Bound," in *The Changing Policy Landscape,* Federal Reserve Bank of Kansas City.

7 Figures and Tables

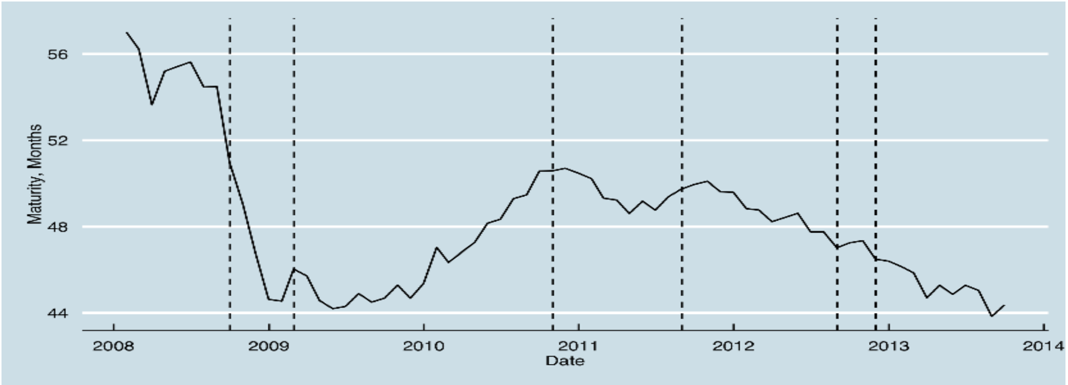

Fig 1: Maturity of outstanding govt debt adjusted for reserves issued by the Federal Reserve

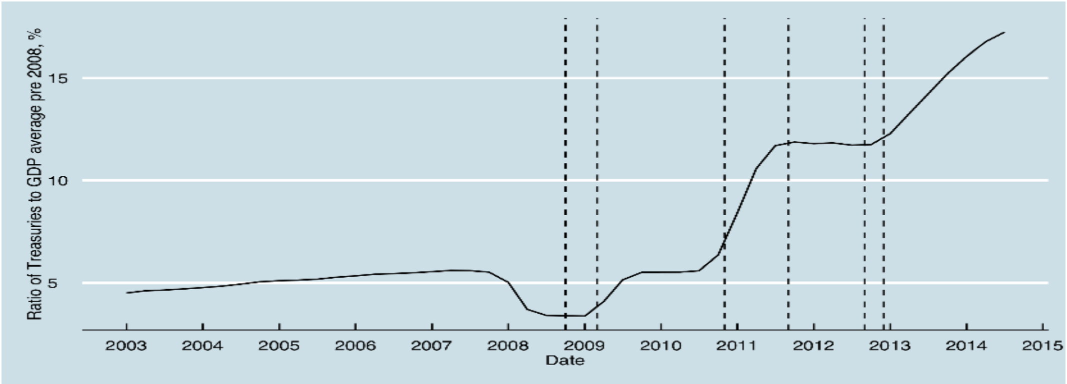

Fig 2: Ratio of Federal Reserve's holdings of Treasuries to pre-crisis average GDP

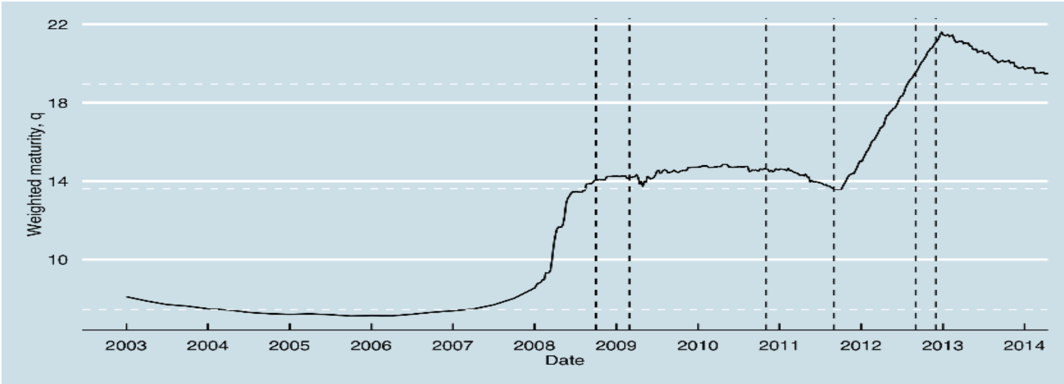

Fig 3: Maturity of Federal Reserve's holdings of Treasuries

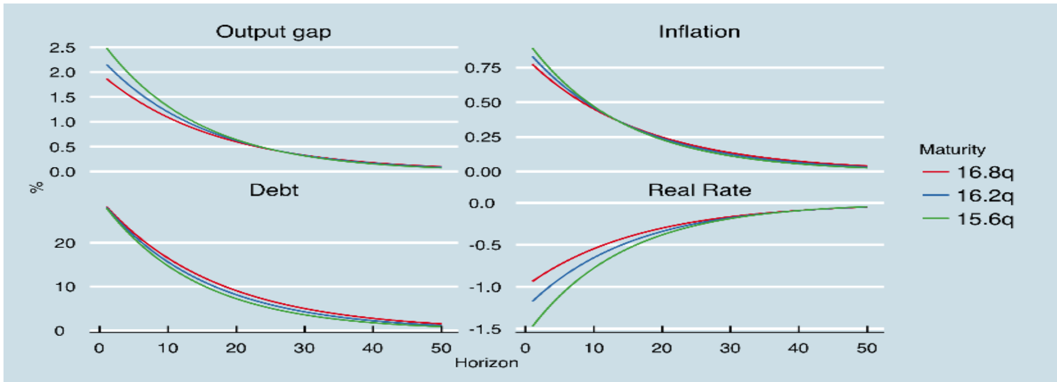

Fig 4: Transition dynamics to an increase in debt outstanding at different levels of duration of debt

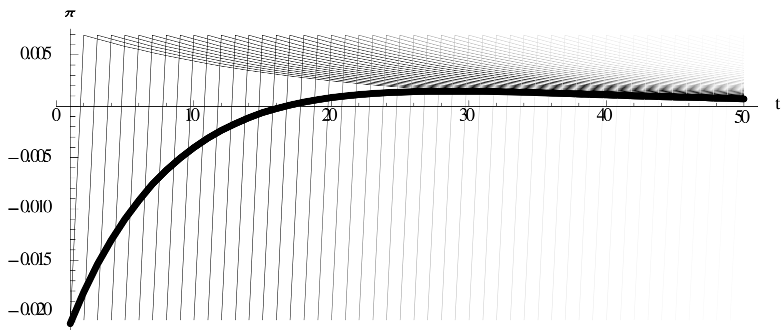

Fig 5: Response of inflation when the duration of government debt is 16.87 quarters. Each thin line represents the response when the efficient rate of interest returns to its steady-state in that period, while the dark line is the probability weighted path.

Fig 6: Response of output when the duration of government debt is 16.87 quarters.

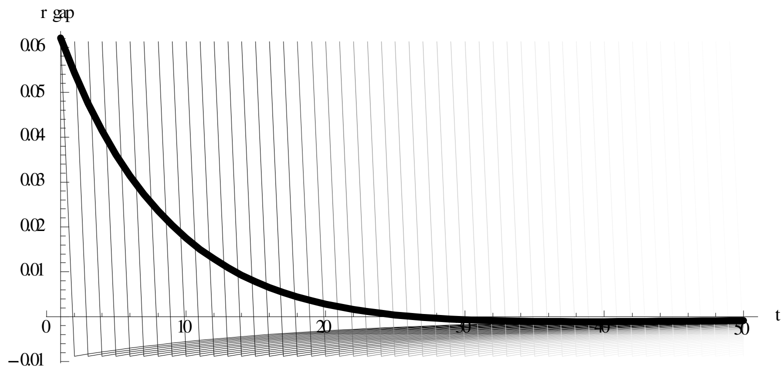

Fig 7: Response of the real interest rate gap when the duration of government debt is 16.87 quarters.

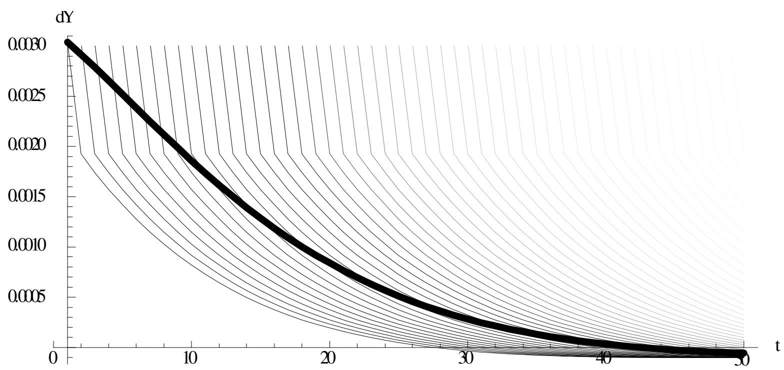

Fig 8: Change in output when the duration of government debt is reduced from 16.87 to 16.2 quarters.

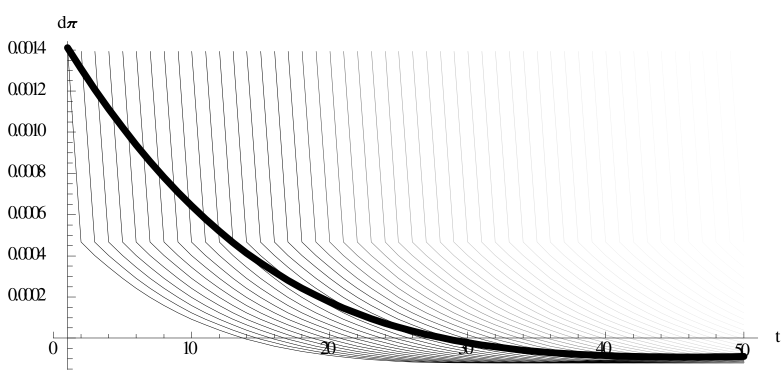

Fig 9: Change in inflation when the duration of government debt is reduced from 16.87 to 16.2 quarters.

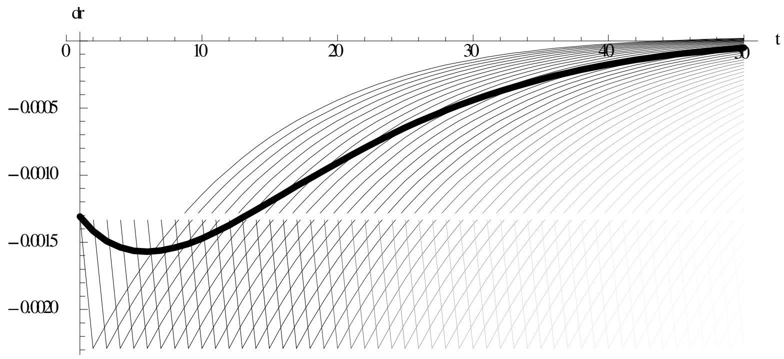

Fig 10: Change in the real interest rate when the duration of government debt is reduced from 16.87 to 16.2 quarters.

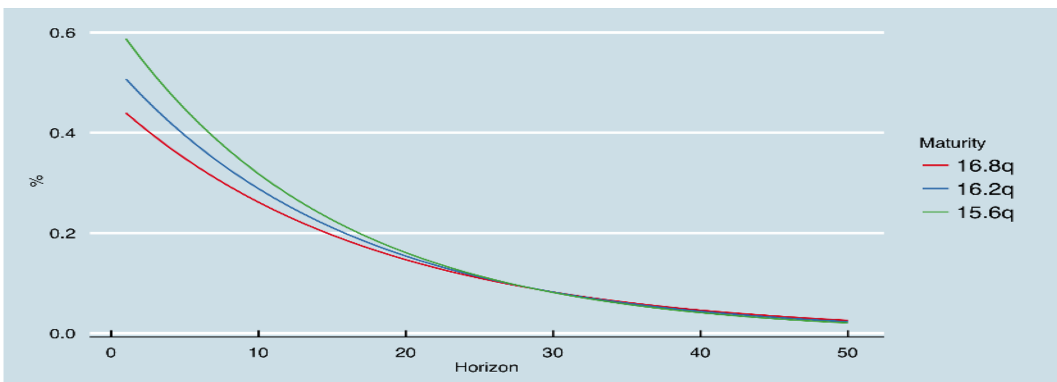

Fig 11: Increase in taxes from reneging on optimal policy

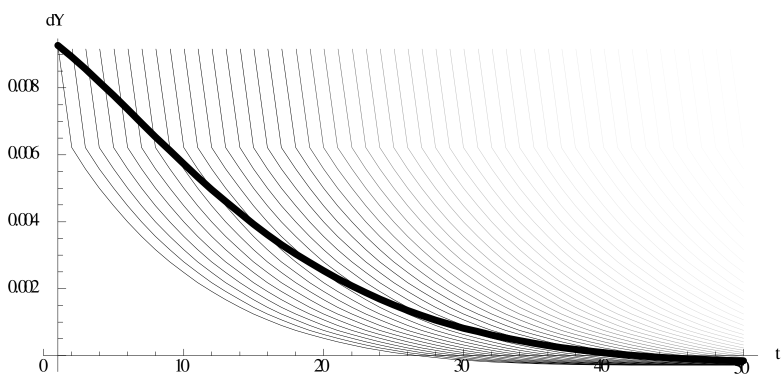

Fig 12: Change in output when the duration of government debt is reduced to 15.07 quarters.

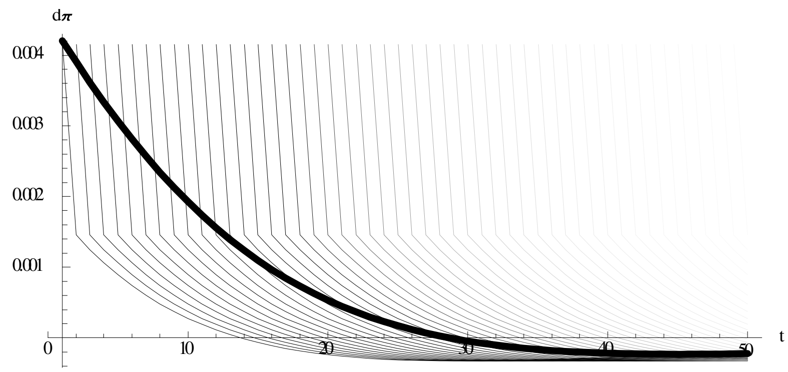

Fig 13: Change in inflation when the duration of government debt is reduced to 15.07 quarters.

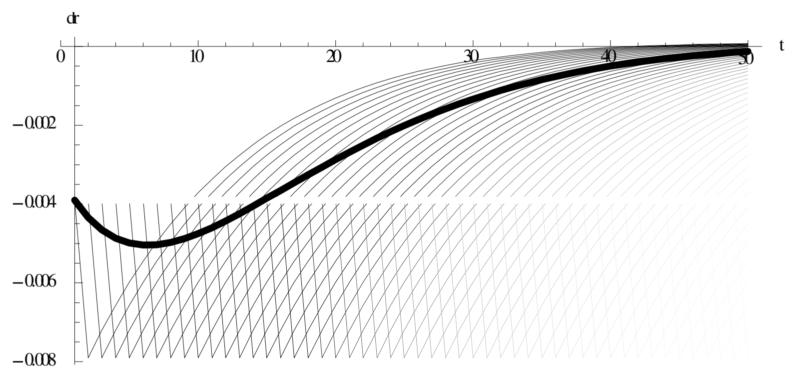

Fig 14: Change in the real interest rate when the duration of government debt is reduced to 15.07 quarters.

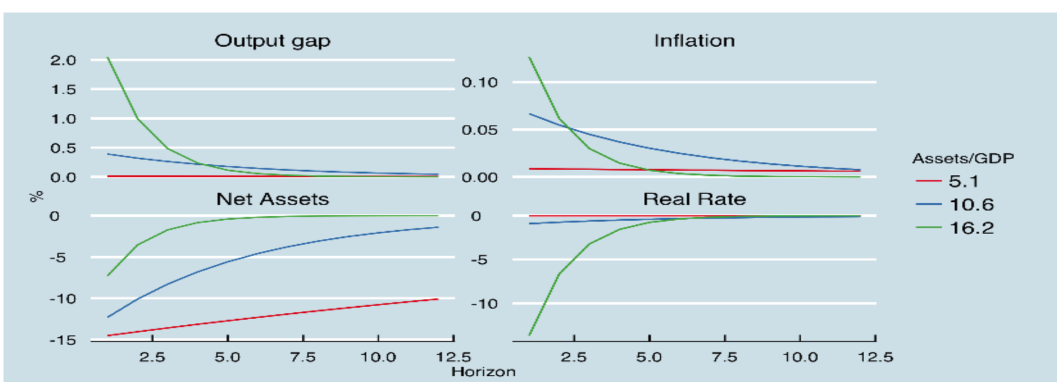

Fig 15: Transition dynamics to a decrease in net assets of the central bank at different sizes of the central bank balance sheet

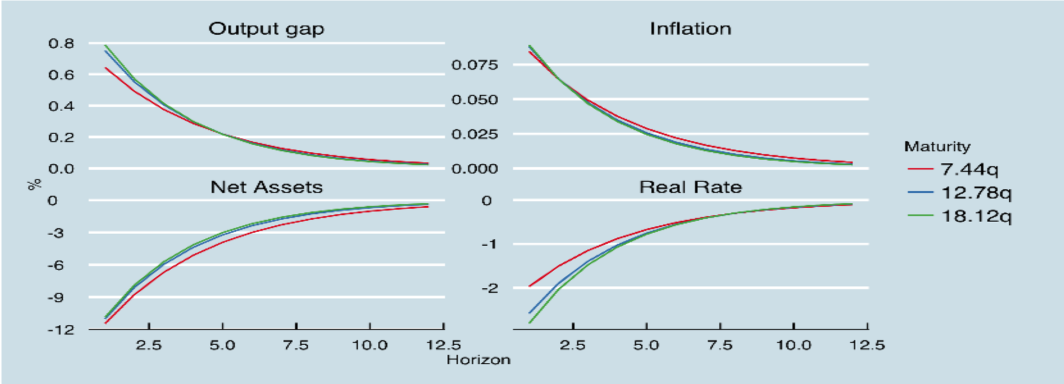

Fig 16: Transition dynamics to a decrease in net assets of the central bank at different levels of duration of the net assets

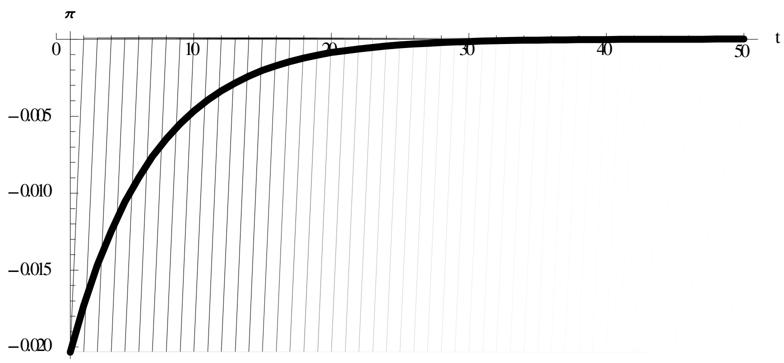

Fig 17: Response of inflation at the initial level of central bank balance sheet size and net asset duration.

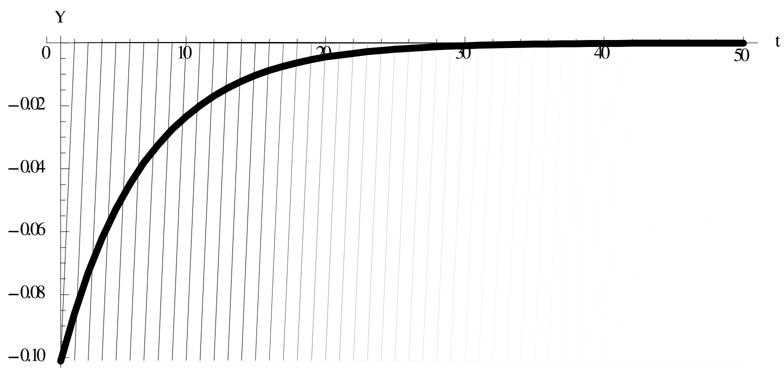

Fig 18: Response of output at the initial level of central bank balance sheet size and net asset duration.

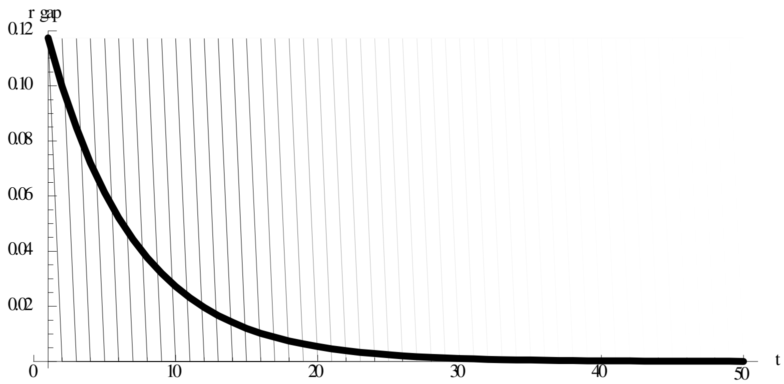

Fig 19: Response of the real interest rate gap at the initial level of central bank balance sheet size and net asset duration.

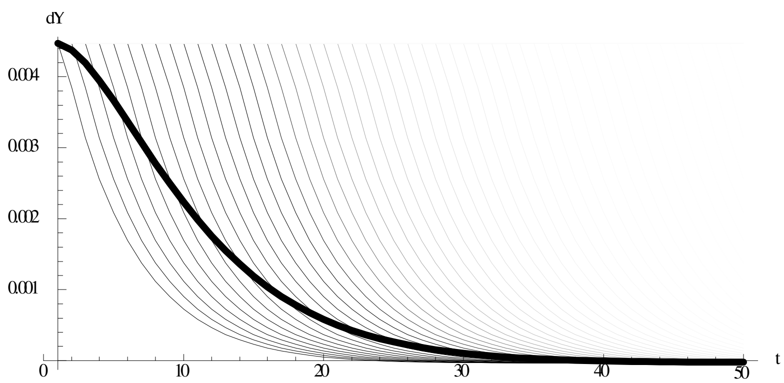

Fig 20: Change in output with an increased central bank balance sheet size and initial level of net asset duration.

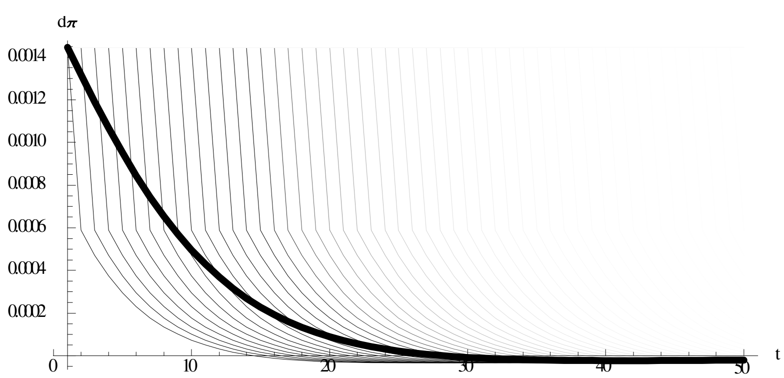

Fig 21: Change in inflation with an increased central bank balance sheet size and initial level of net asset duration.

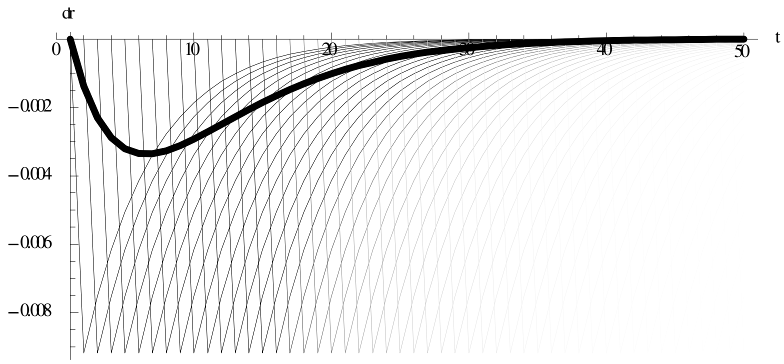

Fig 22: Change in real interest rate with an increased central bank balance sheet size and initial level of net asset duration.

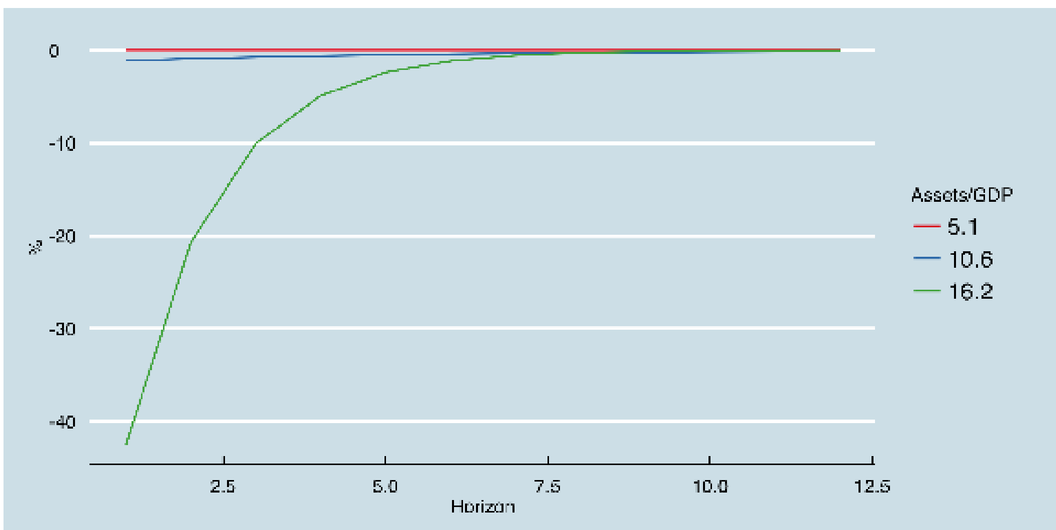

Fig 23: Decrease in transfers to treasury from reneging on optimal policy at different levels of sizes of the central bank balance sheet

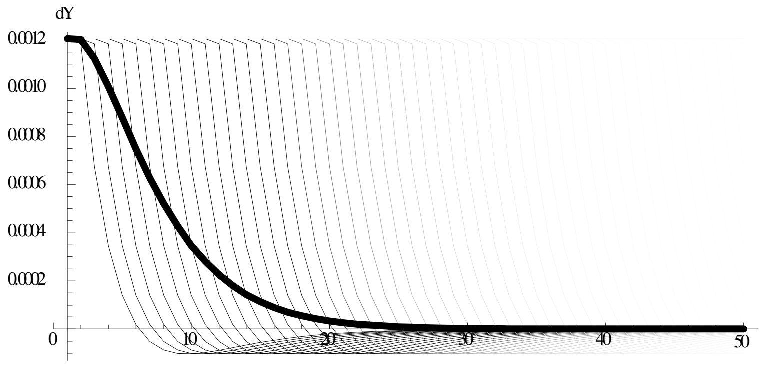

Fig 24: Change in output with an initial size of the central bank balance sheet size and increased net asset duration.

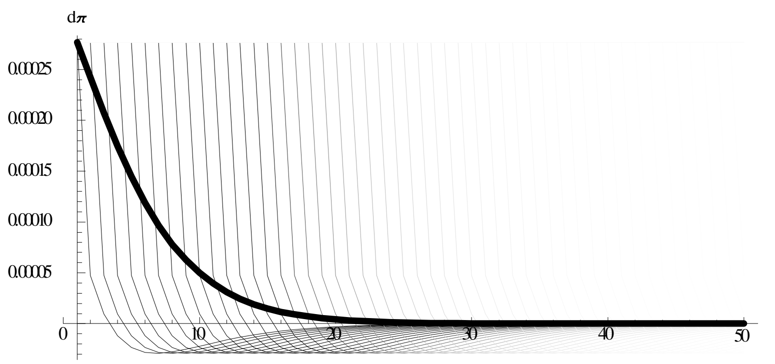

Fig 25: Change in inflation with an initial size of the central bank balance sheet size and increased net asset duration.

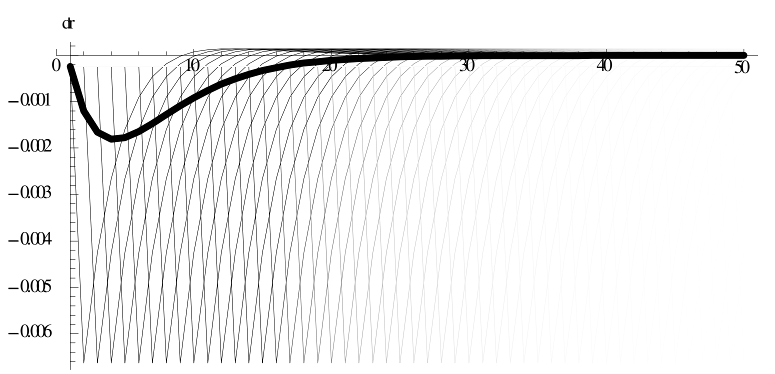

Fig 26: Change in the real interest rate with an initial size of the central bank balance sheet size and increased net asset duration.

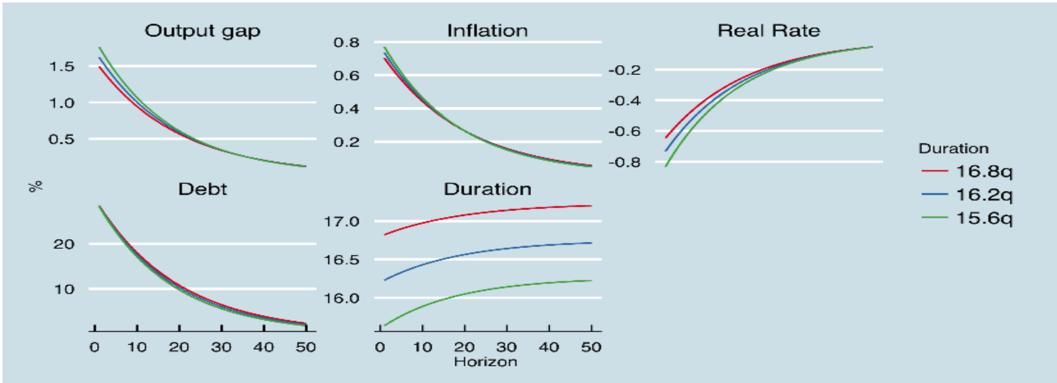

Fig 27: Transition dynamics to an increase in debt outstanding at different initial levels of duration of debt in a model with optimal time-varying duration of debt

Table 1: Calibration/estimation of model parameters for the consolidated government model

Parameter	Value
β	0.99
σ	0.3202
κ	0.0089
ε	8
λ_T	0.17593
$\frac{bS}{T}$	7.2
r_L^e	0.021
μ	0.89
b_L	0.30

Table 2: Calibration/estimation of model parameters for the independent central bank model

Parameter	Value
β	0.99
σ	0.308
κ	0.008
ε	8
λ_V	1.27×10^{-6}
$\frac{Ql}{V}$	111.47
r_L^e	0.035
μ	0.85
$b_L^{N,CB}$	-0.15

8 Appendix

8.1 Model

8.1.1 The perpetuity nominal bond and the flow budget constraint

Following Woodford (2001), the perpetuity issued in period t pays ρ^j dollars $j+1$ periods later, for each $j \geq 0$ and some decay factor $0 \leq \rho < \beta^{-1}$. The implied steady-state duration of this bond is then $(1-\beta\rho)^{-1}$.

Let the price of a newly issued bond in period t be $S_t(\rho)$. Given the existence of the unique stochastic discount factor $Q_{t,t+j}$, we can write this price as

$$S_t(\rho) = E_t \sum_{j=1}^{\infty} Q_{t,t+j}\rho^{j-1}.$$

Now consider the period $t+1$ price of such a bond that was issued in period t. We can then write the price $S^O_{t+1}(\rho)$

$$S^O_{t+1}(\rho) = E_{t+1} \sum_{j=2}^{\infty} Q_{t+1,t+j}\rho^{j-1}.$$

Note first that

$$S^O_{t+1}(\rho) = \rho S_{t+1}(\rho) \tag{41}$$

since

$$S_{t+1}(\rho) = E_{t+1} \sum_{j=2}^{\infty} Q_{t+1,t+j}\rho^{j-2}.$$

This is highly convenient since it implies that one needs to keep track, at each point in time, of the equilibrium price of only one type of bond.

Next, we derive an arbitrage condition between this perpetuity and a one-period bond. By simple expansion of the infinite sums above and manipulation of the terms, one gets

$$S_t(\rho) = [E_t Q_{t,t+1}] + E_t\left[Q_{t,t+1}S^O_{t+1}(\rho)\right].$$

Since

$$E_t Q_{t,t+1} = \frac{1}{1+i_t}$$

we get

$$S_t(\rho) = \frac{1}{1+i_t} + E_t\left[Q_{t,t+1}S^O_{t+1}(\rho)\right].$$

Substituting further for $S^O_{t+1}(\rho) = \rho S_{t+1}(\rho)$, we then derive

$$S_t(\rho) = \frac{1}{1+i_t} + \rho E_t\left[Q_{t,t+1}S_{t+1}(\rho)\right]. \tag{42}$$

Finally, consider the flow budget constraint of the government

$$B^S_t + S_t(\rho)B_t = (1+i_{t-1})B^S_{t-1} + \left(\rho^0 + S^O_t(\rho)\right)B_{t-1} + P_t(F_t - T_t).$$

This can be simplified using $S^O_{t+1}(\rho) = \rho S_{t+1}(\rho)$ to

$$B^S_t + S_t(\rho)B_t = (1+i_{t-1})B^S_{t-1} + (1+\rho S_t(\rho))B_{t-1} + P_t(F_t - T_t).$$

This is the form in which we write down the flow budget constraint of the household and the government in the main text.

8.1.2 Functional forms

We make the following functional form assumptions on preferences and technology

$$u(C,\xi) = \xi \bar{C}^{\frac{1}{\sigma}} \frac{C^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}$$

$$v(h(i),\xi) = \xi \lambda \frac{h(i)^{1+\phi}}{1+\phi}$$

$$g(G,\xi) = \xi \bar{G}^{\frac{1}{\sigma}} \frac{G^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}$$

$$y(i) = h(i)^{\kappa}$$

$$d(\Pi) = d_1 (\Pi - 1)^2$$

$$S(\bar{T}) = s_1 (T - T)^2$$

where we only consider a discount factor shock ξ. Note that $\xi = 1$ in steady-state and that in steady state, we scale hours such that $Y = 1$ as well. This implies that we can derive

$$\tilde{v}(Y,\xi) = \frac{1}{1+\phi} \lambda \xi Y^{\frac{1+\phi}{\kappa}}.$$

8.2 Efficient equilibrium

As benchmark, we first derive the efficient allocation.

Using $G_t = F_t - s(T_t - T) = F - s(T_t - T)$, the social planner's problem can be written as

$$\max u(C_t,\xi_t) + g(F - s(T_t - T)) - \tilde{v}(Y_t)$$

st

$$Y_t = C_t + F.$$

Formulate the Lagrangian

$$\begin{aligned} L_t &= u(C_t,\xi_t) + g(F - s(T_t - T)) - \tilde{v}(Y_t) \\ &\quad + \phi_{1t}(Y_t - C_t - F) \end{aligned}$$

FOCs (where all the derivatives are to be equated to zero)

$$\begin{aligned} \frac{\partial L_t}{\partial Y_t} &= -\tilde{v}_Y + \phi_{1t} \\ \frac{\partial L_t}{\partial C_t} &= u_C + \phi_{1t}[-1] \\ \frac{\partial L_t}{\partial T_t} &= g_G\left(-s'(T_t - T)\right) \end{aligned}$$

Eliminating the Lagrange multiplier gives

$$u_C = \tilde{v}_Y$$

$$g_G\left(-s'(T_t - T)\right) = 0.$$

Note that we make the following functional form assumptions on the tax collection cost

$$s(0) = 0;\ s'(0) = 0.$$

Thus, when taxes are at steady state, that is, $T_t = T$, then $s(T_t - T) = s'(T_t - T) = 0$. But note that we will allow for $s''(0) > 0$.

Efficient allocation thus requires

$$u_C = \tilde{v}_Y$$

$$T_t = T.$$

In steady state, without aggregate shocks, we have

$$Y = C + F$$

$$u_C = \tilde{v}_Y$$

$$T_t = T.$$

8.3 Non-linear markov equilibrium

8.3.1 Optimal policy under discretion

The policy problem can be written as

$$J\left(b_{t-1}, \xi_t\right) = \max\left[U\left(\Lambda_t, \xi_t\right) + \beta E_t J\left(b_t, \xi_{t+1}\right)\right]$$

st

$$S_t(\rho) b_t = \left(1 + \rho S_t(\rho)\right) b_{t-1} \Pi_t^{-1} + \left(F - T_t\right).$$

$$1 + i_t = \frac{u_C\left(C_t, \xi_t\right)}{\beta f_t^e}$$

$$i_t \geq 0$$

$$S_t(\rho) = \frac{1}{u_C\left(C_t, \xi_t\right)} \beta g_t^e$$

$$\varepsilon Y_t\left[\frac{\varepsilon - 1}{\varepsilon}\left(1 + s\right) u_C\left(C_t, \xi_t\right) - \tilde{v}_y\left(Y_t, \xi_t\right)\right] + u_C\left(C_t, \xi_t\right) d'\left(\Pi_t\right) \Pi_t = \beta h_t^e$$

$$Y_t = C_t + F + d\left(\Pi_t\right)$$

$$f_t^e = E_t\left[u_C\left(C_{t+1}, \xi_{t+1}\right) \Pi_{t+1}^{-1}\right] = \bar{f}^e\left(b_t, \xi_t\right)$$

$$g_t^e = E_t\left[u_C\left(C_{t+1}, \xi_{t+1}\right) \Pi_{t+1}^{-1}\left(1 + \rho S_{t+1}(\rho)\right)\right] = \bar{g}^e\left(b_t, \xi_t\right)$$

$$h_t^e = E_t\left[u_C\left(C_{t+1}, \xi_{t+1}\right) d'\left(\Pi_{t+1}\right) \Pi_{t+1}\right] = \bar{h}^e\left(b_t, \xi_t\right)$$

Formulate the period Lagrangian

$$\begin{aligned}
L_t = {} & u\left(C_t, \xi_t\right) + g\left(F - s(T_t - T)\right) - \tilde{v}\left(Y_t\right) + \beta E_t J\left(b_t, \xi_{t+1}\right) \\
& + \phi_{1t}\left(S_t(\rho) b_t - \left(1 + \rho S_t(\rho)\right) b_{t-1} \Pi_t^{-1} - \left(F - T_t\right)\right) \\
& + \phi_{2t}\left(\beta f_t^e - \frac{u_C\left(C_t, \xi_t\right)}{1 + i_t}\right) \\
& + \phi_{3t}\left(\beta g_t^e - u_C\left(C_t, \xi_t\right) S_t(\rho)\right) \\
& + \phi_{4t}\left(\beta h_t^e - \varepsilon Y_t\left[\frac{\varepsilon - 1}{\varepsilon}\left(1 + s\right) u_C\left(C_t, \xi_t\right) - \tilde{v}_y\left(Y_t, \xi_t\right)\right] - u_C\left(C_t, \xi_t\right) d'\left(\Pi_t\right) \Pi_t\right) \\
& + \phi_{5t}\left(Y_t - C_t - F - d\left(\Pi_t\right)\right) \\
& + \psi_{1t}\left(f_t^e - \bar{f}^e\left(b_t, \xi_t\right)\right) \\
& + \psi_{2t}\left(g_t^e - \bar{g}^e\left(b_t, \xi_t\right)\right) \\
& + \psi_{3t}\left(h_t^e - \bar{h}^e\left(b_t, \xi_t\right)\right) \\
& + \gamma_{1t}\left(i_t - 0\right)
\end{aligned}$$

First-order conditions (where all the derivatives should be equated to zero)

$$\frac{\partial L_s}{\partial \Pi_t} = \phi_{1t}\left[\left(1+\rho S_t(\rho)\right) b_{t-1}\Pi_t^{-2}\right] + \phi_{4t}\left[-u_C d^{''}\Pi_t - u_C d^{'}\right] + \phi_{5t}\left[-d'\right]$$
$$\frac{\partial L_s}{\partial Y_t} = -\tilde{v}_Y + \phi_{4t}\left[-\varepsilon\left(\frac{\varepsilon-1}{\varepsilon}\left(1+s\right)u_C\right) + \varepsilon Y_t \tilde{v}_{yy} + \varepsilon \tilde{v}_y\right] + \phi_{5t}$$
$$\frac{\partial L_s}{\partial i_t} = \phi_{2t}\left[u_C\left(1+i_t\right)^{-2}\right] + \gamma_{1t}$$
$$\frac{\partial L_s}{\partial S_t} = \phi_{1t}\left[b_t - \rho b_{t-1}\Pi_t^{-1}\right] + \phi_{3t}\left[-u_C\right]$$
$$\frac{\partial L_s}{\partial C_t} = u_C + \phi_{2t}\left[-u_{CC}\left(1+i_t\right)^{-1}\right] + \phi_{3t}\left[-u_{CC}S_t(\rho)\right] + \phi_{4t}\left[-\varepsilon Y_t\frac{\varepsilon-1}{\varepsilon}\left(1+s\right)u_{CC} - u_{CC}d'\Pi_t\right] + \phi_{5t}\left[-1\right]$$
$$\frac{\partial L_s}{\partial T_t} = g_G\left(-s'(T_t - T)\right) + \phi_{1t}$$
$$\frac{\partial L_s}{\partial b_t} = \beta E_t J_b\left(b_t, \xi_{t+1}\right) + \phi_{1t}\left[S_t(\rho)\right] + \psi_{1t}\left[-\bar{f}_b^e\right] + \psi_{2t}\left[-\bar{g}_b^e\right] + \psi_{3t}\left[-\bar{h}_b^e\right]$$
$$\frac{\partial L_s}{\partial f_t^e} = \beta\phi_{2t} + \psi_{1t}$$
$$\frac{\partial L_s}{\partial g_t^e} = \beta\phi_{3t} + \psi_{2t}$$
$$\frac{\partial L_s}{\partial h_t^e} = \beta\phi_{4t} + \psi_{3t}$$

The complementary slackness conditions are

$$\gamma_{1t} \geq 0,\ i_t \geq 0,\ \gamma_{1t} i_t = 0$$

While the envelope condition is

$$J_b\left(b_{t-1}, \xi_t\right) = \phi_{1t}\left[-\left(1+\rho S_t(\rho)\right)\Pi_t^{-1}\right]$$

This also implies that

$$\beta E_t J_b\left(b_t, \xi_{t+1}\right) = \beta E_t \phi_{1t+1}\left[-\left(1+\rho S_{t+1}(\rho)\right)\Pi_{t+1}^{-1}\right]$$

8.3.2 Steady-state

A Markov-perfect steady-state is non-trivial to characterize because generally, we need to take derivatives of an unknown function, as is clear from the FOCs. Here, we will rely on the fact that given an appropriate production subsidy, the Markov-perfect steady-state will be the same as the efficient steady-state derived above.

First, note that this requires no resource loss from price-adjustment costs, which in turn requires

$$d\left(\Pi\right) = 0$$

and thereby ensures

$$Y = C + F$$

This means that we need

$$\Pi = 1.$$

Also, this implies

$$d'\left(\Pi\right) = 0.$$

Next, note from the Phillips curve that this means, we need

$$\frac{\varepsilon-1}{\varepsilon}\left(1+s\right)u_C - \tilde{v}_y = 0.$$

Now, since the efficient steady-state has $u_C = \tilde{v}_Y$, the production subsidy then has to satisfy

$$\frac{\varepsilon - 1}{\varepsilon}(1+s) = 1.$$

We will be looking at a steady-state with positive interest rates

$$1 + i = \frac{1}{\beta}$$

which means that

$$\gamma_1 = 0$$

and that from the FOC wrt i_t we have

$$\phi_2 = 0.$$

Also, given that taxes are at steady-state, $g_G\left(-s'(T_t - T)\right) = 0$, from the FOC wrt T_t

$$\phi_1 = 0.$$

Given this, in turn, we have from the FOC wrt S_t

$$\phi_3 = 0.$$

Then, given that $d' = 0$ in steady-state and d'' is not, and since $\phi_1 = 0$, it gives from the FOC wrt to Π_t

$$\phi_4 = 0.$$

Note that this is highly convenient since these Lagrange multipliers being zero implies

$$\psi_1 = \psi_2 = \psi_3 = 0.$$

Thus, we do not need to worry about the derivatives of the unknown functions.

This proposed steady-state is consistent with other FOCs. For example, the FOC wrt Y_t is now given by

$$\phi_5 = \tilde{v}_Y$$

and that the FOC wrt C_t is given by

$$u_C = \phi_5$$

which implies

$$\tilde{v}_Y = u_C.$$

Finally, FOC wrt b_t implies

$$\beta J_b = \beta\phi_1\left[-\left(1+\rho S(\rho)\right)\Pi^{-1}\right] = 0$$

which is also consistent with the conjectured guess.

Finally, the guess of the steady-state is also consistent with the other model equilibrium conditions, with $S(\rho)$ given by

$$S(\rho) = \beta\left[\left(1+\rho S(\rho)\right)\right]$$

that is

$$S(\rho) = \frac{\beta}{1-\rho\beta}.$$

Then b and F are linked by

$$S(\rho)b = \left(1+\rho S(\rho)\right)b + (F - T)$$

that is

$$T = F + \frac{1-\beta}{1-\rho\beta}b.$$

8.3.3 First-order approximation

We now take a log-linear approximation of the Markov perfect FOCs and the private sector equilibrium conditions around the steady-state above. Also, lets normalize the scale of the economy (with appropriate scaling of hours) so that $\bar{Y} = 1$. This implies $\bar{C} = 1 - F$. Also the shock ξ_t takes a value of 1 in steady-state.

Private sector equilibrium conditions We first start with the private sector equilibrium conditions. We denote variables that are in log-deviations from their respective steady-states by hats, except for $\hat{\imath}_t$.We denote variables in steady-state by bars.

First,

$$Y_t = C_t + F + d\left(\Pi_t\right)$$

gives

$$\hat{Y}_t = \bar{C}\hat{C}_t$$

Second,

$$\varepsilon Y_t \left[\frac{\varepsilon - 1}{\varepsilon}\left(1+s\right) u_C\left(C_t, \xi_t\right) - \tilde{v}_y\left(Y_t, \xi_t\right)\right] + u_C\left(C_t, \xi_t\right) d'\left(\Pi_t\right)\Pi_t = \beta E_t\left[u_C\left(C_{t+1}, \xi_{t+1}\right) d'\left(\Pi_{t+1}\right)\Pi_{t+1}\right]$$

gives

$$\bar{u}_C d''\hat{\pi}_t + \varepsilon\bar{u}_{CC}\bar{C}\hat{C}_t - \varepsilon\bar{v}_{yy}\hat{Y}_t - \varepsilon\bar{v}_{y\xi}\hat{\xi}_t + \varepsilon\bar{u}_{C\xi}\hat{\xi}_t = \beta\bar{u}_C d'' E_t\hat{\pi}_{t+1}$$

which can be simplified by making use of the log-linearized resource constraint above to yield

$$\bar{u}_C d''\hat{\pi}_t + \varepsilon\left(\bar{u}_{CC} - \bar{v}_{yy}\right)\hat{Y}_t = \beta\bar{u}_C d'' E_t\hat{\pi}_{t+1}$$

$$\hat{\pi}_t = \beta E_t\hat{\pi}_{t+1} + \frac{\varepsilon\left(\bar{v}_{yy} - \bar{u}_{CC}\right)}{\bar{u}_C d''}\hat{Y}_t.$$

Third,

$$1 + i_t = \frac{u_C\left(C_t, \xi_t\right)}{\beta E_t\left[u_C\left(C_{t+1}, \xi_{t+1}\right)\Pi_{t+1}^{-1}\right]}$$

gives

$$\bar{u}_{CC}\bar{C}\hat{C}_t + \bar{u}_{C\xi}\hat{\xi}_t = \bar{u}_C\hat{\imath}_t + \bar{u}_{CC}E_t\bar{C}\hat{C}_{t+1} + \bar{u}_{C\xi}E_t\hat{\xi}_{t+1} - \bar{u}_C E_t\hat{\pi}_{t+1}$$

which can be simplified by making use of the log-linearized resource constraint above to yield

$$\bar{u}_{CC}\hat{Y}_t + \bar{u}_{C\xi}\hat{\xi}_t = \bar{u}_C\hat{\imath}_t + \bar{u}_{CC}E_t\hat{Y}_{t+1} + \bar{u}_{C\xi}E_t\hat{\xi}_{t+1} - \bar{u}_C E_t\hat{\pi}_{t+1}$$

and

$$\hat{Y}_t = E_t\hat{Y}_{t+1} + \frac{\bar{u}_C}{\bar{u}_{CC}}\left[\hat{\imath}_t - E_t\hat{\pi}_{t+1}\right] + \frac{\bar{u}_{C\xi}}{\bar{u}_{CC}}\left[E_t\hat{\xi}_{t+1} - \hat{\xi}_t\right]$$

Note here that this implies that the efficient rate of interest is given by

$$r_t^e = -\frac{\bar{u}_{C\xi}}{\bar{u}_C}\left[E_t\hat{\xi}_{t+1} - \hat{\xi}_t\right].$$

Fourth,

$$S_t(\rho) = \frac{1}{u_C\left(C_t, \xi_t\right)}\beta E_t\left[u_C\left(C_{t+1}, \xi_{t+1}\right)\Pi_{t+1}^{-1}\left(1 + \rho S_{t+1}(\rho)\right)\right]$$

gives

$$\bar{S}\bar{u}_{CC}\bar{C}\hat{C}_t + \bar{S}\bar{u}_{C\xi}\hat{\xi}_t + \bar{u}_C\bar{S}\hat{S}_t = \beta\left(1+\rho\bar{S}\right)\bar{u}_{CC}E_t\bar{C}\hat{C}_{t+1} + \beta\left(1+\rho\bar{S}\right)\bar{u}_{C\xi}E_t\hat{\xi}_{t+1} - \beta\left(1+\rho\bar{S}\right)\bar{u}_C E_t\hat{\pi}_{t+1} + \beta\rho\bar{S}\bar{u}_C E_t\hat{S}_{t+1}$$

which can be simplifies by making use of the log-linearized resource constraint above to yield

$$\bar{S}\bar{u}_{CC}\hat{Y}_t + \bar{S}\bar{u}_{C\xi}\hat{\xi}_t + \bar{u}_C\bar{S}\hat{S}_t = \beta\left(1+\rho\bar{S}\right)\bar{u}_{CC}E_t\hat{Y}_{t+1} + \beta\left(1+\rho\bar{S}\right)\bar{u}_{C\xi}E_t\hat{\xi}_{t+1} - \beta\left(1+\rho\bar{S}\right)\bar{u}_C E_t\hat{\pi}_{t+1} + \beta\rho\bar{S}\bar{u}_C E_t\hat{S}_{t+1}.$$

Note here that by using the log-linearized Euler equation above, one can further simplify as

$$\bar{S}\left[\bar{u}_C\hat{\imath}_t + \bar{u}_{CC}E_t\hat{Y}_{t+1} + \bar{u}_{C\xi}E_t\hat{\xi}_{t+1} - \bar{u}_C E_t\hat{\pi}_{t+1}\right] + \bar{u}_C\bar{S}\hat{S}_t$$
$$= \beta\left(1+\rho\bar{S}\right)\bar{u}_{CC}E_t\hat{Y}_{t+1} + \beta\left(1+\rho\bar{S}\right)\bar{u}_{C\xi}E_t\hat{\xi}_{t+1} - \beta\left(1+\rho\bar{S}\right)\bar{u}_C E_t\hat{\pi}_{t+1} + \beta\rho\bar{S}\bar{u}_C E_t\hat{S}_{t+1}$$

or

$$\left[\bar{u}_C\hat{\imath}_t + \bar{u}_{CC}E_t\hat{Y}_{t+1} + \bar{u}_{C\xi}E_t\hat{\xi}_{t+1} - \bar{u}_C E_t\hat{\pi}_{t+1}\right] + \bar{u}_C\hat{S}_t$$
$$= \frac{\beta\left(1+\rho\bar{S}\right)}{\bar{S}}\bar{u}_{CC}E_t\hat{Y}_{t+1} + \frac{\beta\left(1+\rho\bar{S}\right)}{\bar{S}}\bar{u}_{C\xi}E_t\hat{\xi}_{t+1} - \frac{\beta\left(1+\rho\bar{S}\right)}{\bar{S}}\bar{u}_C E_t\hat{\pi}_{t+1} + \beta\rho\bar{u}_C E_t\hat{S}_{t+1}.$$

Moreover, since

$$\frac{\beta\left(1+\rho\bar{S}\right)}{\bar{S}} = 1$$

we have finally as the asset-pricing condition

$$\hat{\imath}_t + \hat{S}_t = \beta\rho E_t\hat{S}_{t+1}.$$

Fifth,

$$S_t(\rho)b_t = \left(1+\rho S_t(\rho)\right)b_{t-1}\Pi_t^{-1} + \left(F - T_t\right)$$

gives

$$\hat{b}_t + \hat{S}_t = \rho\hat{S}_t + \frac{\left(1+\rho\bar{S}\right)}{\bar{S}}\hat{b}_t - \frac{\left(1+\rho\bar{S}\right)}{\bar{S}}\hat{\pi}_t - \frac{\bar{T}}{\bar{S}\bar{b}}\hat{T}_t$$

which is simplified further as

$$\hat{b}_t = \beta^{-1}\hat{b}_t - \beta^{-1}\hat{\pi}_t - (1-\rho)\hat{S}_t - \frac{\bar{T}}{\bar{S}\bar{b}}\hat{T}_t.$$

Then, finally, the expectation functions are given by

$$\hat{f}_t^e = E_t\left[u_C\left(C_{t+1},\xi_{t+1}\right)\Pi_{t+1}^{-1}\right] = \bar{u}_{CC}E_t\hat{Y}_{t+1} + \bar{u}_{C\xi}E_t\hat{\xi}_{t+1} - \bar{u}_C\hat{\pi}_{t+1}$$

$$\hat{g}_t^e = E_t\left[u_C\left(C_{t+1},\xi_{t+1}\right)\Pi_{t+1}^{-1}\left(1+\rho S_{t+1}(\rho)\right)\right] = \left(1+\rho\bar{S}\right)\bar{u}_{CC}E_t\hat{Y}_{t+1} + \left(1+\rho\bar{S}\right)\bar{u}_{C\xi}E_t\hat{\xi}_{t+1} - \left(1+\rho\bar{S}\right)\bar{u}_C E_t\hat{\pi}_{t+1} + \rho\bar{S}\bar{u}_C E_t\hat{S}_{t+1}$$

$$\hat{h}_t^e = E_t\left[u_C\left(C_{t+1},\xi_{t+1}\right)d'\left(\Pi_{t+1}\right)\Pi_{t+1}\right] = \bar{u}_C d'' E_t\hat{\pi}_{t+1}$$

Markov-perfect FOCs Here, note that since all the Lagrange multipliers except one are zero in steady-state, what we mean by hats will in fact only be deviations from steady-state for all the Lagrange multipliers (as opposed to log-deviations).

First,

$$\phi_{1t}\left[\left(1+\rho S_t(\rho)\right)b_{t-1}\Pi_t^{-2}\right] + \phi_{4t}\left[-u_C d^{''}\Pi_t - u_C d^{'}\right] + \phi_{5t}\left[-d'\right] = 0$$

gives

$$\left(1+\rho\bar{S}\right)\bar{b}\hat{\phi}_{1t} - \bar{u}_C d^{''}\hat{\phi}_{4t} - \bar{\phi}_5 d''\hat{\pi}_t = 0$$

and since $\bar{\phi}_5 = \tilde{v}_y = \bar{u}_C$

$$\left(1+\rho\bar{S}\right)\bar{b}\hat{\phi}_{1t} - \bar{u}_C d^{''}\hat{\phi}_{4t} - \bar{u}_C d''\hat{\pi}_t = 0.$$

Second,

$$-\tilde{v}_Y + \phi_{4t}\left[-\varepsilon\left(\frac{\varepsilon-1}{\varepsilon}\left(1+s\right)u_C\right) + \varepsilon Y_t\tilde{v}_{yy} + \varepsilon\tilde{v}_y\right] + \phi_{5t} = 0$$

gives

$$-\bar{v}_{YY}\bar{Y}\hat{Y}_t + \left[\varepsilon\bar{Y}\bar{v}_{yy}\right]\hat{\phi}_{4t} + \hat{\phi}_{5t} = 0.$$

Third,

$$\phi_{2t}\left[u_C\left(1+i_t\right)^{-2}\right] + \gamma_{1t} = 0$$

gives

$$\bar{u}_C \beta^2 \hat{\phi}_{2t} + \hat{\gamma}_{1t} = 0.$$

Fourth,

$$\phi_{1t}\left[b_t - \rho b_{t-1}\Pi_t^{-1}\right] + \phi_{3t}\left[-u_C\right] = 0$$

gives

$$\left[(1-\rho)\,\bar{b}\right]\hat{\phi}_{1t} - \bar{u}_C\hat{\phi}_{3t} = 0.$$

Fifth,

$$u_C + \phi_{2t}\left[-u_{CC}\left(1+i_t\right)^{-1}\right] + \phi_{3t}\left[-u_{CC}S_t(\rho)\right] + \phi_{4t}\left[-\varepsilon Y_t \frac{\varepsilon-1}{\varepsilon}\left(1+s\right)u_{CC} - u_{CC}d'\Pi_t\right] + \phi_{5t}\left[-1\right] = 0$$

gives

$$\hat{Y}_t + \frac{\bar{u}_{C\xi}}{\bar{u}_{CC}}\hat{\xi}_t - \beta\hat{\phi}_{2t} - \bar{S}\hat{\phi}_{3t} - \varepsilon\hat{\phi}_{4t} - \frac{1}{\bar{u}_{CC}}\hat{\phi}_{5t} = 0.$$

Sixth,

$$g_G\left(-s'(T_t - \bar{T})\right) + \phi_{1t} = 0$$

gives

$$-\bar{g}_G s''\bar{T}\hat{T}_t + \hat{\phi}_{1t} = 0.$$

Seventh (after some replacements),

$$\beta E_t\phi_{1t+1}\left[-\left(1+\rho S_{t+1}(\rho)\right)\Pi_{t+1}^{-1}\right] + \phi_{1t}\left[S_t(\rho)\right] + \beta\phi_{2t}\left[\bar{f}_b^e\right] + \beta\phi_{3t}\left[\bar{g}_b^e\right] + \beta\phi_{4t}\left[\bar{h}_b^e\right] = 0$$

gives

$$-\bar{S}E_t\hat{\phi}_{1t+1} + \bar{S}\hat{\phi}_{1t} + \beta\bar{f}_b\hat{\phi}_{2t} + \beta\bar{g}_b\hat{\phi}_{3t} + \beta\bar{h}_b\hat{\phi}_{4t} = 0.$$

8.4 Linear-quadratic approach

8.4.1 Linear approximation of equilibrium conditions

We approximate around an efficient non-stochastic steady-state where $\Pi = 1$. For simplicity, from here on we will assume that the only shock that hits the economy is a discount factor shock given by ψ. Standard manipulations that are prevalent in the literature, for example in Woodford (2003), and as shown above in the Markov perfect equilibrium, give (22) and (23) where $\sigma = \tilde{\sigma}\frac{C}{Y}$ and $k = \varepsilon\frac{(\sigma^{-1}+\phi)}{d''}$. Here we again detail the derivations of (24) and (25). Given

$$S_t(\rho) = \frac{1}{u_C\left(C_t,\xi_t\right)}\beta E_t\left[u_C\left(C_{t+1},\xi_{t+1}\right)\Pi_{t+1}^{-1}\left(1+\rho S_{t+1}(\rho)\right)\right],$$

and

$$1+i_t = \frac{u_C\left(C_t,\xi_t\right)}{\beta E_t\left[u_C\left(C_{t+1},\xi_{t+1}\right)\Pi_{t+1}^{-1}\right]}$$

and the functional form assumptions above together with in steady state $1+i=\beta^{-1}$, log-linearization gives immediately

$$\hat{S}_t = -\hat{\imath}_t + \rho\beta E_t\hat{S}_{t+1}.$$

Next, given

$$S_t(\rho)b_t = \left(1+\rho S_t(\rho)\right)b_{t-1}\Pi_t^{-1} + \left(F_t - T_t\right)$$

and that we assume $F_t = F$ and have from steady state $S = \frac{\beta}{1-\rho\beta}$, $T = F + \frac{1-\beta}{1-\rho\beta}b$, log-linearization gives immediately

$$\hat{b}_t = \beta^{-1}\hat{b}_{t-1} - \beta^{-1}\pi_t - (1-\rho)\hat{S}_t - \psi\hat{T}_t.$$

Note that also the following relationship holds in steady state $F = G$. We finally derive an expression for r_t^e, the efficient rate of interest

$$r_t^e = \tilde{\sigma}^{-1}\left(\psi_t - E_t\psi_{t+1}\right).$$

8.4.2 Quadratic approximation of household utility

For household utility, we need to approximate the following three components

$$u\left(Y_t - F - d(\Pi_t), \xi_t\right); \; g\left(F - S(T_t - T), \xi_t\right); \; v\left(Y_t, \xi_t\right).$$

Standard manipulations that are prevalent in the literature, for example in Woodford (2003), give as a second-order approximation to household utility

$$\frac{1}{2}\frac{-\sigma\left(\hat{T}_t^2\left(F-\bar{Y}\right)s''\left(\bar{T}\right)+Fd''(1)\hat{\Pi}_t^2\right)+2\sigma\hat{Y}_t\left(-\bar{Y}+F\hat{\xi}_t+F\right)+\hat{Y}_t^2}{\sigma\left(F-\bar{Y}\right)}+$$
$$\frac{1}{2}\bar{Y}\left(\frac{d''(1)\hat{\Pi}_t^2}{F-\bar{Y}}+\hat{\xi}_t\left(\frac{2\sigma}{\sigma-1}-\frac{2\hat{Y}_t}{F-\bar{Y}}\right)+\frac{2\sigma}{\sigma-1}\right)-\frac{1}{2}2\lambda\left(\hat{\xi}_t+1\right)\hat{Y}_t\bar{Y}^{\phi}-\frac{1}{2}\frac{2\lambda\left(\hat{\xi}_t+1\right)\bar{Y}^{\phi+1}}{\phi+1}-\frac{1}{2}\lambda\phi\hat{Y}_t^2\bar{Y}^{\phi-1}+tip$$

which is in turn given as

$$\frac{1}{2}\left(\hat{Y}_t^2\left(\frac{1}{F\sigma-\sigma\bar{Y}}-\lambda\phi\bar{Y}^{\phi-1}\right)+\hat{T}_t^2\left(-s''\left(\bar{T}\right)\right)-2\left(\hat{\xi}_t+1\right)\hat{Y}_t\left(\lambda\bar{Y}^{\phi}-1\right)-d''(1)\hat{\Pi}_t^2\right)+tip.$$

Now lets multiply everything by $\frac{1}{\phi+\tilde{\sigma}^{-1}}$ and consider efficient equilibrium in steady-state ($u_C = \tilde{v}_Y$), together with the scaling that $\lambda\bar{Y}^{\phi} = 1$ and $\bar{Y} = \bar{C} + F = 1$, to get

$$-\frac{\tilde{\sigma}\hat{T}_t^2 s''\left(\bar{T}\right)}{2\left(\phi\tilde{\sigma}+1\right)}-\frac{\tilde{\sigma}d''(1)\hat{\Pi}_t^2}{2\left(\phi\tilde{\sigma}+1\right)}-\frac{\hat{Y}_t^2}{2}$$

So, finally, we get as approximation

$$-\left[\lambda_\pi\pi_t^2+\hat{Y}_t^2+\lambda_T\hat{T}_t^2\right]$$

where

$$\lambda_T = \frac{s''\left(\bar{T}\right)}{\phi+\tilde{\sigma}^{-1}}$$

$$\lambda_\pi = \frac{d''(1)}{\left(\phi+\tilde{\sigma}^{-1}\right)} = \frac{\varepsilon}{\kappa}.$$

8.4.3 Markov-perfect equilibrium at positive interest rates

Given the Lagrangian where the expectation functions are substituted and the shocks are suppressed

$$L_t = \frac{1}{2}(\lambda_\pi\pi_t^2+\hat{Y}_t^2+\lambda_T\hat{T}_t^2)+\beta E_t V(b_t, r_{t+1}^e)+\phi_{1t}[\hat{Y}_t - Y_b b_t+\sigma\hat{\imath}_t-\sigma\pi_b b_t-\sigma r_t^e]$$
$$+\phi_{2t}[\pi_t-\kappa\hat{Y}_t-\beta\pi_b b_t]+\phi_{3t}[b_t-\beta^{-1}b_{t-1}+\beta^{-1}\pi_t+(1-\rho)\hat{S}_t+\psi\hat{T}_t]+\phi_{4t}[\hat{S}_t+\hat{\imath}_t-\rho\beta S_b b_t]$$

the first-order necessary conditions are given by

$$\frac{\partial L}{\partial \pi_t} = \lambda_\pi\pi_t+\phi_{2t}+\beta^{-1}\phi_{3t} = 0$$

$$\frac{\partial L}{\partial Y_t} = Y_t+\phi_{1t}-\kappa\phi_{2t} = 0$$

$$\frac{\partial L}{\partial T_t} = \lambda_T T_t + \psi\phi_{3t} = 0$$
$$\frac{\partial L}{\partial i_t} = \sigma\phi_{1t} + \phi_{4t} = 0$$
$$\frac{\partial L}{\partial S_t} = \phi_{3t}(1-\rho) + \phi_{4t} = 0$$
$$\frac{\partial L}{\partial b_t} = \beta E_t V_b(b_t, r^e_{t+1}) - (Y_b + \sigma\pi_b)\phi_{1t} - \beta\pi_b\phi_{2t} + \phi_{3t} - \rho\beta S_b\phi_{4t} = 0$$

while the envelope condition is given by

$$V_b(b_{t-1}, r^e_t) = -\beta^{-1}\phi_{3t}$$

which implies

$$E_t V_b(b_t, r^e_{t+1}) = -\beta^{-1}E_t\phi_{3t+1}.$$

We can then combine the envelope condition with the last FOC to yield

$$-E_t\phi_{3t+1} - (Y_b + \sigma\pi_b)\phi_{1t} - \beta\pi_b\phi_{2t} + \phi_{3t} - \rho\beta S_b\phi_{4t} = 0.$$

To summarize, we have

$$\lambda_\pi\pi_t + \phi_{2t} + \beta^{-1}\phi_{3t} = 0$$

$$Y_t + \phi_{1t} - \kappa\phi_{2t} = 0$$
$$\lambda_T T_t + \psi\phi_{3t} = 0$$
$$\sigma\phi_{1t} + \phi_{4.t} = 0$$
$$\phi_{3t}(1-\rho) + \phi_{4t} = 0$$
$$-E_t\phi_{3t+1} - (Y_b + \sigma\pi_b)\phi_{1t} - \beta\pi_b\phi_{2t} + \phi_{3t} - \rho\beta S_b\phi_{4t} = 0$$
$$\hat{Y}_t = E_t\hat{Y}_{t+1} - \sigma(\hat{\imath}_t - E_t\pi_{t+1} - r^e_t)$$
$$\pi_t = \kappa\hat{Y}_t + \beta E_t\pi_{t+1}$$
$$b_t = \beta^{-1}b_{t-1} - \beta^{-1}\pi_t - (1-\rho)\hat{S}_t - \psi\hat{T}_t$$
$$\hat{S}_t = -\hat{\imath}_t + \rho\beta E_t\hat{S}_{t+1}$$

which can be simplified to get

$$\lambda_\pi\pi_t + \kappa^{-1}Y_t = [\kappa^{-1}(1-\rho)\sigma^{-1} + \beta^{-1}]\frac{bS}{T}\lambda_T\hat{T}_t$$
$$[1 - \beta\pi_b\kappa^{-1}(1-\rho)\sigma^{-1} - (Y_b + \sigma\pi_b)(1-\rho)\sigma^{-1} + \rho\beta S_b(1-\rho)]\hat{T}_t = -(\psi)\lambda_T^{-1}\beta\pi_b\kappa^{-1}Y_t + E_t\hat{T}_{t+1}$$
$$\hat{Y}_t = E_t\hat{Y}_{t+1} - \sigma(\hat{\imath}_t - E_t\pi_{t+1} - r^e_t)$$
$$\pi_t = \kappa\hat{Y}_t + \beta E_t\pi_{t+1}$$
$$b_t = \beta^{-1}b_{t-1} - \beta^{-1}\pi_t - (1-\rho)\hat{S}_t - \psi\hat{T}_t$$
$$\hat{S}_t = -\hat{\imath}_t + \rho\beta E_t\hat{S}_{t+1}.$$

The final step is then to match coefficients after replacing the conjectured solutions

$$\lambda_\pi\pi_b b_{t-1} + \kappa^{-1}(Y_b b_{t-1}) = [\kappa^{-1}(1-\rho)\sigma^{-1} + \beta^{-1}]\frac{bS}{T}\lambda_T(T_b b_{t-1})$$
$$[1 - \beta\pi_b\kappa^{-1}(1-\rho)\sigma^{-1} - (Y_b + \sigma\pi_b)(1-\rho)\sigma^{-1} + \rho\beta S_b(1-\rho)][T_b b_{t-1}] = -(\psi)\lambda_T^{-1}\beta\pi_b\kappa^{-1}[Y_b b_{t-1}] + [T_b(b_b b_{t-1})]$$
$$Y_b b_{t-1} = Y_b[b_b b_{t-1}] - \sigma((i_b b_{t-1}) - [\pi_b[b_b b_{t-1}]])$$

$$\pi_b b_{t-1} = \kappa \left[Y_b b_{t-1}\right] + \beta \pi_b \left[b_b b_{t-1}\right]$$

$$b_b b_{t-1} = \beta^{-1} b_{t-1} - \beta^{-1} \left[\pi_b b_{t-1}\right] - (1-\rho) \left[S_b b_{t-1}\right] - \psi \left[T_b b_{t-1}\right]$$

$$S_b b_{t-1} = -\left(i_b b_{t-1}\right) + \rho \beta S_b \left[b_b b_{t-1}\right]$$

which in turn can be simplified to get

$$\lambda_\pi \pi_b + \kappa^{-1} Y_b = [\kappa^{-1}(1-\rho)\sigma^{-1} + \beta^{-1}] \frac{bS}{T} \lambda_T T_b \tag{43}$$

$$\left[(1-\rho)^{-1} - \beta \pi_b \kappa^{-1} \sigma^{-1} - (Y_b + \sigma \pi_b)\sigma^{-1} + \rho \beta S_b\right] (1-\rho) T_b = -(\psi) \lambda_T^{-1} \beta \pi_b \kappa^{-1} Y_b + T_b b_b \tag{44}$$

$$Y_b = Y_b b_b - \sigma(i_b - \pi_b b_b) \tag{45}$$

$$\pi_b = \kappa Y_b + \beta \pi_b b_b \tag{46}$$

$$b_b = \beta^{-1} - \beta^{-1} \pi_b - (1-\rho) S_b - \psi T_b \tag{47}$$

$$S_b = -i_b + \rho \beta S_b b_b. \tag{48}$$

We now show some properties of π_b analytically. First, note that we will be restricting to stationary solutions, that is one where $\mid b_b \mid < 1$. Manipulations of (43)-(48) above lead to the following closed-form expression for π_b

$$\pi_b = \frac{(1-\beta b_b)}{\beta \left[\frac{T}{lV} \chi \left[\lambda_\pi + \kappa^{-1} \kappa^{-1} (1-\beta b_b)\right] + \left[(1-\rho)\sigma^{-1} \kappa^{-1} (1-b_b) + \beta^{-1}\right] (1-\beta b_b) (1-\rho \beta b_b)^{-1}\right]}$$

$$\text{where } \chi = \left[[\kappa^{-1} \sigma^{-1} (1-\rho) + \beta^{-1}] \frac{bS}{T} \lambda_T \right]^{-1}.$$

Since $\mid b_b \mid < 1$, it is clear that $\pi_b > 0$ for all $\rho < 1$. What happens when$\rho > 1$? Numerically, we have found that often $\pi_b > 0$ still, but sometimes in fact it can hit zero (and then actually go negative if ρ is increased further). It is in fact possible to pin-down analytically when $\pi_b = 0$. Note that from (46), if $\pi_b = 0$, then $Y_b = 0$. This then implies that for this to be supported as a solution for all possible values of T_b, from (43), it must be the case that

$$[\kappa^{-1}(1-\rho)\sigma^{-1} + \beta^{-1}] = 0$$

which in turn implies that

$$\rho = 1 + \beta^{-1} \sigma \kappa.$$

In this knife-edge case, note that one needs $\rho > 1$. Moreover, since the upper bound on ρ is β^{-1}, one needs to ensure that

$$1 + \beta^{-1} \sigma \kappa < \beta^{-1} \text{ or } \sigma \kappa < 1 - \beta.$$

This is a fairly restrictive parameterization (not fulfilled in our baseline, for example). Still, it is instructive to note that in this case of when π_b does reach 0, then it is indeed possible to show analytically that π_b declines as duration is increased while comparing $\rho = 0$ with $\rho = 1 + \beta^{-1} \sigma \kappa$ or $\rho = 1$ with $\rho = 1 + \beta^{-1} \sigma \kappa$.

8.5 Equivalence of the two approaches

We now show the equivalence of the linearized dynamic systems for non-linear and linear-quadratic approaches.

Consider the system of equations describing private sector equilibrium and optimal government policy

$$\hat{Y}_t = E_t \hat{Y}_{t+1} + \frac{\bar{u}_C}{\bar{u}_{CC}} \left[\hat{\imath}_t - E_t \hat{\pi}_{t+1} - r_t^e\right]$$

$$\hat{\pi}_t = \frac{\varepsilon \left(\bar{v}_{yy} - \bar{u}_{CC}\right)}{\bar{u}_C d''} \hat{Y}_t + \beta E_t \hat{\pi}_{t+1}.$$

$$\hat{\imath}_t + \hat{S}_t = \beta \rho E_t \hat{S}_{t+1}.$$

$$\hat{b}_t = \beta^{-1}\hat{b}_t - \beta^{-1}\hat{\pi}_t - (1-\rho)\,\hat{S}_t - \frac{\bar{T}}{\bar{S}\bar{b}}\hat{T}_t.$$

$$\left(1+\rho\bar{S}\right)\bar{b}\hat{\phi}_{1t} - \bar{u}_C d^{''}\hat{\phi}_{4t} - \bar{\phi}_5 d''\hat{\pi}_t = 0$$

$$-\bar{v}_{YY}\overline{Y}\hat{Y}_t + \left[\varepsilon\bar{Y}\bar{v}_{yy}\right]\hat{\phi}_{4t} + \hat{\phi}_{5t} = 0$$

$$\bar{u}_C\beta^2\hat{\phi}_{2t} + \hat{\gamma}_{1t} = 0$$

$$\left[(1-\rho)\,\bar{b}\right]\hat{\phi}_{1t} - \bar{u}_C\hat{\phi}_{3t} = 0.$$

$$\hat{Y}_t + \frac{\bar{u}_{C\xi}}{\bar{u}_{CC}}\hat{\xi}_t - \beta\hat{\phi}_{2t} - \bar{S}\hat{\phi}_{3t} - \varepsilon\hat{\phi}_{4t} - \frac{1}{\bar{u}_{CC}}\hat{\phi}_{5t} = 0.$$

$$-\bar{g}_G s^{''}\bar{T}\hat{T}_t + \hat{\phi}_{1t} = 0.$$

$$-\bar{S}E_t\hat{\phi}_{1t+1} + \bar{S}\hat{\phi}_{1t} + \beta\bar{f}_b\hat{\phi}_{2t} + \beta\bar{g}_b\hat{\phi}_{3t} + \beta\bar{h}_b\hat{\phi}_{4t} = 0.$$

Under additional functional assumptions outlined in previous sections we get

$$\bar{v}_{YY} = \phi\overline{Y}^{-1} = \phi$$

$$\bar{u}_{CC} = -\sigma^{-1}\bar{C}^{-1} = -\tilde{\sigma}^{-1}$$

$$\bar{u}_{C\xi} = 1$$
$$\bar{u}_C = 1$$

$$\bar{g}_G = 1$$

$$\bar{\phi}_5 = 1$$

$$\bar{Y} = 1$$

The first four equations are equivalent to their counterparts in the LQ-approach once one use the functional form assumptions to get

$$\frac{\bar{u}_C}{\bar{u}_{CC}} = -\tilde{\sigma}$$

and introduce new notation

$$\frac{\varepsilon\left(\bar{v}_{yy} - \bar{u}_{CC}\right)}{\bar{u}_C d''} = \kappa.$$

The latter relation implies

$$\frac{\varepsilon\left(\phi + \tilde{\sigma}^{-1}\right)}{\kappa} = d''.$$

Let us guess solutions for all variables for the case when the ZLB is slack as a linear function of $\hat{b}_{t-1}$ and $\hat{r}^e_t$.

Then the expectations will take form

$$\hat{f}_t^e = \bar{u}_{CC} E_t \hat{Y}_{t+1} + \bar{u}_{C\xi} E_t \hat{\xi}_{t+1} - \bar{u}_C E_t \hat{\pi}_{t+1} = \hat{f}_b^e \bar{b} \hat{b}_{t-1} + \hat{f}_r^e \ \hat{r}_t^e$$

$$\hat{g}_t^e = \left(1+\rho\bar{S}\right) \bar{u}_{CC} E_t \hat{Y}_{t+1} + \left(1+\rho\bar{S}\right) \bar{u}_{C\xi} E_t \hat{\xi}_{t+1} - \left(1+\rho\bar{S}\right) \bar{u}_C E_t \hat{\pi}_{t+1} + \rho\bar{S}\bar{u}_C E_t \hat{S}_{t+1} = \hat{g}_b^e \bar{b} \hat{b}_{t-1} + \hat{g}_r^e \ \hat{r}_t^e$$

$$\hat{h}_t^e = \bar{u}_C d'' E_t \hat{\pi}_{t+1} = \hat{h}_b^e \bar{b} \hat{b}_{t-1} + \hat{h}_r^e \ \hat{r}_t^e$$

where

$$\hat{f}_b^e = -\bar{b}^{-1}\left(\tilde{\sigma}^{-1} Y_b + \pi_b\right)$$

$$\hat{g}_b^e = -\tilde{\sigma}^{-1}\bar{b}^{-1}\left(1+\rho\bar{S}\right) Y_b - \left(1+\rho\bar{S}\right)\bar{b}^{-1}\pi_b + \rho\bar{S}S_b\bar{b}^{-1}$$

$$\hat{h}_b^e = \frac{\varepsilon\left(\phi+\tilde{\sigma}^{-1}\right)}{\kappa}\pi_b \bar{b}^{-1}$$

Also under the assumption about the process $\hat{\xi}_t$ we get

$$E_t \hat{\xi}_{t+1} = \mu \hat{\xi}_t$$

and

$$\hat{r}_t^e = -\frac{\bar{u}_{C\xi}}{\bar{u}_C}\left[\mu\hat{\xi}_t - \hat{\xi}_t\right] = \hat{\xi}_t(1-\mu).$$

When the economy is out of ZLB $\hat{r}_t^e = 0$.

Now we can find explicit representation for the Lagrange multipliers

$$\hat{\phi}_{1t} = s^{''}\bar{T}\hat{T}_t.$$

$$\hat{\phi}_{2t} = 0$$

$$\hat{\phi}_{3t} = \left[(1-\rho)\right] s^{''}\bar{T}\hat{T}_t.$$

$$\hat{\phi}_{4t} = \frac{\kappa}{\varepsilon\left(\phi+\tilde{\sigma}^{-1}\right)}\left(1+\rho\bar{S}\right)\bar{b}s^{''}\bar{T}\hat{T}_t - \hat{\pi}_t$$

$$\hat{\phi}_{5t} = \phi\hat{Y}_t - \phi\frac{\kappa}{\left(\phi+\tilde{\sigma}^{-1}\right)}\left(1+\rho\bar{S}\right)s^{''}\bar{T}\hat{T}_t + \varepsilon\phi\hat{\pi}_t$$

So the last two equations take the form

$$\varepsilon\hat{\pi}_t = -\hat{Y}_t + \frac{1}{\left(\phi+\tilde{\sigma}^{-1}\right)}\left[\tilde{\sigma}^{-1}\bar{S}\left(1-\rho\right) + \kappa\left(1+\rho\bar{S}\right)\right]\bar{b}s^{''}\bar{T}\hat{T}_t.$$

$$\bar{S}s^{''}\bar{T}\hat{T}_t + \left(\rho\beta S_b - \tilde{\sigma}^{-1}Y_b - \pi_b\right)\bar{b}^{-1}\left(1-\rho\right)\bar{S}\bar{b}s^{''}\bar{T}\hat{T}_t + \bar{b}^{-1}\pi_b\bar{S}\bar{b}s^{''}\bar{T}\hat{T}_t - \beta\frac{\varepsilon\left(\phi+\tilde{\sigma}^{-1}\right)}{\kappa}\pi_b\bar{b}^{-1}\hat{\pi}_t = s^{''}\bar{T}\bar{S}E_t\hat{T}_{t+1}.$$

which after straightforward manipulations become

$$\frac{\varepsilon}{\kappa}\hat{\pi}_t + \hat{Y}_t\kappa^{-1} = \left[\tilde{\sigma}^{-1}\left(1-\rho\right)\kappa^{-1} + \beta^{-1}\right]\frac{\bar{b}\bar{S}}{\bar{T}}\frac{\bar{T}^2 s^{''}}{\left(\phi+\tilde{\sigma}^{-1}\right)}\hat{T}_t.$$

$$\left(1+\left(\rho\beta S_b - \tilde{\sigma}^{-1}Y_b - \pi_b\right)\left(1-\rho\right) + \pi_b\right)\frac{\bar{S}\bar{b}}{\bar{T}}\frac{s^{''}\bar{T}^2}{\left(\phi+\tilde{\sigma}^{-1}\right)}\hat{T}_t - \beta\frac{\varepsilon}{\kappa}\pi_b\hat{\pi}_t = \frac{s^{''}\bar{T}^2}{\left(\phi+\tilde{\sigma}^{-1}\right)}\frac{\bar{b}\bar{S}}{\bar{T}}E_t\hat{T}_{t+1}.$$

These two equations are equivalent for the last two equations from the dynamic system obtained in LQ–approach once we use the derived weights from the quadratic approximation of the loss function

$$\lambda_T = \frac{s^{''}\bar{T}^2}{\left(\phi+\tilde{\sigma}^{-1}\right)}$$

$$\lambda_\pi = \kappa^{-1}\varepsilon.$$

8.6 Details of the solution at positive interest rates

Provided that the expectation functions are differentiable, the solution of the model is of the form

$$\begin{aligned}\pi_t &= \pi_b \hat{b}_{t-1} + \pi_r r_t^e,\ \hat{Y}_t = Y_b \hat{b}_{t-1} + Y_r r_t^e,\ \hat{S}_t = S_b \hat{b}_{t-1} + S_r r_t^e, \\ \hat{\imath}_t &= i_b \hat{b}_{t-1} + i_r r_t^e,\ \hat{T}_t = T_b \hat{b}_{t-1} + T_r r_t^e,\ \text{and } \hat{b}_t = b_b \hat{b}_{t-1} + b_r r_t^e\end{aligned} \tag{49}$$

where $\pi_b,\ Y_b,\ S_b,\ i_b, b_b, T_b,\ \pi_r,\ Y_r,\ S_r,\ i_r,\ b_r$, and T_r are unknown coefficients to be determined. We make the assumption that the exogenous process r_t^e satisfies $E_t r_{t+1}^e = \rho_r r_t^e$ where $0 < \rho_r < 1$. Then (49) implies that the expectations are given by

$$E_t \pi_{t+1} = \pi_b \hat{b}_t + \pi_r \rho_r r_t^e,\ E_t \hat{Y}_{t+1} = Y_b \hat{b}_t + Y_r \rho_r r_t^e,\ \text{and } E_t \hat{S}_{t+1} = S_b \hat{b}_t + S_r \rho_r r_t^e. \tag{50}$$

We can then formulate the Lagrangian of the government problem. We substitute out for the expectation function using (50) and suppress the shock for simplicity

$$\begin{aligned}L_t &= \frac{1}{2}(\lambda_\pi \pi_t^2 + \hat{Y}_t^2 + \lambda_T \hat{T}_t^2) + \beta E_t V(\hat{b}_t, r_{t+1}^e) \\ &+ \phi_{1t}[\hat{Y}_t - Y_b \hat{b}_t + \sigma \hat{\imath}_t - \sigma \pi_b \hat{b}_t] + \phi_{2t}[\pi_t - \kappa \hat{Y}_t - \beta \pi_b \hat{b}_t] \\ &+ \phi_{3t}[\hat{b}_t - \beta^{-1}\hat{b}_{t-1} + \beta^{-1}\pi_t + (1-\rho)\hat{S}_t + \psi \hat{T}_t] + \phi_{4t}[\hat{S}_t + \hat{\imath}_t - \rho\beta S_b \hat{b}_t].\end{aligned}$$

For now, we are not analyzing the effects of the shock, and hence not carrying around π_r, Y_r, S_r, i_r, b_r, and T_r since our key area of interest at this state is not the effect of the shock at positive interest rates. Instead, we will start focusing on the shock once the zero bound becomes binding.

The associated first order necessary conditions of the Lagrangian problem above and the envelope condition of the minimization problem of the government above are provided in the appendix. Our first substantiative result is that the equilibrium conditions can be simplified, in particular by eliminating the Lagrange multipliers $\phi_{1t} - \phi_{4t}$, to get

$$\lambda_\pi \pi_t + \kappa^{-1}\hat{Y}_t = [\kappa^{-1}(1-\rho)\sigma^{-1} + \beta^{-1}]\frac{1}{\psi}\lambda_T \hat{T}_t \tag{51}$$

$$[1 - \beta\pi_b\kappa^{-1}(1-\rho)\sigma^{-1} - (Y_b + \sigma\pi_b)(1-\rho)\sigma^{-1} + \rho\beta S_b(1-\rho)]\hat{T}_t = -(\psi)\lambda_T^{-1}\beta\pi_b\kappa^{-1}\hat{Y}_t + E_t\hat{T}_{t+1} \tag{52}$$

which along with (22)-(25) define the equilibrium in the approximated economy. The final step to computing the solution is then to plug in the conjectured solution and to match coefficients on various variables in (22)-(25), (51), and (52), along with the requirement that expectation are rational, to determine π_b, Y_b, S_b, i_b, b_b, and T_b. The details of this step are in the appendix.

The most important relationships emerging from our analysis are captured by (51) and (52). (51) is the so called "targeting rule" of our model. That represents the equilibrium (static) relationship among the three target variables π_t, $\hat{Y}_t$, and $\hat{T}_t$ that emerges from the optimization problem of the government. (51) thus captures how target variables are related in equilibrium as governed both by the weights they are assigned in the loss function (λ_π and λ_T) as well as the trade-offs among them as given by the private sector equilibrium conditions (κ^{-1} and $[\kappa^{-1}(1-\rho)\sigma^{-1} + \beta^{-1}]\frac{1}{\psi}$). Note in particular that κ^{-1} represents the trade-off between π_t and $\hat{Y}_t$ as given by (23) while $[\kappa^{-1}(1-\rho)\sigma^{-1} + \beta^{-1}]\frac{1}{\psi}$ represents the trade-off between π_t and $\hat{Y}_t$ vs. $\hat{T}_t$ as given by the combination of (22), (23), and (24).

(52) is another optimality condition characterizing the Markov-perfect equilibrium and represents the "tax-smoothing objective" of the government. In contrast to similar expressions following the work of Barro (1979), which would lead to taxes being a martingale, output appears in (52) because of sticky-prices, which makes output endogenous. Finally, because of the dynamic nature of (52), as opposed to (51), the unknown coefficients that are critical for expectations of variables, π_b, Y_b, and S_b, appear in (52). As we shall see – and this is again in contrast to

the classic tax smoothing result in which debt is a random walk – the government will in general have an incentive to pay down public debt if it is above steady-state due to strategic reasons.

8.6.1 Polar cases on price rigidities

To clarify what is going on in the model, we first find it helpful to consider two special cases.

a) Fully flexible prices When prices are fully flexible, then as is well-known, monetary policy cannot control the (ex-ante) real interest rate $\hat{r}_t$. Then, the only way monetary policy can affect the economy is through surprise inflation as debt is nominal. In fact there is a well-known literature that addresses the issue of how the duration of nominal debt in a flexible price environment affects allocations under time consistent optimal monetary policy. For example, Calvo and Guidotti (1990 and 1992) address optimal maturity of nominal government debt in a flexible price environment while Sims (2013) explores how the response of inflation to fiscal shocks depends on the maturity of government debt under optimal monetary policy. We conduct a complementary exercise here and want to characterize how inflation incentives depend on the duration of debt under optimal monetary and fiscal policy under discretion. Thus, we are interested in how π_b depends on duration of debt.

For this exercise, we can think of this special case of fully flexible prices as $\kappa \to \infty$. Note however, that from the two optimality conditions (51) and (52), while under flexible prices $T_b = 0$, there is indeterminacy in terms of inflation and nominal interest rate dynamics.[44] This is a well-known result in monetary economics under discretion in a flexible price environment, and comes about because the government cannot affect output and taxes can be put to zero with various combinations of inflation and interest rate choices. To show our result on the role of the duration of debt, we follow the literature such as Calvo and Guidotti (1990 and 1992) and Sims (2013) and include a (very small) aggregate social cost of inflation that is independent of the level of price stickiness. Then, the objective of the government under flexible prices will be given by

$$U_t = -\left[\lambda_\pi^{'}\pi_t^2 + \lambda_T \hat{T}_t^2\right]$$

where $\lambda_\pi^{'}$ parameterizes the cost of inflation that is independent of sticky prices. Using $\lambda_\pi^{'} = 0.001$ and the rest of the parameter values from Table 1, Fig A1 shows how π_b depends on duration of debt (quarters). We see that with shorter duration, there is more of an incentive to use current inflation. The intuition for this result is that from (24), we see that everything else the same, when ρ (and thereby, duration) decreases, then there will be a greater period by period incentive to increase $\hat{S}_t$, that is, keep nominal interest rates low to manage the debt burden. This, however, will increase current inflation further *in equilibrium* according to our result. As a consequence – and perhaps a little counterintuitively – equilibrium nominal interest rate will generally *increase* as well with lower duration since the real rate is exogenously given under flexible prices. In particular, observe that $\hat{\imath}_t = E_t\pi_{t+1}$ (since $\hat{Y}_t = 0$ and there

[44] Note that $\lambda_\pi = \frac{\varepsilon}{k}$.

are no shocks) and hence i_t is increasing one-to-one with expected inflation.

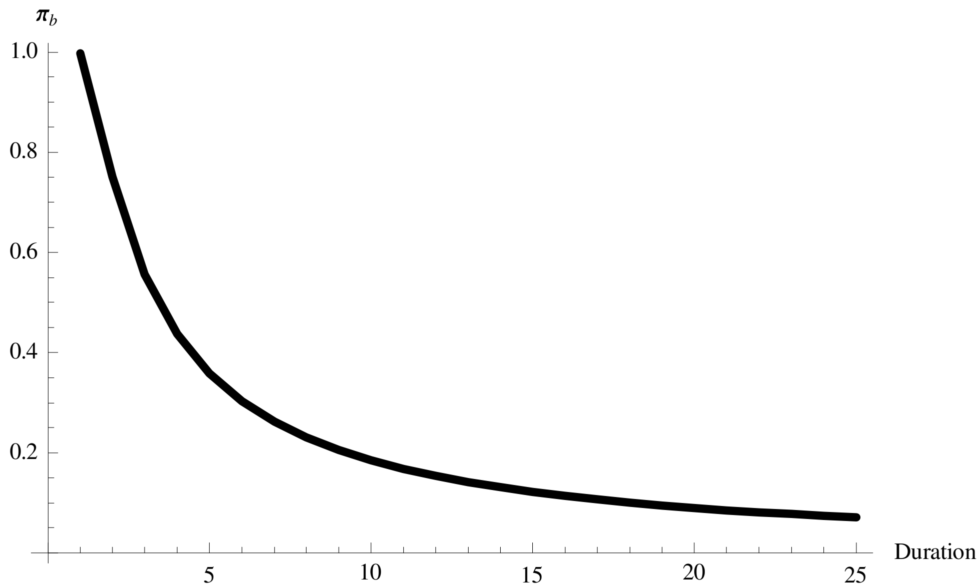

Fig A1: Inflation response coefficients at different duration of debt under flexible prices

b) Fully rigid prices Next we can consider the other extreme case: that of fully rigid prices. In this case, inflation is zero in equilibrium and hence $\pi_b = 0$. Then, one can directly consider the effects on the ex-ante real interest rate by analyzing the effect on the nominal interest rate since $r_b = i_b - \pi_b b_b = i_b$. Thus, we are interested in how i_b depends on the duration of debt. Using the parameters from Table 1, Fig A2 shows how i_b depends on duration of debt (quarters). We see that with shorter duration, there is more of an incentive to keep the nominal interest rate lower. The intuition for this result is again that from (24), when ρ (and thereby, duration) decreases, then there will be more of an incentive to increase $\hat{S}_t$, that is keep interest rates low, to manage the debt burden.

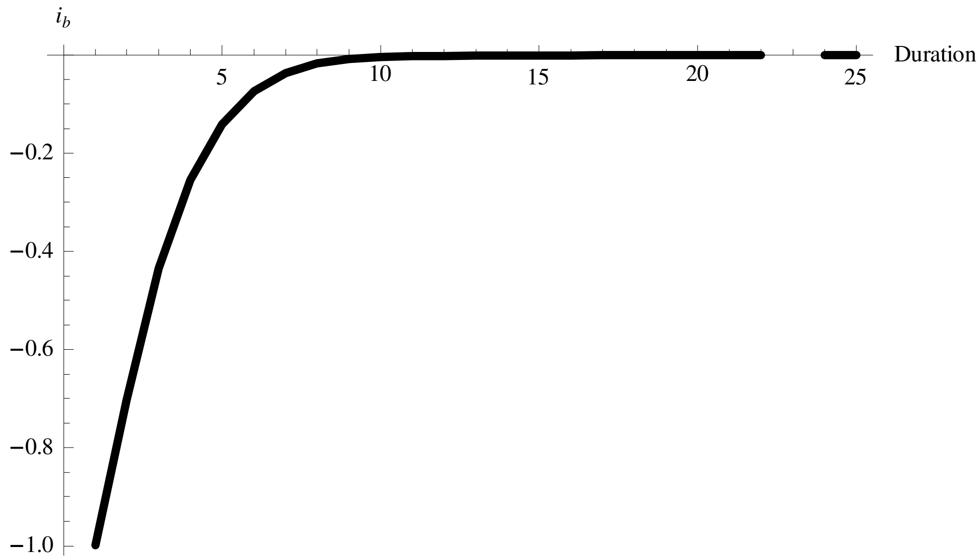

Fig A2: Nominal interest rate response coefficients at different duration of debt under rigid prices

We now move to discussing in detail properties of the solution for the partially rigid price cases that underlies our baseline calibration.

8.6.2 Real interest rate incentives and duration of outstanding debt

How does the duration of debt affect the real interest rate incentives of the central bank for the partially rigid cases that underlies our baseline calibration? The effect on the real interest rate of lower duration is at the heart of the matter because what influences output is eventually the real interest rate and the aim in a zero lower bound situation is precisely to be able to decrease the short-term real interest rate (today and in future) even when the current short-term nominal interest rate is stuck at zero. That is, we are interested in the properties of

$$\hat{r}_t = \hat{\imath}_t - E_t\pi_{t+1} = (i_b - \pi_b b_b)\,\hat{b}_{t-1} = r_b\hat{b}_{t-1}$$

where $\hat{r}_t$ is the short-term real interest rate. This is important because in a liquidity trap situation, as is well-known, decreasing future real interest rate is key to mitigating negative effects on output, and so, if r_b depends negatively on duration, then we are able to provide a theoretical rationale for quantitative easing actions by the government.

Having established the results in the two special cases above, we now move on to the main mechanism in our paper in the intermediate and quantitatively relevant case of partially rigid prices. Fig A3 shows how r_b depends on duration of debt (quarters). When the duration of debt in the hands of the public is shorter, it unambiguously provides an incentive to the government to keep the short-term real interest rate lower in future. The intuition again is that by doing so, it reduces the cost of rolling over the debt period by period. That is, if the debt is short-term, then current real short rates will more directly affect the cost of rolling the debt over period by period, while the cost of rolling over long-term debt is not affected in the same way period by period.

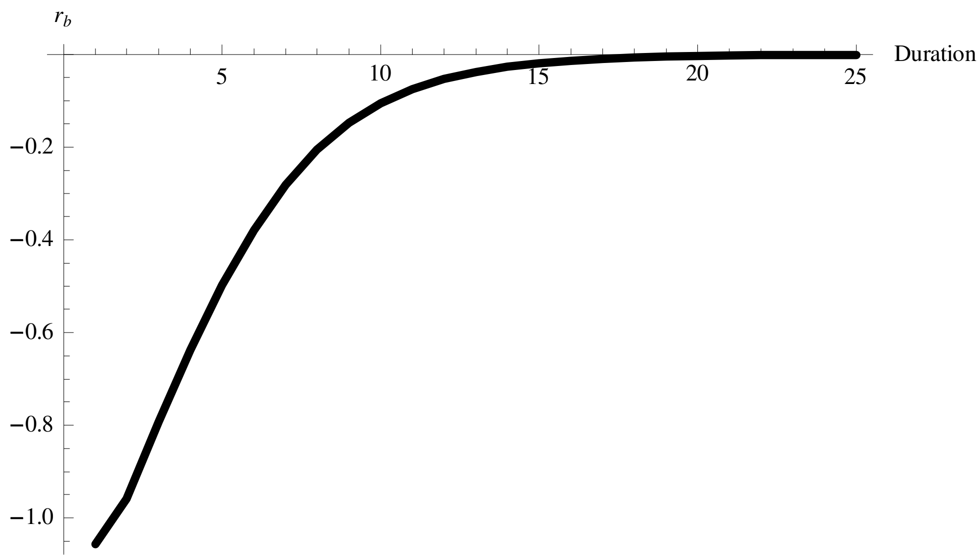

Fig A3: Real interest rate response coefficients at different duration of debt

8.6.3 Inflation incentives and duration of outstanding debt

While the dependence of real interest rate incentive of the government on the duration of debt is the main mechanism of our paper, given the attention inflation incentives receive in the literature, we now study how π_b varies with the duration of outstanding government debt?[45] Moreover, it helps us emphasize a point that just focusing on inflation incentives might not be sufficient to understand the nature of optimal monetary and fiscal policy in a liquidity trap situation.

While an analytical expression for π_b is available in the appendix and it is possible to show that $\pi_b > 0$ for all $\rho < 1$, a full analytical characterization of the comparative statics with respect to the duration of debt is not available and so we rely on numerical results. Fig A4 below shows how π_b depends on duration of debt (quarters) at different levels of κ. As expected, π_b is positive at all durations in the figure.[46] More importantly, note that that at our baseline parameterization of $\kappa = 0.02$, π_b does decrease with duration over a wide range of maturity.[47] At the same time, however, there is a hump-shaped behavior, with π_b increasing when duration is increased at very short durations.

[45] For some recent discussion and analysis of how inflation dynamics in sticky-price models could depend on duration of government debt, see Sims (2011).

[46] Again, it is possible to show analytically that for all $\rho < 1$, $\pi_b > 0$. In a very similar model but one with no steady-state debt, Eggertsson (2006) proved that $\pi_b > 0$ for $\rho = 0$ (that is, for one period debt).

[47] Some limited analytical results on the properties of π_b with respect to ρ are available in the appendix. For example, it can be shown that π_b is positive for all $\rho < 1$ and that when $\rho = 1 + \beta^{-1}\sigma\kappa$ (and thus, $\rho > 1$), $\pi_b = 0$. In this sense, for a specific case, one can show that π_b is declining in ρ by comparing some extreme cases (such as $\rho = 0$ with $\rho = 1 + \beta^{-1}\sigma\kappa$). Please see the appendix for details. Note also here that the upper bound on ρ is β^{-1}. So this case of $\pi_b = 0$ is not necessarily always reached.

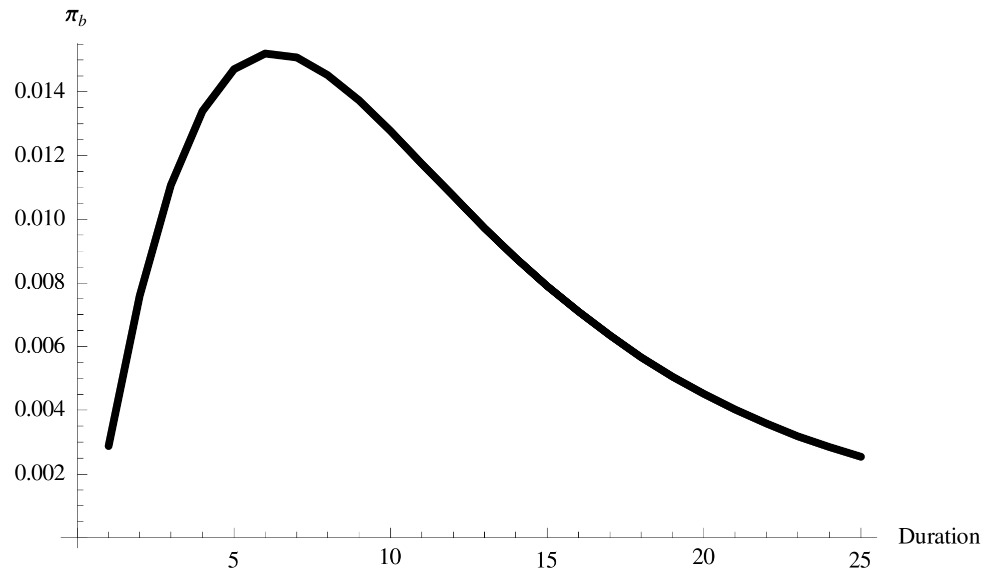

Fig A4: Inflation response coefficients at different duration of debt

What drives this result? First, note from (24) that everything else the same, when ρ (and thereby, duration) decreases, then there will be an incentive to decrease $\hat{S}_t$, that is keep interest rates low to manage the debt burden. This will then increase inflation in equilibrium. At the same time however, the government's incentives on inflation are not fully/only captured by this reasoning. This is because what ultimately matters for the cost of debt is the real interest rate, which because of sticky prices, is endogenous and under the control of the central bank, and which we have seen before robustly depends negatively on the duration of debt. Therefore, to understand the overall effect on the government's inflation incentives, it is critical to analyze the targeting rule, as discussed above and given by (51). Here, the term $[\kappa^{-1}(1-\rho)\sigma^{-1}+\beta^{-1}]$ plays a key role as it captures the trade-off between π_t and $\hat{Y}_t$ vs. $\hat{T}_t$. Note first that β^{-1} here simply captures the role of surprise inflation in reducing the debt burden as government debt is nominal. This term would be present even when prices are completely flexible. The term $\kappa^{-1}(1-\rho)\sigma^{-1}$ however, appears because of sticky prices. This reflects how the real interest rate is affected by manipulation of the nominal interest rate and how it in turn affects output.

Thus, in situations where either κ^{-1}, σ^{-1}, or $(1-\rho)$ is high, this term can dominate and it can be the case that decreasing the duration (or decreasing ρ) actually leads to a lower π_b. This means, for example, that when κ^{-1} is low (or when prices are more flexible), the hump-shaped behavior of π_b gets restricted to very short maturities only as the channel coming from sticky-prices is not that influential. We in fact showed this above in the case of fully flexible prices, where inflation response depends negatively throughout on duration of debt.

8.6.4 Debt dynamics and duration of outstanding debt

While the primary focus so far is on properties of r_b as it determines the real interest rate incentives of the central bank, it is also interesting to consider the properties of b_b, the parameter governing the persistence of government debt. This exercise is interesting in its own right, but more importantly, it is also worth exploring because as explained before, what is critical is the behavior of the real interest rate, and that gets affected by b_b as $E_t\pi_{t+1}=\pi_b\hat{b}_t=\pi_b b_b\hat{b}_{t-1}$. Unlike for π_b, it is not possible to show a tractable analytical solution (or any property) for b_b as it is generally a root of a fourth-order polynomial equation. We thus rely fully on numerical solutions.

Fig. A5 shows how b_b depends on duration of debt. It is clear that the persistence of debt increases monotonically as the duration increases.[48] In fact, for a high enough duration, debt dynamics approach that of a random walk ($b_b = 1$), as in the analysis of Barro (1979). The persistence of debt increases with duration mainly because the response of the short-term real interest rate decreases, as we discussed above. Some of this effect is reflected in the response of inflation decreasing as also discussed above. Thus, the existence of long-term nominal debt has an important impact on the dynamics of debt under optimal policy under discretion.

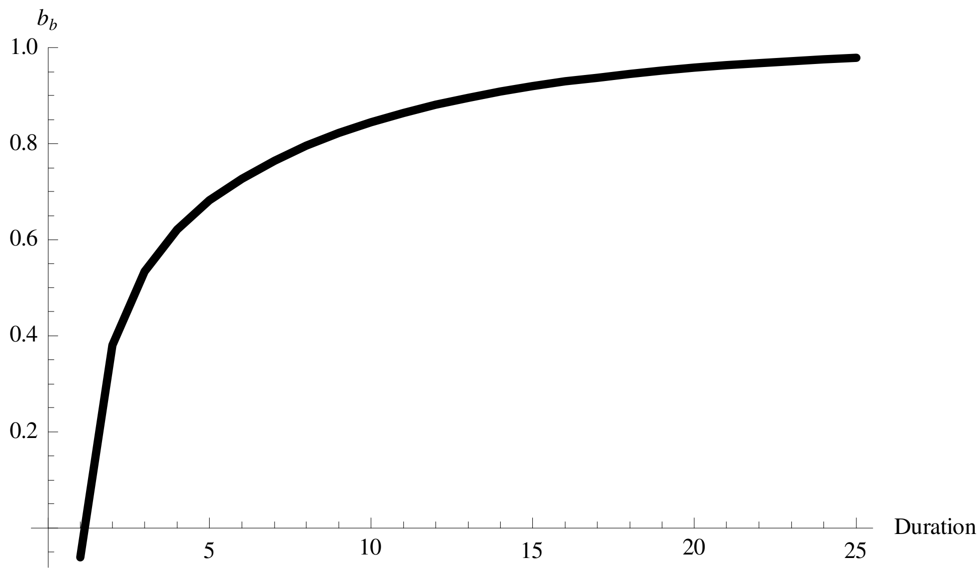

Fig A5: Debt response coefficients at different duration of debt

8.7 Computation at ZLB

In our experiment the debt is kept fixed at the zero lower bound at b_L . Moreover, at the zero lower bound, $\hat{\imath}_t = 1 - \beta^{-1}$. Given the specific assumptions on the two-state Markov shock process, the equilibrium is described by the system of equations

$$\begin{aligned}
\hat{Y}_t &= -\sigma\left(-\pi_b(1-\mu)\hat{b}_t - r_t^e - \mu\pi_L + \hat{\imath}_t\right) + (1-\mu)\hat{b}_t Y_b + \mu Y_L, \\
\hat{\pi}_t &= \beta\left(\pi_b(1-\mu)\hat{b}_t + \mu\pi_L\right) + \kappa Y_t, \\
\hat{\imath}_t &= 1 - \beta^{-1}, \\
\hat{b}_t &= b_L, \\
\hat{S}_t &= -\hat{\imath}_t + \rho\beta\left((1-\mu)S_b b_L + \mu S_L\right).
\end{aligned}$$

[48] There is thus, no hump-shaped pattern, unlike for inflation. The reason is that what matters directly for persistence of debt is the real interest rate and there is no hump-shaped pattern there as we show later. Note also that for one-period debt, often b_b is negative (even here, recall that π_b is still positive). A negative b_b is not very interesting empirically as it implies oscillatory behavior of debt.

where variables with a b subscript denote the solution we compute at positive interest rates while variables with a L subscript denote values at the ZLB. The solution to this system is

$$
\begin{aligned}
\hat{Y}_t = Y_L =& \frac{\beta\pi_b(\mu-1)b_L}{\kappa} - \frac{(1-\beta\mu)\left(\beta\pi_b(1-\mu)^2 b_L - \kappa\left(b_L\left(\pi_b(\mu-1)\sigma + (\mu-1)Y_b\right) - \sigma r_t^e + \sigma i_L\right)\right)}{\kappa(\kappa\mu\sigma - (1-\mu)(1-\beta\mu))}, \\
\hat{\pi}_t = \pi_L =& -\frac{\beta\pi_b(1-\mu)^2 b_L - \kappa\left(b_L\left(\pi_b(\mu-1)\sigma + (\mu-1)Y_b\right) - \sigma r_t^e + \sigma i_L\right)}{\kappa\mu\sigma - (1-\mu)(1-\beta\mu)}, \\
\hat{\imath}_t = i_L =& 1-\beta^{-1},\ \hat{b}_t = b_L, \\
\hat{S}_t = S_L =& \frac{\rho\beta(1-\mu)S_b b_L - \hat{\imath}_t}{1-\rho\beta\mu}
\end{aligned}
$$

where the solution for variables with a b subscript has already been provided above.

8.8 Non-linear Markov equilibrium of extended model

We now consider a model where the duration of government debt is time-varying and chosen optimally by the government.

The policy problem can be written as

$$
J\left(b_{t-1}, \rho_{t-1}, \xi_t\right) = \max\left[U\left(\Lambda_t, \xi_t\right) + \beta E_t J\left(b_t, \rho_t, \xi_{t+1}\right)\right]
$$

st

$$
\begin{gathered}
S_t(\rho_t)b_t = \left(1 + \rho_{t-1}W_t(\rho_{t-1})\right)b_{t-1}\Pi_t^{-1} + \left(F - T_t\right). \\
1 + i_t = \frac{u_C\left(C_t, \xi_t\right)}{\beta f_t^e} \\
i_t \geq 0 \\
S_t(\rho_t) = \frac{1}{u_C\left(C_t, \xi_t\right)}\beta g_t^e \\
W_t(\rho_{t-1}) = \frac{1}{u_C\left(C_t, \xi_t\right)}\beta j_t^e \\
\varepsilon Y_t\left[\frac{\varepsilon-1}{\varepsilon}\left(1+s\right)u_C\left(C_t, \xi_t\right) - \tilde{v}_y\left(Y_t, \xi_t\right)\right] + u_C\left(C_t, \xi_t\right)d'\left(\Pi_t\right)\Pi_t = \beta h_t^e \\
Y_t = C_t + F + d\left(\Pi_t\right) \\
f_t^e = E_t\left[u_C\left(C_{t+1}, \xi_{t+1}\right)\Pi_{t+1}^{-1}\right] = \bar{f}^e\left(b_t, \rho_t, \xi_t\right) \\
g_t^e = E_t\left[u_C\left(C_{t+1}, \xi_{t+1}\right)\Pi_{t+1}^{-1}\left(1 + \rho_t S_{t+1}(\rho_t)\right)\right] = \bar{g}^e\left(b_t, \rho_t, \xi_t\right) \\
h_t^e = E_t\left[u_C\left(C_{t+1}, \xi_{t+1}\right)d'\left(\Pi_{t+1}\right)\Pi_{t+1}\right] = \bar{h}^e\left(b_t, \rho_t, \xi_t\right) \\
\\
j_t^e = E_t\left[u_C\left(C_{t+1}, \xi_{t+1}\right)\Pi_{t+1}^{-1}\left(1 + \rho_{t-1}W_{t+1}(\rho_{t-1})\right)\right] = \bar{j}^e\left(b_t, \rho_t, \xi_t\right)
\end{gathered}
$$

Formulate the period Lagrangian

$$
\begin{aligned}
L_t = {} & u\left(C_t,\xi_t\right) + g\left(F - s(T_t - T)\right) - \tilde{v}\left(Y_t\right) + \beta E_t J\left(b_t,\rho_t,\xi_{t+1}\right) \\
& + \phi_{1t}\left(S_t(\rho_t)b_t - \left(1+\rho_{t-1}W_t(\rho_{t-1})\right)b_{t-1}\Pi_t^{-1} - (F - T_t)\right) \\
& + \phi_{2t}\left(\beta f_t^e - \frac{u_C\left(C_t,\xi_t\right)}{1+i_t}\right) \\
& + \phi_{3t}\left(\beta g_t^e - u_C\left(C_t,\xi_t\right)S_t(\rho_t)\right) \\
& + \phi_{4t}\left(\beta j_t^e - u_C\left(C_t,\xi_t\right)W_t(\rho_{t-1})\right) \\
& + \phi_{5t}\left(\beta h_t^e - \varepsilon Y_t\left[\frac{\varepsilon-1}{\varepsilon}\left(1+s\right)u_C\left(C_t,\xi_t\right) - \tilde{v}_y\left(Y_t,\xi_t\right)\right] - u_C\left(C_t,\xi_t\right)d'\left(\Pi_t\right)\Pi_t\right) \\
& + \phi_{6t}\left(Y_t - C_t - F - d\left(\Pi_t\right)\right) \\
& + \psi_{1t}\left(f_t^e - \bar{f}^e\left(b_t,\rho_t,\xi_t\right)\right) \\
& + \psi_{2t}\left(g_t^e - \bar{g}^e\left(b_t,\rho_t,\xi_t\right)\right) \\
& + \psi_{3t}\left(h_t^e - \bar{h}^e\left(b_t,\rho_t,\xi_t\right)\right) \\
& + \psi_{4t}\left(j_t^e - \bar{j}^e\left(b_t,\rho_t,\xi_t\right)\right) \\
& + \gamma_{1t}\left(i_t - 0\right)
\end{aligned}
$$

First-order conditions (where all the derivatives should be equated to zero)

$$
\begin{aligned}
\frac{\partial L_s}{\partial \Pi_t} &= \phi_{1t}\left[\left(1+\rho_{t-1}W_t(\rho_{t-1})\right)b_{t-1}\Pi_t^{-2}\right] + \phi_{5t}\left[-u_C d^{''}\Pi_t - u_C d^{'}\right] + \phi_{6t}\left[-d'\right] \\
\frac{\partial L_s}{\partial Y_t} &= -\tilde{v}_Y + \phi_{5t}\left[-\varepsilon\left(\frac{\varepsilon-1}{\varepsilon}\left(1+s\right)u_C\right) + \varepsilon Y_t\tilde{v}_{yy} + \varepsilon\tilde{v}_y\right] + \phi_{6t} \\
\frac{\partial L_s}{\partial i_t} &= \phi_{2t}\left[u_C\left(1+i_t\right)^{-2}\right] + \gamma_{1t} \\
\frac{\partial L_s}{\partial S_t} &= \phi_{1t}\left[b_t\right] + \phi_{3t}\left[-u_C\right] \\
\frac{\partial L_s}{\partial W_t} &= \phi_{1t}\left[\rho_{t-1}b_{t-1}\Pi_t^{-1}\right] + \phi_{4t}\left[-u_C\right] \\
\frac{\partial L_s}{\partial C_t} &= u_C + \phi_{2t}\left[-u_{CC}\left(1+i_t\right)^{-1}\right] + \phi_{3t}\left[-u_{CC}S_t(\rho_t)\right] + \phi_{4t}\left[-u_{CC}W_t(\rho_{t-1})\right] + \phi_{5t}\left[-\varepsilon Y_t\frac{\varepsilon-1}{\varepsilon}\left(1+s\right)u_{CC} - u_{CC}d'\Pi_t\right] + \phi_{6t}\left[-1\right] \\
\frac{\partial L_s}{\partial T_t} &= g_G\left(-s'(T_t - T)\right) + \phi_{1t} \\
\frac{\partial L_s}{\partial b_t} &= \beta E_t J_b\left(b_t,\rho_t,\xi_{t+1}\right) + \phi_{1t}\left[S_t(\rho_t)\right] + \psi_{1t}\left[-\bar{f}_b^e\right] + \psi_{2t}\left[-\bar{g}_b^e\right] + \psi_{3t}\left[-\bar{h}_b^e\right] + \psi_{4t}\left(-\bar{j}_b^e\right) \\
\frac{\partial L_s}{\partial \rho_t} &= \beta E_t J_\rho\left(b_t,\rho_t,\xi_{t+1}\right) \\
\frac{\partial L_s}{\partial f_t^e} &= \beta\phi_{2t} + \psi_{1t} \\
\frac{\partial L_s}{\partial g_t^e} &= \beta\phi_{3t} + \psi_{2t} \\
\frac{\partial L_s}{\partial j_t^e} &= \beta\phi_{4t} + \psi_{4t} \\
\frac{\partial L_s}{\partial h_t^e} &= \beta\phi_{5t} + \psi_{3t}
\end{aligned}
$$

The complementary slackness conditions are

$$\gamma_{1t} \geq 0,\ i_t \geq 0,\ \gamma_{1t}i_t = 0$$

While the envelope conditions are

$$J_b\left(b_{t-1}, \rho_{t-1}, \xi_t\right) = \phi_{1t}\left[-\left(1 + \rho_{t-1} W_t(\rho_{t-1})\right)\Pi_t^{-1}\right]$$

$$J_\rho\left(b_{t-1}, \rho_{t-1}, \xi_t\right) = \phi_{1t}\left[-W_t(\rho_{t-1}) b_{t-1} \Pi_t^{-1}\right].$$